Banach J. Math. Anal. 11 (2017), no. 2, 311–334
http://dx.doi.org/10.1215/17358787-0000005X
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

COMPOSITION OPERATORS ON THE BLOCH SPACE OF THE UNIT BALL OF A HILBERT SPACE

OSCAR BLASCO,[1] PABLO GALINDO,[1]
MIKAEL LINDSTRÖM,[2] and ALEJANDRO MIRALLES[3*]

Communicated by K. Jarosz

Abstract. Every analytic self-map of the unit ball of a Hilbert space induces a bounded composition operator on the space of Bloch functions. Necessary and sufficient conditions for compactness of such composition operators are provided, as well as some examples that clarify the connections among such conditions.

1. Introduction

Let E be a complex Hilbert space of arbitrary dimension, and denote B_E its open unit ball. The space $\mathcal{B}(B_E)$ of Bloch functions was introduced in [1]. There it was shown that it can be endowed with a (modulo the constant functions) norm that is invariant under the automorphisms of B_E (see Section 3 below for the basics). This article studies composition operators acting on $\mathcal{B}(B_E)$, that is, self-maps of $\mathcal{B}(B_E)$ defined according to $C_\varphi(f) = f \circ \varphi$ for a given analytic map $\varphi : B_E \to B_E$. As in the finite-dimensional case, every composition operator is bounded actually of norm not greater than 1 for the invariant norm if the symbol vanishes at 0, and also the hyperbolic metric on B_E measures the distance between evaluations in the dual space. We also study the compactness of composition operators providing necessary and sufficient conditions. There are two common

---

Received Mar. 14, 2016; Accepted May 30, 2016.
[*] Corresponding author.
2010 *Mathematics Subject Classification.* Primary 30D45; Secondary 46E50, 46G20.
Keywords. composition operator, Bloch function in the ball, infinite-dimensional holomorphy.

requirements for both the necessity and the sufficiency:

$$\lim_{\|\varphi(z)\|\to 1} \frac{(1-\|z\|^2)\|\mathcal{R}\varphi(z)\|}{\sqrt{1-\|\varphi(z)\|^2}} = 0 \quad \text{and}$$

$$\lim_{\|\varphi(z)\|\to 1} \frac{(1-\|z\|^2)|\langle \varphi(z), \mathcal{R}\varphi(z)\rangle|}{1-\|\varphi(z)\|^2} = 0,$$

where $\mathcal{R}\varphi(z)$ denotes the radial derivative at z. The fact that, for all $0 < \delta < 1$, $\varphi(\delta B_E)$ and $\{(1-\|z\|^2)\mathcal{R}\varphi(z) : z \in B_E\}$ are relatively compact completes a necessary condition, while the additional assumption of $\varphi(B_E) \cap \delta B_E$ and $\{(1-\|z\|^2)\mathcal{R}\varphi(z) : \|\varphi(z)\| < \delta\}$ being relatively compact provides a sufficient one. Such compactness requirements are trivially satisfied in the finite-dimensional case, and thus the two limits above yield an apparently new characterization.

Some of our techniques are inspired by Dai's paper [4]. However, there are some obstacles to avoid when allowing an infinite number of variables, such as the lack of relative compactness of the ball, the number of components of the symbol, or the use of the invariant Laplacian, and still a major one: uniform convergence on compact sets does not imply uniform convergence on compact sets of the derivatives; this only happens in the finite-dimensional setting (see [3]). Such an obstacle causes the lengthy proof of our main result Theorem 4.13. In the final section we present several examples that discuss the relations among the conditions we have found.

2. BACKGROUND

Let $(e_k)_{k\in\Gamma}$ be an orthonormal basis of E that we fix at once. Then every $z \in E$ can be written as $z = \sum_{k\in\Gamma} z_k e_k$, and we write $\overline{z} = \sum_{k\in\Gamma} \overline{z_k} e_k$.

Given an analytic function $\varphi : B_E \to B_E$, we write $\varphi(x) = \sum_{k\in\Gamma} \varphi_k(x) e_k$, $\varphi'(x) : E \to E$ its derivative at x, and $\mathcal{R}\varphi(x) = \varphi'(x)(x)$ its radial derivative at x.

We denote by φ_a the Möbius transforms for Hilbert spaces. For each $a \in B_E$, $\varphi_a : B_E \to B_E$ is defined by

$$\varphi_a(x) = (s_a Q_a + P_a)\big(m_a(x)\big),$$

where $s_a = \sqrt{1-\|a\|^2}$, $m_a : B_E \to B_E$ is the analytic function

$$m_a(x) = \frac{a-x}{1-\langle x, a\rangle}$$

and $P_a = \frac{1}{\|a\|^2} a \otimes a$, where $u \otimes v(x) = \langle x, u\rangle v$ and $Q_a = \mathrm{Id} - P_a$ are the orthogonal projection on the 1-dimensional subspace generated by a and on its orthogonal complement, respectively. Since $\varphi_a \circ \varphi_a(x) = x$, one has $(\varphi_a)^{-1} = \varphi_a$ and $\varphi_a'(a) = (\varphi_a'(0))^{-1}$.

Actually,

$$\varphi_a'(0) = -s_a^2 P_a - s_a Q_a, \tag{2.1}$$

and

$$\varphi_a'(a) = -\frac{1}{s_a^2} P_a - \frac{1}{s_a} Q_a \tag{2.2}$$

(see, for instance, [1, Lemma 3.2]). The pseudohyperbolic and hyperbolic metrics on B_E are respectively defined by

$$\rho_E(x,y) := \big\|\varphi_x(y)\big\| \quad \text{and} \quad \beta_E(x,y) := \frac{1}{2}\log\frac{1+\rho_E(x,y)}{1-\rho_E(x,y)}.$$

It is known (see [6, p. 99]) that

$$\big\|\varphi_x(y)\big\|^2 = 1 - \frac{(1-\|x\|^2)(1-\|y\|^2)}{|1-\langle x,y\rangle|^2}. \tag{2.3}$$

Also,

$$\rho_E(x,y) = \sup\big\{\rho\big(f(x),f(y)\big) : f\in H^\infty(B_E), \|f\|_\infty \le 1\big\}, \tag{2.4}$$

where ρ is the pseudohyperbolic metric on the open unit disk $\mathbb{D}$ in the complex plane given by $\rho(z,w) = |\frac{z-w}{1-\bar{z}w}|$ and $H^\infty(B_E)$ denotes the Banach space of bounded analytic functions on B_E endowed with the sup-norm.

Since $(s+t)/(1+st)$ is an increasing function of s and t for $0 \le s,\, t \le 1$, the sharpened form of the triangle inequality for $\rho(z,w)$ easily yields the same inequality for $\rho_E(x,y)$:

$$\rho_E(x,y) \le \frac{\rho_E(x,u)+\rho_E(u,y)}{1+\rho_E(x,u)\rho_E(u,y)}, \quad x,u,y\in B_E. \tag{2.5}$$

The following estimate holds (see [1, Lemma 4.1]):

$$\rho_E(x,y) \le \frac{\|x-y\|}{|1-\langle x,y\rangle|}, \quad x,y\in B_E. \tag{2.6}$$

The open unit ball of $H^\infty(B_E)$ is invariant under postcomposition with conformal self-maps of $\mathbb{D}$. By composing f with a conformal self-map of $\mathbb{D}$ that maps $f(y)$ to 0, one obtains

$$\rho_E(x,y) = \sup\big\{\big|f(x)\big| : f\in H^\infty(B_E), \|f\|_\infty\le 1, f(y)=0\big\}. \tag{2.7}$$

Recall that if $f: B_E\to\mathbb{C}$ is analytic, then we have $f'(x)(y) = \langle y, \overline{\nabla f(x)}\rangle$ and $(f\circ\varphi_x)'(0)(y) = \langle y, \overline{\widetilde{\nabla} f(x)}\rangle$, where $\widetilde{\nabla} f(x)$ denotes the invariant gradient of f at $x\in B_E$ given by

$$\widetilde{\nabla} f(x) = \nabla(f\circ\varphi_x)(0).$$

The following result gives an explicit formula to compute the invariant gradient. It is a modification of Lemma 3.5 in [1] in a form that fits our purposes.

Lemma 2.1. *Let $f: B_E\to\mathbb{C}$ be an analytic function, and let $x\in B_E$. Then*

$$\big\|\widetilde{\nabla} f(x)\big\| = \sup_{w\neq 0}\frac{|\langle\nabla f(x),\overline{w}\rangle|(1-\|x\|^2)}{\sqrt{(1-\|x\|^2)\|w\|^2+|\langle w,x\rangle|^2}}. \tag{2.8}$$

Proof. For the linear functional $w\in E\mapsto\langle\varphi_x'(0)(w),\overline{\nabla f(x)}\rangle$, we have

$$\big\|\widetilde{\nabla} f(x)\big\| = \sup_{w\neq 0}\frac{|\langle\varphi_x'(0)(w),\overline{\nabla f(x)}\rangle|}{\|w\|} = \sup_{w\neq 0}\frac{|\langle\nabla f(x),\overline{\varphi_x'(0)(w)}\rangle|}{\|w\|}.$$

Now we can replace w by $\varphi_x'(0)^{-1}(w)$ in the above formula, and so

$$\|\widetilde{\nabla} f(x)\| = \sup_{w\neq 0} \frac{|\langle \nabla f(x), \overline{w}\rangle|}{\|\varphi_x'(0)^{-1}(w)\|}.$$

In the proof of Lemma 3.5 in [1], it is shown that

$$\|\varphi_x'(0)^{-1}(w)\| = \frac{\sqrt{(1-\|x\|^2)\|w\|^2 + |\langle w, x\rangle|^2}}{1-\|x\|^2},$$

and so the statement follows. □

Throughout this article, $\varphi : B_E \to B_E$ denotes an analytic map, and, given $y \in E \setminus \{0\}$ and $w \in E$ with $\|w\| \leq 1$, we write

$$\varphi_{y,w}(\lambda) = \Big\langle \varphi\Big(\lambda \frac{y}{\|y\|}\Big), \overline{w}\Big\rangle, \quad |\lambda| < 1. \tag{2.9}$$

The following version of the Schwarz–Pick lemma will be needed later. The analogue of these results in several variables was proved in [2].

Lemma 2.2. *Let $\varphi : B_E \to B_E$ be an analytic map, and let $y \in B_E$. Then*

$$\big|\big\langle \mathcal{R}\varphi(y), \varphi(y)\big\rangle\big| \leq \|y\|\|\varphi(y)\| \frac{1-\|\varphi(y)\|^2}{1-\|y\|^2}, \tag{2.10}$$

$$\frac{(1-\|y\|^2)}{\|y\|}\|\mathcal{R}\varphi(y)\| + \|\varphi(y)\|^2 \Big|\Big\langle \frac{\mathcal{R}\varphi(y)}{\|\mathcal{R}\varphi(y)\|}, \frac{\varphi(y)}{\|\varphi(y)\|}\Big\rangle\Big|^2 \leq 1, \tag{2.11}$$

$$\|\mathcal{R}\varphi(y)\| \leq 2\frac{(1-\|\varphi(y)\|^2)^{1/2}}{1-\|y\|^2}, \tag{2.12}$$

Furthermore, if $\varphi(0) = 0$, *then* $\|\varphi(y)\| \leq \|y\|$. (2.13)

Proof. Let us fix $y \in B_E \setminus \{0\}$, $\varphi(y) \neq 0$, and $w \in E$ with $\|w\| \leq 1$. We apply the classical Schwarz lemma to $\varphi_{y,w}$, and we get for any $|\lambda| < 1$ that

$$|\varphi_{y,w}'(\lambda)| \leq \frac{1-|\varphi_{y,w}(\lambda)|^2}{1-|\lambda|^2}.$$

Now if $\lambda \neq 0$, then we have $\varphi_{y,w}'(\lambda) = \frac{1}{\lambda}\langle \mathcal{R}\varphi(\lambda\frac{y}{\|y\|}), \overline{w}\rangle$. Hence, for $\lambda = \|y\|$, it follows that

$$\big|\big\langle \mathcal{R}\varphi(y), \overline{w}\big\rangle\big| \leq \|y\| \frac{1-|\langle \varphi(y), \overline{w}\rangle|^2}{1-\|y\|^2}.$$

This shows (2.10) and (2.11) by choosing $w = \frac{\overline{\varphi(y)}}{\|\varphi(y)\|}$ and $w = \frac{\overline{\mathcal{R}\varphi(z)}}{\|\mathcal{R}\varphi(z)\|}$, respectively.

To get (2.12), we use the estimate

$$\big|\big\langle \mathcal{R}\varphi(y), \overline{w}\big\rangle\big| \leq 2\|y\| \frac{1-|\langle \varphi(y), \overline{w}\rangle|}{1-\|y\|^2}.$$

In particular, for any $\theta \in [-\pi, \pi)$ and $\|w\| = 1$, we see that

$$\Big|\Big\langle \frac{(1-\|y\|^2}{2\|y\|}\mathcal{R}\varphi(y) + e^{i\theta}\varphi(y), \overline{w}\Big\rangle\Big| \leq \frac{1-\|y\|^2}{2\|y\|}\big|\big\langle \mathcal{R}\varphi(y), \overline{w}\big\rangle\big| + \big|\big\langle \varphi(y), \overline{w}\big\rangle\big| \leq 1.$$

Hence

$$\left\| \frac{1-\|y\|^2}{2\|y\|}\mathcal{R}\varphi(y) + e^{i\theta}\varphi(y) \right\| \leq 1 \quad \text{for } \theta \in [-\pi, \pi).$$

Now, integrating over θ, we obtain

$$\frac{(1-\|y\|^2)^2}{4\|y\|^2}\left\|\mathcal{R}\varphi(y)\right\|^2 + \left\|\varphi(y)\right\|^2 \leq 1.$$

In the case $\varphi(0) = 0$ using $\varphi_{y,w}(0) = 0$ and the scalar Schwarz lemma, we obtain

$$\left|\varphi_{y,w}(\lambda)\right| \leq |\lambda|$$

for all $y \in B_E \setminus \{0\}$, $\varphi(y) \neq 0$, and $w \in E$. This implies (2.13) choosing again $\lambda = \|y\|$ and $w = \frac{\overline{\varphi(y)}}{\|\varphi(y)\|}$. This completes the proof. □

For background on analytic (or holomorphic) mappings on infinite-dimensional complex spaces, we refer to [3].

3. The Bloch space

The classical Bloch space $\mathcal{B}$ is the space of analytic functions on the open unit disk $f : \mathbb{D} \to \mathbb{C}$ such that the seminorm $\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1-|z|^2)|f(z)|$ is bounded; it becomes a Banach space when endowed with the norm $\|f\|_{\text{Bloch}} = |f(0)|+\|f\|_{\mathcal{B}}$ (see [10] for general background on the classical Bloch space). The Bloch space of functions defined on the finite-dimensional Euclidean ball was introduced by Timoney in [8] (see [9] for further information).

A function $f : B_E \to \mathbb{C}$ is a Bloch function if

$$\|f\|_{\mathcal{B}(B_E)} = \sup_{x\in B_E}\left(1-\|x\|^2\right)\left\|f'(x)\right\| < \infty.$$

The space of Bloch functions is denoted by $\mathcal{B}(B_E)$, and it has been studied in [1]. As in the finite-dimensional case, the space $H^\infty(B_E)$ is strictly contained in $\mathcal{B}(B_E)$ (see [1, Corollary 4.3]), and the following inequality holds for any $f \in H^\infty(B_E)$:

$$\|f\|_{\mathcal{B}(B_E)} \leq \|f\|_\infty. \tag{3.1}$$

An equivalent seminorm for the space of Bloch functions is given by

$$\|f\|_{\text{inv}} = \sup_{x\in B_E}\left\|\widetilde{\nabla}f(x)\right\| < \infty.$$

This seminorm satisfies $\|f\circ\varphi\|_{\text{inv}} = \|f\|_{\text{inv}}$ for any $f \in \mathcal{B}(B_E)$ and any automorphism φ of B_E. The space $\mathcal{B}(B_E)$ is usually endowed with the norm $\|f\|_{\text{Bloch}(B_E)} = |f(0)| + \|f\|_{\text{inv}}$, and then it becomes a Banach space.

Another equivalent seminorm is given by

$$\|f\|_{\mathrm{rad}} = \sup_{x\in B_E} \big(1-\|x\|^2\big)\big|\mathcal{R}f(x)\big|,$$

where $\mathcal{R}f(x) = f'(x)(x)$ is the radial derivative of f at x.

We refer to [1, Theorem 3.8] for all the equivalences of these seminorms. In particular, we have the following inequalities:

$$\|f\|_{\mathcal{B}(B_E)} \le \|f\|_{\mathrm{inv}} \le \Big(1+\frac{\sqrt{31}}{2}\Big)\|f\|_{\mathcal{B}(B_E)}. \tag{3.2}$$

The following result extends Theorem 5.5 in [10] to an infinite-dimensional Hilbert space E.

Theorem 3.1. *Let $f : B_E \to \mathbb{C}$ be an analytic function. Then*

$$\|f\|_{\mathrm{inv}} = \sup\Big\{\frac{|f(x)-f(y)|}{\beta_E(x,y)} : x,y\in B_E, x\ne y\Big\}.$$

Proof. First, we prove that

$$\|f\|_{\mathrm{inv}} \ge M := \sup\Big\{\frac{|f(x)-f(y)|}{\beta_E(x,y)} : x,y\in B_E, x\ne y\Big\}.$$

If $\|f\|_{\mathrm{inv}} = \infty$, then we are done, and so take $f\in\mathcal{B}(B_E)$ and $x,y\in B_E$. Then

$$\begin{aligned}\big|f(x)-f(0)\big| &= \Big|\Big(\int_0^1 f'(xt)\,dt\Big)(x)\Big| \le \|x\|\Big\|\int_0^1 \frac{f'(xt)(1-\|xt\|^2)}{1-\|xt\|^2}\,dt\Big\| \\ &\le \|f\|_{\mathcal{B}(B_E)}\int_0^1 \frac{\|x\|}{1-\|x\|^2|t|^2}\,dt = \|f\|_{\mathcal{B}(B_E)}\frac{1}{2}\log\frac{1+\|x\|}{1-\|x\|}.\end{aligned}$$

Consider $f\circ\varphi_y\in\mathcal{B}(B_E)$. By the inequality above, (3.2), and bearing in mind that $\|f\circ\varphi\|_{\mathrm{inv}} = \|f\|_{\mathrm{inv}}$ for any automorphism φ, we have

$$\big|f\circ\varphi_y(z) - f\circ\varphi_y(0)\big| \le \|f\circ\varphi_y\|_{\mathrm{inv}}\frac{1}{2}\log\frac{1+\|z\|}{1-\|z\|} = \|f\|_{\mathrm{inv}}\frac{1}{2}\log\frac{1+\|z\|}{1-\|z\|}.$$

Selecting $z=\varphi_y(x)$, we have

$$\begin{aligned}\big|f(x)-f(y)\big| &\le \|f\circ\varphi_y\|_{\mathrm{inv}}\frac{1}{2}\log\frac{1+\|\varphi_y(x)\|}{1-\|\varphi_y(x)\|} \\ &= \|f\|_{\mathrm{inv}}\frac{1}{2}\log\frac{1+\rho_E(x,y)}{1-\rho_E(x,y)} = \|f\|_{\mathrm{inv}}\beta_E(x,y).\end{aligned}$$

Hence $\|f\|_{\mathrm{inv}}\ge M$.

Now we prove that $\|f\|_{\mathrm{inv}}\le M$. Notice that

$$\big|f(x)-f(0)\big| \le M\beta_E(x,0) = \frac{M}{2}\log\frac{1+\|x\|}{1-\|x\|},$$

and so

$$\frac{|f(x)-f(0)|}{\|x\|} \le \frac{M}{2\|x\|}\log\frac{1+\|x\|}{1-\|x\|}$$

for all $x \in B_E \backslash \{0\}$. For a unit vector $u \in E$, we consider the directional derivative $D_u f(0)$ given by

$$D_u f(0) = \lim_{t \to 0} \frac{f(0+tu) - f(0)}{t} = \nabla f(0)(u).$$

If $x = tu$, and by taking limits when $\|x\| \to 0$, we have

$$\big|\nabla f(0)(u)\big| \leq M \lim_{\|x\| \to 0} \frac{1}{2\|x\|} \log \frac{1+\|x\|}{1-\|x\|} = M$$

since $\lim_{r \to 0} \frac{1}{r} \log \frac{1+r}{1-r} = 2$, and so $\|\nabla f(0)\| \leq M$. Notice that, for any automorphism φ on B_E, it is clear that

$$M = \sup\Big\{ \frac{|(f \circ \varphi)(x) - (f \circ \varphi)(y)|}{\beta_E(x,y)} : x, y \in B_E, x \neq y \Big\}$$

since $\beta_E(\varphi(x), \varphi(y)) = \beta(x,y)$. Hence, for any $x \in B_E$, we have

$$\|f\|_{\text{inv}} = \sup_{x \in B_E} \big\|\nabla(f \circ \varphi_x)(0)\big\| \leq M,$$

and we are done. □

Corollary 3.2. *If $\delta_x(f) = f(x)$, then we have that $\delta_x \in \mathcal{B}(B_E)^*$ and $\|\delta_x\| \leq L_x$, where*

$$L_x = \max\Big\{ \frac{1}{2} \log \frac{1+\|x\|}{1-\|x\|}, 1 \Big\}.$$

Proof. From Theorem 3.1, we have for any $x \in B_E$

$$\big|f(x) - f(0)\big| \leq \frac{1}{2} \|f\|_{\mathcal{B}(B_E)} \log \frac{1+\|x\|}{1-\|x\|}. \tag{3.3}$$

Also,

$$\begin{aligned} \big|\delta_x(f)\big| &\leq \big|f(x) - f(0)\big| + \big|f(0)\big| \leq \frac{1}{2} \|f\|_{\mathcal{B}(B_E)} \log \frac{1+\|x\|}{1-\|x\|} + \big|f(0)\big| \\ &\leq \max\Big\{ \frac{1}{2} \log \frac{1+\|x\|}{1-\|x\|}, 1 \Big\} \big(\|f\|_{\mathcal{B}(B_E)} + \big|f(0)\big| \big) = L_x \|f\|_{\text{Bloch}(B_E)}. \end{aligned}$$

□

Remark 3.3. For $x, y \in B_E$, we have

$$\frac{1}{2}\|x-y\| \leq \rho_E(x,y) \leq \|\delta_x - \delta_y\| \leq \beta_E(x,y). \tag{3.4}$$

In particular, we observe that the norm topology of $\mathcal{B}(B_E)$ is finer than the compact open topology *co*.

As consequence of Theorem 3.1, we have the following.

Corollary 3.4. *An analytic function $f : B_E \to \mathbb{C}$ belongs to $\mathcal{B}(B_E)$ if and only if there exists a constant $C > 0$ such that*

$$\big|f(x) - f(y)\big| \leq C \beta_E(x,y).$$

Notice that the metric $\beta_E(x,y)$ can be also recovered from the Bloch seminorm $\|f\|_{\text{inv}}$.

Corollary 3.5. *For any $x, y \in B_E$, we have*

$$\beta_E(x,y) = \sup\big\{\big|f(x) - f(y)\big| : \|f\|_{\mathrm{inv}} \le 1\big\}.$$

Proof. By Theorem 3.1 we have $|f(x) - f(y)| \le \|f\|_{\mathrm{inv}}\beta_E(x,y)$ for all $f \in \mathcal{B}(B_E)$ and $x, y \in B_E$. Hence $\sup\{|f(x) - f(y)| : \|f\|_{\mathrm{inv}} \le 1\} \le \beta_E(x,y)$.

To check the other inequality, follow the same pattern as in Theorem 3.9 in [9], and recall [1, Lemma 3.3]. □

4. Composition operators

4.1. Boundedness.

As it occurs in the finite-dimensional case, every composition operator on $\mathcal{B}(B_E)$ is bounded.

Theorem 4.1. *Every analytic map $\varphi : B_E \to B_E$ induces a bounded composition operator $C_\varphi : \mathcal{B}(B_E) \to \mathcal{B}(B_E)$.*

Proof. Let $\varphi : B_E \to B_E$ be analytic, and consider for any $f \in \mathcal{B}(B_E)$ the seminorm $\|f \circ \varphi\|_{\mathrm{inv}}$. By Theorem 3.1, we have

$$\begin{aligned}\|f \circ \varphi\|_{\mathrm{inv}} &= \sup\Big\{\frac{|(f \circ \varphi)(x) - (f \circ \varphi)(y)|}{\beta_E(x,y)} : x, y \in B_E, x \ne y\Big\} \\ &\le \sup\Big\{\frac{|(f(\varphi(x)) - (f(\varphi(y))|}{\beta_E(\varphi(x), \varphi(y))} : x, y \in B_E, \varphi(x) \ne \varphi(y)\Big\},\end{aligned}$$

where the last inequality holds because $\rho_E(x,y)$ is contractive for analytic maps $\varphi : B_E \to B_E$ and $h(t) = \frac{1}{2}\log\frac{1+t}{1-t}$ is nondecreasing. Since $\varphi(B_E) \subset B_E$, we get the estimate

$$\|f \circ \varphi\|_{\mathrm{inv}} \le \sup\Big\{\frac{|f(x) - f(y)|}{\beta_E(x,y)} : x, y \in B_E, x \ne y\Big\} = \|f\|_{\mathrm{inv}}.$$

Further, using Corollary 3.2,

$$\begin{aligned}\big\|C_\varphi(f)\big\|_{\mathrm{Bloch}(B_E)} &= \|f \circ \varphi\|_{\mathrm{inv}} + \big|f\big(\varphi(0)\big)\big| \le \|f\|_{\mathrm{inv}} + L_{\varphi(0)}\|f\|_{\mathrm{Bloch}(B_E)} \\ &\le \|f\|_{\mathrm{inv}} + \big|f(0)\big| + L_{\varphi(0)}\|f\|_{\mathrm{Bloch}(B_E)} = (1 + L_{\varphi(0)})\|f\|_{\mathrm{Bloch}(B_E)},\end{aligned}$$

and we conclude that C_φ is bounded. □

We provide another proof that relies on magnitudes that will appear further on.

Proof. Let $\|f\|_{\mathrm{inv}} \le 1$. Since $\mathcal{R}(f \circ \varphi)(z) = \langle \nabla f(\varphi(z)), \overline{\mathcal{R}\varphi(z)}\rangle$, we use Lemma 2.1, and obtain

$$\big|\mathcal{R}(f \circ \varphi)(z)\big|^2 \le \big\|\widetilde{\nabla} f(\varphi(z)\big\|^2 \frac{(1 - \|\varphi(z)\|^2)\|\mathcal{R}\varphi(z)\|^2 + |\langle \mathcal{R}\varphi(z), \varphi(z)\rangle|^2}{(1 - \|\varphi(z)\|^2)^2}.$$

By combining this with Lemma 2.2, we conclude that

$$\big|\mathcal{R}(f \circ \varphi)(z)\big|\big(1 - \|z\|^2\big) \le \sqrt{5}.$$

Thus the boundedness of C_φ is immediate if we assume that $\varphi(0) = 0$.

If $\varphi(0) = x \neq 0$, then we consider the mapping $\psi = \varphi_x \circ \varphi$, for which $\psi(0) = 0$, and the bounded operator C_ψ. Since $\|f \circ \varphi_x\|_{\mathrm{inv}} = \|f\|_{\mathrm{inv}}$, it follows using Corollary 3.2 as well that C_{φ_x} is continuous. Hence $C_\varphi = C_\psi \circ C_{\varphi_x}$ is continuous. $\square$

Remark 4.2. It is clear that if $\varphi(0) = 0$, then $\|C_\varphi\| \leq 1$.

4.2. **Compactness.** Now we proceed to discuss necessary and sufficient conditions for a composition operator on $\mathcal{B}(B_E)$ to be compact. We begin with some necessary ones.

Recall that $H(\mathbb{D})$ denotes the space of analytic functions $f : \mathbb{D} \to \mathbb{C}$ and $H(B_E)$ denotes the space of analytic functions $f : B_E \to \mathbb{C}$.

4.2.1. Necessary conditions. The following result is a little improvement of a result due to Dai [4, proof of Theorem 3.2] for finitely many variables. From now on, $\varphi : B_E \to B_E$ is a fixed analytic map.

Lemma 4.3. *For each $z \in B_E$ with $\varphi(z) \neq 0$, there is $\eta(z) \in E$, $\|\eta(z)\| = 1$ with $\langle \varphi(z), \eta(z) \rangle = 0$ such that, for $\xi = \varphi(z) + \sqrt{1 - \|\varphi(z)\|^2}\eta(z)$, one has*

$$\big|\langle \mathcal{R}\varphi(z), \xi \rangle\big| \geq \sqrt{1 - \big\|\varphi(z)\big\|^2}\big\|\mathcal{R}\varphi(z)\big\| - \Big(1 + \frac{\sqrt{1 - \|\varphi(z)\|^2}}{\|\varphi(z)\|}\Big)\big|\langle \mathcal{R}\varphi(z), \varphi(z) \rangle\big|.$$

Proof. We use the projection theorem for Hilbert spaces, and so for each $z \in B_E$ with $\varphi(z) \neq 0$ there is $\eta(z) \in E$, $\|\eta(z)\| = 1$ with $\langle \varphi(z), \eta(z) \rangle = 0$ such that

$$\mathcal{R}\varphi(z) = \alpha \frac{\varphi(z)}{\|\varphi(z)\|} + \beta \eta(z),$$

where $\alpha = \frac{\langle \mathcal{R}\varphi(z), \varphi(z) \rangle}{\|\varphi(z)\|}$ and $\beta = \langle \mathcal{R}\varphi(z), \eta(z) \rangle$. Clearly, $\|\xi\| = 1$, $\langle \varphi(z), \xi \rangle = \|\varphi(z)\|^2$, and $\langle \mathcal{R}\varphi(z), \xi \rangle = \langle \mathcal{R}\varphi(z), \varphi(z) \rangle + \sqrt{1 - \|\varphi(z)\|^2}\beta$. Moreover, $|\alpha|^2 + |\beta|^2 = \|\mathcal{R}\varphi(z)\|^2$, and so

$$\begin{aligned}
\big|\langle \mathcal{R}\varphi(z), \xi \rangle\big| &\geq \sqrt{1 - \big\|\varphi(z)\big\|^2}|\beta| - \big|\langle \mathcal{R}\varphi(z), \varphi(z) \rangle\big| \\
&\geq \sqrt{1 - \big\|\varphi(z)\big\|^2}\big(\big|\mathcal{R}\varphi(z)\big| - |\alpha|\big) - \big|\langle \mathcal{R}\varphi(z), \varphi(z) \rangle\big| \\
&= \sqrt{1 - \big\|\varphi(z)\big\|^2}\big\|\mathcal{R}\varphi(z)\big\| \\
&\quad - \Big(1 + \frac{\sqrt{1 - \|\varphi(z)\|^2}}{\|\varphi(z)\|}\Big)\big|\langle \mathcal{R}\varphi(z), \varphi(z) \rangle\big|.
\end{aligned}$$ $\square$

Lemma 4.4. *The composition operator $C_\varphi : \mathcal{B}(B_E) \to \mathcal{B}(B_E)$ is compact if and only if for each bounded net (f_α) in $\mathcal{B}(B_E)$ such that $f_\alpha \to 0$ in $(\mathcal{B}(B_E), co)$ it follows that $\|C_\varphi(f_\alpha)\|_{\mathcal{B}(B_E)} \to 0$.*

Proof. Suppose that $C_\varphi : \mathcal{B}(B_E) \to \mathcal{B}(B_E)$ is compact, and let (f_α) be a bounded net in $\mathcal{B}(B_E)$ such that $f_\alpha \to 0$ in $(\mathcal{B}(B_E), co)$. Then also $C_\varphi(f_\alpha) \to 0$ in $(\mathcal{B}(B_E), co)$, and the norm closure of the set $\{C_\varphi(f_\alpha), 0\}$ is compact in $\mathcal{B}(B_E)$. Therefore, $\|C_\varphi(f_\alpha)\|_{\mathcal{B}(B_E)} \to 0$.

If C_φ is noncompact, then there are $\varepsilon > 0$ and a sequence (f_n) in $\mathcal{B}(B_E)$ such that $\|f_n\|_{\mathcal{B}(B_E)} = 1$ and

$$\big\|C_\varphi(f_n) - C_\varphi(f_m)\big\|_{\mathcal{B}(B_E)} \geq \varepsilon \quad \text{for each } n \neq m.$$

Now, by Montel's theorem (see [3, Theorem 17.21]), there is a subnet $(f_{n(\alpha)})$ of (f_n) that converges uniformly on compact subsets of B_E in $H(B_E)$. For each $n(\alpha)$, choose $n(\beta) > n(\alpha)$ such that $f_{n(\alpha)} \neq f_{n(\beta)}$, and let $g_{n(\alpha)} = f_{n(\alpha)} - f_{n(\beta)}$. Then $g_{n(\alpha)} \to 0$ in $(\mathcal{B}(B_E), co)$, but $\|C_\varphi(g_{n(\alpha)})\|_{\mathcal{B}(B_E)} \geq \varepsilon > 0$. □

Theorem 4.5. *Assume that $C_\varphi : \mathcal{B}(B_E) \to \mathcal{B}(B_E)$ is a compact operator. Then*

$$\varphi(\delta B_E) \text{ is relatively compact for each } 0 < \delta < 1, \tag{4.1}$$

$$\lim_{\|\varphi(z)\|\to 1} \frac{(1-\|z\|^2)\|\mathcal{R}\varphi(z)\|}{\sqrt{1-\|\varphi(z)\|^2}} = 0, \quad \text{and} \tag{4.2}$$

$$\lim_{\|\varphi(z)\|\to 1} \frac{(1-\|z\|^2)|\langle \varphi(z), \mathcal{R}\varphi(z)\rangle|}{1-\|\varphi(z)\|^2} = 0. \tag{4.3}$$

Proof. First we prove (4.1). Indeed, since the set $\{\delta_z : \|z\| \leq \delta\} \subset (\mathcal{B}(B_E))^*$ is bounded and C_φ^* is compact, $\{C_\varphi^*(\delta_z) : \|z\| \leq \delta\}$ is relatively compact in $\mathcal{B}(B_E)^*$. The fact that $C_\varphi^*(\delta_z) = \delta_{\varphi(z)}$ allows us to conclude that $\varphi(\delta B_E)$ is relatively compact by appealing to (3.4).

Let (n_k) be an increasing sequence in $\mathbb{N}$, and let (ξ_k) be a sequence in E with $\|\xi_k\| \leq 1$. According to [1, Corollary 4.3], the family $\{\langle z, \xi\rangle^{n_k} : \|\xi\| = 1\}$ is bounded in $\mathcal{B}(B_E)$. Furthermore, the resulting sequence $\{\langle z, \xi_k\rangle^{n_k}\}$ converges to zero in $(\mathcal{B}(B_E), co)$, and therefore the compactness of $C_\varphi : \mathcal{B}(B_E) \to \mathcal{B}(B_E)$ implies (according to Lemma 4.4) that

$$\lim_k \big\|\langle \varphi, \xi_k\rangle^{n_k}\big\|_{\text{rad}} \to 0 \quad \text{when } k \to \infty. \tag{4.4}$$

We have

$$\big\|\langle \varphi, \xi_k\rangle^{n_k}\big\|_{\text{rad}} = \sup_{z\in B_E} \big(1-\|z\|^2\big) n_k \big|\langle \varphi(z), \xi_k\rangle\big|^{n_k-1} \big|\langle \mathcal{R}\varphi(z), \xi_k\rangle\big|.$$

Let us first show (4.3). We suppose that there exist $\varepsilon > 0$ and a sequence $(z_k) \in B_E$ such that $\|\varphi(z_k)\| \to 1$ and, for each k,

$$\frac{1-\|z_k\|^2}{1-\|\varphi(z_k)\|^2} \big|\langle \varphi(z_k), \mathcal{R}\varphi(z_k)\rangle\big| \geq \varepsilon. \tag{4.5}$$

Let n_k be the integer part of $\frac{1}{1-\|\varphi(z_k)\|}$, and choose $\xi_k = \frac{\varphi(z_k)}{\|\varphi(z_k)\|}$. Since $\lim_k (1 - \|\varphi(z_k)\|) n_k = 1$ and $\lim_k \|\varphi(z_k)\|^{n_k-2} = \frac{1}{e}$, it follows from (4.4) that

$$\begin{aligned} 0 &= \lim_{k\to\infty} \frac{1-\|z_k\|^2}{1-\|\varphi(z_k)\|^2} \big\|\varphi(z_k)\big\|^{n_k-2} \big|\langle \mathcal{R}\varphi(z_k), \varphi(z_k)\rangle\big| \\ &= \frac{1}{e} \lim_{k\to\infty} \frac{1-\|z_k\|^2}{1-\|\varphi(z_k)\|^2} \big|\langle \mathcal{R}\varphi(z_k), \varphi(z_k)\rangle\big|, \end{aligned}$$

which gives a contradiction if (4.5) holds. Thus (4.3) holds.

Let us now show (4.2). As above, we suppose that there exist $\varepsilon > 0$ and a sequence $(z_k) \in B_E$ such that

$$\frac{1-\|z_k\|^2}{\sqrt{1-\|\varphi(z_k)\|^2}}\big\|\mathcal{R}\varphi(z_k)\big\| \geq \varepsilon. \tag{4.6}$$

Let n_k be the integer part of $\frac{1}{1-\|\varphi(z_k)\|}$ and $\xi_k = \varphi(z_k) + \sqrt{1-\|\varphi(z_k)\|^2}\eta(z_k)$ with $\|\eta(z_k)\| = 1$ and $\langle \varphi(z_k), \eta(z_k)\rangle = 0$; we obtain from (4.4)

$$\begin{aligned}0 &= \lim_{k\to\infty} \frac{1-\|z_k\|^2}{1-\|\varphi(z_k)\|^2}\big(\|\varphi(z_k)\|^{n_k-1}\big)^2\big|\langle \mathcal{R}\varphi(z_k), \xi_k\rangle\big| \\ &= \frac{1}{e^2}\lim_{k\to\infty} \frac{1-\|z_k\|^2}{1-\|\varphi(z_k)\|^2}\big|\langle \mathcal{R}\varphi(z_k), \xi_k\rangle\big|.\end{aligned}$$

This together with condition (4.3) and Lemma 4.3 yields a contradiction to (4.6), and so (4.2) holds. □

Remark 4.6. Realize that conditions (4.2) and (4.3) hold trivially true in the case $\varphi(B_E) \subset rB_E$ for some $0 \leq r < 1$.

Remark 4.7. Note that $\varphi(z) = z$ satisfies (4.2) and fails (4.3). Also observe that

$$\frac{(1-\|z\|^2)\langle \mathcal{R}\varphi(z), \varphi(z)\rangle}{1-\|\varphi(z)\|^2} = \frac{(1-\|z\|^2)\|\mathcal{R}\varphi(z)\|}{\sqrt{1-\|\varphi(z)\|^2}}\frac{\langle \frac{\mathcal{R}\varphi(z)}{\|\mathcal{R}\varphi(z)\|}, \varphi(z)\rangle}{\sqrt{1-\|\varphi(z)\|^2}}.$$

Hence (4.3) implies (4.2) if there exists $\delta > 0$ such that

$$\inf_{\|\varphi(z)\|\geq\delta} \frac{|\langle \frac{\mathcal{R}\varphi(z)}{\|\mathcal{R}\varphi(z)\|}, \varphi(z)\rangle|}{\sqrt{1-\|\varphi(z)\|^2}} > 0.$$

Proposition 4.8. *Let $\varphi : B_E \to B_E$ be analytic such that $C_\varphi : \mathcal{B}(B_E) \to \mathcal{B}(B_E)$ is a compact operator. Then $\{\mathcal{R}\varphi(z) : \|z\| \leq \delta\}$ is relatively compact for all $\delta < 1$ as well as $\{(1-\|z\|^2)\mathcal{R}\varphi(z) : z \in B_E\}$.*

Proof. For $z \in B_E$ and $w \in E$, we consider the linear functional $\lambda_{z,w}$ acting on $f \in \mathcal{B}(B_E)$ according to $\lambda_{z,w}(f) = f'(z)(w) = \langle w, \overline{\nabla f(z)}\rangle$. It is continuous since $|\lambda_{z,w}(f)| \leq \frac{\|w\|}{1-\|z\|^2}\|f\|_{\mathcal{B}(B_E)}$. Realize that

$$C_\varphi^*(\lambda_{z,w})(f) = \lambda_{z,w}(f\circ\varphi) = \big\langle \varphi'(z)w, \overline{\nabla f\big(\varphi(z)\big)}\big\rangle,$$

and thus that $C_\varphi^*(\lambda_{z,z}) = \lambda_{\varphi(z),\mathcal{R}\varphi(z)}$.

Notice that $\mathcal{R}\varphi(\delta B_E)$ is a bounded subset of E by (2.12) in Lemma 2.2. Since C_φ^* is compact and $\sup\{\|\lambda_{z,z}\| : \|z\| \leq \delta\} < \infty$, then

$$\big\{C_\varphi^*(\lambda_{z,z}) : \|z\| \leq \delta\big\} = \big\{\lambda_{\varphi(z),\mathcal{R}\varphi(z)} : \|z\| \leq \delta\big\}$$

is relatively compact in $\mathcal{B}(B_E)^*$. Now we conclude that $\mathcal{R}\varphi(\delta B_E)$ is relatively compact because for the function $e_u(z) = \langle z, u\rangle$ we have $\mathcal{R}C_\varphi(e_u)(z) = \langle \mathcal{R}\varphi(z), u\rangle = \lambda_{\varphi(z),\mathcal{R}\varphi(z)}(e_u)$, and hence

$$\big\|\mathcal{R}\varphi(z) - \mathcal{R}\varphi(z')\big\| = \sup_{\|u\|\leq 1}\big|\langle \mathcal{R}\varphi(z) - \mathcal{R}\varphi(z'), u\rangle\big| \leq \|\lambda_{\varphi(z),\mathcal{R}\varphi(z)} - \lambda_{\varphi(z'),\mathcal{R}\varphi(z')}\|.$$

Moreover, $\{(1-\|z\|^2)\lambda_{z,w} : z, w \in B_E\}$ is also a bounded set in $\mathcal{B}(B_E)^*$, and thus

$$\left\{C_\varphi^*\big((1-\|z\|^2)\lambda_{z,z}\big) : \|z\| < 1\right\} = \left\{(1-\|z\|^2)\lambda_{\varphi(z),\mathcal{R}\varphi(z)} : \|z\| < 1\right\}$$

is a relatively compact set. Then the compactness of $\{(1-\|z\|^2)\mathcal{R}\varphi(z) : z \in B_E\}$ follows as above. □

There are also necessary conditions in terms of the components of the symbol φ. Recall that $(e_k)_{k\in\Gamma}$ is an orthonormal basis of E and that $\varphi = \sum_{k\in\Gamma} \varphi_k(x)e_k$. Here, $\varphi_k = \langle \varphi, e_k\rangle$.

Proposition 4.9. *Assume that* $C_\varphi : \mathcal{B}(B_E) \to \mathcal{B}(B_E)$ *is a compact operator. Then*

$$C_{\varphi_{k,l}} : \mathcal{B} \to \mathcal{B} \quad \textit{is compact} \tag{4.7}$$

for all $k, l \in \Gamma$, *where* $\varphi_{k,l}(\lambda) := \varphi_k(\lambda e_l)$, $\lambda \in \mathbb{D}$. *Also,*

$$\lim_{k\in\Gamma} \sup_{z\in B_E} \frac{(1-\|z\|^2)|\mathcal{R}\varphi_k(z)|}{1-|\varphi_k(z)|^2} = 0. \tag{4.8}$$

In particular, $\lim_{k\in\Gamma} \|\varphi_k\|_{\mathcal{B}(B_E)} = 0$. *And further,*

$$\lim_{|\varphi_n(z)|\to 1} \frac{(1-\|z\|^2)|\mathcal{R}\varphi_n(z)|}{1-|\varphi_n(z)|^2} = 0, \quad n \in \Gamma. \tag{4.9}$$

Proof. Let $y \in E\setminus\{0\}$, and let $\|\xi\| \le 1$. We write $F^\xi(x) = F(\langle x, \bar{\xi}\rangle)$, $x \in B_E$ for each $F \in H(\mathbb{D})$, and we write $f_y(\lambda) = f(\lambda\frac{y}{\|y\|})$, $\lambda \in \mathbb{D}$ for each $f \in H(B_E)$.

Consider $F \in \mathcal{B}$. Since $\nabla F^\xi(x) = F'(\langle x, \bar{\xi}\rangle)\xi$, then $F^\xi \in \mathcal{B}(B_E)$ and

$$\big(1-\|x\|^2\big)\big\|\nabla F^\xi(x)\big\| \le \|\xi\|\|F\|_{\mathcal{B}}\frac{1-\|x\|^2}{1-|\langle x,\bar{\xi}\rangle|^2} \le \|F\|_{\mathcal{B}}.$$

Hence the operator $E_\xi : F \in \mathcal{B} \mapsto F^\xi \in \mathcal{B}(B_E)$ is continuous.

If $f \in \mathcal{B}(B_E)$ and $\|y\| \le 1$, then it is an easy calculation that $f_y \in \mathcal{B}$ and $\|f_y\|_{\mathcal{B}} \le \|f\|_{\mathcal{B}(B_E)}$. Hence the operator $R_y : f \in \mathcal{B}(B_E) \mapsto f_y \in \mathcal{B}$ is continuous. For each $y, \xi \in B_E$ and $F \in \mathcal{B}$, we can write

$$\big(C_\varphi(F^\xi)\big)_y(\lambda) = F^\xi\Big(\varphi\Big(\lambda\frac{y}{\|y\|}\Big)\Big) = F\Big(\Big\langle\varphi\Big(\lambda\frac{y}{\|y\|}\Big), \bar{\xi}\Big\rangle\Big) = C_{\varphi_{y,\xi}}(F)(\lambda),$$

and so $C_{\varphi_{y,\xi}} = R_y \circ C_\varphi \circ E_\xi$ is compact. Then (4.7) follows because $\varphi_{k,l} = \varphi_{e_k,e_l}$.

Let us now show (4.8). Given a weakly null net $(\xi_k)_{k\in\kappa} \in E$ with $\|\xi_k\| \le 1$, we consider $f_k(z) = \log(\frac{1}{1-\langle z,\xi_k\rangle})$. According to [1, Corollary 4.4], $f_k \in \mathcal{B}(B_E)$ and $\|f_k\|_{\mathcal{B}(B_E)} \le \|\log(\frac{1}{1-\lambda})\|_{\mathcal{B}}$. Thus the net $\{f_k : k \in \kappa\}$ is bounded on compact subsets in B_E, and hence a *coh*-relatively compact set by Montel's theorem. Since $\lim_{k\in\kappa} f_k(z) = 0$, it follows that $\{f_k : k \in \kappa\}$ converges to zero uniformly on compact sets of B_E. Hence $\lim_{k\in\kappa}\|C_\varphi(f_k)\|_{\mathcal{B}(B_E)} = 0$. Now notice that $\mathcal{R}(C_\varphi(f_k))(z) = \frac{\langle\mathcal{R}\varphi(z),\xi_k\rangle}{1-\langle\varphi(z),\xi_k\rangle}$. Therefore,

$$\lim_{k\in\kappa} \sup_{\|z\|<1} \frac{(1-\|z\|^2)|\langle\mathcal{R}\varphi(z),\xi_k\rangle|}{|1-\langle\varphi(z),\xi_k\rangle|} = 0. \tag{4.10}$$

Assume now that (4.8) does not hold. Then there exist $\varepsilon > 0$ and a subnet (n_k) such that, for every n_k, there is z_k with

$$\frac{(1-\|z_k\|^2)|\mathcal{R}\varphi_{n_k}(z_k)|}{1-|\varphi_{n_k}(z_k)|^2} \geq \varepsilon. \tag{4.11}$$

Selecting now $\xi_k = e_{n_k}\overline{\varphi_{n_k}(z_k)}$, we get a weakly null net for which thus (4.10) holds. Then

$$\sup_{\|z\|<1} \frac{(1-\|z\|^2)|\mathcal{R}\varphi_{n_k}(z)||\varphi_{n_k}(z_k)|}{|1-\varphi_{n_k}(z)\overline{\varphi_{n_k}(z_k)}|} \to 0, \quad k \to \infty,$$

which contradicts (4.11).

Finally, we prove (4.9). Let $n \in \Gamma$, and assume that (4.9) does not hold; that is, there are $\varepsilon > 0$ and a sequence (z_l) with $\lim_{l\to\infty} |\varphi_n(z_l)| = 1$ and

$$\frac{(1-\|z_l\|^2)|\mathcal{R}\varphi_n(z_l)|}{1-|\varphi_n(z_l)|^2} \geq \varepsilon. \tag{4.12}$$

Let $F_l(\lambda) = \log \frac{1}{1-\lambda\overline{\varphi_n(z_l)}}$, and let $g_l(x) = F_l(\langle x, e_n\rangle) = \log \frac{1}{1-\langle x,e_n\rangle\overline{\varphi_n(z_l)}}$.

We may assume that $\varphi_n(z_l)$ converges to some $w_0, |w_0| = 1$. This means that (g_l) *co*-converges to $g_0(x) = F_0(\langle x, e_n\rangle) = \log \frac{1}{1-\langle x,e_n\rangle\overline{w_0}}$, where $F_0(\lambda) = \log \frac{1}{1-\lambda\overline{w_0}}$. Next, notice that $C_\varphi(g_l)(x) = F_l(\langle \varphi(x), e_n\rangle) = F_l \circ \varphi_n(x)$.

The compactness of C_φ yields that $\lim_l \|C_\varphi(g_l) - C_\varphi(g_0)\|_{\text{rad}} = 0$. However,

$$\begin{aligned}
&\big\|C_\varphi(g_l) - C_\varphi(g_0)\big\|_{\text{rad}} \\
&\quad= \|F_l \circ \varphi_n - F_0 \circ \varphi_n\|_{\text{rad}} \\
&\quad= \sup_{x\in B_E} \big(1-\|x\|^2\big)\big|\mathcal{R}(F_l\circ\varphi_n)(x) - \mathcal{R}(F_0\circ\varphi_n)(x)\big| \\
&\quad= \sup_{x\in B_E} \big(1-\|x\|^2\big)\big|F_l'\big(\varphi_n(x)\big)\mathcal{R}\varphi_n(x) - F_0'\big(\varphi_n(x)\big)\mathcal{R}\varphi_n(x)\big| \\
&\quad= \sup_{x\in B_E} \big(1-\|x\|^2\big)\big|\mathcal{R}\varphi_n(x)\big|\big|F_l'\big(\varphi_n(x)\big) - F_0'\big(\varphi_n(x)\big)\big| \\
&\quad= \sup_{x\in B_E} \big(1-\|x\|^2\big)\big|\mathcal{R}\varphi_n(x)\big|\left|\frac{\overline{\varphi_n(z_l)}}{1-\overline{\varphi_n(z_l)}\varphi_n(x)} - \frac{\overline{w_0}}{1-\overline{w_0}\varphi_n(x)}\right| \\
&\quad\geq \big(1-\|z_l\|^2\big)\big|\mathcal{R}\varphi_n(z_l)\big|\left|\frac{\overline{\varphi_n(z_l)}}{1-\overline{\varphi_n(z_l)}\varphi_n(z_l)} - \frac{\overline{w_0}}{1-\overline{w_0}\varphi_n(z_l)}\right| \\
&\quad= \frac{(1-\|z_l\|^2)}{1-|\varphi_n(z_l)|^2}\big|\mathcal{R}\varphi_n(z_l)\big|\left|\frac{\overline{\varphi_n(z_l)}-\overline{w_0}}{1-\varphi_n(z_l)\overline{w_0}}\right| \geq \varepsilon,
\end{aligned}$$

a contradiction. □

4.2.2. Compactness criteria.

Lemma 4.10. *Let $f : B_E \to \mathbb{C}$ be analytic, and let $x \in B_E$. Then*

$$\big(1-\|x\|^2\big)\mathcal{R}f(x) = \frac{-1}{2\pi i}\int_{|\xi|=1} f\big(\varphi_x(\xi x)\big)\frac{d\xi}{\xi^2}. \tag{4.13}$$

Proof. Observe that since φ_x is self-inverse, $f = (f \circ \varphi_x) \circ \varphi_x$. Hence, for $y \in B_E$,

$$\begin{aligned}
\big\langle y, \overline{\nabla f(x)} \big\rangle &= f'(x)(y) = (f \circ \varphi_x)'(0) \circ (\varphi_x)'(x)(y) \\
&= (f \circ \varphi_x)'(0) \Big[\Big(-\frac{1}{s_x^2} P_x - \frac{1}{s_x} Q_x \Big)(y) \Big] \\
&= -\frac{1}{s_x^2} (f \circ \varphi_x)'(0) \big[P_x(y) \big] - \frac{1}{s_x} (f \circ \varphi_x)'(0) \big[Q_x(y) \big] \\
&= -\frac{1}{s_x^2} \big\langle P_x(y), \overline{\widetilde{\nabla} f(x)} \big\rangle - \frac{1}{s_x} \big\langle Q_x(y), \overline{\widetilde{\nabla} f(x)} \big\rangle \\
&= -\frac{1}{s_x^2} \big\langle P_x(y), \overline{\widetilde{\nabla} f(x)} \big\rangle - \frac{1}{s_x} \big\langle y - P_x(y), \overline{\widetilde{\nabla} f(x)} \big\rangle,
\end{aligned}$$

and, using the fact that P_x is self-adjoint,

$$\begin{aligned}
\big\langle y, \overline{\nabla f(x)} \big\rangle &= -\frac{1}{s_x^2} \big\langle y, P_x \big(\overline{\widetilde{\nabla} f(x)} \big) \big\rangle - \frac{1}{s_x} \big\langle y, \overline{\widetilde{\nabla} f(x)} \big\rangle + \frac{1}{s_x} \big\langle y, P_x \big(\overline{\widetilde{\nabla} f(x)} \big) \big\rangle \\
&= \Big(-\frac{1}{s_x^2} + \frac{1}{s_x} \Big) \Big\langle y, \frac{\overline{\langle \widetilde{\nabla} f(x), x \rangle}}{\|x\|^2} x \Big\rangle - \frac{1}{s_x} \big\langle y, \overline{\widetilde{\nabla} f(x)} \big\rangle \\
&= \Big(-\frac{1}{s_x^2} + \frac{1}{s_x} \Big) \frac{\langle x, \overline{\widetilde{\nabla} f(x)} \rangle}{\|x\|^2} \langle y, x \rangle - \frac{1}{s_x} \big\langle y, \overline{\widetilde{\nabla} f(x)} \big\rangle.
\end{aligned}$$

Therefore,

$$s_x^2 \big\langle y, \overline{\nabla f(x)} \big\rangle = (s_x - 1) \frac{\langle x, \overline{\widetilde{\nabla} f(x)} \rangle}{\|x\|^2} \langle y, x \rangle - s_x \big\langle y, \overline{\widetilde{\nabla} f(x)} \big\rangle. \tag{4.14}$$

By the Cauchy formula, we have

$$\begin{aligned}
\big\langle x, \overline{\widetilde{\nabla} f(x)} \big\rangle &= (f \circ \varphi_x)'(0)(x) = \frac{1}{2\pi i} \int_{|\xi|=1} f \circ \varphi_x(\xi x) \frac{d\xi}{\xi^2} \quad \text{and} \\
\big\langle y, \overline{\widetilde{\nabla} f(x)} \big\rangle &= (f \circ \varphi_x)'(0)(y) = \frac{1}{2\pi i} \int_{|\xi|=1} f \circ \varphi_x(\xi y) \frac{d\xi}{\xi^2}.
\end{aligned}$$

Thus equality (4.14) becomes

$$\begin{aligned}
s_x^2 \big\langle y, \overline{\nabla f(x)} \big\rangle = (s_x - 1) &\frac{1}{2\pi i} \int_{|\xi|=1} f \circ \varphi_x(\xi x) \frac{d\xi}{\xi^2} \frac{\langle y, x \rangle}{\|x\|^2} \\
&- s_x \frac{1}{2\pi i} \int_{|\xi|=1} f \circ \varphi_x(\xi y) \frac{d\xi}{\xi^2},
\end{aligned} \tag{4.15}$$

and we conclude by taking $y = x$. $\square$

Remark 4.11. From (4.14) we deduce the following identity that might be of independent interest:

$$s_x^2 \nabla f(x) + s_x \widetilde{\nabla} f(x) = (s_x - 1) \frac{\langle \widetilde{\nabla} f(x), \bar{x} \rangle}{\|x\|^2} \bar{x}. \tag{4.16}$$

Lemma 4.12. *For every* $0<\delta<1$, *there exists* $C_\delta>0$ *such that*

$$\big|\langle y,\overline{\nabla f(x)}\rangle-\langle y',\overline{\nabla f(x')}\rangle\big|\le C_\delta\|f\|_{\mathcal{B}}\Big(\|x-x'\|+\frac{1-\delta}{2}\|y-y'\|\Big) \tag{4.17}$$

whenever $x,x'\in\delta B_E$ *and* $\|y\|\le 1$, $\|y'\|\le 1$, *and* $f\in\mathcal{B}(B_E)$.

Proof. Let $\varepsilon=\frac{1-\delta}{2}$. Since $\max\{\|x+\varepsilon\xi y\|,\|x'+\varepsilon\xi y'\|:|\xi|=1\}\le\frac{1+\delta}{2}$, we conclude by taking $u=0$ in (2.5) that

$$\rho_E(x+\varepsilon\xi y,x'+\varepsilon\xi y')\le\frac{4(1+\delta)}{4+(1+\delta)^2}<1.$$

Since $\frac{1}{2}\log\frac{1+r}{1-r}\le\frac{r}{1-r}$ for all $0<r<1$, we have

$$\beta_E(x+\varepsilon\xi y,x'+\varepsilon\xi y')\le\frac{\rho_E(x+\varepsilon\xi y,x'+\varepsilon\xi y')}{1-\frac{4(1+\delta)}{4+(1+\delta)^2}},$$

and so it follows that, for some constant C'_δ depending only on δ,

$$\beta_E(x+\varepsilon\xi y,x'+\varepsilon\xi y')\le C'_\delta\rho_E(x+\varepsilon\xi y,x'+\varepsilon\xi y').$$

Next, using the Cauchy formula, we have for $x,x'\in\delta B_E$, $\|y\|\le 1$, $\|y'\|\le 1$,

$$\langle y,\overline{\nabla f(x)}\rangle-\langle y',\overline{\nabla f(x')}\rangle=\frac{1}{\varepsilon}\frac{1}{2\pi i}\int_{|\xi|=1}f(x+\varepsilon\xi y)-f(x'+\varepsilon\xi y')\frac{d\xi}{\xi^2}.$$

From this, Theorem 3.1, and the equivalence of the seminorms, we get that, for some constant $C>0$,

$$\begin{aligned}\big|\langle y,\overline{\nabla f(x)}\rangle-\langle y',\overline{\nabla f(x')}\rangle\big|&\le\frac{1}{\varepsilon}\int_0^{2\pi}\big|f(x+\varepsilon e^{it}y)-f(x'+\varepsilon e^{it}y')\big|\frac{dt}{2\pi}\\&\le\frac{1}{\varepsilon}C\|f\|_{\mathcal{B}(B_E)}\int_0^{2\pi}\beta_E(x+\varepsilon e^{it}y,x'+\varepsilon e^{it}y')\frac{dt}{2\pi}.\end{aligned}$$

Applying (2.6), we find a constant $C_\delta>0$ depending only on δ such that

$$\begin{aligned}&\big|\langle y,\overline{\nabla f(x)}\rangle-\langle y',\overline{\nabla f(x')}\rangle\big|\\&\le\frac{1}{\varepsilon}C\cdot C'_\delta\|f\|_{\mathcal{B}(B_E)}\int_0^{2\pi}\rho_E(x+\varepsilon e^{it}y,x'+\varepsilon e^{it}y')\frac{dt}{2\pi}\\&\le C_\delta\|f\|_{\mathcal{B}(B_E)}\int_0^{2\pi}\big\|(x+\varepsilon e^{it}y)-(x'+\varepsilon e^{it}y')\big\|\frac{dt}{2\pi}\\&\le C_\delta\|f\|_{\mathcal{B}(B_E)}\big(\|x-x'\|+\varepsilon\|y-y'\|\big)\\&=C_\delta\|f\|_{\mathcal{B}(B_E)}\Big(\|x-x'\|+\frac{1-\delta}{2}\|y-y'\|\Big).\end{aligned}$$

□

Theorem 4.13. *Let* $\varphi:B_E\to B_E$ *be analytic. Assume that*

(i) $\{\varphi(z):\|\varphi(z)\|\le\delta\}$ *and* $\{(1-\|z\|^2)\mathcal{R}\varphi(z):\|\varphi(z)\|\le\delta\}$ *are relatively compact for all* $0<\delta<1$,

(ii) $$\lim_{\|\varphi(z)\|\to1}\frac{(1-\|z\|^2)\|\mathcal{R}\varphi(z)\|}{\sqrt{1-\|\varphi(z)\|^2}}=0,$$ *and*

$$\text{(iii)}\qquad \lim_{\|\varphi(z)\|\to 1} \frac{(1-\|z\|^2)|\langle \varphi(z), \mathcal{R}\varphi(z)\rangle|}{1-\|\varphi(z)\|^2} = 0.$$

Then $C_\varphi : \mathcal{B}(B_E) \to \mathcal{B}(B_E)$ *is a compact operator.*

Proof. We are going to apply Lemma 4.4. Let (f_α) be a bounded net in $\mathcal{B}(B_E)$ converging to zero uniformly on compact sets. Recall that

$$\mathcal{R}(f_\alpha \circ \varphi)(z) = \big\langle \nabla f_\alpha\big(\varphi(z)\big), \overline{\mathcal{R}\varphi(z)}\big\rangle.$$

Let $\varepsilon > 0$. By (ii) and (iii) there exists $\delta < 1$ such that, for $\|\varphi(z)\| > \delta$, we have

$$\big(1-\|z\|^2\big)\frac{\sqrt{(1-\|\varphi(z)\|^2)\|\mathcal{R}\varphi(z)\|^2 + |\langle \varphi(z), \mathcal{R}\varphi(z)\rangle|^2}}{1-\|\varphi(z)\|^2} < \varepsilon,$$

and hence, using Lemma 2.1, we have

$$\big(1-\|z\|^2\big)\big|\mathcal{R}(f_\alpha\circ\varphi)(z)\big| \le \sup_\alpha \big\|\widetilde{\nabla} f_\alpha\big(\varphi(z)\big)\big\|\varepsilon \le \sup_\alpha \|f_\alpha\|_{\mathrm{inv}}\varepsilon. \tag{4.18}$$

Denote $A_\delta = \{z \in B_E : \|\varphi(z)\| \le \delta\}$. For $z \in A_\delta$, we use formula (4.15) obtained in the proof of Lemma 4.10 to have

$$\begin{aligned}
&\Big\langle \frac{\mathcal{R}\varphi(z)}{2\|\mathcal{R}\varphi(z)\|}, \overline{\nabla f\big(\varphi(z)\big)}\Big\rangle \\
&\quad = \frac{1}{s_{\varphi(z)}}\Big(1-\frac{1}{s_{\varphi(z)}}\Big)\frac{1}{2\pi i}\int_{|\xi|=1} f\big(\varphi_{\varphi(z)}\big(\xi\varphi(z)\big)\big)\frac{d\xi}{\xi^2}\frac{\langle \mathcal{R}\varphi(z), \varphi(z)\rangle}{2\|\mathcal{R}\varphi(z)\|\|\varphi(z)\|^2} \\
&\qquad - \frac{1}{s_{\varphi(z)}}\frac{1}{2\pi i}\int_{|\xi|=1} f\Big(\varphi_{\varphi(z)}\Big(\xi\frac{\mathcal{R}\varphi(z)}{2\|\mathcal{R}\varphi(z)\|}\Big)\Big)\frac{d\xi}{\xi^2}.
\end{aligned}$$

Hence, for each $z \in A_\delta$,

$$\begin{aligned}
&\big(1-\|z\|^2\big)\big|\big\langle \nabla f_\alpha\big(\varphi(z)\big), \overline{\mathcal{R}\varphi(z)}\big\rangle\big| \\
&\quad \le \frac{(1-\|z\|^2)}{\|\varphi(z)\|}\frac{1}{s_{\varphi(z)}}\Big(\frac{1}{s_{\varphi(z)}}-1\Big)\big\|\mathcal{R}\varphi(z)\big\|\int_0^{2\pi}\big|f_\alpha\big(\varphi_{\varphi(z)}\big(e^{it}\varphi(z)\big)\big)\big|\frac{dt}{2\pi} \\
&\qquad + \frac{2(1-\|z\|^2)}{s_{\varphi(z)}}\|\mathcal{R}\varphi(z)\|\int_0^{2\pi}\Big|f_\alpha\Big(\varphi_{\varphi(z)}\Big(e^{it}\frac{\mathcal{R}\varphi(z)}{2\|\mathcal{R}\varphi(z)\|}\Big)\Big)\Big|\frac{dt}{2\pi}.
\end{aligned}$$

Bearing in mind (2.12) in Lemma 2.2 and the fact that $\lim_{\epsilon\to 0}\frac{1}{\epsilon}\big(\frac{1}{\sqrt{1-\epsilon^2}}-1\big) = 0$, there is $C > 0$ such that, for $\|\varphi(z)\| \le \delta$, we have

$$\frac{(1-\|z\|^2)}{\|\varphi(z)\|}\frac{1}{s_{\varphi(z)}}\Big(\frac{1}{s_{\varphi(z)}}-1\Big)\big\|\mathcal{R}\varphi(z)\big\| \le \frac{2}{\|\varphi(z)\|}\Big(\frac{1}{(1-\|\varphi(z)\|^2)^{1/2}}-1\Big) \le C.$$

In particular, for each $\delta < 1$, there exists $C_\delta > 0$ such that

$$\begin{aligned}
&\big(1-\|z\|^2\big)\big|\big\langle \nabla f_\alpha\big(\varphi(z)\big), \overline{\mathcal{R}\varphi(z)}\big\rangle\big| \\
&\quad \le C_\delta\Big(\int_0^{2\pi}\big|f_\alpha\big(\varphi_{\varphi(z)}\big(e^{it}\varphi(z)\big)\big)\big|\frac{dt}{2\pi} + \int_0^{2\pi}\Big|f_\alpha\Big(\varphi_{\varphi(z)}\Big(e^{it}\frac{\mathcal{R}\varphi(z)}{2\|\mathcal{R}\varphi(z)\|}\Big)\Big)\Big|\frac{dt}{2\pi}\Big)
\end{aligned}$$

when $\|\varphi(z)\| \leq \delta$. Therefore, since

$$\{\varphi_{\varphi(z)}\big(\xi\varphi(z)\big) : \xi \in \mathbb{T}\} \cup \Big\{\varphi_{\varphi(z)}\Big(\xi\frac{\mathcal{R}\varphi(z)}{2\|\mathcal{R}\varphi(z)\|}\Big) : \xi \in \mathbb{T}\Big\}$$

is compact in B_E, we have for each $z \in A_\delta$ that

$$\big(1 - \|z\|^2\big)\big|\big\langle\nabla f_\alpha\big(\varphi(z)\big), \overline{\mathcal{R}\varphi(z)}\big\rangle\big| \to 0. \tag{4.19}$$

Now bearing in mind (2.11) to observe that $s_y^2\|\mathcal{R}\varphi(y)\| \leq 1$, we may use Lemma 4.12 to have, for each $z, z' \in A_\delta$,

$$\begin{aligned}&\big|\big\langle\nabla f_\alpha\big(\varphi(z)\big), s_z^2\overline{\mathcal{R}\varphi(z)}\big\rangle - \big\langle\nabla f_\alpha\big(\varphi(z')\big), s_{z'}^2\overline{\mathcal{R}\varphi(z')}\big\rangle\big| \\ &\quad\leq C_\delta\|f_\alpha\|_{\mathcal{B}}\Big(\big\|\varphi(z) - \varphi(z')\big\| + \frac{1-\delta}{2}\big(\big\|s_z^2\overline{\mathcal{R}\varphi(z)} - s_{z'}^2\overline{\mathcal{R}\varphi(z')}\big\|\big)\Big).\end{aligned}$$

To finish the proof, we use the fact that both $\varphi(A_\delta)$ and $\{(1 - \|z\|^2)\mathcal{R}\varphi(z) : z \in A_\delta\}$ are relatively compact, and thus also the set $\{(\varphi(z), (1 - \|z\|^2)\mathcal{R}\varphi(z)) : z \in A_\delta\} \subset E \times E$ is relatively compact. Then, given $\varepsilon > 0$, there exists a finite family of points $\{z_k : 1 \leq k \leq N\} \subset A_\delta$ such that, for each $z \in A_\delta$, there exists z_k for which $\|\varphi(z) - \varphi(z_k)\| + \frac{1-\delta}{2}(\|s_z^2\overline{\mathcal{R}\varphi(z)} - s_{z_k}^2\overline{\mathcal{R}\varphi(z_k)}\|) < \varepsilon$. Hence

$$\sup_{z \in A_\delta}\big|\big\langle\nabla f_\alpha\big(\varphi(z)\big), s_z^2\overline{\mathcal{R}\varphi(z)}\big\rangle\big| \leq C'2\varepsilon + \max_{1 \leq k \leq n}\big|\big\langle\nabla f_\alpha\big(\varphi(z_k)\big), s_{z_k}^2\overline{\mathcal{R}\varphi(z_k)}\big\rangle\big|.$$

The proof is then complete using (4.19). □

Corollary 4.14. *Assume that* $\{\varphi(z) : \|\varphi(z)\| \leq \delta\}$ *and* $\{(1 - \|z\|^2)\mathcal{R}\varphi(z) : \|\varphi(z)\| \leq \delta\}$ *are relatively compact for all* $\delta < 1$*. Then* $C_\varphi : \mathcal{B}(B_E) \to \mathcal{B}(B_E)$ *is a compact operator if and only if*

$$\text{(i)} \quad \lim_{\|\varphi(z)\| \to 1} \frac{(1 - \|z\|^2)\|\mathcal{R}\varphi(z)\|}{\sqrt{1 - \|\varphi(z)\|^2}} = 0, \quad \textit{and}$$

$$\text{(ii)} \quad \lim_{\|\varphi(z)\| \to 1} \frac{(1 - \|z\|^2)|\langle\varphi(z), \mathcal{R}\varphi(z)\rangle|}{1 - \|\varphi(z)\|^2} = 0.$$

Corollary 4.15. *Assume that* $\|\varphi\|_\infty < 1$*. The composition operator* C_φ *is compact if* $\varphi(B_E)$ *is relatively compact.*

Proof. It is enough to check that the set $\{(1 - \|z\|^2)\mathcal{R}\varphi(z) : z \in B_E\}$ is relatively compact. Lemma 4.10 applied to $\mu \circ \varphi$ for all $\mu \in E^*$ yields $(1 - \|z\|^2)\mathcal{R}\varphi(z) = \frac{-1}{2\pi i}\int_{|\xi|=1}\varphi(\varphi_x(\xi x))\frac{d\xi}{\xi^2}$. Hence $(1 - \|z\|^2)\mathcal{R}\varphi(z)$ belongs to the weak closure of the balanced convex hull of the compact set $\overline{\{\frac{1}{\xi^2 2\pi}\varphi(B_E) : |\xi| = 1\}} \subset E$ that is also a compact set. □

Example 4.16. Let $\{e_n\}$ be a sequence in the given basis $\{e_k\}$. If $\{\varphi_n\}$ is a sequence in $H^\infty(B_E)$ such that $\sum_{n=1}^\infty \|\varphi_n\|_\infty^2 < 1$, then the mapping $\varphi(z) := \sum_n \varphi_n(z)e_n$ yields a compact composition operator C_φ on $\mathcal{B}(B_E)$.

In particular, for $\varphi_n(z) = \prod_{j=n}^{2n}\langle z, e_j\rangle$, C_φ is compact on $\mathcal{B}(B_E)$.

Proof. Note that $\sup_{\|z\|<1}\|\varphi(z)\|^2 \leq \left(\sum_{n=1}^{\infty}\sup_{\|z\|<1}|\varphi_n(z)|^2\right) < 1$. Moreover, $\varphi(B_E)$ is relatively compact since it lies inside the Hilbert cube given by the sequence $(\|\varphi_n\|_\infty)$. Now apply Corollary 4.15.

To verify the particular case, we use the inequality between geometric and arithmetic means, namely,

$$\big|\varphi_n(z)\big| = \prod_{j=n}^{2n}\big|\langle z, e_j\rangle\big| \leq \Big(\frac{1}{n+1}\sum_{j=n}^{2n}\big|\langle z, e_j\rangle\big|\Big)^{n+1} \leq (n+1)^{-\frac{n+1}{2}}\|z\|,$$

which produces the estimate $\sum_{n=1}^{\infty}\|\varphi_n\|_\infty^2 \leq \sum_{n=1}^{\infty}(n+1)^{-(n+1)} < 1$. □

Next, we introduce a class of symbols φ that allows a characterization of the compactness of C_φ. We say that the analytic mapping $\varphi : B_E \to B_E$ belongs to $\mathcal{B}_0(B_E, B_E)$ if

$$\lim_{\|z\|\to 1}\big(1-\|z\|^2\big)\big\|\mathcal{R}\varphi(z)\big\| = 0. \tag{4.20}$$

In particular, any map with bounded radial derivative satisfies (4.20). It is easy to produce examples of maps in $\mathcal{B}_0(B_E, B_E)$.

Proposition 4.17. *Let $\{e_n\}$ be a sequence in the given basis $\{e_k\}$. If $\{\varphi_n\}_n \subset \mathcal{B}(B_E)$ is such that*

$$\lim_{\|z\|\to 1}\big(1-\|z\|^2\big)\big|\mathcal{R}\varphi_n(z)\big| = 0 \quad \textit{for all } n\in\mathbb{N} \textit{ and } \sum_{n=1}^{\infty}\|\varphi_n\|_{\mathcal{B}(B_E)}^2 < \infty,$$

then $\varphi(z) = \sum_{n=1}^{\infty}\varphi_n(z)e_n \in \mathcal{B}_0(B_E, B_E)$.

Proof. Given $\varepsilon > 0$, there exist $N \in \mathbb{N}$ and $0 < \delta_j < 1$ for $j = 1, \dots, N$ such that

$$\big(1-\|z\|^2\big)^2\big\|\mathcal{R}\varphi(z)\big\|^2 \leq \sum_{n=1}^{N}\big(1-\|z\|^2\big)^2\big|\mathcal{R}\varphi_n(z)\big|^2 + \varepsilon^2/2$$

and

$$\big(1-\|z\|^2\big)\big|\mathcal{R}\varphi_j(z)\big| < \varepsilon/\sqrt{2N}, \qquad \|z\| > \delta_j, \quad j = 1, \dots, N.$$

Hence, if $\|z\| > \max_{1\leq j\leq N}\{\delta_j\}$, then $(1-\|z\|^2)\|\mathcal{R}\varphi(z)\| < \varepsilon$. □

Proposition 4.18. *Let $\varphi \in \mathcal{B}_0(B_E, B_E)$ with $\varphi(0) = 0$. Then*

(i) $$\limsup_{\|z\|\to 1}\frac{(1-\|z\|^2)\|\mathcal{R}\varphi(z)\|}{\sqrt{1-\|\varphi(z)\|^2}} = \limsup_{\|\varphi(z)\|\to 1}\frac{(1-\|z\|^2)\|\mathcal{R}\varphi(z)\|}{\sqrt{1-\|\varphi(z)\|^2}},$$

(ii) $$\limsup_{\|z\|\to 1}\frac{(1-\|z\|^2)|\langle\varphi(z), \mathcal{R}\varphi(z)\rangle|}{1-\|\varphi(z)\|^2} = \limsup_{\|\varphi(z)\|\to 1}\frac{(1-\|z\|^2)|\langle\varphi(z), \mathcal{R}\varphi(z)\rangle|}{1-\|\varphi(z)\|^2}.$$

Proof. In the case $\|\varphi\|_\infty < 1$, both right-hand-side limits are null, and both left-hand-side limits vanish according to the assumption.

Since $\|\varphi(z)\| \leq \|z\|$ by Lemma 2.2, the limits on the right-hand side are not greater than those on the left-hand side. Now, in the case $\|\varphi\|_\infty = 1$, there is a sequence $(z_n) \subset B_E$ such that $\|z_n\| \to 1$ and $\limsup_{\|z\|\to 1}\frac{(1-\|z\|^2)\|\mathcal{R}\varphi(z)\|}{\sqrt{1-\|\varphi(z)\|^2}} =$

$\lim_n \frac{(1-\|z_n\|^2)\|\mathcal{R}\varphi(z_n)\|}{\sqrt{1-\|\varphi(z_n)\|^2}}$. From the bounded sequence $(\|\varphi(z_n)\|)$ we get a convergent subsequence that we denote the same. If $\lim_n \|\varphi(z_n)\| = 1$, then we have $\limsup_{\|\varphi(z)\|\to 1} \frac{(1-\|z\|^2)\|\mathcal{R}\varphi(z)\|}{\sqrt{1-\|\varphi(z)\|^2}} \geq \lim_n \frac{(1-\|z_n\|^2)\|\mathcal{R}\varphi(z_n)\|}{\sqrt{1-\|\varphi(z_n)\|^2}}$ that leads to the equality (i), while if $\lim_n \|\varphi(z_n)\| < 1$, then $\limsup_{\|z\|\to 1} \frac{(1-\|z\|^2)\|\mathcal{R}\varphi(z)\|}{\sqrt{1-\|\varphi(z)\|^2}} = 0$, and so (i) holds as well, the analogous argument for (ii). ☐

In the following result we replace condition (i) in Theorem 4.13 by the weaker one given by (4.1), and we replace conditions (4.2) and (4.3) by the stronger ones given by taking $\lim_{\|\varphi(z)\|\to 1}$ instead of $\lim_{\|z\|\to 1}$. Since the proof follows the same arguments as in Theorem 4.13, it will only be sketched.

Proposition 4.19. *Let $\varphi : B_E \to B_E$ be analytic with $\varphi(0) = 0$. If φ satisfies (4.1),*

$$\lim_{\|z\|\to 1} \frac{(1-\|z\|^2)\|\mathcal{R}\varphi(z)\|}{\sqrt{1-\|\varphi(z)\|^2}} = 0 \quad \text{and} \tag{4.21}$$

$$\lim_{\|z\|\to 1} \frac{(1-\|z\|^2)|\langle \varphi(z), \mathcal{R}\varphi(z)\rangle|}{1-\|\varphi(z)\|^2} = 0, \tag{4.22}$$

then C_φ is compact on $\mathcal{B}(B_E)$.

Proof. By Lemma 2.2, we have $\|\varphi(z)\| \leq \|z\|$. The analogous estimate to (4.18) holds for $\|z\| > \delta$.

In the remaining case $\|z\| \leq \delta$, and also $\|\varphi(z)\| \leq \delta$ so that the estimates in the proof of Theorem 4.13 hold; that is, if $\|z\| \leq \delta$, then

$$\big(1-\|z\|^2\big)\big|\big\langle \nabla f_\alpha\big(\varphi(z)\big), \overline{\mathcal{R}\varphi(z)}\big\rangle\big| \to 0. \tag{4.23}$$

Now the final argument in the proof of Theorem 4.13 relies on the relative compactness of $\varphi(\{\|z\| \leq \delta\})$ that holds by assumption and that of $\{\mathcal{R}\varphi(z) : \|z\| \leq \delta\}$, which follows from the Cauchy formula. Indeed, $\mathcal{R}\varphi(z) = \varphi'(z)(z) = \frac{1}{2\pi i}\int_{|\lambda|=r} \frac{\varphi(z+\lambda z)}{\lambda^2}\, d\lambda$ for $0 < r < 1$ such that $\delta + r < 1$. Therefore, $\mathcal{R}\varphi(z)$ belongs to the weak closure of the balanced convex hull of the compact set $\overline{\{\mu\varphi((\delta+r)B_E) : |\mu| = r^{-2}\}} \subset E$ that is also a compact set. ☐

Let us mention that (4.21) implies that $\varphi \in \mathcal{B}_0(B_E, B_E)$ and that, combining the necessary condition obtained in Theorem 4.5 and Proposition 4.18, we get the following corollary.

Corollary 4.20. *Let $\varphi \in \mathcal{B}_0(B_E, B_E)$ with $\varphi(0) = 0$. Then C_φ is compact in $\mathcal{B}(B_E)$ if and only if φ satisfies (4.1), (4.21), and (4.22).*

5. EXAMPLES

In this section we provide a number of examples to discuss the relations among the various conditions we have found above.

Example 5.1. Consider $(\xi_n) \subset B_E$ such that

$$\sup_{\|z\|\le 1} \sum_n \left|\langle z, \xi_n\rangle\right|^2 \le 1. \tag{5.1}$$

Define $\varphi_n(z) = \langle z, \xi_n\rangle$, and define $\varphi(z) = \sum_n \varphi_n(z) e_n$, where $\{e_n\}$ is an orthonormal sequence in E. Then φ satisfies (4.21). In particular, $\varphi \in \mathcal{B}_0(B_E, B_E)$.

Moreover, if (ξ_n) is an orthogonal system, then we have that

- (i) φ satisfies (4.3) whenever $\sup_n \|\xi_n\| < 1$,
- (ii) φ fails (4.3) whenever there exists n_0 with $\|\xi_{n_0}\| = 1$,
- (iii) $\varphi(B_E)$ is relatively compact whenever $\sum_n \|\xi_n\|^2 < \infty$, and
- (iv) φ fails (4.1) whenever $\limsup_{n\to\infty} \|\xi_n\| > 0$.

Proof. Assumption (5.1) guarantees that φ is analytic and maps B_E to B_E. Since $\varphi(0) = 0$, by Lemma 2.2, we have $\|\varphi(z)\| \le \|z\|$ for any $z \in B_E$. Notice that $\mathcal{R}\varphi(z) = \sum_n \mathcal{R}\varphi_n(z) e_n = \varphi(z)$, and using the fact that $\alpha \mapsto \frac{\alpha}{\sqrt{1-\alpha^2}}$ is increasing for $0 < \alpha < 1$, we have

$$\frac{(1-\|z\|^2)\|\mathcal{R}\varphi(z)\|}{\sqrt{1-\|\varphi(z)\|^2}} = \frac{(1-\|z\|^2)\|\varphi(z)\|}{\sqrt{1-\|\varphi(z)\|^2}} \le \sqrt{1-\|z\|^2}.$$

In particular, φ satisfies (4.21).

Since $\langle \varphi(z), \mathcal{R}\varphi(z)\rangle = \|\varphi(z)\|^2$, we get that φ satisfies (4.3) if and only if $\lim_{\|\varphi(z)\|\to 1} \frac{1-\|z\|^2}{1-\|\varphi(z)\|^2} = 0$.

Assume now that (ξ_n) is an orthogonal system. Hence

$$\left\|\varphi(z)\right\|^2 = \sum_n \left|\langle z, \xi_n\rangle\right|^2 \le \sup_n \|\xi_n\|^2 \sum_n \left|\left\langle z, \frac{\xi_n}{\|\xi_n\|}\right\rangle\right|^2 \le \sup_n \|\xi_n\|^2 \|z\|^2.$$

Assuming $\sup_n \|\xi_n\|^2 < 1$, we have $\varphi(B_E) \subset \delta B_E$ for some $\delta < 1$, and (4.3) trivially holds, which shows (i).

Assume now that $\|\xi_{n_0}\| = 1$. Selecting $z = \lambda \xi_{n_0}$, we have $\varphi(z) = \lambda e_{n_0}$ and

$$\lim_{\|\varphi(z)\|\to 1} \frac{1-\|z\|^2}{1-\|\varphi(z)\|^2} \left\|\varphi(z)\right\|^2 = 1.$$

This gives (ii).

Now (iii) follows using that $|\varphi_n(z)| \le \|\xi_n\|$ for each n. Hence $\varphi(B_E)$ is contained in the Hilbert cube given by the sequence $(\|\xi_n\|)$.

Finally, to show (iv), assume $\limsup_{n\to\infty} \|\xi_n\| > 0$. Hence there exist $\varepsilon > 0$ and indices m_n such that $\|\xi_{m_n}\| \ge \varepsilon$. For each $0 < \delta < 1$, we have $\varphi(\delta \frac{\xi_n}{\|\xi_n\|}) = \delta \|\xi_n\| e_n$. Hence $\{\delta \|\xi_{m_n}\| e_{m_n} : n \in \mathbb{N}\} \subset \varphi(\delta B_E)$, which gives that $\varphi(\delta B_E)$ is not relatively compact in B_E. $\square$

In [5] it was shown that $\varphi(z) = \sum_{n=1}^{\infty} z_n^n e_n$ satisfies (4.1). Such φ is a particular choice in the following example.

Example 5.2. Let $\{e_k\}$ be an orthonormal sequence in E. Let $F_k : \mathbb{D} \to \mathbb{D}$ be a sequence of analytic functions such that $F_k(0) = 0$. Define

$$\varphi(z) := \sum_{k=1}^{\infty} F_k\big(\langle z, e_k\rangle\big) e_k.$$

(i) If $\|F_k\|_\infty < 1$ for all $k \in \mathbb{N}$, then φ satisfies (4.7).
(ii) If $F_k \in \mathcal{B}_0$, the little Bloch space, and $\|F_k\|_\infty < 1$ for all $k \in \mathbb{N}$, then φ satisfies (4.9).
(iii) If there exists $n_0 \in \mathbb{N}$ such that $C_{F_{n_0}}$ is noncompact on $\mathcal{B}$, then φ fails (4.3).
(iv) If $\sup_k \|F_k\|_\infty < 1$, then φ satisfies $\varphi(B_E) \subset \delta B_E$ for some $0 < \delta < 1$. In particular, φ satisfies (4.2) and (4.3).
(v) If $\sum_k \|F_k\|_\infty^2 < \infty$, then $\varphi(B_E)$ is relatively compact in B_E.
(vi) If $\sum_k \|F_k\|_{\mathcal{B}}^2 < \infty$, then φ satisfies (4.1).

Proof. Notice that, since $|F_k(\lambda)| \le |\lambda|$, we have that φ maps B_E into B_E. Actually, one has

$$\big\|\varphi(z)\big\|^2 = \sum_{k=1}^{\infty}\big|F_k\big(\langle z, e_k\rangle\big)\big|^2 \le \sum_{k=1}^{\infty}\|F_k\|_\infty^2\big|\langle z, e_k\rangle\big|^2 \le \|z\|^2 \quad \text{and, further,}$$

$$\big\|\varphi(z)\big\| \le \big(\sup_k \|F_k\|_\infty\big)\|z\|. \tag{5.2}$$

Since $\varphi'(z)(u) = \sum_{k=1}^{\infty} F_k'(\langle z, e_k\rangle)\langle u, e_k\rangle e_k$, then

$$\mathcal{R}\varphi(z) = \sum_{k=1}^{\infty} F_k'\big(\langle z, e_k\rangle\big)\langle z, e_k\rangle e_k, \quad \text{and so}$$

$$\big\langle \varphi(z), \mathcal{R}\varphi(z)\big\rangle = \sum_{k=1}^{\infty} F_k\big(\langle z, e_k\rangle\big)\overline{F_k'\big(\langle z, e_k\rangle\big)\langle z, e_k\rangle}.$$

Statement (i) follows since $\varphi_{k,l}(\lambda) = F_k(\lambda)\delta_{k,l}$ and $\|F_k\|_\infty < 1$ implies compactness of C_{F_k}.

To verify (ii), notice that

$$\frac{|\mathcal{R}\varphi_n(z)|}{1 - |\varphi_n(z)|^2} = \frac{|F_n'(\langle z, e_n\rangle)||\langle z, e_n\rangle|}{1 - |F_n(\langle z, e_n\rangle)|^2},$$

from which we conclude that

$$\frac{(1 - \|z\|^2)|\mathcal{R}\varphi_n(z)|}{1 - |\varphi_n(z)|^2} \le \frac{(1 - |\langle z, e_n\rangle|^2)|F_n'(\langle z, e_n\rangle)|}{1 - \|F_n\|_\infty^2},$$

which shows (4.9).

Concerning (iii), since $C_{F_{n_0}}$ is noncompact, then by Theorem 2 in [7] there exists $(\lambda_n) \subset \mathbb{D}$ for which $|F_{n_0}(\lambda_n)| \to 1$ (in particular, $|\lambda_n| \to 1$) and

$$\lim_n \frac{(1 - |\lambda_n|^2)|F_{n_0}'(\lambda_n)|}{1 - |F_{n_0}(\lambda_n)|^2} \ne 0.$$

Selecting the sequence $\xi_n = \lambda_n e_{n_0}$, we have

$$\left\|\varphi(\xi_n)\right\|^2 = \left|F_{n_0}(\lambda_n)\right|^2, \qquad \left\|\mathcal{R}\varphi(\xi_n)\right\| = \left|F'_{n_0}(\lambda_n)\|\lambda_n\right|,$$

and $\langle \mathcal{R}\varphi(\xi_n), \varphi(\xi_n)\rangle = \overline{F_{n_0}(\lambda_n)} F'_{n_0}(\lambda_n)\lambda_n$. Therefore, φ fails (4.3).

To check (iv), choose $\delta = \sup_k \|F_k\|_\infty$, and use (5.2).

Since $\varphi(B_E)$ is contained in the Hilbert cube given by the sequence $(\|F_k\|_\infty)$, it is relatively compact. Thus (v) holds.

Finally, to show (vi), we use the estimate for analytic functions $F : \mathbb{D} \to \mathbb{D}$ with $F(0) = 0$ given by $|F(\lambda)| \le \|F\|_{\mathcal{B}} \beta(0, \lambda)$ to obtain that $\varphi(\delta B_E)$ is contained in the Hilbert cube given by the sequence $(\|F_k\|_{\mathcal{B}} \beta(0, \delta))$. This gives (4.1). □

Example 5.3. Let $\{e_k\}$ be an orthonormal sequence in E. Let us consider $\varphi(z) = \sum_k \varphi_k(z) e_k$, where

$$\varphi_k(z) = \langle z, e_k\rangle^k. \tag{5.3}$$

Then φ satisfies (4.1) and fails (4.8). In particular, C_φ is noncompact on $\mathcal{B}(B_E)$.

Proof. Notice that $\varphi(z) \in B_E$ for each $z \in B_E$ because

$$\sum_{k=1}^{\infty} \left|\varphi_k(z)\right|^2 \le \sum_{k=1}^{\infty} \left|\langle z, e_k\rangle\right|^2 \le \|z\|^2.$$

It is clear that $\mathcal{R}\varphi_k(z) = k\varphi_k(z)$.

To show (4.1), just observe that $\sup_{\|z\|\le\delta} |\varphi_k(z)| \le \delta^k$. Denote

$$A_k = \sup_{z\in B_E} \frac{(1 - \|z\|^2)|\mathcal{R}\varphi_k(z)|}{1 - |\varphi_k(z)|^2}.$$

Let $z = \lambda e_k$, and estimate

$$A_k \ge \sup_{0<\lambda<1} \frac{(1-\lambda^2)k\lambda^k}{1-\lambda^{2k}} \ge \sup_k \left(1 - \frac{1}{k}\right)^{\frac{k}{2}} > 0.$$

□

Example 5.4. Let $\{e_k\}$ be an orthonormal sequence in E. Let $(n_k)_{k\in\mathbb{N}}$ be an increasing sequence of natural numbers with $n_0 = 0$, and define $\varphi(z) = \sum_k \varphi_k(z) e_k$ and $\psi(z) = \sum_k \psi_k(z) e_k$, where

$$\varphi_k(z) = \sum_{j=n_{k-1}+1}^{n_k} z_j^{2k} \quad \text{and} \quad \psi_k(z) = \Bigl(\sum_{j=n_{k-1}+1}^{n_k} z_j^2\Bigr)^k. \tag{5.4}$$

Then φ and ψ satisfy (4.1) but fail (4.7). Hence C_φ and C_ψ are noncompact on $\mathcal{B}(B_E)$.

Proof. Notice that $\varphi(z), \psi(z) \in B_E$ for each $z \in B_E$ because

$$\max\left\{\left|\varphi_k(z)\right|^2, \left|\psi_k(z)\right|^2\right\} \le \Bigl(\sum_{j=n_{k-1}+1}^{n_k} |z_j|^2\Bigr)^k \le \sum_{j=n_{k-1}+1}^{n_k} |z_j|^2.$$

Condition (4.1) follows from the estimate $\max\{|\varphi_k(z)|^2, |\psi_k(z)|^2\} \le \|z\|^{2k}$.

It is immediate to see that $\mathcal{R}\varphi_k(z) = 2k\varphi_k(z)$ and $\mathcal{R}\psi_k(z) = 2k\psi_k(z)$, and for each $k, m \in \mathbb{N}$, we have

$$\psi_{k,m}(\lambda) = \varphi_{k,m}(\lambda) = \lambda^{2k}, \quad n_k \leq m \leq n_{k+1}$$

and $\psi_{k,m} = \varphi_{k,m} = 0$ otherwise.

We see that $C_{\varphi_{k,m}}$ is noncompact on $\mathcal{B}$ because

$$\lim_{|\lambda^{2k}|\to 1} \frac{(1-|\lambda|^2)2k|\lambda|^{2k-1}}{1-|\lambda|^{4k}} \neq 0$$

due to the estimate $1-|\lambda|^{4k} \leq 2k(1-|\lambda|^2)$, $|\lambda| < 1$. $\square$

Acknowledgments. Blasco's work was partially supported by Ministerio de Economía y Competitividad (MINECO) grant MTM2011-23164. Galindo and Lindström's work was partially supported by MINECO grant MTM2014-53241-P. Miralles's work was partially supported by MINECO grant MTM2014-53241-P, Universitat Jaume I project P1-1B2014-35, and Generalitat Valenciana project AICO/2016/030.

REFERENCES

1. O. Blasco, P. Galindo, and A. Miralles, *Bloch functions on the unit ball of an infinite dimensional Hilbert space*, J. Funct. Anal. **267** (2014), no. 4, 1188–1204. Zbl 1293.32010. MR3217061. DOI 10.1016/j.jfa.2014.04.018. 311, 313, 314, 315, 316, 318, 320, 322
2. O. Blasco, M. Lindström, and J. Taskinen, *Bloch-to-BMOA compositions in several complex variables*, Complex Var. Theory Appl. **50** (2005), no. 14, 1061–1080. Zbl 1093.47025. MR2175841. DOI 10.1080/02781070500277672. 314
3. S. B. Chae, *Holomorphy and Calculus in Normed Spaces*, Pure and Appl. Math, **92**, Marcel Dekker, New York, 1985. Zbl 0571.46031. MR0788158. 312, 315, 320
4. J. Dai, *Compact composition operators on the Bloch space of the unit ball*, J. Math. Anal. Appl. **386** (2012), no. 1, 294–299. Zbl 1225.32014. MR2834885. DOI 10.1016/j.jmaa.2011.07.067. 312, 319
5. D. García, M. Maestre, and P. Sevilla-Peris, *Composition operators between weighted spaces of holomorphic functions on Banach spaces*, Ann. Acad. Sci. Fenn. Math. **29** (2004), no. 1, 81–98. Zbl 1064.47022. MR2041700. 330
6. K. Goebel and S. Reich, *Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Pure Appl. Math. **83**, Marcel Dekker, New York, 1984. Zbl 0537.46001. MR0744194. 313
7. K. Madigan and A. Matheson, *Compact composition operators on the Bloch space*, Trans. Amer. Math. Soc. **347** (1995), no. 7, 2679–2687. Zbl 0826.47023. MR1273508. DOI 10.2307/2154848. 331
8. R. M. Timoney, Bloch functions in several complex variables, I, *Bull. Lond. Math. Soc.* **12** (1980), no. 4, 241–267. Zbl 0416.32010. MR0576974. DOI 10.1112/blms/12.4.241. 315
9. K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Grad. Texts in Math. **226**, Springer, New York, 2005. Zbl 1067.32005. MR2115155. 315, 318
10. K. Zhu, *Operator Theory in Function Spaces*, 2nd ed, Math. Surveys Monogr. **138**, American Mathematical Society, Providence, RI, 2007. Zbl 1123.47001. MR2311536. DOI 10.1090/surv/138. 315, 316

[1]Departamento de Análisis Matemático, Universidad de Valencia, Valencia, Spain.

E-mail address: `oscar.blasco@uv.es`; `pablo.galindo@uv.es`

[2]Department of Mathematics, Abo Akademi University, Abo, Finland.
E-mail address: `mlindstr@abo.fi`

[3]Departament de Matemàtiques and Instituto Universitario de Matemáticas y Aplicaciones de Castellón (IMAC), Universitat Jaume I de Castelló (UJI), Castelló, Spain.
E-mail address: `mirallea@uji.es`