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Inferentialism and Quantification

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Abstract Logical inferentialists contend that the meanings of the logical constants are given by their inference rules. Not just any rules are acceptable, however: inferentialists should demand that inference rules must reflect reasoning in natural language. By this standard, I argue, the inferentialist treatment of quantification fails. In particular, the inference rules for the universal quantifier contain free variables, which find no answer in natural language. I consider the most plausible natural language correlate to free variables—the use of variables in the language of informal mathematics—and argue that it lends inferentialism no support.

1 Introduction

Logical inferentialists contend that the meanings of the logical constants are given by their inference rules.[1] Not just any rules are acceptable, however: inferentialists generally place constraints on admissible rules. I will focus on the constraint of *answerability*, which demands that the deductive system within which these rules are articulated must reflect reasoning in natural language. By this standard, I argue, the inferentialist treatment of quantification fails. In particular, the inference rules for the universal quantifier contain free variables, which find no answer in natural language. I consider the most plausible natural language correlate to free variables—the use of variables in the language of informal mathematics—and argue that it lends inferentialism no support.

2 Inferentialism and Answerability

Logical inferentialists should accept what Steinberger [8, p. 335] calls the *principle of answerability*:

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> *Principle of answerability* Only such deductive systems are permissible as can be seen to be suitably connected to our ordinary deductive inferential practices.

The inferential powers of a logical constant are represented by its introduction and elimination rules, as articulated within a given deductive system. But not just any inference rules whatsoever can succeed in conferring meaning on a logical constant. As [8, p. 335] makes clear, "it is the practice represented, not the formalism as such, that confers meanings." The thought is that inferentialists are out to capture the meaning-conferring aspects of our practice in their formalism, so it is reasonable to demand that an inferentialist's formalism does accurately represent these features of practice.

It may be objected that the principle of answerability is too demanding, because many of the usual logical connectives, for example, the material conditional, or even conjunction, do not adequately capture the whole meaning of natural language indicative conditionals and conjunctions. These are not, however, the expressions that must match natural language, according to the principle of answerability. Rather, these are the expressions whose meanings the logical inferentialist *determines* by using their formalism. It is the *deductive system* within which the inference rules are formulated that must be answerable to the meaning-conferring practices, since it is the deductive system that represents what a speaker has to grasp to understand an expression.

Steinberger discusses answerability in the context of a discussion of multiple-conclusion systems, arguing that multiple-conclusion arguments do not occur in natural language and so, given answerability, inferentialists should not make use of multiple-conclusion logical systems.

I do not need to take a stance on multiple-conclusion systems, but will argue that the inferentialist's rules for the universal quantifier fail to satisfy the principle of answerability. The meaning of the universal quantifier, they claim, is given by, for example, the following pair of rules:[2]

$$\frac{A(y/x)}{\forall x A}\ \forall\mathrm{I} \qquad \frac{\forall x A \qquad \begin{matrix}[A(t/x)]\\ \vdots \\ C\end{matrix}}{C}\ \forall\mathrm{E}$$

In the $\forall$I rule, $A(y/x)$ is the result of replacing every instance of x in A with y. Intuitively, $\forall x A$ can be derived just when $A(y/x)$ has been derived *for arbitrary* y. As is familiar, this arbitrariness is ensured by placing a variable restriction on the rule: y must not occur free in any assumption that $A(y/x)$ depends on, nor in $\forall x A$. In this way, y is a free variable. In $\forall$E, t is a term.

3 Informal Mathematics

If the inferentialist is to find natural language counterparts to free variables, the obvious place to look is the language of informal mathematics, where constructions such as "consider a pair of integers x and y" or "consider an arbitrary triangle ABC" are common. If natural language counterexamples to free variables are to be found anywhere, it seems they will be found in the language of informal mathematics. For these examples to serve the inferentialist's needs, however, their logical behavior must match—to a reasonable degree—that of the free variables in the formalism. I will now argue that this is not the case.

First, we know from Kleene [3, Section 32] that there are two ways in which free variables are used in informal mathematics, which he calls the *generality* interpretation and the *conditional* interpretation.[3] Consider the following pair of equations:

(1) $(x + y)^2 = x^2 + 2xy + y^2$;
(2) $\sin^2 x + \cos^2 x = 1$.

The claims made in (1) and (2) are true for *any* numerical substitution of the variables x and y. The most natural interpretation of the variables is, therefore, the *generality* interpretation, which means that for any value, the claim holds.

In contrast, consider the following equation:

(3) $x^2 + 2 = 3x$.

In this case, (3) is not true for *every* numerical substitution of x. Rather, the variable is functioning as what Rosser [7] and others call an *unknown*. Equation (3) is true for *some* replacements of x but not all, and as such a *conditional* interpretation is more appropriate: *if* x takes some value, *then* the claim holds.

On both the generality and conditional interpretations, the concept of *logical truth* remains the same: for an open sentence A with x free, $\models A$ means that every valuation of x satisfies A. The choice of interpretation makes a significant difference, however, to the concept of *logical consequence*. On the generality interpretation,

(Generality): $A \models B$ means that, *if* every valuation of x satisfies A, then every valuation of x satisfies B.

On the conditional interpretation,

(Conditional): $A \models B$ means that *any* valuation of x that satisfies A also satisfies B.

One important upshot of this distinction is that, by (Generality), A is logically equivalent to $\forall x A$. By (Conditional), however, $\forall x A \models A$, but $A \not\models \forall x A$. This is because A may be satisfied by one value of x but not another. In this way, the conditional interpretation is *stronger* than the generality interpretation. The inferentialist must decide, therefore, whether it is the generality or the conditional interpretation of free variables in informal mathematics that their formalism represents. I will take them in turn.

First, the inferentialist could suggest that free variables on the *conditional* interpretation are the most plausible natural language correlates to their formalism. This does not seem like an attractive option, however, as on the conditional interpretation, A does not entail $\forall x A$. It is for this reason that open sentences are not logically equivalent to their universally quantified counterparts on the conditional interpretation. But of course the proof theorist *needs* open sentences to imply their universally quantified counterparts if they are to match the behavior of their formalism.

It seems, then, that the proof theorist's best option is to appeal to the generality interpretation of free variables.[4] On the generality reading, $A \models \forall x A$ and $\forall x A \models A$, so the inferential powers *do* match the formalism. The problem, however, is that the open sentences in the proof theorist's rules are now *implicitly general*. If we asked the proof theorist to explain the meaning of an open sentence, they would have to do so with reference to that sentence's universally closed counterpart, for this is precisely what the generality interpretation requires. But if we then asked the proof theorist for the meaning of a universally quantified sentence, they would have to—by their inferentialism—appeal to the inference rules for the universal quantifier. But, as we have seen, these inference rules involve open sentences. Thus, on the generality interpretation, any explanation of meaning is circular: the universal quantifier's meaning is explained with reference to an open sentence, which is in turn explained with reference to a universal quantifier. We are not, therefore, given any satisfactory account of the meaning of the universal quantifier, which the inferentialist requires.[5]

The inferentialist cannot, therefore, appeal to the role of free variables in informal mathematics to satisfy the principle of answerability. Such variables must be interpreted on the conditional or the generality approach, but neither is successful: the former fails to match the formalism, and the latter renders any meaning explanations circular.

4 Inferentialist Responses

First, it may be objected that, although most inferentialists do use free variables in the inference rules for the quantifiers, they *need* not. Rather, the introduction rule for the universal quantifier could be given by, for example,

$$\frac{A(a)}{\forall x A(x)}\ \forall\mathrm{I}'$$

The suggestion is that, in $\forall\mathrm{I}'$, a is not a free variable but an *arbitrary name*. If we formulate universal introduction in this way, then there are no free variables involved, only bound variables and arbitrary names. As such, we do not need to find readings of free variables in natural language. Rather, we need to find natural language counterparts to the expression $F(a)$, with a as an arbitrary name. Quite plausibly, something like "F is true of an arbitrary a" is an English counterpart to this. So, the *arbitrary name* response goes, we avoid the problem of free variables by eliminating them and replacing them with arbitrary names for which English readings can be found.

It should first be noted that, despite occasional appearances to the contrary, this is not usually the way that inferentialists formulate the rule of universal introduction. Prawitz [5]—though no other time slices of Prawitz—does express the rule syntactically in this way, as does Dummett [2]. Both writers, however, are clear that a is still being treated as a free variable: they are merely using letters from the start of the alphabet for free variables and those from the end of the alphabet for bound variables (see [5, p. 7] and [2, p. 259]).

In any case, the arbitrary name response is not one that can help the inferentialist at this point. The problem comes with how the formula containing an arbitrary name, $F(a)$, is to be understood. If a functions as a name, then the resulting introduction rule that supposedly gives the meaning of the universal quantifier fails: the introduction rule should not feature any *particular* name, because the meaning of the universal quantifier does not depend on any particular name. Rather, the universal quantifier must be in some sense *schematic*: it must somehow range over all of the particular closed instances. After all, it is clearly not sufficient grounds to assert $\forall x Fx$ that we have established F to hold of some name in particular; we must have established F to hold of every name. So it must not be that a in $\forall\mathrm{I}'$ involves any particular name, as the rule must be in some way schematic. What more must be added to this rule to ensure that a really is arbitrary?

It would have to be the case that a does not occur in $\forall x A(x)$ nor in any assumption on which $A(a)$ depends. In other words, even though a is supposed to be functioning as an arbitrary name, rather than as a free variable, they come to the same. It is functioning as a variable because, to achieve the generality required, it must have a range of instances. And the range of instances is having to be limited by just the same restrictions that have to be in play for any rule that attempts to achieve generality in this way. And it is a free variable because it is not bound by any quantifier. The arbitrary name response fails, therefore, because the sort of arbitrary name that would have to be in place for the rule to be plausible would make it indistinguishable from a free variable.

The inferentialist may respond at this point that, although free variables do feature ineliminably in the introduction rule for the universal quantifier, this is unproblematic because the universal quantifier is itself eliminable. In particular, it can be defined in terms of the existential quantifier by using $\neg\exists\neg$, and the introduction rule for the existential quantifier does not feature any free variables:

$$\frac{A(t/x)}{\exists x A}\ \exists\mathrm{I}$$

In the introduction rule, t is not a free variable, since it does not range over all values: it is not the case that to assert $\exists x A$, we must have established that A holds of *every* name. And, the response continues, since the introduction rules *typically* give the meaning of a logical constant on the inferentialist approach, providing an acceptable introduction rule for the existential quantifier and taking the universal quantifier as defined is enough.

This response is not, however, an attractive one for the inferentialist to make because, even though existential introduction typically does not involve open sentences, existential elimination typically does:

$$\frac{\exists x A \qquad \begin{array}{c}[A(y/x)]\\ \vdots \\ C\end{array}}{C}\ \exists\mathrm{E}$$

So, although the open sentences have moved from the introduction to the elimination rule, the open sentences remain and have no natural language counterpart. So making this response would amount to the inferentialist admitting that they have gotten the extension of logical consequence wrong: their valid open arguments have no natural language counterpart, so their concept of logical consequence has overgenerated.

Another possible response from the logical inferentialist is that they rely on a prior *metalinguistic* understanding of universal quantification. For example, in the case of conjunction, the following metalinguistic clause may be given:

(C) $A \wedge B$ can be deduced once we have deduced A and we have deduced B.

We may worry that there is circularity involved at the metalevel in the case of conjunction because (C) uses the expression "and." But (C) is nevertheless generally allowed as an elucidation of conjunction. Why is the case of universal quantification different?[6]

My reply is that *even if* the appeal to a prior metalinguistic understanding is legitimate *and* is unproblematic in the case of conjunction, the analogy between conjunction and the universal quantifier breaks down.

Steinberger [8, Section 7] discusses the issue of circularity and conjunction. He notes that it may *seem* circular to offer a clause such as (C) as an elucidation of the meaning of conjunction, but that this is unproblematic. We do not need a prior understanding of "and" to be able to assert A and to be able to assert B. What we learn when we master the meaning of $\wedge$ is that when we are warranted to assert A, and when we are warranted to assert B, we are warranted to assert $A \wedge B$, and vice versa. But no understanding of "and" is presupposed by our knowing how to assert A and our knowing how to assert B.

This is the sort of reason we may have for thinking that an apparent circularity in the case of conjunction is unproblematic. But, even if something like this story is correct, the analogous story cannot be told in the case of the universal quantifier. The logical inferentialist could offer the following clause, along with the usual restrictions:

(Q) $\forall x A$ can be deduced once we have deduced $A(y/x)$.

But, if the free variables in $A(y/x)$ are understood on the generality interpretation, as I have argued they should, then universal quantification is being presupposed once again. In the case of universal quantification, therefore, the metalinguistic move on the part of the logical inferentialist has merely pushed the problem up a level. This was not the case with conjunction because, as we have seen, no prior understanding of "and" is required to assert conjunctions by using the inferentialist's rules. But, in the case of the universal quantifier, a prior understanding of universal quantification *is* being presupposed when the generality interpretation of free variables is in place.

The analogy with conjunction therefore breaks down: even if the metalinguistic move is a legitimate one *and* it is unproblematic in the case of conjunction, the same move cannot be made in the case of the universal quantifier. The inferentialist is able to appeal to a prior metalinguistic understanding of conjunction, therefore, because no circularity is involved. But the same move cannot be made in the case of the universal quantifier.

5 Conclusion

Inferentialists should accept the principle of answerability. But this creates problems for their treatment of quantification: the inference rules for the quantifiers make use of free variables, and there is no answer to these in natural language. The obvious answer would be the language of informal mathematics, but here they must choose between Kleene's conditional and generality readings: on the former, the language does not logically match the formalism; on the latter, any meaning explanations are circular.

Notes

1. *Logical* inferentialism is generally contrasted with the *global* inferentialism of, for example, [1], who holds that the meanings of *all* expressions can be given by introduction and elimination rules.

2. This exact formulation is found in [4, p. 64].

3. I am assuming that Kleene's interpretations are *exhaustive*, since it is not obvious what other interpretation is possible, and I am not aware of any other interpretation in the literature.

4. The generality interpretation is implicit in the work of most logical inferentialists. Read [6, p. 137] is explicit, however, that he intends the generality interpretation of free variables.

5. I do not want to imply that we do not understand reasoning about arbitrary objects in informal mathematics. Rather, I believe that we *do* understand such reasoning, so it is a failure of logical inferentialism that it cannot account for this understanding. What is needed instead, I suggest, is an alternative approach to the meanings of the logical constants in terms of *truth-conditions*.

6. I am grateful to an anonymous reviewer for making this response.

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