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Concrete Fibrations

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Abstract As far as we know, no notion of concrete fibration is available. We provide one such notion in adherence to the foundational attitude that characterizes the adoption of the fibrational perspective in approaching fundamental subjects in category theory and discuss it in connection with the notion of concrete category and the notions of locally small and small fibrations. We also discuss the appropriateness of our notion of concrete fibration for fibrations of small maps, which is relevant to algebraic set theory.

1 Introduction

A concrete category is a pair $(\mathbb{C}, F)$ where $\mathbb{C}$ is a category and $F : \mathbb{C} \to \mathbb{Sets}$ is a faithful functor, with $\mathbb{Sets}$ the category of small sets and the functions between them. As far as we know, no notion of concrete fibration is available. The aim of this paper is to give one such notion by pursuing the foundational attitude that characterizes the adoption of the fibrational perspective in approaching fundamental subjects in category theory, as much as possible in the spirit of Bénabou [5]. Accordingly, fibered category theory over a fixed base category $\mathbb{B}$ should be thought of as category theory over the "set-theoretic" universe which is $\mathbb{B}$, whose strength depends on the structural requirements made on $\mathbb{B}$. Thus, from the perspective of fibered category theory any "set-theoretic" property that could be asked to hold for a fibration should be categorially expressible with respect to its base universe, that is, intrinsically without reference to any previously assigned set-metatheoretic framework. In category theory this is obtained by substituting the set-theoretic formulation of a notion of interest by an equivalent reformulation in purely categorial terms, typically by means of a categorial universal property of the form "for all ... there exists a unique ...," namely, as an elementary, that is, first-order, statement about arrows and objects that have to be explicitly constructed. This way of proceeding is ubiquitous in category

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theory. It goes from the formulation of adjunctions to the elementary axiomatization of toposes (see McLarty [21]) to the formulation of local smallness for fibrations (see Bénabou [3] for instance). The notion of concrete category depends on an assigned external set-theoretic framework, and a reformulation of it in purely fibrational terms by means of an elementary categorial universal property is not evident. The point on which the whole paper hinges is the existence of a characterization of concreteness amounting to the satisfaction of the so-called *Isbell condition*, introduced and proved to be necessary by Isbell in [14] and proved to be sufficient by Freyd in [11]. Roughly, a category satisfies the Isbell condition if, for every pair of objects in the category, the class of spans between the objects modulo a suitable equivalence relation admits a system of representatives which is a set, rather than a proper class, so that each span in the class is equivalent to exactly one span in the set. The Isbell condition is nothing but an application of the axiom of choice for classes in which, rather than the existence of a system of representatives which could be a proper class, the existence of a system of representatives which is a set is required. The Isbell condition depends on an assigned external set-metatheoretic framework as much as concreteness itself, but it turns out that it can be brought back to the representability of a certain $\mathbb{Cls}$-valued functor, with $\mathbb{Cls}$ the category of classes and the functions between them. This means that such a functor has a universal element enjoying a categorial universal property, allowing an equivalent purely categorial reformulation of the Isbell condition itself, which is suitable for fibrations. All this is discussed in detail in Section 3. This state of things provided reasons for starting this research with the purpose declared at the beginning of this section.

1.1 On the adequacy of our definition As anticipated, it is in Section 3 that we define what it means for a fibration to satisfy the Isbell condition. We consider this as the counterpart to concreteness for fibrations on which we base the rest of the paper. Evidence in support of the adequacy of our definition is produced in terms of the specific goals that it should meet. Because of the foundational attitude that we are adopting, such goals have to be identified by the extent to which our definition corresponds to the notion of concrete category and the notions of locally small and small fibrations.

1.2 Comparison with the notion of concrete category Generally speaking, new notions for fibrations should be corroborated by verifying that they allow one to recover a corresponding typically well-known notion for categories by considering their instantiation for the fibration which is the naive indexing of a category (see Example 2.12), since naive indexing of ordinary category theory is the way to embed it in the fibered one. As a leading example, one has that a category is locally small if and only if its naive indexing is locally small as a fibration, which is, for a general fibered category, a different condition from being fiberwise locally small (see Example 4.11). For concrete fibrations, we prove Proposition 3.11, which, on the basis of the just-mentioned general goodness criterion, is an expected and reassuring result. Example 4.11 also shows that there are fibrations which are fiberwise concrete in the ordinary sense while not concrete in our sense, so that the two notions are different, analogous to what we have just observed for the local smallness of fibrations. The same example shows that, in general, a category, concrete by means of a faithful fibration to $\mathbb{Sets}$, is not a concrete fibration in our sense. This reaffirms that the concreteness of categories is recovered through their naive indexing, again as in the case of their local smallness.

1.3 Comparison with the notion of locally small fibration Further facts in support of our notion of concrete fibration come from the discussion of its connection to the notion of locally small fibration. Section 4 is dedicated to this. It is not evident that a concrete fibration is locally small, and in fact, there is no need for it to be so. This affirmation should not be a surprise since the adoption of the fibrational perspective often requires the abandonment of ordinary intuition about the notions under investigation (see [5]). Nonetheless, we investigated how to recover a suitable counterpart to the ordinary state of things. We obtained that, among the concrete fibrations over a finitely complete base category, those which are locally small are exactly those in which isomorphisms are definable (see Corollary 4.9).

1.4 Comparison with the notion of small fibration Among the locally small categories which are concrete, there are the small ones (see Eilenberg and MacLane [8]). Section 5 deals with the concreteness of internal categories, that is, small fibrations. It turns out that small fibrations are concrete in our sense without any appeal to equivalence classes of spans, which is very nice, expected, and reassuring (see Proposition 5.3).

1.5 Relevance to algebraic set theory The setting provided by algebraic set theory could be thought of as one for which the concreteness of fibrations should have been seriously taken into account and, as far as we know, has not been so far. In algebraic set theory, categories equipped with a class of distinguished morphisms are considered. Such distinguished morphisms are referred to as small maps and conceived as if their fibers were sets, rather than proper classes. Now, concrete categories are nothing but subcategories of $\mathbb{Sets}$ essentially, so it seemed natural to us to investigate the concreteness of (sub)fibrations of small maps in accordance with the point of view that we are proposing. We do this in Sections 6 and 7.

2 Preliminaries

Throughout the paper we work within the metatheoretic framework provided by a theory of sets and classes such as Göbel–Bernays–von Neumann. In particular, we assume the availability of the axiom of choice for classes: for every class X and equivalence relation $E \subseteq X \times X$, there exists a class $Y \subseteq X$ such that for every $x \in X$ there exists a unique $y \in Y$ with $(x, y) \in E$. Moreover, we assume that the reader is acquainted with the basics of ordinary and fibered category theory, but see MacLane [20], McLarty [21], and Bénabou [5] to begin with. Nonetheless, some of the relevant notions will be briefly recalled in Sections 2.1 and 2.2, as necessary for the rest of the paper and also for establishing notation and terminology. Other relevant notions will be recalled in due course.

2.1 Basics of concrete categories In this section, we briefly recall some basics concerning the notion of concrete categories. For further details see Adámek [1], Adámek, Herrlich, and Strecker [2], Freyd [10], [11], Isbell [14], Velebilová [26], and Borceux [6].

Let $\mathbb{Sets}$ be the category of small sets and the functions between them.

Definition 2.1 A *concrete category* is a pair $(\mathbb{C}, F)$ where $\mathbb{C}$ is a category and $F : \mathbb{C} \to \mathbb{S}\mathrm{ets}$ is a faithful functor. A category $\mathbb{C}$ is *concretizable* if it can be equipped with a faithful functor $F : \mathbb{C} \to \mathbb{S}\mathrm{ets}$.

Remark 2.2 It is well known that not all categories are concretizable (see Freyd [9]).

Terminology 2.3 *For all objects* A, B *in a category* $\mathbb{C}$*, a* span *from* A *to* B*, or* (A, B)-span*, is a diagram* $A \xleftarrow{f} X \xrightarrow{g} B$*, whereas an* (A, B)-cospan *is an* (A, B)*-span in* $\mathbb{C}^{op}$*.*

Notation 2.4 When no confusion is likely to arise, spans and cospans will be also briefly written as triplets (f, X, g). For all objects A, B in a category $\mathbb{C}$, $\mathrm{Span}(\mathbb{C})(A, B)$ will denote the class of all (A, B)-spans in $\mathbb{C}$.

Definition 2.5 In a category $\mathbb{C}$, two (A, B)-spans (f, X, g) and (f', X', g') are *equivalent* if, for every (A, B)-cospan (h, Z, k), $hf = kg$ if and only if $hf' = kg'$, which will also be written more briefly as $(f, X, g) \sim (f', X', g')$.

Definition 2.6 A category $\mathbb{C}$ satisfies the *Isbell condition* if, for all objects A, B in $\mathbb{C}$, there exists a *choice set* $C_{A,B}$ of (A, B)-spans such that every (A, B)-span is equivalent to exactly one span in $C_{A,B}$.

Remark 2.7 If a category $\mathbb{C}$ satisfies the Isbell condition, then for every pair (A, B) of objects of $\mathbb{C}$, $\mathrm{Span}(\mathbb{C})(A, B)/\!\sim$ is a set rather than a proper class. If, in a category $\mathbb{C}$, for some pair (A, B) of objects, $\mathrm{Span}(\mathbb{C})(A, B)$ is a proper class, then there exists a pair of different and equivalent (A, B)-spans, that is, at least one element of $\mathrm{Span}(\mathbb{C})(A, B)/\!\sim$ is not a singleton. Thus, if a category $\mathbb{C}$ satisfies the Isbell condition and for every pair (A, B) of objects of $\mathbb{C}$ all the $\sim$-equivalence classes are singletons, that is, any two equivalent (A, B)-spans are equal, then $\mathrm{Span}(\mathbb{C})(A, B)$ is a set.

Remark 2.8 In a category $\mathbb{C}$, for every $f, g : A \to B$, if $(\mathrm{id}_A, A, f) \sim (\mathrm{id}_A, A, g)$, then $f = g$, since for the (A, B)-cospan (f, B, id_B), in particular, one has that the condition "$f = f$ if and only if $f = g$" holds if and only if $f = g$ holds. From this and Remark 2.7 it follows that a category $\mathbb{C}$ satisfying the Isbell condition is locally small.

Remark 2.9 In a category $\mathbb{C}$, for every $f : A \to B$ and (A, B)-span (a, X, b) if $(\mathrm{id}_A, A, f) \sim (a, X, b)$, then $b = fa$, since for the (A, B)-cospan (f, B, id_B), in particular, one has that the condition "$f = f$ if and only if $fa = b$" holds if and only if $fa = b$ holds.

In [14] it is shown that a concretizable category must necessarily satisfy the Isbell condition, whereas in [11] it is shown that a category satisfying the Isbell condition must be concretizable, even without the explicit exhibition of a faithful functor to $\mathbb{S}\mathrm{ets}$, which has been exhibited in Vinárek [27].

Theorem 2.10 *A category* $\mathbb{C}$ *is concretizable if and only if it satisfies the Isbell condition.*

Proof See [11], [14], and [27]. □

2.2 Basics on fibered categories In this section we briefly recall some basics about the notion of fibered categories. For further details see Bénabou [3], [5], Grothendieck [12], Jacobs [15], Streicher [24], and Borceux [6].

Definition 2.11 Let $P : \mathbb{X} \to \mathbb{B}$ be a functor, and let $\varphi : X \to Y$ be a morphism in $\mathbb{X}$. The morphism φ is P-*Cartesian* if, for every morphism $v : K \to PX$ in $\mathbb{B}$, for every morphism $g : Z \to Y$ in $\mathbb{X}$ such that $Pg = P\varphi \circ v$, there exists a unique morphism $h : Z \to X$ such that $\varphi \circ h = g$ and $Ph = v$. Diagrammatically:

$$\begin{tikzcd} Z \arrow[dr, dashed, "h"'] \arrow[drr, bend left, "g"] & & \\ & X \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

$$\begin{tikzcd} K \arrow[r, "v"'] & PX \arrow[r, "P\varphi"'] & PY \end{tikzcd}$$

The functor P is a *fibration*, or a *fibered category*, if, for every object Y in $\mathbb{X}$ and for every morphism $u : I \to PY$ in $\mathbb{B}$, there exists a P-Cartesian morphism $\varphi : X \to Y$ such that $P\varphi = u$. The domain of a fibration is its *total category*, whereas its codomain is its *base category*.

In the case in which $P : \mathbb{X} \to \mathbb{B}$ is a fibration, the Cartesian morphism $\varphi : X \to Y$, which is required to exist and fit in a situation such as

$$\begin{tikzcd} X \arrow[r, "\varphi"] & Y \end{tikzcd}$$

$$\begin{tikzcd} I \arrow[r, "u=P\varphi"'] & PY \end{tikzcd}$$

for an object Y in $\mathbb{X}$ and a morphism u in $\mathbb{B}$, is a P-*Cartesian lifting* of Y along u. One may briefly say that a fibration has *enough Cartesian liftings*. For J in $\mathbb{B}$, the *fiber category of* P *at* J is the category P_J identified by the morphisms of $\mathbb{X}$ whose image under P is id_J. A P-*vertical morphism* is a morphism of $\mathbb{X}$ whose image under P is identical in $\mathbb{B}$. Cartesian liftings are identified up to a unique vertical isomorphism. By means of the axiom of choice for classes, Cartesian liftings can be chosen. A chosen Cartesian lifting of an object Y along a morphism u will be henceforth denoted as $y_u : Y \cdot u \to Y$. A choice of Cartesian liftings is a *cleavage*, and a fibration equipped with a cleavage is *cloven*. A cloven fibration has *enough lifting functors*, also referred to as *reindexing*, *substitution*, or *restriction* functors, between its fibers, in the sense that, for every morphism $u : I \to J$ in $\mathbb{B}$ and for every Y in P_J, the assignment $Y \mapsto Y \cdot u$ uniquely extends to a functor $(-)\cdot u : P_J \to P_I$, as shown in the diagram

$$\begin{tikzcd} Y\cdot u \arrow[r, "y_u"] \arrow[d, dashed, "f\cdot u"'] & Y \arrow[d, "f"] \\ Z\cdot u \arrow[r, "z_u"'] & Z \end{tikzcd}$$

$$I \xrightarrow[u]{} J$$

where the vertical morphism $f \cdot u$ is uniquely induced because of the P-Cartesianness of z_u. A cleavage is in general just pseudofunctorial, that is, functorial up to unique vertical isomorphisms, rather than strictly functorial, namely, up to identical isomorphisms. In other words, in general, for $P : \mathbb{X} \to \mathbb{B}$ a cloven fibration, Y in $\mathbb{X}$, $u : I \to PY$, and $v : K \to I$ in $\mathbb{B}$, it holds that $\mathrm{id}_Y \simeq y_{\mathrm{id}_{PY}}$ by means of a unique isomorphism in P_{PY}, and $y_{u\circ v} \simeq y_u \circ y_v$ by means of a unique isomorphism in P_K. When a fibration is equipped with a strictly functorial cleavage, the fibration is *split*. A cloven fibration $P : \mathbb{X} \to \mathbb{B}$ identifies a $\mathbb{B}$-indexed category, as a pseudofunctor

$$\mathbb{B}^{op} \xrightarrow{P} \mathrm{CAT}$$

$$\begin{tikzcd} J \arrow[r, mapsto] \arrow[d, "u"'] & P_J \arrow[d, "(-)\cdot u"] \\ I \arrow[r, mapsto] & P_I \end{tikzcd}$$

where, avoiding questions of size, we mentioned the category of all categories CAT, which is not a class. In order to conversely give rise to a cloven fibration, a pseudofunctor such as the previous one has to satisfy some suitable coherence conditions, which we do not bother to mention. This correspondence can be made precise by means of the so-called *Grothendieck construction*. For further details we refer the reader to the references cited at the beginning of this section.

Example 2.12 For every category $\mathbb{C}$, let $\mathrm{Fam}(\mathbb{C})$ be the category of set-indexed families of objects and morphisms of $\mathbb{C}$: its objects are families $(A_i)_{i\in I}$ with I a set and, for every $i \in I$, A_i an object of $\mathbb{C}$, whereas its morphisms are pairs $(u, f) : (A_i)_{i\in I} \to (B_j)_{j\in J}$, with $u : I \to J$ a function and $f = (f_i : A_i \to B_{u(i)})_{i\in I}$ an I-indexed family of morphisms of $\mathbb{C}$. For every object $(A_i)_{i\in I}$ in $\mathrm{Fam}(\mathbb{C})$, the assignment $(A_i)_{i\in I} \mapsto I$ extends to a fibration $P_{\mathbb{C}} : \mathrm{Fam}(\mathbb{C}) \to \mathbb{S}\mathbf{ets}$. Indeed, it can be verified that the lifting

$$(B_{u(i)})_{i\in I} \xrightarrow{(u,\mathrm{id})} (B_j)_{j\in J}$$

$$I \xrightarrow[u]{} J$$

with $\mathrm{id} = (\mathrm{id}_{B_{u(i)}})_{i\in I}$, is a $P_{\mathbb{C}}$-Cartesian lifting of $(B_j)_{j\in J}$ along u. Moreover, such a choice of Cartesian lifting is strictly functorial, thus making $P_{\mathbb{C}}$ a split fibration. It can be verified that, for every set I, the $P_{\mathbb{C}}$-fiber at I is equivalent to $\mathbb{C}^I$. For a category $\mathbb{C}$, $P_{\mathbb{C}}$ is often referred to as the naive indexing of $\mathbb{C}$. It is the fibration which is very often implicitly involved in the categorial arguments that more or less secretly rely on set theory.

Example 2.13 For every category $\mathbb{B}$, one can consider the category $\mathbb{B}^{\to}$ of morphisms and the commutative squares in $\mathbb{B}$: its objects are the morphisms of $\mathbb{B}$, whereas a morphism from $a : A \to I$ to $b : B \to J$ is a pair of morphisms of $\mathbb{B}$, $u : I \to J$ and $f : A \to B$, with $ua = bf$ in $\mathbb{B}$. We have that $\mathbb{B}^{\to}$ is connected to $\mathbb{B}$ by means of an obvious codomain functor $\mathrm{cod}_{\mathbb{B}} : \mathbb{B}^{\to} \to \mathbb{B}$. It can be verified that, for $b : B \to J$ and $u : I \to J$ in $\mathbb{B}$, a $\mathrm{cod}_{\mathbb{B}}$-Cartesian lifting of b along u is exactly a pullback of b along u, so that in turn $\mathrm{cod}_{\mathbb{B}}$ is a fibration if and only if $\mathbb{B}$ is a category with pullbacks. If this is the case, then $\mathrm{cod}_{\mathbb{B}}$ is not a split fibration in general, since a category does not come equipped with a functorial choice of pullback diagrams, in general.

Definition 2.14 Let $\mathbb{B}$ be a category, and let $P : \mathbb{X} \to \mathbb{B}$, $Q : \mathbb{Y} \to \mathbb{B}$ be fibrations. A *fibered functor* F from P to Q, $F : P \to Q$, is a functor $F : \mathbb{X} \to \mathbb{Y}$ such that $QF = P$, mapping P-Cartesian morphisms to Q-Cartesian morphisms. In the case in which P and Q are cloven, F is not required to map chosen Cartesian liftings to chosen Cartesian liftings.

Remark 2.15 For P, Q fibrations as in Definition 2.14, it can be verified that, for every object I in $\mathbb{B}$, a fibered functor $F : P \to Q$ restricts to a *fiber functor* $F_I : P_I \to Q_I$ so that F is faithful (resp., full; resp., an equivalence) if and only if it is fiberwisely so.

3 Isbell Condition for Fibrations

In this section we introduce the notions that lead toward the formulation of the Isbell condition for fibrations in Definition 3.4. After that, we reformulate it equivalently for a cloven fibration, in Proposition 3.9, as an elementary, namely, first-order, categorial universal property involving data to be constructed in the base $\mathbb{B}$ of the fibration, that is, relative to the “set-theoretic” universe which is $\mathbb{B}$, thus intrinsically to the fibration itself. Furthermore, under the putative working hypothesis that allows one to reintroduce a connection with an external set-theoretic framework, in Remark 3.10 we explain why such a universal property can be read as a representability condition for a suitable $\mathbb{Cls}$-valued functor, with $\mathbb{Cls}$ the category of classes and functions between them, in much the same way that the notion of local smallness for fibrations is treated in [3]. This is more or less a standard way of presenting things, but considering a presentation of such things in the reverse order may help to more clearly explain the perspective that has motivated the introduction of the Isbell condition for fibrations in the form that we will be dealing with. In general, the fact that a certain $\mathbb{Cls}$-valued functor is representable is equivalent to the fact that it has a universal element. Thus, it is possible to substitute the representability of such a functor as expressed in set-theoretic terms by the universal property which characterizes the universal element, that is, in terms of a first-order statement that has nothing to do with set theory. This way of proceeding is typical in category theory, and it is just what is going to be applied in the present section.

Definition 3.1 Let $P : \mathbb{X} \to \mathbb{B}$ be a fibration, and let A, B be objects in $\mathbb{X}$ in the same P-fiber. An (A, B)-span in P is an (A, B)-span (f, X, g) in $\mathbb{X}$ with

$Pf = Pg$. A vertical (A, B)-span in P is an (A, B)-span in P whose components are P-vertical morphisms.

Definition 3.2 Let $P : \mathbb{X} \to \mathbb{B}$ be a fibration, and let A, B be objects in $\mathbb{X}$ in the same P-fiber. Two (A, B)-spans (f, X, g), (f', X', g') in P are P-*equivalent* if $Pf = Pf'$ and $(\overline{f}, X, \overline{g}) \sim (\overline{f'}, X', \overline{g'})$ in P_X, with $\overline{f}, \overline{f'}$ (resp., $\overline{g}, \overline{g'}$) the vertical components of a (P-vertical, P-Cartesian) factorization of f, f' (resp., g, g') through a P-Cartesian lifting of A along Pf (resp., of B along Pg). When no confusion is likely to arise we will also write $(f, X, g) \sim_P (f', X', g')$.

Remark 3.3 Definition 3.4 below is basically the one on which the whole paper depends. In giving it for a fibration $P : \mathbb{X} \to \mathbb{B}$ we have preemptively employed the axiom of choice for classes to identify in each P-fiber, for every pair of objects in it, a choice class of spans to let a P-vertical span be $\sim$-equivalent to exactly one span in the choice class in the same P-fiber. Chosen vertical spans are not assumed to be stable under P-reindexing, in the sense that the P-reindexing of a chosen vertical span is not required to be a chosen vertical span.

Definition 3.4 Let $P : \mathbb{X} \to \mathbb{B}$ be a fibration. Then P satisfies the *Isbell condition* if, for all objects A, B in $\mathbb{X}$ in the same P-fiber, there exists an (A, B)-span (α, R, β) in P such that, for every (A, B)-span (f, X, g), there exists a P-Cartesian morphism $\theta : S \to R$, unique up to a uniquely determined P-vertical isomorphism, such that $(f, X, g) \sim_P (\alpha\theta, S, \beta\theta)$.

Remark 3.5 The intuition that the previous definition wants to capture is the following: we think of the span (α, R, β) as a $P\alpha$-indexed family of choice sets of (A, B)-spans with respect to the notion of "set" provided by the "set-theoretic" universe which is $\mathbb{B}$, so as to be able for every (A, B)-span (f, X, g) to "extract" from it, by means of θ, the unique one equivalent to (f, X, g) in the choice class of vertical spans preemptively determined in P_X, in accordance with Remark 3.3.

Remark 3.6 We think of the satisfaction of the Isbell condition on behalf of a fibration as the fibrational counterpart to concreteness for ordinary categories. Accordingly, Proposition 3.11 below is an expected and reassuring result that should be thought of as the fibrational counterpart to Theorem 2.10.

Remark 3.7 In the ordinary case, a category satisfying the Isbell condition is necessarily locally small (see Remark 2.8); however, in the case of fibrations the analogous counterpart to this fact does not hold evidently. In principle, a fibration could satisfy the Isbell condition without being locally small. This possibility should not surprise us, since often working in fibered category theory requires the abandonment of the ordinary intuition about the notions under investigation. In Section 4 we will properly address the question by appealing to a suitable form of the fibrational separation axiom which is referred to as definability (see [5]).

Remark 3.8 Let $P : \mathbb{X} \to \mathbb{B}$ be a cloven fibration, and let A, B be objects in $\mathbb{X}$ in the same P-fiber. Every (A, B)-span (f, X, g) in P identifies a vertical

$(A \cdot Pf, B \cdot Pf)$-span $(\overline{f}, X, \overline{g})$ in P_X as in the commutative diagram

$$\begin{tikzcd} A \cdot Pf \arrow[r, "a_{Pf}"] & A \\ X \arrow[u, "\overline{f}"] \arrow[ur, "f"'] \arrow[dr, "g"] \arrow[d, "\overline{g}"'] & \\ B \cdot Pf \arrow[r, "b_{Pf}"'] & B \end{tikzcd} \tag{1}$$

where $f = a_{Pf} \circ \overline{f}$ and $g = b_{Pf} \circ \overline{g}$ are (P-vertical, P-Cartesian) factorizations of f and g, respectively.

For a cloven fibration, the Isbell condition can be equivalently formulated in terms of an explicit first-order categorial universal property involving data to be constructed in its base (see Proposition 3.9 below). It is worth pointing this out because it is a way to let the Isbell condition for fibrations be solely ascribable to the fibrational structure in accordance with the point of view pursued in [5].

Proposition 3.9 *Let $P : \mathbb{X} \to \mathbb{B}$ be a cloven fibration. Then P satisfies the Isbell condition if and only if, for every object I in $\mathbb{B}$ and objects A, B in P_I, there exist a morphism $\pi : C_{A,B} \to I$ and a vertical $(A \cdot \pi, B \cdot \pi)$-span (p_A, R, p_B) such that for every morphism $u : J \to I$ and vertical $(A \cdot u, B \cdot u)$-span (f, X, g) there exists a unique morphism $\overline{u} : J \to C_{A,B}$ such that $\pi \circ \overline{u} = u$ and $(f, X, g) \sim (p_A \cdot \overline{u}, R \cdot \overline{u}, p_B \cdot \overline{u})$ in P_X.*

Proof Let P satisfy the Isbell condition with (α, R, β) as in Definition 3.4. Let I be an object of $\mathbb{B}$, and let A, B be objects in P_I. Put $\pi \doteq P\alpha$, and let (p_A, R, p_B) be the vertical $(A \cdot \pi, B \cdot \pi)$-span identified by (α, R, β). Now, for every morphism $u : J \to I$ and vertical $(A \cdot u, B \cdot u)$-span (f, X, g) as in

$$\begin{tikzcd} A \cdot u \arrow[r, "a_u"] & A \\ X \arrow[u, "f"] \arrow[d, "g"'] & \\ B \cdot u \arrow[r, "b_u"'] & B \end{tikzcd}$$

there exists a P-Cartesian morphism $\theta : S \to R$, unique up to a uniquely determined P-vertical isomorphism, such that $(\alpha\theta, R, \beta\theta) \sim_P (a_u f, X, b_u g)$, which implies $P(\alpha\theta) = P(a_u f)$. That is, $\pi \circ P\theta = u$ in the first place, so put $\overline{u} \doteq P\theta$, and $(f, X, g) \sim (p_A \cdot \overline{u}, R \cdot \overline{u}, p_B \cdot \overline{u})$ in the second place, up to a uniquely determined vertical isomorphism $(A \cdot \pi) \cdot \overline{u} \simeq A \cdot u$ in P_X.

Conversely, let A, B be objects in the same P-fiber. We show that $(a_\pi p_A, R, b_\pi p_B)$ is universal as in Definition 3.4. Let (f, X, g) be an (A, B)-span, and let $(\overline{f}, X, \overline{g})$ be the vertical $(A \cdot Pf, B \cdot Pf)$-span it determines. Let $\overline{u} : PX \to C_{A,B}$ be the unique morphism such that $\pi \circ \overline{u} = Pf$ and $(p_A \cdot \overline{u}, R \cdot \overline{u}, p_B \cdot \overline{u}) \sim (\overline{f}, X, \overline{g})$ in P_X. Put $\theta \doteq r_{\overline{u}} : R \cdot \overline{u} \to R$. Now, $(a_\pi p_A \theta, R \cdot \overline{u}, b_\pi p_B \theta) \sim_P (f, X, g)$ since

$Pa_\pi p_A\theta = \pi \circ \overline{u} = Pf$ and $(p_A\cdot\overline{u}, R\cdot\overline{u}, p_B\cdot\overline{u})$ is the vertical $(A\cdot Pf, B\cdot Pf)$-span identified by $(a_\pi p_A\theta, R\cdot\overline{u}, b_\pi p_B\theta)$. $\square$

In Remark 3.10 below, we show that under a suitable set-theoretic hypothesis the Isbell condition for a cloven fibration as formulated in Proposition 3.9 can be brought back to the representability of a certain $\mathbb{Cls}$-valued functor. The hypothesis that we consider allows the recovery of the universal property in Proposition 3.9 up to equality, rather than up to $\sim$-equivalence. This is a minor aspect since the argument we present is, for the sake of reasoning, directed toward the production of evidence in favor of the adoption of a pure categorial perspective by the abandonment of a set-theoretic point of view, so as to illuminate the understanding of the motivations for the introduction of the Isbell condition for fibrations in the form that we are dealing with, as suggested at the beginning of this section.

Remark 3.10 Consider a cloven fibration $P : \mathbb{X} \to \mathbb{B}$, let I be an arbitrary object of $\mathbb{B}$, and let A, B be arbitrary objects in P_I. For every morphism $u : J \to I$ consider the assignment

$$u \mapsto C_{A\cdot u, B\cdot u}$$

with $C_{A\cdot u,B\cdot u}$ a choice class of $(A\cdot u, B\cdot u)$-spans in P_J. By avoiding questions of size, let $\mathbb{Cls}$ be the category of classes and the functions between them. Now, contrary to what has been assumed in Remark 3.3, suppose for the sake of argument that chosen spans in the fibers of P are stable under P-reindexing and, moreover, that they have been chosen functorially, namely, so that $C_{A\cdot \mathrm{id}_I, B\cdot\mathrm{id}_I} = C_{A,B}$ and, for every morphism $v : K \to J$, $C_{A\cdot u\cdot v, B\cdot u\cdot v} = C_{A\cdot u\circ v, B\cdot u\circ v}$. Under the putative validity of these working hypotheses, P-reindexing allows one to extend the previous assignment to a functor

$$F_{A,B} : (\mathbb{B}/I)^{op} \to \mathbb{Cls}$$

which is representable if and only if P satisfies the Isbell condition in the sense of Proposition 3.9, up to equality, because $F_{A,B}$ is representable if and only if it has a universal element, that is, an object π in $\mathbb{B}/I$ together with an element $(p_A, R, p_B) \in C_{A\cdot\pi,B\cdot\pi}$ such that, for every object u in $\mathbb{B}/I$ and element $(f, X, g) \in C_{A\cdot u,B\cdot u}$, there exists a unique morphism $\overline{u} : \pi \to u$ in $(\mathbb{B}/I)^{op}$ such that $(p_A\cdot\overline{u}, R\cdot\overline{u}, p_B\cdot\overline{u}) = (f, X, g)$.

Proposition 3.11 *Let $\mathbb{C}$ be a category. The following facts are equivalent.*

(i) $\mathbb{C}$ *satisfies the Isbell condition.*
(ii) $P_{\mathbb{C}} : \mathrm{Fam}(\mathbb{C}) \to \mathbb{Sets}$ *satisfies the Isbell condition.*

Proof To prove (i) $\Longrightarrow$ (ii), we use Proposition 3.9, considering that $P_{\mathbb{C}}$ is a split fibration. Let I be a set, and let $A \doteq (A_i)_{i\in I}$, $B \doteq (B_i)_{i\in I}$ be I-indexed families of objects of $\mathbb{C}$. Since $\mathbb{C}$ satisfies the Isbell condition, for every $i \in I$ there exists a choice set C_{A_i,B_i}, whose elements we indicate as (p_{A_i}, R, p_{B_i}). Put $C_{A,B} \doteq \bigsqcup_{i\in I} C_{A_i,B_i}$, and let $\pi : C_{A,B} \to I$ be the evident projection. We claim that the vertical $(A\cdot\pi, B\cdot\pi)$-span

$$\big((\mathrm{id}_{C_{A,B}}, p_A), \mathcal{R}, (\mathrm{id}_{C_{A,B}}, p_B)\big),$$

with

$$\mathcal{R} = (R)_{(i,(p_{A_i},R,p_{B_i}))\in C_{A,B}},$$
$$p_A = (p_{A_i} : R \to A_i)_{(i,(p_{A_i},R,p_{B_i}))\in C_{A,B}},$$

and

$$p_B = (p_{B_i} : R \to B_i)_{(i,(p_{A_i},R,p_{B_i}))\in C_{A,B}},$$

is universal as in Proposition 3.9: for every morphism $u : J \to I$ and vertical $(A \cdot u, B \cdot u)$-span $((\mathrm{id}_J, f), (X_j)_{j\in J}, (\mathrm{id}_J, g))$ with $f = (f_j : X_j \to A_{u(j)})_{j\in J}$ and $g = (g_j : X_j \to B_{u(j)})_{j\in J}$, for every $j \in J$, put $\overline{u}(j) \doteq (u(j), (p_{A_{u(j)}}, R, p_{B_{u(j)}}))$ with $(p_{A_{u(j)}}, R, p_{B_{u(j)}}) \in C_{A_{u(j)},B_{u(j)}}$ the unique span equivalent to (f_j, X_j, g_j) in $\mathbb{C}$, and let $\overline{u} : J \to C_{A,B}$ be the resulting function. Now, $\pi \circ \overline{u} = u$ holds,

$$\begin{aligned}&\big((\mathrm{id}_{C_{A,B}}, p_A) \cdot \overline{u}, \mathcal{R} \cdot \overline{u}, (\mathrm{id}_{C_{A,B}}, p_B) \cdot \overline{u}\big)\\ &\quad = \big((\mathrm{id}_J, (p_{A_{u(j)}})_{j\in J}\big), (R)_{(u(j),(p_{A_{u(j)}},R,p_{B_{u(j)}}))\in C_{A,B}}, \big(\mathrm{id}_J, (p_{B_{u(j)}})_{j\in J}\big)\big),\end{aligned}$$

and

$$\begin{aligned}&\big((\mathrm{id}_J, (p_{A_{u(j)}})_{j\in J}\big), (R)_{(u(j),(p_{A_{u(j)}},R,p_{B_{u(j)}}))\in C_{A,B}}, \big(\mathrm{id}_J, (p_{B_{u(j)}})_{j\in J}\big)\big)\\ &\quad \sim \big((\mathrm{id}_J, f), (X_j)_{j\in J}, (\mathrm{id}_J, g)\big)\end{aligned}$$

in $\mathbb{C}^J$.

We now consider (ii) $\Longrightarrow$ (i). We note that $\mathbb{C}$ satisfies the Isbell condition because $P_{\mathbb{C}}$ satisfies the Isbell condition with respect to 1-indexed families of objects and spans of $\mathbb{C}$ in particular. □

Example 3.12 Let **HTop** be the category of topological spaces and the homotopy classes of continuous functions between them. By virtue of [9] and Proposition 3.11, we can say that $P_{\mathbf{HTop}}$ does not satisfy the Isbell condition; hence, it is not concrete in our sense.

4 Concreteness and Local Smallness of Fibrations

In this section we employ the definability of isomorphisms (see Definition 4.1 below) to address in the spirit of Remark 3.6 the question anticipated in Remark 3.7. More precisely, the aim of this section is to show that under the definability of isomorphisms a fibration satisfying the Isbell condition is locally small (see Definition 4.4 below). Roughly, definability is the fibrational counterpart to the set-theoretic axiom of separation, and it is one of the most important fibrational notions. We refer the reader to [5] for a thorough discussion of it from a foundational perspective.

Definition 4.1 Let $P : \mathbb{X} \to \mathbb{B}$ be a fibration. We say that P is a *fibration with definable isomorphisms*, or *isomorphisms are definable in* P, if for every P-vertical morphism $f : X \to Y$ there exists a universal P-Cartesian morphism $\varphi : Z \to X$ with $f \circ \varphi$ a P-Cartesian morphism and φ universal in the following sense: for every P-Cartesian morphism $\psi : W \to X$ with $f \circ \psi$ a P-Cartesian morphism, there exists a unique (necessarily P-Cartesian) morphism $\theta : W \to Z$ with $\varphi\theta = \psi$.

Proposition 4.2 *Let* $P : \mathbb{X} \to \mathbb{B}$ *be a cloven fibration. Isomorphisms are definable in* P *if and only if for every* P*-vertical morphism* $f : X \to Y$ *there exists a morphism* $\iota : I \to PX$ *such that* $f \cdot \iota$ *is an isomorphism in* P_I *and, for every morphism* $u : K \to PX$*, if* $f \cdot u$ *is an isomorphism in* P_K*, then there exists a unique morphism* $\overline{u} : K \to I$ *with* $\iota \circ \overline{u} = u$*.*

Proof The proof is straightforward. □

Remark 4.3 The morphism ι mentioned in the statement of Proposition 4.2 is necessarily a monic morphism. By thinking of $f : X \to Y$ as a PX-indexed family of morphisms $(f_x)_{x \in PX}$, one has $\iota : \{x \in PX \mid f_x \text{ is an isomorphism}\} \hookrightarrow PX$ by analogy.

Because of its relevance for what will follow, we recall the following definition.

Definition 4.4 Let $P : \mathbb{X} \to \mathbb{B}$ be a fibration. The fibration P is *locally small* if, for all objects A, B in $\mathbb{X}$ in the same P-fiber, there exists an (A, B)-span (φ, X, f) with φ a P-Cartesian morphism such that, for every (A, B)-span (ψ, Y, g) with ψ a P-Cartesian morphism, there exists a unique, necessarily Cartesian, mediating morphism $\theta : (\psi, Y, g) \to (\varphi, X, f)$, that is, a P-Cartesian $\theta : Y \to X$ with $\psi\theta = \varphi$ and $g\theta = f$.

Remark 4.5 Let $P : \mathbb{X} \to \mathbb{B}$ be a cloven fibration, and let A, B be objects in $\mathbb{X}$ in the same P-fiber. We observe that an (A, B)-span (φ, X, f) in P, with φ a P-Cartesian morphism, identifies a vertical $(A \cdot Pf, B \cdot Pf)$-span as in diagram (1) whose component $\overline{\varphi}$ is a vertical isomorphism that can be taken to be id_X. Because of this we think of (φ, X, f) as the graph of the (P-vertical component of the) morphism $f : X \to B$. Moreover, for every (φ, X, f), (φ', X', g) with φ and φ' P-Cartesian, one has $(\varphi, X, f) \sim_P (\varphi', X', g)$ provided $P\varphi = P\varphi'$ in the first place, so that $X \simeq X'$ vertically and up to a uniquely determined isomorphism commuting with φ and φ', so we assume $\varphi = \varphi'$. Thus, one has $(\varphi, X, f) \sim_P (\varphi, X, g)$ only if $f = g$, thanks to Remark 2.8.

If P is a fibration satisfying the Isbell condition, then for every (A, B)-span (φ, X, f) with φ a P-Cartesian morphism, one has the commutative diagram

$$\begin{tikzcd}
X \arrow[rrr, "\varphi"] & & & A \\
X \arrow[u, "\mathrm{id}_X"] \arrow[d, "\overline{f}"'] & S \arrow[ul, "a"'] \arrow[r, "\theta"] \arrow[dl, "\overline{f}a"] & R \arrow[ur, "\alpha"'] \arrow[dr, "\beta"] & \\
B \cdot Pf \arrow[rrr, "b_{Pf}"'] & & & B
\end{tikzcd} \tag{2}$$

in which a is vertical and uniquely determined, and $(\mathrm{id}_X, X, \overline{f}) \sim (a, S, \overline{f}a)$ in P_X, by virtue of Remark 2.9. Now, Definition 3.4 generalizes Definition 4.4 in the following sense: if in diagram (2) α were P-Cartesian, then a would be P-Cartesian as well and, hence, an isomorphism. Thus, $\theta \circ a^{-1} : X \to R$ would be mediating from (φ, X, f) to (α, R, β), Cartesian, and completely determined by $P\theta$, as in Definition 4.4. Conversely, if a were an isomorphism, then θa^{-1} and $\alpha \circ (\theta a^{-1})$ would be P-Cartesian, which does not imply that α is so in general.

Remark 4.6 In accordance with Remarks 3.3 and 4.5 and the previous observations, in the $\sim_P$-equivalence class of a span (φ, X, f) with P-Cartesian first component, we assume, without loss of generality, to choose the representative, which is (φ, X, f) itself. In other words, we assume that θ in diagram (2) exists so that $\theta : X \to R$ is mediating from (φ, X, f) to (α, R, β).

Proposition 4.7 *Let $P : \mathbb{X} \to \mathbb{B}$ be a fibration with definable isomorphisms. If P satisfies the Isbell condition, then it is locally small.*

Proof Consider the commutative diagram

$$\begin{tikzcd}
 & & \bullet \arrow[r, "a"'] & A \\
X \arrow[r, dashed, "x"] \arrow[urrr, bend left, "\varphi"] \arrow[rr, bend right, "\theta"'] \arrow[drrr, bend right, "f"'] & T \arrow[r, "\mu"] \arrow[ur, "p_A\mu"] & R \arrow[u, "p_A"'] \arrow[ur, "\alpha"'] \arrow[dr, "\beta"] & \\
 & & & B
\end{tikzcd}$$

and refer to it in following the argument below. The morphism a is a P-Cartesian lifting of A along $P\alpha$, so that $\alpha = ap_A$ is a (P-vertical, P-Cartesian) factorization of α. Let μ be the universal P-Cartesian morphism that makes $p_A\mu$ a P-Cartesian morphism by virtue of the definability of isomorphisms in P. We claim that the (A, B)-span $(ap_A\mu, T, \beta\mu)$ is universal as required in Definition 4.4. For every (A, B)-span (φ, X, f) with φ a P-Cartesian morphism, let θ be a P-Cartesian morphism such that $\alpha\theta = \varphi$ and $\beta\theta = f$, thanks to Remark 4.6. Now, $a(p_A\theta) = \varphi$ allows one to conclude that $p_A\theta$ is P-Cartesian, so there exists a unique P-Cartesian morphism x such that $\mu x = \theta$. Hence, $ap_A\mu x = ap_A\theta = \varphi$, $\beta\mu x = \beta\theta = f$, and x is completely determined by Px as the uniquely induced (necessarily P-Cartesian) morphism such that $ap_A\mu x = \varphi$. □

We recall the following well-known result.

Proposition 4.8 *Let $P : \mathbb{X} \to \mathbb{B}$ be a fibration with $\mathbb{B}$ a finitely complete category. If P is locally small, then isomorphisms are definable in P.*

Proof See [15]. □

Corollary 4.9 *Let $P : \mathbb{X} \to \mathbb{B}$ be a fibration satisfying the Isbell condition, with $\mathbb{B}$ a finitely complete category. The fibration P is locally small if and only if isomorphisms are definable in it.*

Example 4.10 As an example of a locally small fibration that does not satisfy the Isbell condition, consider the fibration $P_{\mathbf{HTop}}$ introduced in Example 3.12.

Example 4.11 Let **Top** be the category of topological spaces and the continuous functions between them. It is well known that the forgetful functor $P : \mathbf{Top} \to \mathbb{Sets}$ is a fibration: for every topological space (X, τ) and function $f : Y \to X$, a P-Cartesian lifting of (X, τ) along f is

$$\begin{tikzcd}
(Y, f^*\tau) \arrow[r, "f"] & (X, \tau) \\
Y \arrow[r, "f"'] & X
\end{tikzcd}$$

with $f^*\tau$ the smallest topology that makes f continuous. Precisely, $O \in f^*\tau$ if and only if O is the counterimage along f of some $V \in \tau$. Moreover, it is also well known that P is not locally small in the sense of Definition 4.4 (see Johnstone [17]), but fiberwisely so in the ordinary sense, since it is fiberwise a poset. It turns out that isomorphisms are definable in P: let $\mathrm{id}_X : (X, \tau) \to (X, \sigma)$ be a continuous P-vertical morphism in **Top**. This means that $\sigma \subseteq \tau$. Now, put

$$A \doteq \{x \in X \mid \forall V \in \tau . \exists U \in \sigma . (x \in V \Leftrightarrow x \in U)\},$$

and let $\tau \cap A$ and $\sigma \cap A$ be the smallest topologies on A that make it a subspace of (X, τ) and (X, σ), respectively. One has that $\tau \cap A \subseteq \sigma \cap A$ and, consequently, that $\mathrm{id}_A : (A, \tau \cap A) \to (A, \sigma \cap A)$, obtained by P-reindexing id_X along the inclusion $A \hookrightarrow X$, is a homeomorphism. Now, let $f : Y \to X$ be a function such that $\mathrm{id}_Y : (Y, f^*\tau) \to (Y, f^*\sigma)$, obtained by P-reindexing id_X along f, is a homeomorphism, namely, that $f^*\tau \subseteq f^*\sigma$. This last condition explicitly means that, for every $y \in Y$ and for every $V \in \tau$, there exists a $U \in \sigma$ such that $f(y) \in V$ if and only if $f(y) \in U$, which in turn means that f factorizes through $A \hookrightarrow X$. This example shows that P cannot be concrete in our sense, since if it were, then by virtue of the previous discussion and Corollary 4.9, it would be a locally small fibration, which is not the case, as already observed. Furthermore, this example shows that a category which is concrete in the ordinary sense may well not be so in our sense, as is also the case for local smallness. As already pointed out, the right way to recover the ordinary notions of local smallness and concreteness of categories in a fibrational perspective is by means of their naive indexing. Finally, this example shows that not only local smallness in the fibrational sense and fiberwise local smallness in the ordinary sense are different, but also fiberwise concreteness and concreteness in our fibrational sense are different, since it is clear that P is fiberwise concrete.

5 Concreteness of Internal Categories

With reference to Definition 3.4, we observe that if the Cartesian morphism θ were mediating from (f, X, g) to (α, R, β), then automatically $(f, X, g) \sim_P (\alpha\theta, X, \beta\theta)$. Thus, it would be nice to find a class of fibrations satisfying the Isbell condition in Definition 3.4 by means of mediating θ's. This happens for the split fibrations which are the externalization of an internal category, whose concreteness in the sense of Definition 3.4 is the matter of interest of this section. So, let $\mathbb{B}$ be a category with pullbacks. If not otherwise specified, any internal category referred to in the rest of this section is in $\mathbb{B}$.

To fix the data we will be working with, we briefly recall the definitions of internal category and externalization of an internal category.

Definition 5.1 An *internal category* in $\mathbb{B}$ is a 6-tuple $\mathbf{C} = (C_0, C_1, d_0, d_1, c, i)$ where $d_0, d_1 : C_1 \to C_0$, $c : C_1 \times_{C_0} C_1 \to C_1$, $i : C_0 \to C_1$ are morphisms in $\mathbb{B}$ with

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C_1 \arrow[d, "d_1"] \\ C_1 \arrow[r, "d_0"'] & C_0 \end{tikzcd}$$

a pullback, $d_0 \circ i = d_1 \circ i = \mathrm{id}_{C_0}$, $c \circ (c \times_{C_0} C_1) = c \circ (C_1 \times_{C_0} c)$, and

$$c \circ \langle i \circ d_1, \mathrm{id}_{C_1} \rangle = \mathrm{id}_{C_1} = c \circ \langle \mathrm{id}_{C_1}, i \circ d_0 \rangle.$$

A *small category* is an internal category in $\mathbb{S}\mathrm{ets}$.

Notation 5.2 For every $f, g : I \to C_1$ with $d_0 f = d_1 g$, we write $f \bullet g$ for the composite $c \circ \langle f, g \rangle$.

For $\mathbf{C}$ an internal category, let $\mathrm{Fam}(\mathbf{C})$ be the category identified by the following data:

objects: pairs (I, X) where I is an object of $\mathbb{B}$ and $X : I \to C_0$ is in $\mathbb{B}$;
morphisms: pairs $(u, f) : (I, X) \to (J, Y)$ where $u : I \to J$ and $f : I \to C_1$ in $\mathbb{B}$ such that $d_0 f = X$ and $d_1 f = Y \circ u$;
composition: given by the rule

$$\begin{tikzcd} (I,X) \arrow[r, "(u,f)"] \arrow[rr, bend right, "(v\circ u,(g\circ u)\bullet f)"'] & (J,Y) \arrow[r, "(v,g)"] & (K,Z) \end{tikzcd}$$

identity: the identity morphism at, say, an object (I, X) is $(\mathrm{id}_I, i \circ X)$.

For every object (I, X) of $\mathrm{Fam}(\mathbf{C})$, the assignment $(I, X) \mapsto I$ extends to a split fibration $P_{\mathbf{C}} : \mathrm{Fam}(\mathbf{C}) \to \mathbb{B}$ referred to as an *externalization* of $\mathbf{C}$. For every morphism $u : I \to J$ and object (J, Y), a $P_{\mathbf{C}}$-Cartesian lifting of (J, Y) along u is

$$(I, Y \circ u) \xrightarrow{(u,\, i\circ Y\circ u)} (J, Y)$$

$$I \xrightarrow[u]{} J$$

Proposition 5.3 *For every internal category* $\mathbf{C}$*, the fibration* $P_{\mathbf{C}}$ *satisfies the Isbell condition.*

Proof For every (I, A), (I, B) in $\mathrm{Fam}(\mathbf{C})$, construct the diagram

$$\begin{tikzcd} C_{A,B} \arrow[r, "h"] \arrow[dd, "\pi"'] & S \arrow[r, "\sigma"] \arrow[d, "\langle s_1,s_2\rangle"] & C_0 \arrow[d, "\Delta_0"] \\ & C_1 \times C_1 \arrow[r, "d_0\times d_0"'] \arrow[d, "d_1\times d_1"] & C_0 \times C_0 \\ I \arrow[r, "\langle A,B\rangle"'] & C_0 \times C_0 & \end{tikzcd} \tag{3}$$

in which all the quadrilaterals are pullbacks. We claim that the $P_{\mathbf{C}}$-vertical $((I, A) \cdot \pi, (I, B) \cdot \pi)$-span that fits in the diagram

$$\begin{tikzcd} (C_{A,B}, A\circ\pi) \arrow[r, "(\pi,i\circ A\circ\pi)"] & (I,A) \\ (C_{A,B}, \sigma\circ h) \arrow[u, "(\mathrm{id},s_1\circ h)"] \arrow[d, "(\mathrm{id},s_2\circ h)"'] & \\ (C_{A,B}, B\circ\pi) \arrow[r, "(\pi,i\circ B\circ\pi)"'] & (I,B) \end{tikzcd} \tag{4}$$

$$C_{A,B} \xrightarrow[\pi]{} I$$

is universal as described in Proposition 3.9. Indeed, for every morphism $u : J \to I$ and for every $P_{\mathbf{C}}$-vertical $((I, A) \cdot u, (I, B) \cdot u)$-span

$$(J, A \circ u) \xleftarrow{(\mathrm{id}, f)} (J, X) \xrightarrow{(\mathrm{id}, g)} (J, B \circ u)$$

there exists a unique morphism $\tilde{u} : J \to S$ with $\sigma \circ \tilde{u} = X$ and $\langle s_1, s_2 \rangle \circ \tilde{u} = \langle f, g \rangle$. Thus, in turn there exists a unique morphism $\overline{u} : J \to C_{A,B}$ with $\pi \circ \overline{u} = u$ and $h \circ \overline{u} = \tilde{u}$. Finally, the diagram

$$\begin{tikzcd}
(J, A \circ u) \arrow[r, dashed, "(\overline{u}, i \circ A \circ \pi \circ \overline{u})"] & (C_{A,B}, A \circ \pi) \\
(J, X) \arrow[u, "(\mathrm{id}, f)"] \arrow[d, "(\mathrm{id}, g)"'] \arrow[r, "(\overline{u}, i \circ \sigma \circ h \circ \overline{u})"] & (C_{A,B}, \sigma \circ h) \arrow[u, "(\mathrm{id}, s_1 \circ h)"'] \arrow[d, "(\mathrm{id}, s_2 \circ h)"] \\
(J, B \circ u) \arrow[r, dashed, "(\overline{u}, i \circ B \circ \pi \circ \overline{u})"'] & (C_{A,B}, B \circ \pi) \\
J \arrow[r, "\overline{u}"'] & C_{A,B}
\end{tikzcd}$$

commutes over $\overline{u}$. □

Remark 5.4 The proof of Proposition 5.3 is directed toward the construction of the span

$$(I, B) \xleftarrow{(\pi, s_2 h)} (C_{A,B}, \sigma h) \xrightarrow{(\pi, s_1 h)} (I, A), \tag{5}$$

which is universal as required in Definition 3.4, because for every $((I, A), (I, B))$-span $((u, f), (J, X), (u, g))$, the diagram

$$\begin{tikzcd}
 & (I, A) \\
(J, X) \arrow[ur, bend left, "(u, f)"] \arrow[dr, bend right, "(u, g)"'] \arrow[r, "(\overline{u}, i \circ \sigma \circ h \circ \overline{u})"] & (C_{A,B}, \sigma \circ h) \arrow[u, "(\pi, s_1 \circ h)"'] \arrow[d, "(\pi, s_2 \circ h)"] \\
 & (I, B)
\end{tikzcd}$$

commutes.

6 Concrete Fibrations of Small Maps

Concrete categories abstractly refer to categories of structured sets and structure-preserving functions in which the structure referred to remains unspecified or otherwise specified by distinguishing the structured sets and functions of interest organizing them in a category—concrete indeed. In this perspective, an approach to a notion of concrete fibration such as the one we pursued so far turns out to be applicable within the framework of algebraic set theory (see Joyal and Moerdijk [18]). Roughly, in algebraic set theory, categories equipped with a class of distinguished morphisms, in the totality of theirs, are considered. Such distinguished morphisms are referred to as small maps and conceived as if their fibers were sets, rather than proper classes, or better as (internal) class-indexed families of sets, rather than class-indexed families of classes more generally. Typically, small maps are required to satisfy a list of axioms to provide a flexible categorial set-theoretic framework as necessary. For further details, the reader is invited to consult Joyal and Moerdijk [18], Simpson [22], and Butz [7], for instance.

Definition 6.1 A *category with small maps* is a pair $(\mathbb{B}, \mathcal{S})$ where $\mathbb{B}$ is a category with pullbacks and $\mathcal{S}$ is a class of pullback-stable morphisms of $\mathbb{B}$, which are referred to as *small maps*.

Remark 6.2 To various aims, categories equipped with a class of distinguished morphisms have already been considered (see, e.g., Bénabou [4], Hyland and Pitts [13], Streicher [23], Taylor [25]). The distinguished morphisms of interest are always required to be stable under pullbacks, since the characteristic that distinguishes them is of their fibers, even if the ambient category is not required to have all of them: the pullback of a distinguished morphism along a morphism of the ambient category is required to exist and the ensuing opposite morphism is required to be a distinguished one. To our aims, in Definition 6.1, the ambient category $\mathbb{B}$ was required to be a category with pullbacks because this equivalently corresponds to the availability of the fibration $\mathrm{cod}_{\mathbb{B}}$ as the ambient fibration of class-indexed families of classes (see Example 2.13).

Henceforth, even if not explicitly mentioned, the small maps referred to in the rest of this section have to be considered as equipment furnished with a fixed category with pullbacks $\mathbb{B}$ in the sense of Definition 6.1.

Small maps identify a full subcategory $\mathcal{S}_{\mathbb{B}} \hookrightarrow \mathbb{B}^{\to}$ and in turn a full subfibration $\mathrm{cod}_{\mathcal{S}} : \mathcal{S}_{\mathbb{B}} \to \mathbb{B}$ of $\mathrm{cod}_{\mathbb{B}}$, of class-indexed families of sets.

Notation 6.3 For every object I of $\mathbb{B}$, the $\mathrm{cod}_{\mathcal{S}}$-fiber at I will be henceforth denoted $\mathcal{S}_I$.

Beyond pullback-stability, the further axioms that the small maps may be required to satisfy can be motivated by pursuing a set-theoretic intuition, whereas they also correspond to suitable categorial properties that the fibration $\mathrm{cod}_{\mathcal{S}}$ may enjoy. For instance, if singleton classes should be sets, then the small maps should satisfy the following axiom.

Axiom 6.4 (Unit) All the identity morphisms of $\mathbb{B}$ are small maps.

This provides terminal objects in the fibers of $\mathrm{cod}_{\mathcal{S}}$. If set-indexed disjoint unions of sets should be sets, then small maps should satisfy the following axiom.

Axiom 6.5 (Sum) The small maps are closed under composition.

Together with pullback-stability, this would provide binary products in the fibers of $\mathrm{cod}_{\mathcal{S}}$.

Proposition 6.6 *Let* $(\mathbb{B}, \mathcal{S})$ *be a category with small maps. The identity and sum axioms hold if and only if* $\mathrm{cod}_{\mathcal{S}}$ *has fiberwise finite products which are stable under pullback along any morphism of* $\mathbb{B}$*, that is,* $\mathrm{cod}_{\mathcal{S}}$*-reindexing stable.*

Proof The proof is straightforward. □

In accordance with the line of investigation that we have pursued so far, in the rest of this section we will be interested in the axioms that make $\mathrm{cod}_{\mathcal{S}}$ locally small or concrete relative to $\mathcal{S}$. So, consider the following axiom.

Axiom 6.7 (Local smallness) For every pair of small maps $a : A \to I$ and $b : B \to I$, there exists a span

$$\begin{tikzcd} A \arrow[d, "a"'] \arrow[dr, phantom, "(*)"] & \bullet \arrow[l] \arrow[r, "f"] \arrow[d, "x"] & B \arrow[d, "b"] \\ I & J \arrow[l, "u"] \arrow[r, "u"'] & I \end{tikzcd} \tag{6}$$

in which u is a small map and $(*)$ is a pullback, such that for every span

$$\begin{tikzcd} A \arrow[d, "a"'] \arrow[dr, phantom, "(+)"] & \bullet \arrow[l] \arrow[r] \arrow[d, "y"] & B \arrow[d, "b"] \\ I & K \arrow[l, "v"] \arrow[r, "v"'] & I \end{tikzcd} \tag{7}$$

in which v is a small map and $(+)$ is a pullback, there exists a unique mediating pullback

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "y"'] & \bullet \arrow[d, "x"] \\ K \arrow[r] & J \end{tikzcd}$$

making the span (7) factor through the span (6).

Remark 6.8 If $\mathrm{cod}_{\mathcal{S}}$ were cloven, then for every pair of small maps a, b as above, the local smallness axiom expresses in elementary terms the representability of the functor extending the assignment

$$F_{a,b} : (\mathcal{S}_I)^{op} \ni u : J \to I \mapsto \mathcal{S}_J(a \cdot u, b \cdot u) \in \mathbb{Cls},$$

which is nothing but the local smallness of $\mathrm{cod}_{\mathcal{S}}$ relative to $\mathcal{S}$, in accordance with the notion of locally small fibration as originally given in [3] relative to a class of so-called *proper maps*.

Axiom 6.9 (Concreteness) For every pair of small maps $a : A \to I$ and $b : B \to I$, there exists a span

$$\begin{tikzcd} A \arrow[d, "a"'] & \bullet \arrow[l, "\alpha"'] \arrow[r, "\beta"] \arrow[d, "x"] & B \arrow[d, "b"] \\ I & C_{a,b} \arrow[l, "\pi"] \arrow[r, "\pi"'] & I \end{tikzcd} \tag{8}$$

in which π is a small map, such that for every span

$$\begin{tikzcd} A \arrow[d, "a"'] & \bullet \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "y"] & B \arrow[d, "b"] \\ I & K \arrow[l, "v"] \arrow[r, "v"'] & I \end{tikzcd} \tag{9}$$

in which v is a small map, there exists a unique morphism $w : K \to C_{a,b}$ with $\pi \circ w = v$ and a pullback

$$\begin{tikzcd} \bullet \arrow[r, "\overline{w}"] \arrow[d, "\overline{x}"'] & \bullet \arrow[d, "x"] \\ K \arrow[r, "w"'] & C_{a,b} \end{tikzcd}$$

making the (a, b)-spans $((v, f), y, (v, g))$ and $((v, \alpha\overline{w}), \overline{x}, (v, \beta\overline{w}))$ $\mathrm{cod}_{\mathcal{S}}$-equivalent (see Definition 3.4).

We can say that the concreteness axiom is not among the usually required ones for fibrations of small maps, and as far as we know, it is new. As pointed out in Remark 3.7 and by virtue of the discussion carried out in Section 4, concreteness and local smallness are unrelated in general or related under the definability of isomorphisms in $\mathrm{cod}_{\mathcal{S}}$ relative to $\mathcal{S}$. Consequently, we feel entitled to also consider the following axiom.

Axiom 6.10 (Iso-definability) For every small map $y : Y \to J$ and for every morphism $f : X \to Y$ with yf a small map, there exist a monic small map ι and a commutative diagram

$$\begin{tikzcd} \bullet \arrow[r, "\bar{\iota}"] \arrow[d, "i"'] \arrow[dr, phantom, "(*)"] & X \arrow[r, "f"] \arrow[d, "yf"] & Y \arrow[dl, bend left, "y"] \\ W \arrow[r, "\iota"'] & J & \end{tikzcd} \tag{10}$$

in which $(*)$ is a pullback that makes the exterior quadrilateral a pullback of y along ι, such that for every commutative diagram

$$\begin{tikzcd} \bullet \arrow[r, "\overline{v}"] \arrow[d, "n"'] \arrow[dr, phantom, "(+)"] & X \arrow[r, "f"] \arrow[d, "yf"] & Y \arrow[dl, bend left, "y"] \\ K \arrow[r, "v"'] & J & \end{tikzcd}$$

in which $(+)$ is a pullback that makes the exterior quadrilateral a pullback of y along v, there exists a unique pullback

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "n"'] & \bullet \arrow[d, "i"] \\ K \arrow[r] & W \end{tikzcd}$$

making $(+)$ factorize through $(*)$.

Now, by virtue of the previous observations, we give the counterpart to Proposition 4.7.

Proposition 6.11 *Let $(\mathbb{B}, \mathcal{S})$ be a category with small maps. If the sum, iso-definability, and concreteness axioms hold, then the local smallness axiom holds.*

Proof The proof is a suitable counterpart to that for Proposition 4.7 in the present setting. The sum axiom must be taken into account since at a certain point the composition $\pi \circ \iota : W \to I$ (see diagrams (8) and (10)) must be a small map. $\square$

We dedicate the rest of this section to recovering a suitable transposition of Corollary 4.9 in the present setting.

Definition 6.12 Let $(\mathbb{B}, \mathcal{S})$ be a category with small maps. We say that $\mathbb{B}$ has *small evaluation spans* if every diagram

$$\begin{tikzcd} A \arrow[d, "a"'] & \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

in which a and u are small maps, can be completed to a pullback diagram

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "\varepsilon"'] & E \arrow[dd, "e"] \\ A \arrow[d, "a"'] & \\ I \arrow[r, "u"'] & J \end{tikzcd} \tag{11}$$

in which e is a small map, such that for every pullback diagram

$$\begin{tikzcd} P' \arrow[r, "\varphi'"] \arrow[d, "a\circ\varepsilon'"'] & E' \arrow[d, "e'"] \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

in which e' is a small map, there exists a unique morphism $\beta : e' \to e$ in $\mathcal{S}_J$ such that in the diagram

$$\begin{tikzcd} P' \arrow[r, "\varphi'"] \arrow[dr, dashed, "\overline{\beta}"] \arrow[ddr, bend right, "\varepsilon'"'] & E' \arrow[dr, dashed, "\beta"] & \\ & P \arrow[r, "\varphi"] \arrow[d, "\varepsilon"'] & E \arrow[dd, "e"] \\ & A \arrow[d, "a"'] & \\ & I \arrow[r, "u"'] & J \end{tikzcd}$$

$\varepsilon \circ \overline{\beta} = \varepsilon'$, with $\overline{\beta}$ the uniquely induced morphism to the pullback diagram (11), which, in view of all this, is referred to as a small evaluation span of a along u.

Definition 6.13 Let $(\mathbb{B}, \mathcal{S})$ be a category with small maps. We say that $\mathbb{B}$ is *locally Cartesian closed relative to* $\mathcal{S}$ if $\mathrm{cod}_{\mathcal{S}}$ is Cartesian closed; that is, it has fiberwise finite products and exponents both stable under pullback along any morphism of $\mathbb{B}$, that is, $\mathrm{cod}_{\mathcal{S}}$-reindexing stable.

Proposition 6.14 *Let* $(\mathbb{B}, \mathcal{S})$ *be a category with small maps. The following are equivalent.*

(i) $\mathbb{B}$ *is locally Cartesian closed relative to* $\mathcal{S}$.
(ii) *The identity and sum axioms hold, and* $\mathrm{cod}_{\mathcal{S}}$ *is locally small relative to* $\mathcal{S}$*; that is, the local smallness axiom holds as well.*

Proof To prove (i) $\Longrightarrow$ (ii), we use Proposition 6.6 to obtain that the identity and sum axioms hold. Now, for every pair of small maps $u : I \to J$, $b : B \to J$, the exponent small map $u \Rightarrow b : E \to J$ is the one occurring in a small evaluation span

$$\begin{tikzcd}
P \arrow[rr, "\varphi"] \arrow[d, "\varepsilon"'] & & E \arrow[dd, "u\Rightarrow b"] \\
\bullet \arrow[r, "\overline{u}"] \arrow[dr, "\overline{b}"'] & B \arrow[dr, "b"] \arrow[d, phantom, "(*)"] & \\
& I \arrow[r, "u"'] & J
\end{tikzcd} \tag{12}$$

of $\overline{b}$ along u, where $(*)$ is a pullback. The whole quadrilateral (12) is a pullback whose diagonal is $u \wedge u \Rightarrow b$. Moreover, $\overline{u} \circ \varepsilon : u \wedge u \Rightarrow b \to b$ in $\mathcal{S}_J$ is universal as needed. It can be verified that the span

$$\begin{tikzcd}
I \arrow[dd, "u"'] & \bullet \arrow[l, "\overline{b}"'] \arrow[d, "\overline{u}"] & P \arrow[l, "\varepsilon"'] \arrow[r, "\overline{u}\circ\varepsilon"] \arrow[dd, "\varphi"] & B \arrow[dd, "b"] \\
& B \arrow[dl, "b"] & & \\
J & & E \arrow[ll, "u\Rightarrow b"] \arrow[r, "u\Rightarrow b"'] & J
\end{tikzcd}$$

whose left-hand component is nothing but the evaluation span (12), is universal as required in the local smallness axiom.

To prove (ii) $\Longrightarrow$ (i), we use Proposition 6.6 to obtain that $\mathrm{cod}_{\mathcal{S}}$ has $\mathrm{cod}_{\mathcal{S}}$-reindexing stable fiberwise finite products. Now refer to the universal span (6). It can be verified that $f : a \wedge u \to b$ in $\mathcal{S}_J$ is a universal morphism from the functor $a \wedge - : \mathcal{S}_J \to \mathcal{S}_J$ to b. It is well known (see Lawvere [19]) that the stability of exponents under $\mathrm{cod}_{\mathcal{S}}$-reindexing is guaranteed by the following Frobenius reciprocity: for all small maps $u : I \to J$, $a : A \to I$, and $b : B \to J$

$$u \circ (a \wedge b \cdot u) \simeq (u \circ a) \wedge b$$

in $\mathcal{S}_J$, which holds since the sum axiom holds. □

In the particular case in which all the morphisms of $\mathbb{B}$ are considered to be small, as a consequence of the previous proposition, one has the following well-known result.

Proposition 6.15 *Let* $\mathbb{B}$ *be a category with pullbacks. The fibration* $\mathrm{cod}_{\mathbb{B}}$ *is locally small if and only if* $\mathbb{B}$ *is a locally Cartesian closed category.*

We end this section by formulating the counterpart to Corollary 4.9 in the present setting.

Corollary 6.16 *Let* $\mathbb{B}$ *be a finitely complete category, and let* $(\mathbb{B}, \mathcal{S})$ *be a category with small maps. Let* $\mathrm{cod}_{\mathcal{S}}$ *satisfy the concreteness axiom. Then* $\mathbb{B}$ *is locally Cartesian closed relative to* $\mathcal{S}$ *if and only if* $\mathrm{cod}_{\mathcal{S}}$ *satisfies the iso-definability axiom.*

7 Concreteness of Subfibrations

The notion of concrete fibration that we introduced and discussed in the previous sections is founded on a suitable counterpart to the Isbell condition (see Definition 2.6) in the fibered setting (see Definition 3.4). As already pointed out, this plan was motivated by the desire to pursue the foundational attitude that characterizes the adoption of the fibrational point of view in approaching fundamental subjects in category theory in the spirit of [5]. In this section we discuss the concreteness of fibrations in terms of Proposition 7.1 below—so maybe in more intuitive terms. Actually, we do not do this for general fibrations but for fibrations of small maps, as these turn out to be more manageable in the new perspective we are adopting. In due course, the notions of definability and stack are cited. We already encountered definability in Section 4. A *stack* is a fibered category with sheaflike gluing properties that allow good control on its fibers. We will not go into the details of its description, considering that it will play just an auxiliary role in support of the general discussion and also that it will appear in a very specific form as an explicit axiom to be satisfied by the small maps. For more details, the interested reader is referred to [15], [24], and Janelidze and Tholen [16].

Proposition 7.1 *A category is concretizable if and only if it is isomorphic to a subcategory of* $\mathbb{S}\mathrm{ets}$.

Proof This result is obtained by constructing a faithful and injective-on-objects functor, that is, an embedding, out of a faithful one (see [1]). □

As in Section 3, the discussion of the naive case is enlightening. As opposed to the approach that we have pursued so far, one might be tempted to define a fibration $P : \mathbb{X} \to \mathbb{S}\mathrm{ets}$ as concrete if there is a faithful fibered functor $F : P \to P_{\mathbb{S}\mathrm{ets}}$, but then one wants to understand to what extent fibrations that are concrete in this sense can be identified with subfibrations of $P_{\mathbb{S}\mathrm{ets}}$, that is, to what extent one can go in parallel with Proposition 7.1. To face the problem it seems indispensable, or at least very useful in the first place, to have a characterization of the subfibrations of $P_{\mathbb{S}\mathrm{ets}}$ which correspond to subcategories of $\mathbb{S}\mathrm{ets}$ and to have conditions ensuring the possibility of constructing a fibered embedding out of a faithful fibered functor in the second place. Moreover, it seems natural to ask for such conditions to allow the implementation of this construction fiberwise, starting from the fiber-functor $F_1 : P_1 \to \mathbb{S}\mathrm{ets}$, where 1 is a terminal set, also recalling that a fibered functor is faithful if and only if it is fiberwisely so (see Remark 2.15).

A characterization of the subfibrations of $P_{\mathbb{S}\mathrm{ets}}$ which correspond to subcategories of $\mathbb{S}\mathrm{ets}$ is provided by the following well-known result.

Proposition 7.2 *Let* $\mathbb{S}$ *be a category. Every subcategory of* $\mathbb{S}$ *determines and is determined by a definable subfibration of* $P_{\mathbb{S}} : \mathrm{Fam}(\mathbb{S}) \to \mathbb{S}\mathrm{ets}$.

Proof A subcategory $\mathbb{C} \hookrightarrow \mathbb{S}$ determines the definable subfibration $P_{\mathbb{C}} \hookrightarrow P_{\mathbb{S}}$. A definable subfibration $Q \hookrightarrow P_{\mathbb{S}}$ determines a subcategory $\mathbb{C} \hookrightarrow \mathbb{S}$, where $\mathbb{C}$ is equivalent to Q_1. Moreover, one has a fibered equivalence $Q \equiv P_{\mathbb{C}}$ over $\mathbb{S}\mathrm{ets}$. □

With reference to a faithful fibered functor $F : P \to P_{\mathbb{S}\mathrm{ets}}$, the possibility of constructing a fibered embedding out of it from the single faithful fiber-functor F_1 amounts to the possibility of recovering, for every set I, P_I from P_1 and then to the possibility of reconstructing $F_I : P_I \to \mathbb{S}\mathrm{ets}^I$ from F_1. These seem to be the kind of possibilities which are available if suitable gluing conditions hold, which is the case when it is possible to see P as a stack over $\mathbb{S}\mathrm{ets}$ considered as a *site*, that is, a category equipped with the assignment of a class of covering families to each of its objects (see Johnstone [17], Vistoli [28]). Actually, if to every set I is assigned the class of covering families which is the singleton $\{(i : 1 \to I)_{i \in I}\}$, then, for every category $\mathbb{C}$, $P_{\mathbb{C}}$ is a stack, essentially because one has the series of equivalences $(P_{\mathbb{C}})_I \equiv \mathbb{C}^I \equiv \prod_{i\in I} \mathbb{C} \equiv \prod_{i \in I}(P_{\mathbb{C}})_1$ pointing out that the $P_{\mathbb{C}}$-fiber at a set I is reconstructible from the $P_{\mathbb{C}}$-fiber at 1, allowing one to conclude that in general $P : \mathbb{X} \to \mathbb{S}\mathrm{ets}$ is a stack, with respect to the covering under consideration, if and only if $P \equiv P_{P_1}$ as fibrations over $\mathbb{S}\mathrm{ets}$. Thus, Proposition 7.2 says that the definable subfibrations of a naive indexing $P_{\mathbb{S}}$ are exactly its substacks. There are examples of subfibrations of a naive indexing that are not definable and, thus, not substacks (see [6], [24]). This discussion puts in evidence that, already under a naive fibrational perspective, the notion of concrete fibration, as one equipped with a faithful functor to $P_{\mathbb{S}\mathrm{ets}}$, and that of subfibration of $P_{\mathbb{S}\mathrm{ets}}$ can be significantly different. They coincide for those fibrations $P : \mathbb{X} \to \mathbb{S}\mathrm{ets}$ which are stacks, namely, those of the form $P_{\mathbb{C}}$ for some concrete category $\mathbb{C}$, which is reassuring.

Now, let $(\mathbb{B}, \mathcal{S})$ be a category with small maps (see Definition 6.1). By virtue of Proposition 7.2 and the previous discussion, we can say that the fibration of small maps $\mathrm{cod}_{\mathcal{S}}$ (see Section 6) is concrete as a subfibration of $\mathrm{cod}_{\mathbb{B}}$ if it is a definable subfibration of $\mathrm{cod}_{\mathbb{B}}$, which amounts to the satisfaction of the following axiom.

Axiom 7.3 (Definability) For every morphism $a : A \to I$ in $\mathbb{B}$, there exist a monic morphism $u : K \to I$ and a pullback diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d, "b"'] & A \arrow[d, "a"] \\ K \arrow[r, "u"'] & I \end{tikzcd}$$

in which b is a small map, such that for every morphism $v : W \to I$ if a pullback of a along v is a small map, then v uniquely factors through u.

We recall that a morphism v in a category $\mathbb{B}$ is a *cover* if it is surjective, namely, that if $v = m \circ u$ with m a monic morphism, then m is an isomorphism. If $\mathbb{B}$ is considered as a site whose covering families are singletons of covers, then it is well known that $\mathrm{cod}_{\mathcal{S}}$ is a stack with respect to these covering families if and only if the following axiom holds.

Axiom 7.4 (Descent) For every pullback diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\pi"'] & B \arrow[d, "b"] \\ K \arrow[r, "v"'] & J \end{tikzcd}$$

in $\mathbb{B}$, with v a cover, if π is a small map, then b is a small map.

Now we recall the following well-known result.

Proposition 7.5 *Let $(\mathbb{B}, \mathcal{S})$ be a category with small maps. If the definability axiom holds, then the descent axiom holds.*

Proof The proof is straightforward. □

Finally, we can give the following definition.

Definition 7.6 Let $(\mathbb{B}, \mathcal{S})$ be a category with small maps. Then $\mathbb{B}$ is *concrete relative to $\mathcal{S}$* if the identity, sum, concreteness, iso-definability, and definability axioms hold.

Remark 7.7 Definition 7.6 comprises the ordinary situation as follows. Let $\mathbb{C} \hookrightarrow \mathbb{S}\mathrm{ets}$ be a wide subcategory, that is, one completely determined by its morphisms. Consider all the morphisms of $\mathbb{C}$ to be small maps. The identity and sum axioms are clearly satisfied. By virtue of Proposition 7.2, $P_{\mathbb{C}}$ is a definable subfibration of $P_{\mathbb{S}\mathrm{ets}}$ which is locally small. Hence, the definability axiom holds. The iso-definability axiom holds because $P_{\mathbb{C}}$ is a locally small fibration, as a definable subfibration of a locally small one (well known), and by virtue of Proposition 4.8. Finally, the concreteness axiom holds by virtue of Proposition 3.11.

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