Notre Dame Journal of Formal Logic
Volume 58, Number 1, 2017

Indiscernible Extraction and Morley Sequences

Sebastien Vasey

Abstract We present a new proof of the existence of Morley sequences in simple theories. We avoid using the Erdős–Rado theorem and instead use only Ramsey's theorem and compactness. The proof shows that the basic theory of forking in simple theories can be developed using only principles from "ordinary mathematics," answering a question of Grossberg, Iovino, and Lessmann, as well as a question of Baldwin.

1 Introduction

Shelah [12, Lemma 9.3] has shown that, in a simple first-order theory T, Morley sequences exist for every type. The proof proceeds by first building an independent sequence of length $\beth_{(2^{|T|})^+}$ for the given type and then using the Erdős–Rado theorem together with Morley's method to extract the desired indiscernibles.

After slightly improving on the length of the original independent sequence in [5, Appendix A], Grossberg, Iovino, and Lessmann observed that, in contrast, most of the theory of forking in a stable first-order theory T does not need the existence of such "big" cardinals. These authors then asked whether the same could be said about simple theories; in particular, they asked whether there was another way to build Morley sequences there. Baldwin (see [2] and [3, Question 3.9]) similarly asked[1] whether the equivalence between forking and dividing in simple theories had an alternative proof.

We answer those questions in the affirmative by showing how to extract a Morley sequence from any infinite independent sequence. Our construction relies on a property of forking we call *dual finite character*. We show that it holds in simple theories and that the converse is also true. (The latter statement was noticed by Itay Kaplan.)

Received May 3, 2014; accepted June 9, 2014
First published online December 2, 2016
2010 Mathematics Subject Classification: Primary 03C45; Secondary 03B30, 03E30
Keywords: forking, Morley sequences, dual finite character, simple theories
 10.1215/00294527-3800865

2 Preliminaries

For the rest of this paper, fix a complete first-order theory T in a language $L(T)$ and work inside its monster model $\mathfrak{C}$. We write $|T|$ for $|L(T)| + \aleph_0$. We denote by $\mathrm{Fml}(L(T))$ the set of first-order formulas in the language $L(T)$. If A is a set, we say a formula is *over* A if it has parameters from A. For a tuple $\bar{a}$ in $\mathfrak{C}$ and ϕ a formula, we write $\models \phi[\bar{a}]$ instead of $\mathfrak{C} \models \phi[\bar{a}]$.

When I is a linearly ordered set, $(\bar{a}_i)_{i \in I}$ are tuples, and $i \in I$, we write $\bar{a}_{<i}$ for $(\bar{a}_j)_{j<i}$. It is often assumed without comment that all the $\bar{a}_i$'s have the same (finite) arity.

We assume the reader is familiar with forking. We will use the combinatorial definition stated, for example, in [12, Definition 1.2]. It turns out that our construction of Morley sequences does not rely on this exact definition, but only on abstract properties of forking such as invariance, extension, and symmetry.

Recall also the definition of a Morley sequence.

Definition 1 Let I be a linearly ordered set. Let $\mathbf{I} := \langle \bar{a}_i \mid i \in I \rangle$ be a sequence of finite tuples of the same arity. Let $A \subseteq B$ be sets, and let $p \in S(B)$ be a type that does not fork over A. We say that $\mathbf{I}$ is an *independent sequence for p over A* if

1. for all $i \in I$, $\bar{a}_i \models p$;
2. for all $i \in I$, $\mathrm{tp}(\bar{a}_i / B\bar{a}_{<i})$ does not fork over A.

We say that $\mathbf{I}$ is a *Morley sequence for p over A* if

1. $\mathbf{I}$ is an independent sequence for p over A;
2. $\mathbf{I}$ is indiscernible over B.

3 Morley Sequences in Simple Theories

It is well known that independent sequences can be built by repeated use of the extension property of forking. If the theory is stable, then the existence of Morley sequences follows, because in such theories any sufficiently long sequence contains indiscernibles. The latter fact is no longer true in general, and in fact, there are counterexamples among both simple and dependent theories (see [13, p. 209], [9], resp.). Thus, a different approach is needed in the unstable case. Recall from the Introduction that we do not want to use big cardinals, so Morley's method cannot be used. We can, however, make use of the following variation of the Ehrenfeucht–Mostowski theorem.

Fact 2 ([14, Lemma 5.1.3]) *Let A be a set, and let I be a linearly ordered set. Let $\mathbf{J} := \langle \bar{a}_j \mid j < \omega \rangle$ be a sequence of finite tuples of the same arity. Then there exists a sequence $\mathbf{I} := \langle \bar{b}_i \mid i \in I \rangle$ indiscernible over A such that, for any $i_0 < \cdots < i_{n-1}$ in I, for all* finite *$q \subseteq \mathrm{tp}(\bar{b}_{i_0} \cdots \bar{b}_{i_{n-1}} / A)$, there exist $j_0 < \cdots < j_{n-1} < \omega$ so that $\bar{a}_{j_0} \cdots \bar{a}_{j_{n-1}} \models q$.*

Do we get a Morley sequence if we apply Fact 2 to an independent sequence? In general, we see no reason why it should be true. However, we will see that it *is* true if we assume the following local definability property of forking.

Definition 3 (Dual finite character) Forking is said to have *dual finite character (DFC)* if, whenever $\mathrm{tp}(\bar{c}/A\bar{b})$ forks over A, there is a formula $\phi(\bar{x}, \bar{y})$ over A such that

- $\models \phi[\bar{c}, \bar{b}]$, and
- $\models \phi[\bar{c}, \bar{b}']$ implies $\mathrm{tp}(\bar{c}/A\bar{b}')$ forks over A.

A variation of DFC appears as [11, Property A.7′], but we have not found any other explicit occurrence in the literature. Notice that DFC immediately implies something stronger.

Proposition 4 *Assume that forking has DFC. Assume that $p := \mathrm{tp}(\bar{c}/A\bar{b})$ forks over A, and $\phi(\bar{x}, \bar{y})$ is as given by Definition 3. Then $\mathrm{tp}(\bar{c}'/A) = \mathrm{tp}(\bar{c}/A)$ and $\models \phi[\bar{c}', \bar{b}']$ imply that $\mathrm{tp}(\bar{c}'/A\bar{b}')$ forks over A.*

Proof Assume $\mathrm{tp}(\bar{c}'/A) = \mathrm{tp}(\bar{c}/A)$. Let f be an automorphism of $\mathfrak{C}$ fixing A such that $f(\bar{c}') = \bar{c}$. Assume $\models \phi[\bar{c}', \bar{b}']$. By applying f, $\models \phi[\bar{c}, f(\bar{b}')]$. Since ϕ witnesses DFC, $\mathrm{tp}(\bar{c}/Af(\bar{b}'))$ forks over A. By applying f^{-1} and by using the invariance of forking, $\mathrm{tp}(\bar{c}'/A\bar{b}')$ forks over A. □

Theorem 5 *Assume that forking has DFC. Let $A \subseteq B$ be sets. Let $p \in S(B)$ be a type that does not fork over A. Let I be a linearly ordered set. Then there is a Morley sequence $\mathbf{I} := \langle \bar{b}_i \mid i \in I \rangle$ for p over A.*

Proof By repeated use of the extension property of forking, build an independent sequence $\mathbf{J} := \langle \bar{a}_j \mid j < \omega \rangle$ for p over A. Let $\mathbf{I} := \langle \bar{b}_i \mid i \in I \rangle$ be indiscernible over B as described by Fact 2. We claim $\mathbf{I}$ is as required.

It is indiscernible over B, and for every $i \in I$, every $\bar{b}_i$ realizes p: if $\bar{b}_i \not\models p$, fix a formula $\phi(\bar{x}, \bar{b}) \in p$ so that $\models \neg\phi[\bar{b}_i, \bar{b}]$. By the defining property of $\mathbf{I}$, there exists $j < \omega$ so that $\models \neg\phi[\bar{a}_j, \bar{b}]$, so $\bar{a}_j \not\models p$, which is a contradiction.

It remains to see that, for every $i \in I$, $p_i := \mathrm{tp}(\bar{b}_i/B\bar{b}_{<i})$ does not fork over A. Assume not, and fix $i \in I$ so that p_i forks over A. Fix $\bar{b} \in B$ and $i_0 < \cdots < i_{n-1} < i$ such that $p_i' := \mathrm{tp}(\bar{b}_i/A\bar{b}_{i_0} \cdots \bar{b}_{i_{n-1}}\bar{b})$ forks over A. Fix $\phi(\bar{x}, \bar{b}_{i_0} \cdots \bar{b}_{i_{n-1}}\bar{b}) \in p_i'$, a formula over A witnessing DFC.

Find $j_0 < \cdots < j_n < \omega$ such that $\models \phi[\bar{a}_{j_n}, \bar{a}_{j_0} \cdots \bar{a}_{j_{n-1}}\bar{b}]$. Since it has already been observed that $\mathrm{tp}(\bar{a}_{j_n}/A) = \mathrm{tp}(\bar{b}_i/A) = p \upharpoonright A$, Proposition 4 implies that $\mathrm{tp}(\bar{a}_{j_n}/A\bar{a}_{j_0} \cdots \bar{a}_{j_{n-1}}\bar{b})$ forks over A, contradicting the independence of $\mathbf{J}$. □

We now show that a simple theory has DFC. (This was essentially already observed by Makkai [11].) Recall from [10, Theorem 2.4] that T is simple exactly when forking has the symmetry property. Moreover, the methods of [1] show that the equivalence can be proven without using Morley sequences. The key is [1, Theorem 3.6], which shows (without using Morley sequences) that if the D-rank is bounded, then symmetry holds.

Lemma 6 *Assume that T is simple. Then forking has DFC.*

Proof Assume $p := \mathrm{tp}(\bar{c}/A\bar{b})$ forks over A. By symmetry, $q := \mathrm{tp}(\bar{b}/A\bar{c})$ forks over A. Fix $\psi(\bar{y}, \bar{x})$ over A such that $\psi(\bar{y}, \bar{c}) \in q$ witnesses forking; that is, if $\models \psi[\bar{b}', \bar{c}]$, then $\mathrm{tp}(\bar{b}'/A\bar{c})$ forks over A.

Let $\phi(\bar{x}, \bar{y}) := \psi(\bar{y}, \bar{x})$. Then $\phi(\bar{x}, \bar{b}) \in p$, and if $\models \phi[\bar{c}, \bar{b}']$, then $\models \psi[\bar{b}', \bar{c}]$. Hence, $\mathrm{tp}(\bar{b}'/A\bar{c})$ forks over A, so by symmetry, $\mathrm{tp}(\bar{c}/A\bar{b}')$ forks over A. This shows $\phi(\bar{x}, \bar{y})$ witnesses DFC. □

Corollary 7 (Existence of Morley sequences in simple theories) *Assume that T is simple. Let $A \subseteq B$ be sets. Let $p \in S(B)$ be a type that does not fork over A. Let I*

be a linearly ordered set. Then there is a Morley sequence $\mathbf{I} := \langle \bar{b}_i \mid i \in I \rangle$ *for* p *over* A.

Proof Combine Lemma 6 and Theorem 5. □

We end by closing the loop in our study of DFC: Lemma 6 shows that simplicity implies DFC, but it turns out that they are equivalent! This was pointed out by Itay Kaplan [8] in a personal communication. Definition 9 and the statement that (2) implies (3) implies (1) in Theorem 10 below are due to Kaplan, and I am grateful to him for allowing me to include them here. The key is to observe that symmetry fails very badly when the theory is not simple.

Fact 8 ([4, Lemma 6.16]) *Assume that* T *is* not *simple. Then there are a model* M *and tuples* $\bar{b}, \bar{c}$ *such that* $\text{tp}(\bar{b}/M\bar{c})$ *is finitely satisfiable in* M*, but* $\text{tp}(\bar{c}/M\bar{b})$ *divides over* M.

We are now ready to prove that forking has DFC exactly when the theory is simple. In fact, we only need the following version of DFC.

Definition 9 Forking is said to have *weak DFC* if, whenever M is a model and $\text{tp}(\bar{c}/M\bar{b})$ divides over M, there is a formula $\phi(\bar{x}, \bar{y})$ over M such that

- $\models \phi[\bar{c}, \bar{b}]$, and
- $\models \phi[\bar{c}, \bar{b}']$ implies $\text{tp}(\bar{c}/M\bar{b}')$ is not finitely satisfiable in M.

Theorem 10 *The following are equivalent.*

1. *T is simple.*
2. *Forking has DFC.*
3. *Forking has weak DFC.*

Proof That (1) implies (2) is Lemma 6, and that (2) implies (3) is because finite satisfiability implies nonforking. We show that (3) implies (1). Assume that T is not simple. Fix M, $\bar{b}$, and $\bar{c}$ as given by Fact 8. In particular, $p := \text{tp}(\bar{c}/M\bar{b})$ divides over M. Let $\phi(\bar{x}, \bar{y})$ be a formula over M such that $\models \phi[\bar{c}, \bar{b}]$. By assumption, $\text{tp}(\bar{b}/M\bar{c})$ is finitely satisfiable in M, so in particular there is $\bar{b}' \in M$ such that $\models \phi[\bar{c}, \bar{b}']$. Thus, $\text{tp}(\bar{c}/M\bar{b}') = \text{tp}(\bar{c}/M)$ must be finitely satisfiable over M; hence, $\phi(\bar{x}, \bar{y})$ cannot witness weak DFC for p. Since ϕ was arbitrary, this shows weak DFC fails. □

We end by pointing out that all the results of this paper could be formalized in a weak fragment of ZFC, such as ZFC − Replacement − Power set + "for any set X of size $\leq |T|$, $\mathcal{P}(\mathcal{P}(X))$ exists."[2] To go further, it would be interesting to extend Harnik's [6], [7] work on the reverse mathematics of stability theory by finding the exact proof-theoretic strength of the existence of Morley sequences.

Notes

1. Akito Tsuboi [15] has independently answered this question.
2. Formally, we have to work in a language containing a constant symbol standing for $|T|$.

References

[1] Adler, H., "Thorn-forking as local forking," *Journal of Mathematical Logic*, vol. 9 (2009), pp. 21–38. MR 2665780. DOI 10.1142/S0219061309000823. 129

[2] Baldwin, J. T., "The use of replacement in model theory," Foundations of Mathematics (FOM) mailing list, 28 January 2010, http://www.cs.nyu.edu/pipermail/fom/2010-January/014347.html. 127

[3] Baldwin, J. T., "How big should the monster model be?," pp. 30–51 in *Logic without Borders*, edited by Å. Hirvonen, J. Kontinen, R. Kossak, and A. Villaveces, vol. 5 of *Ontos Mathematical Logic*, de Gruyter, Boston, 2015. DOI 10.1515/9781614516873.31. 127

[4] Chernikov, A., "Theories without the tree property of the second kind," *Annals of Pure and Applied Logic*, vol. 165 (2014), pp. 695–723. MR 3129735. DOI 10.1016/j.apal.2013.10.002. 130

[5] Grossberg, R., J. Iovino, and O. Lessmann, "A primer of simple theories," *Archive for Mathematical Logic*, vol. 41 (2002), pp. 541–80. MR 1923196. DOI 10.1007/s001530100126. 127

[6] Harnik, V., "Stability theory and set existence axioms," *Journal of Symbolic Logic*, vol. 50 (1985), pp. 123–37. MR 0780531. DOI 10.2307/2273795. 130

[7] Harnik, V., "Set existence axioms for general (not necessarily countable) stability theory," *Annals of Pure and Applied Logic*, vol. 34 (1987), pp. 231–43. MR 0899081. DOI 10.1016/0168-0072(87)90002-9. 130

[8] Kaplan, I., personal communication, March 2014. 130

[9] Kaplan, I., and S. Shelah, "A dependent theory with few indiscernibles," *Israel Journal of Mathematics*, vol. 202 (2014), 59–103. MR 3265314. DOI 10.1007/s11856-014-1067-2. 128

[10] Kim, B., "Simplicity, and stability in there," *Journal of Symbolic Logic*, vol. 66 (2001), pp. 822–36. MR 1833481. DOI 10.2307/2695047. 129

[11] Makkai, M., "A survey of basic stability theory, with particular emphasis on orthogonality and regular types," *Israel Journal of Mathematics*, vol. 49 (1984), pp. 181–238. MR 0788268. DOI 10.1007/BF02760649. 129

[12] Shelah, S., "Simple unstable theories," *Annals of Mathematical Logic*, vol. 19 (1980), pp. 177–203. MR 0595012. DOI 10.1016/0003-4843(80)90009-1. 127, 128

[13] Shelah, S., "Monadic logic: Hanf numbers," pp. 203–23 in *Around Classification Theory of Models*, vol. 1182 of *Lecture Notes in Mathematics*, Springer, Berlin, 1986. MR 0850059. DOI 10.1007/BFb0098511. 128

[14] Tent, K., and M. Ziegler, *A Course in Model Theory*, vol. 40 of *Lecture Notes in Logic*, Cambridge University Press, Cambridge, 2012. MR 2908005. DOI 10.1017/CBO9781139015417. 128

[15] Tsuboi, A., "Dividing and forking: A proof of the equivalence," *RIMS Kokyuroku*, vol. 1888 (2014), pp. 23–27. 130

Acknowledgments

This paper was written while working on a Ph.D. dissertation under the direction of Rami Grossberg at Carnegie Mellon University, and I would like to thank Professor Grossberg for his guidance and assistance in my research in general and in this work specifically. I also thank John Baldwin, José Iovino, Itay Kaplan, Alexei Kolesnikov, Anand Pillay, and Akito Tsuboi for valuable comments on earlier versions of this paper.

Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, Pennsylvania 15217
USA
sebv@cmu.edu
http://math.cmu.edu/~svasey