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Improving a Bounding Result That Constructs Models of High Scott Rank

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Abstract Let T be a theory in a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ whose extensions in countable fragments have only countably many types. Sacks proves a bounding theorem that generates models of high Scott rank. For this theorem, a tree hierarchy is developed for T that enumerates these extensions.

In this paper, we effectively construct a predecessor function for formulas defining types in this tree hierarchy as follows. Let $T_\gamma \subseteq T_\delta$ with T_γ- and T_δ-theories on level γ and δ, respectively. Then if $\ulcorner p \urcorner(T_\delta)$ is a formula that defines a type for T_δ, our predecessor function provides a formula for defining its subtype in T_γ.

By constructing this predecessor function, we weaken an assumption for Sacks's result.

1 Introduction

Vaught's conjecture questions the number of models of a complete, countable theory and is one of the questions that have shaped modern model theory. In Morley's groundbreaking paper [3], which gives a positive result toward Vaught's conjecture, he introduces the notion of scattered theories, defined below in Section 1.2. Sacks [4] uses a generalization of these theories, called *weakly scattered theories*, to produce further results. He introduces a tree hierarchy, called the *raw hierarchy*, detailed in Section 1.2, that enumerates the models of a weakly scattered theory.

Within the raw hierarchy, we construct a predecessor function for formulas that define types for theories in the tree. If $T_\gamma \subseteq T_\delta$ with T_γ- and T_δ-theories on level γ and δ, respectively, then if $\ulcorner p \urcorner(T_\delta)$ is a formula that defines a type for T_δ, the predecessor function provides a formula for defining its subtype in T_γ. Using this predecessor function, we improve a bounding result in Sacks [4, Theorem 3.1] for weakly scattered theories.

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In Section 1, we provide an outline of important results from [4] that we will need for the predecessor function, including the small sets lemma and the iterated bounding theorem. We define a weakly scattered theory and provide its related hierarchy, the raw hierarchy. Then we recall the effective recovery theorem and outline its important steps necessary for our construction.

In Section 2, we construct our predecessor function. We conclude, in Section 3, by showing how this predecessor function improves the main bounding result in [4].

1.1 Small Δ_0 sets Throughout this paper, $\mathcal{L}$ is a countable first-order language. We work in $\mathcal{L}_{\omega_1,\omega}$, usually within a countable fragment $\mathcal{L}_A$, where A is a countable admissible set.

We use Barwise compactness extensively throughout, mostly in the form of the small sets lemma (see [4]) given below in Theorem 1.1. Also, Theorem 1.3 is an extremely useful result derived from uniformity hidden in the proof of the small sets lemma.

These results, together with the iterated bounding theorem (Theorem 1.4), establish the Σ_1-nature of the enumeration of models that drives the effective recovery process (Theorem 1.7), and in a similar manner, our predecessor function given in Section 2.

Theorem 1.1 (Small sets lemma [4, Theorem 3.1]) *Let A be a countable admissible set, and let $D(x, y)$ be a Δ_0-formula. For $p, b \in A$, define*

$$S_{p,b} = \{x \mid x \subseteq b \text{ and } D(x, p)\}.$$

Then if $S_{p,b} \notin A$, the cardinality of $S_{p,b}$ is 2^ω.

Proof Let the language $\mathcal{L}$ contain the $\in$ symbol, a constant c_a for all $a \in A$, and a constant c distinct from the c_a. We define the Δ_1^A set of sentences Z as follows:

1. the sets of sentences

$$\{c_a \in c_b \mid \text{if } a \in b \text{ for } a, b \in A\} \qquad \text{and} \qquad \{c_a \notin c_b \mid \text{if } a \notin b \text{ for } a, b \in A\};$$

   that is, the atomic diagram of A;
2. the sentences describing that the constant c is in $S_{p,b}$ but not in A; that is, $(c \subseteq c_b)$, $D(c, c_p)$, and $(c \neq c_a)$ for all $a \in A$.

We claim that Z is consistent by contradiction. So suppose that Z is inconsistent in $\mathcal{L}_A$. Then by Barwise compactness and completeness, there is some $Z_0 \in A$ such that $Z_0 \subseteq Z$ and Z_0 is inconsistent. So Z_0 contains a subset of the atomic diagram, the sentences $(c \subseteq c_b)$ and $D(c, c_p)$, and $\{(c \neq c_a) \mid a \in a_0\}$ for some $a_0 \in A$. But Z_0 is inconsistent. So there is a proof in A that

$$(c \subseteq c_b) \qquad \text{and} \qquad D(c, c_p) \longrightarrow c \in a_0. \tag{1}$$

But then $S_{p,b} \subseteq a_0$, that is, $S_{p,b} \in A$, which is a contradiction.

We now claim that for each sentence ψ in $\mathcal{L}_A$, we have that $Z \cup \psi$ is not complete for $\mathcal{L}_A$. Assume that there is some $\varphi \in \mathcal{L}_A$ such that $Z \cup \varphi$ is complete. Then for each c_a, there is a deduction $D_a \in A$ from $Z \cup \varphi$ of either $(c_a \in c)$ or $(c_a \notin c)$. By the Σ_1-admissibility of A, there is some deduction $D \in A$ that determines which elements of c_b are in c. But then there is some $e \in A$ such that $(c = c_e)$ is deducible from $Z \cup \varphi$, a contradiction since c is distinct from c_e.

Since $Z \cup \psi$ is not complete for all $\psi \in \mathcal{L}_A$, by Barwise [1, Theorem 8.1], there are 2^ω distinct $\mathcal{L}_A$-theories for models of Z. The argument uses a Henkin-style approach where each stage T_n is enlarged to cause incompatible choices for c. $\square$

Corollary 1.2 *Let A be a countable admissible set. For $p, b \in A$, we have that $S_{p,b}$ is countable if and only if $S_{p,b} \in A$.*

Theorem 1.3 *There is a Σ_1^{ZF}-formula $\mathcal{F}(x, y, z)$ such that for any countable Σ_1-admissible set A and any elements $p, b, s \in A$, we have the following.*

1. *If $S_{p,b}$ is countable, then $A \models \exists z \mathcal{F}(p, b, z)$.*
2. *For all $a \in A$, we have that $A \models \mathcal{F}(p, b, a)$ implies that $a = S_{p,b}$.*

Proof We find $\mathcal{F}(x, y, z)$ from the proof of the small sets lemma (Lemma 1.1). The formula $\mathcal{F}(x, y, z)$ says that there exists a w_1 such that w_1 is a subset of the atomic diagram and there is a deduction of equation (1) from w_1 such that

$$z = \{s \mid s \in w_0 \text{ and } s \subseteq y \text{ and } D(s, x)\},$$

where w_0 is given in equation (1).

Thus, by the small sets lemma (Lemma 1.1), Z is inconsistent if and only if $S_{p,b}$ is countable if and only if $S_{p,b} \in A$. So if $S_{p,b}$ is countable, then $A \models \exists z \mathcal{F}(p, b, z)$. And the second part also follows immediately. $\square$

We now recall an iterated bounding result for admissible sets, given in [4]. This theorem is very useful in conjunction with the small sets lemma. Let $\mathcal{B}(x)$ be a Δ_0-formula with parameter p_0. We say that $\mathcal{B}(x)$ is *β-bounded* if

$$\forall a \left[\mathcal{B}(a) \iff L[\beta, p_0; a] \models \mathcal{B}(a)\right].$$

Thus if $a_\beta = a \cap L[\beta, p_0; a]$, then $\mathcal{B}(a_\beta) \iff \mathcal{B}(a)$.

Note that $L[\beta, p_0; a]$ is the result of iterating first-order definability to the βth stage over the transitive closure of the set p_0 and also using the additional atomic predicate $x \in a$.

For all z, let HYP_z be the least Σ_1-admissible set containing z; that is,

$$\mathrm{HYP}_z = L\big(\omega_1^z, tc(z)\big).$$

Theorem 1.4 (Iterated bounding theorem [4, Theorem 7.1]) *Let $\mathcal{B}(x)$ be a β-bounded Δ_0-formula with parameter p_0. Let $\mathcal{F}(u, v)$ be a Σ_1-formula with parameter p_1, and let $p = \{p_0, p_1\}$.*

Suppose that for all a, the following is true:

If $\mathcal{B}(a)$ holds, then there is a unique $\delta_{p,\beta,a_\beta} \in \mathrm{HYP}_{\{p,\beta,a_\beta\}}$ such that $\mathrm{HYP}_{\{p,\beta,a_\beta\}} \models \mathcal{F}(a_\beta, \delta_{p,\beta,a_\beta})$.

Then we have the following.

1. *There exists a uniform bound $\delta_{p,\beta} \in \mathrm{HYP}_{\{p,\beta\}}$ such that for all a, we have that if $\mathcal{B}(a)$ holds, then $\delta_{p,\beta,a} \leq \delta_{p,\beta}$.*
2. *We obtain $\delta_{p,\beta}$ uniformly; that is, $\delta_{p,\beta}$ is determined by a partial function of p and β whose restriction to an admissible A has a uniform Σ_1^A definition.*

1.2 Weakly scattered theories and their corresponding raw tree hierarchy Weakly scattered theories are a generalization of Morley's notion of a scattered theory. Morley introduced scattered theories in [3] to give a positive result toward Vaught's conjecture.

We first define scattered theories as given in [4], which is equivalent to Morley's original definition.

Definition 1.5 Let $\mathcal{L}$ be a countable first-order language, and let $\mathcal{L}_0$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$. Fix $T \subseteq \mathcal{L}_0$, a theory with a model. Let $\mathcal{L}'$ be any arbitrary countable fragment of $\mathcal{L}_{\omega_1,\omega}$ extending $\mathcal{L}_0$, and let $T' \subseteq \mathcal{L}'$ be any finitarily consistent, ω-complete theory extending T (so that T' is complete and has a model). We say that T is *scattered* if the following hold.

1. For all n and all T', the set of all n-types over T', denoted $S_n T'$, is countable.
2. For all $\mathcal{L}'$, the set $\{T' \mid T' \subseteq \mathcal{L}'\}$ is countable.

We say that T is *weakly scattered* if only (1) holds.

We now introduce a tree hierarchy to enumerate all the models of a weakly scattered theory. This notion is introduced in [4] and extends a similar tree hierarchy for scattered theories. Since we are only interested in weakly scattered theories here, we do not develop the scattered version first. Needless to say, the scattered tree hierarchy is considerably more constructive and can be developed inside $L(\omega_1, T)$. However, surprisingly constructive results are obtained for the weakly scattered case.

Definition 1.6 Let $\mathcal{L}$ be a countable first-order theory, and let $\mathcal{L}_0$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$. Let $T \subseteq \mathcal{L}_0$ be a weakly scattered theory with a model. Following the notation in [4], define

$$\delta- = \begin{cases} \delta - 1 & \text{if } \delta \text{ is a successor ordinal,} \\ \delta & \text{otherwise.} \end{cases}$$

We define the *raw hierarchy* for T, denoted $\mathcal{RH}(T)$, as follows.

Level 0. Include every T_0 such that $T \subseteq T_0$ and T_0 is a finitarily consistent, ω-complete theory of $\mathcal{L}_0$. We define $\mathcal{L}_0(T_{0-})$ to be $\mathcal{L}_0$.

Level $\delta + 1$. We first define $\mathcal{L}_{\delta+1}(T_\delta)$. Assume that T_δ extends a unique predecessor $T_{\delta-}$ on level $\delta-$ and that $\mathcal{L}_\delta(T_{\delta-})$ is countable. If T_δ is an atomic theory, then $\mathcal{L}_{\delta+1}(T_\delta)$ is undefined and T_δ has no extensions on level $\delta + 1$. Otherwise, let $\mathcal{L}_{\delta+1}(T_\delta)$ be the least fragment of $\mathcal{L}_{\omega_1,\omega}$ extending $\mathcal{L}_\delta(T_{\delta-})$ and containing the conjunctions

$$\bigwedge\{\varphi(\vec{x}) \mid \varphi(\vec{x}) \in p(\vec{x})\}$$

for each nonprincipal type $p(\vec{x})$ of T_δ. Note that since T is weakly scattered, T_δ is too, and so $\mathcal{L}_{\delta+1}(T_\delta)$ is countable. Now for level $\delta + 1$ of the tree, include every $T_{\delta+1}$ that extends T_δ and is a finitarily consistent, ω-complete theory of $\mathcal{L}_{\delta+1}(T_\delta)$.

Limit level λ. We include the theory T_λ on level λ if there exists a sequence $\langle T_\delta \mid \delta < \lambda\rangle$ such that the following hold:

1. T_δ is a theory on level δ;
2. $T_{\delta_1} \subseteq T_{\delta_2}$ for $\delta_1 < \delta_2 < \lambda$; and
3. $T_\lambda = \bigcup\{T_\delta \mid \delta < \lambda\}$.

Then $\mathcal{L}_\lambda(T_\lambda)$ is $\bigcup\{\mathcal{L}_\delta(T_{\delta-}) \mid \delta < \lambda\}$.

We then define the *raw tree rank* of a model $\mathfrak{A}$ as

$$\mathrm{rtr}(\mathfrak{A}) = (\text{least } \delta)[\mathfrak{A} \text{ is the atomic model of some } T_\delta].$$

It is clear from the definitions that $\mathfrak{A}$ is a countable model of T if and only if there exists some countable δ such that $\mathfrak{A}$ is the atomic model of T_δ.

For a given model $\mathfrak{A}$ of T, we can analyze its path through the raw hierarchy. Thus, following the notation in [4], we define the *raw tree analysis* of $\mathfrak{A}$ to be the following:

1. $\mathcal{L}_{T(0,\mathfrak{A})} = \mathcal{L}_0$;
2. $T(0, \mathfrak{A})$ = the $\mathcal{L}_0$-theory of $\mathfrak{A}$;
3. $\mathcal{L}_{T(\delta+1,\mathfrak{A})} = \mathcal{L}_{\delta+1}(T(\delta, \mathfrak{A}))$, as given in Definition 1.6;
4. $T(\delta + 1, \mathfrak{A})$ = the $\mathcal{L}_{T(\delta+1,\mathfrak{A})}$-theory of $\mathfrak{A}$;
5. $\mathcal{L}_{T(\lambda,\mathfrak{A})} = \bigcup\{\mathcal{L}_{T(\delta,\mathfrak{A})} \mid \delta < \lambda\}$ for λ a limit;
6. $T(\lambda, \mathfrak{A}) = \bigcup\{T(\delta, \mathfrak{A}) \mid \delta < \lambda\}$.

1.3 Effective recovery of the raw hierarchy Since a weakly scattered theory could potentially have continuum many extensions on a given level of the raw hierarchy, it is not generally possible for the raw hierarchy of a given theory T to exist inside $L(\alpha, T)$ when $\alpha \leq \omega_1$ and $L(\alpha, T)$ is Σ_1-admissible. However, it is surprising how much information on the raw hierarchy can be expressed inside $L(\alpha, T)$.

Theorem 1.7 (Effective recovery process [4, p. 22]) *Let $\mathcal{L}$ be a countable first-order language, and let $\mathcal{L}_0$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$. Let $T \subseteq \mathcal{L}_0$ be a weakly scattered theory. Assume, for convenience, that $\mathcal{L}_0$ and $\mathcal{L}$ are effectively recoverable from T.*

Let α be an ordinal such that $\alpha \leq \omega_1$ and $L(\alpha, T)$ is Σ_1 admissible. Let $\delta < \alpha$, and define A_δ to be the set of all theories T_δ on level δ of $\mathcal{RH}(T)$. Then there exists a β such that A_δ is defined by a β-bounded Δ_0^{ZF}-formula, denoted $\ulcorner A_\delta \urcorner$, and $\ulcorner A_\delta \urcorner \in L(\alpha, T)$.

The proof can be found in [4, pp. 22–24]. We list some important properties obtained from the proof here to use them for our construction.

1. Ordinals ρ_δ are found so that $\mathcal{L}_\delta(T_{\delta-})$ is uniformly constructible from $T_{\delta-}$ by using ρ_δ for all $T_{\delta-} \in A_{\delta-}$.
2. Both $\ulcorner A_\delta \urcorner$ and ρ_δ are constructed simultaneously by a $\Sigma_1^{L(\alpha,T)}$-recursion that is uniform in α.
3. Let ST_δ denote the set of all n-types of T_δ in $\mathcal{L}_\delta(T_{\delta-})$ for $n \geq 0$. Let γ_{T_δ} be the least γ such that $ST_\delta \in L(\gamma, T_\delta)$.

   Then there exists a $\gamma_\delta < \alpha$ such that for all $T_\delta \in A_\delta$ we have:
   (a) $\gamma_{T_\delta} \leq \gamma_\delta$;
   (b) $ST_\delta \in L(\gamma_\delta, T_\delta)$;
   (c) γ_δ is a uniform Σ_1 function of δ using the parameters from $\ulcorner A_\delta \urcorner$ and the parameters from the uniform Σ_1 definition of γ_{T_δ}.
4. A special set of first-order ZF definitions is assembled on level γ_δ of $L(\alpha, T)$ that construct all types in all ST_δ, given a theory T_δ.

   First, let

$$\left\{\ulcorner\varphi_j\urcorner(\mathcal{T}_\delta) \;\middle|\; j \in \mathcal{J}_\delta\right\} \tag{2}$$

be the set of all first-order ZF definitions over $L(\gamma, T)$ for all $\gamma < \gamma_\delta$ with parameter $\mathcal{T}_\delta$. Let $\ulcorner\varphi_j\urcorner(T_\delta)$ represent the set constructed from the definition $\ulcorner\varphi_j\urcorner(\mathcal{T}_\delta)$ when the set T_δ is substituted for the parameter $\mathcal{T}_\delta$. Thus $\ulcorner\varphi_j\urcorner(T_\delta) \in L(\gamma_\delta, T_\delta)$.

Let W_δ denote the natural well-ordering of the set in equation (2), where each definition $\ulcorner\varphi_j\urcorner(\mathcal{T}_\delta)$ is first ordered by the level $\gamma < \gamma_\delta$ in which it is constructed and then by its Gödel number $e < \omega$. Let $\ulcorner d_\delta\urcorner(\mathcal{T}_\delta)$ denote the *default type for* $\mathcal{T}_\delta$ and be defined by the following:

$$\begin{aligned} j(\mathcal{T}_\delta) &= (W_\delta\text{-least } j)\big[\ulcorner\varphi_j\urcorner(\mathcal{T}_\delta) \text{ is an } n\text{-type of } \mathcal{T}_\delta \text{ for some } n\big];\\ \ulcorner d_\delta\urcorner(\mathcal{T}_\delta) &= \ulcorner\varphi_{j(\mathcal{T}_\delta)}\urcorner(\mathcal{T}_\delta). \end{aligned}$$

The first-order ZF definitions of the types are then tweaked. Let $\ulcorner p_j\urcorner(\mathcal{T}_\delta)$ be the first-order ZF definition with parameter $\mathcal{T}_\delta$ defined by

$$\ulcorner p_j\urcorner(\mathcal{T}_\delta) = \begin{cases} \ulcorner\varphi_j\urcorner(\mathcal{T}_\delta) & \text{if } \ulcorner\varphi_j\urcorner(\mathcal{T}_\delta) \text{ is an } n\text{-type of } \mathcal{T}_\delta \text{ for some } n;\\ \ulcorner d_\delta\urcorner(\mathcal{T}_\delta) & \text{the default type of } \mathcal{T}_\delta\text{, otherwise.}\end{cases} \tag{3}$$

Let $\mathcal{P}_\delta = \{\ulcorner p_j\urcorner(\mathcal{T}_\delta) \mid j \in \mathcal{J}_\delta\}$, assembled on level $\gamma_\delta + 1$ in $L(\alpha, T)$. Note that we have the following.

(a) For all $T_\delta \in A_\delta$ and all $p(\vec{x}) \in ST_\delta$, there exists a $j \in \mathcal{J}_\delta$ such that $\ulcorner p_j\urcorner(T_\delta)$ defines the type $p(\vec{x})$ on level $\gamma_\delta + 1$ of $L(\alpha, T)$.

(b) For all $T_\delta \in A_\delta$ and all $j \in \mathcal{J}_\delta$, we have that $\ulcorner p_j\urcorner(T_\delta) \in ST_\delta$.

(c) It is possible that for some $T_\delta \in A_\delta$ and some $j, k \in \mathcal{J}_\delta$ that $j \neq k$ but $\ulcorner p_j\urcorner(T_\delta) = \ulcorner p_k\urcorner(T_\delta)$.

2 Predecessor Function for the Raw Hierarchy

In property 4 of the effective recovery process (Theorem 1.7), first-order ZF definitions are assembled on level γ_δ of $L(\alpha, T)$ that construct all types in all ST_δ given a theory T_δ. We call these formulas *type definitions*.

We add to the $\Sigma_1^{L(\alpha,T)}$ definitions of ρ_δ and $\ulcorner A_\delta\urcorner$ in the effective recovery process to recursively define a *predecessor function*

$$f[\ulcorner p\urcorner, \delta] = \big\{f\big(\ulcorner p\urcorner(\mathcal{T}_\delta), \gamma\big)(\mathcal{T}_\gamma) \;\big|\; \gamma < \delta\big\}$$

for type definitions $\ulcorner p\urcorner$ at stage δ. The predecessor function is a partial function such that if $\ulcorner p\urcorner(T_\delta)$ defines an actual type for a given theory T_δ on level δ, then $f(\ulcorner p\urcorner, \gamma)(T_\gamma)$ defines an actual type on level $\gamma < \delta$ that is an actual subtype of the type defined by $\ulcorner p\urcorner(T_\delta) \restriction ST_\gamma$.

To do this, we must alter the type definitions given in the effective recovery process (Theorem 1.7) so that the default type definitions are not used. As such, the type definitions are not necessarily defined for all theories on the type's level anymore; they have a partial domain. However, these new type definitions will be enough to improve the result in the next section.

Note that each stage δ in our construction of type definitions and the predecessor function corresponds to one level higher in the effective recovery process in Theorem 1.7. That is, to construct $\mathcal{L}_{\delta+1}(T_\delta)$ and $T_{\delta+1}$, we need to construct ST_δ.

2.1 Stage 0 Develop the type definitions as in equation (3) of the effective recovery process (Theorem 1.7), except omit the default type clause. That is, let

$$\begin{aligned}\mathcal{P}_0 &= \left\{\ulcorner p_{j_0}\urcorner(\mathcal{T}_0) \;\middle|\; j_0 \in \mathcal{J}_0\right\}\\ &= \left\{\ulcorner \varphi_{j_0}\urcorner(\mathcal{T}_0) \text{ and also asserts that}\right.\\ &\qquad \left.\ulcorner \varphi_{j_0}\urcorner(\mathcal{T}_0) \text{ is an } n\text{-type of } \mathcal{T}_0 \text{ for some } n \;\middle|\; j_0 \in \mathcal{J}_0\right\}.\end{aligned}$$

So that if defined, $\ulcorner p_{j_0}\urcorner(T_0)$ is the definition of a type in ST_0 for at least one (and not necessarily all) $T_0 \in A_0$.

Then let

$$f(\ulcorner p_{j_0}\urcorner, 0)(\mathcal{T}_0) = \ulcorner p_{j_0}\urcorner(\mathcal{T}_0) \quad \text{for all } \ulcorner p_{j_0}\urcorner(\mathcal{T}_0) \in \mathcal{P}_0.$$

2.2 Stage $\delta + 1$ Assume that we have $\{\rho_i \mid i \leq \delta + 1\}$ and $\{\ulcorner A_i\urcorner \mid i \leq \delta + 1\}$. Using these sequences, we reconstruct the set of definitions from the previous stage δ:

$$\mathcal{P}_\delta = \left\{\ulcorner p_{k_\delta}\urcorner(\mathcal{T}_\delta) \;\middle|\; k_\delta \in \mathcal{K}_\delta\right\}.$$

Here $\mathcal{K}_\delta$ is the index set of type definitions developed here at level δ, and $\mathcal{J}_\delta$ is reserved for the original type definitions developed like $\mathcal{J}_0$ on Level 0.

Also when defining $\mathcal{L}_{\delta+1}(T_\delta)$, we want to use the type definitions uniformly to avoid the language having a domain too. To do so, we alter the type definitions slightly. Let

$$\ulcorner r_j\urcorner(\mathcal{T}_\delta) = \begin{cases}\ulcorner p_j\urcorner(\mathcal{T}_\delta) & \text{if } \ulcorner p_j\urcorner(\mathcal{T}_\delta) \text{ is an } n\text{-type of } \mathcal{T}_\delta;\\ x = x & \text{otherwise.}\end{cases}$$

Then

$$\mathcal{L}_{\delta+1}(T_\delta) = \mathcal{L}_\delta(T_{\delta -}) \cup \left\{\bigwedge \ulcorner r_j\urcorner(T_\delta) \;\middle|\; j \in \mathcal{K}_\delta\right\}$$

and closed under finitary operations.

As in the effective recovery process without default types, we assemble the set

$$\begin{aligned}\mathcal{Q}_{\delta+1} &= \left\{\ulcorner p_j\urcorner(\mathcal{T}_{\delta+1}) \;\middle|\; j \in \mathcal{J}_{\delta+1}\right\}\\ &= \left\{\ulcorner \varphi_j\urcorner(\mathcal{T}_{\delta+1}) \text{ and also asserts that}\right.\\ &\qquad \left.\ulcorner \varphi_j\urcorner(\mathcal{T}_{\delta+1}) \text{ is an } n\text{-type of } \mathcal{T}_{\delta+1} \text{ for some } n \;\middle|\; j \in \mathcal{J}_{\delta+1}\right\}\end{aligned}$$

of first-order definitions at level $\gamma_{\delta+1}$ of $L(\alpha, T)$ by using the small sets lemma and the iterated bounding theorem. Also, $\mathcal{Q}_{\delta+1}$ has a natural well-ordering $\mathcal{W}_{\delta+1}$ (as given in the effective recovery process), which is also definable at level $\gamma_{\delta+1}$.

In what follows, we note that given a theory $T_{\delta+1}$ on level $\delta + 1$, we can effectively reconstruct its immediate predecessor T_δ by using the inductive hypotheses. Thus, for a given fixed $k_\delta \in \mathcal{K}_\delta$ and for each $j_{\delta+1} \in \mathcal{J}_{\delta+1}$, let

$$\begin{aligned}&\ulcorner p_{k_\delta, j_{\delta+1}}\urcorner(\mathcal{T}_{\delta+1}) \text{ be } \ulcorner p_{j_{\delta+1}}\urcorner(\mathcal{T}_{\delta+1}) \in \mathcal{Q}_{\delta+1} \text{ and also assert that}\\ &\quad \ulcorner p_{j_{\delta+1}}\urcorner(\mathcal{T}_{\delta+1}) \text{ is an } m\text{-type of } \mathcal{T}_{\delta+1} \text{ and}\\ &\quad \ulcorner p_{j_{\delta+1}}\urcorner(\mathcal{T}_{\delta+1}) \supseteq \ulcorner p_{k_\delta}\urcorner(\mathcal{T}_\delta), \text{ an } m\text{-type of } \mathcal{T}_\delta \text{ for some } m.\end{aligned}$$

Let $\mathcal{P}_{\delta+1,k_\delta}$ be the collection all such type definitions extending the fixed definition $\ulcorner p_{k_\delta}\urcorner(\mathcal{T}_\delta)$, and let

$$\mathcal{P}_{\delta+1} = \left\{\ulcorner p_{k_\delta, j_{\delta+1}}\urcorner(\mathcal{T}_{\delta+1}) \;\middle|\; k_\delta \in \mathcal{K}_\delta \text{ and } j_{\delta+1} \in \mathcal{J}_{\delta+1}\right\}.$$

Thus, $\mathcal{P}_{\delta+1}$ is the union of all such $\mathcal{P}_{\delta+1,k_\delta}$ as these indices range over $\mathcal{K}_\delta$.

We have to check that $\mathcal{P}_{\delta+1}$ is in $L(\alpha, T)$. First, for a given k_δ and $T_{\delta+1}$, we show that

$$\mathcal{P}_{\delta+1,k_\delta}(T_{\delta+1}) \in L\big(\gamma, T; \langle k_\delta, T_{\delta+1}\rangle\big) \quad \text{for some } \gamma < \alpha.$$

But this is true from Σ_1-replacement since we can effectively construct

$$T_\delta, T_{\delta+1}, \mathcal{Q}_{\delta+1}, \mathcal{P}_\delta \in L(\omega_1^{T_{\delta+1}}, T_{\delta+1})$$

by using the inductive hypotheses.

Let

$$\gamma(k_\delta, T_{\delta+1}) = (\text{least } \gamma)\big[\mathcal{P}_{\delta+1,k_\delta}(T_{\delta+1}) \in L\big(\gamma, \langle k_\delta, T_{\delta+1}\rangle\big)\big].$$

By Theorem 1.3, we have that $\gamma(k_\delta, T_{\delta+1})$ as a function of k_δ and $T_{\delta+1}$ is uniformly Σ_1. So by the iterated bounding theorem (Theorem 1.4), there is a uniform bound $\gamma_{\delta+1}$ of all these γ's, and $\gamma_{\delta+1}$ has a uniform Σ_1 definition from $\delta + 1$ and its parameters.

Thus, we have that $\bigcup_{k_\delta \in \mathcal{K}_\delta} \mathcal{P}_{\delta+1,k_\delta} = \mathcal{P}_{\delta+1} \in L(\gamma_{\delta+1} + 1, T)$, and $\gamma_{\delta+1} < \alpha$. Also, since we have just added repetitions of definitions, we still have that

1. for each $T_{\delta+1} \in A_{\delta+1}$ and $p(\vec{x}) \in ST_{\delta+1}$, there is a $k \in \mathcal{K}_{\delta+1}$ such that $\ulcorner p_k \urcorner(T_{\delta+1})$ defines $p(\vec{x})$ at level $\gamma_{\delta+1} + 1$ of $L(\alpha, T)$, and
2. for all $k \in \mathcal{K}_{\delta+1}$, we have that, if defined, $\ulcorner p_k \urcorner(T_{\delta+1}) \in ST_{\delta+1}$ for at least one $T_{\delta+1} \in A_{\delta+1}$.

Since we have kept track of indices of the immediate predecessors, simply define

$$f(\ulcorner p_{k_\delta, j_{\delta+1}} \urcorner, \gamma)(\mathcal{T}_{\delta+1}) = \begin{cases} \ulcorner p_{k_\delta} \urcorner(\mathcal{T}_\delta) & \text{for } \gamma = \delta; \\ f(\ulcorner p_{k_\delta} \urcorner, \gamma)(\mathcal{T}_\gamma) & \text{for } \gamma < \delta. \end{cases}$$

Finally, let $\rho_{\delta+1} < \alpha$ be just large enough to develop the sequence $\langle \rho_i \mid i \leq \delta\rangle$ and the ordinal $\gamma_{\delta+1} + 1$ needed to construct $\mathcal{P}_{\delta+1}$.

2.3 Limit stage We introduce a rank, called the *type definition rank* (TD rank), that is based on the Cantor–Bendixson rank (CB rank). We use certain isolating-formula definitions derived from TD rank to index the type definitions at the λth stage of the construction, for λ a limit ordinal.

Define *type definition rank* for a type definition $\ulcorner p \urcorner(\mathcal{T}_\lambda) \in S\mathcal{T}_\lambda$ as follows.

1. $\mathrm{TDR}(\ulcorner p \urcorner(\mathcal{T}_\lambda)) = 0$ if there exists a formula definition $\ulcorner \varphi \urcorner(\mathcal{T}_\lambda) \in \mathcal{L}(\mathcal{T}_\lambda)$ such that the formula $\ulcorner \varphi \urcorner(T_\lambda)$ isolates $\ulcorner p \urcorner(T_\lambda)$ in ST_λ.
2. $\mathrm{TDR}(\ulcorner p \urcorner(\mathcal{T}_\lambda)) = \beta$ if there exists a formula definition $\ulcorner \varphi \urcorner(\mathcal{T}_\lambda) \in \mathcal{L}(\mathcal{T}_\lambda)$ such that the formula $\ulcorner \varphi \urcorner(T_\lambda)$ isolates
$$\ulcorner p \urcorner(T_\lambda) \in ST_\lambda - \big\{q \in ST_\lambda \bigm| \mathrm{TDR}(q) < \beta\big\}.$$
Thus, $\ulcorner p \urcorner(T_\lambda)$ is the unique type containing
$$\ulcorner \varphi \urcorner(T_\lambda) \in ST_\lambda - \big\{q \bigm| \mathrm{TDR}(q) < \beta\big\}.$$

We define the type definitions at level λ by recursion. At each stage β, we repeat the type definitions $\ulcorner p \urcorner(\mathcal{T}_\lambda)$ and assert that $\ulcorner p \urcorner(\mathcal{T}_\lambda)$ is a type definition of TD rank β, and index it by its isolating-formula definition.

As in the effective recovery process (Theorem 1.7), we develop the initial set of type definitions $\{\ulcorner p_j \urcorner(\mathcal{T}_\lambda) \mid j \in \mathcal{J}_\lambda\}$.

Stage 0. Define the type definitions of TD rank 0:

$$\mathcal{P}_{\lambda,0} = \big\{ \ulcorner p_{\varphi,0} \urcorner(\mathcal{T}_\lambda) \;\big|\; \ulcorner p_{\varphi,0} \urcorner(\mathcal{T}_\lambda) \text{ is type definition } \ulcorner p_j \urcorner(\mathcal{T}_\lambda) \text{ for some } j \in \mathcal{J}_\lambda \text{ and also asserts } \ulcorner \varphi \urcorner(\mathcal{T}_\lambda) \in \mathcal{L}(\mathcal{T}_\lambda) \text{ and isolates } \ulcorner p_j \urcorner(\mathcal{T}_\lambda) \text{ in } S\mathcal{T}_\lambda \big\}.$$

Then for $\ulcorner p_{\varphi,0} \urcorner(T_\lambda) \in \mathcal{P}_{\lambda,0}$, since $\ulcorner \varphi \urcorner(T_\lambda) \in \mathcal{L}_\lambda(T_\lambda)$, there is some $\delta < \lambda$ such that $\ulcorner \varphi \urcorner(T_\delta) \in \mathcal{L}_{\delta+1}(T_\delta)$. Thus, $\ulcorner \varphi \urcorner(T_\lambda)$ isolates $\ulcorner p_{\varphi,0} \urcorner(T_\lambda) \restriction T_\gamma$ for all $\gamma \geq \delta$.

For $\gamma \geq \delta$, let $j(\ulcorner \varphi \urcorner(\mathcal{T}_\lambda), \gamma) \in \mathcal{J}_\gamma$ be the least index j in the sense of W_γ such that $\ulcorner \varphi \urcorner(\mathcal{T}_\lambda) \in \ulcorner p_j \urcorner(\mathcal{T}_\gamma)$, a member of $\mathcal{P}_\gamma$.

The predecessors of $\ulcorner p_{\varphi,0} \urcorner(\mathcal{T}_\lambda)$ are then

$$f\big(\ulcorner p_{\varphi,0} \urcorner(\mathcal{T}_\lambda), \gamma\big)(\mathcal{T}_\lambda) = \begin{cases} \ulcorner p_{j(\ulcorner \varphi \urcorner(\mathcal{T}_\lambda),\gamma)} \urcorner(\mathcal{T}_\gamma) & \text{if } \gamma \geq \delta, \\ f\big(\ulcorner p_{j(\ulcorner \varphi \urcorner(\mathcal{T}_\lambda),\delta)} \urcorner(\mathcal{T}_\delta), \gamma\big) & \text{if } \gamma < \delta. \end{cases}$$

We now show that the set of rank 0 types is in $L[\alpha, T; T_\lambda]$ uniformly for use in the next step.

Let $R_{\lambda,0}$ denote the set of rank 0 types of T_λ on level λ; that is,

$$R_{\lambda,0} = \big\{ x \;\big|\; x \subseteq T_\lambda \text{ and } \exists \varphi \in \mathcal{L}_\lambda(T_\lambda) \forall \psi \in \mathcal{L}(T_\lambda) \big[\psi \in x \Leftrightarrow (\varphi \to \psi) \in T_\lambda \big] \text{ and } x \text{ is complete, finitarily consistent, and } \omega\text{-complete} \big\}.$$

By the small sets lemma (Theorem 1.1), we have that $R_{\lambda,0} \in L(\omega_1^{T_\lambda}, T_\lambda)$. But then by the iterated bounding theorem (Theorem 1.4), there exists a $\gamma_{\lambda,0} < \omega_1^{T_\lambda}$ such that $R_{\lambda,0} \in L(\gamma_{\lambda,0}, T_\lambda)$ for all $T_\lambda \in A_\lambda$. Alter $\gamma_{\lambda,0}$ so that it is large enough to develop all of stage 0 (but still less than α).

Stage $\beta > 0$. Assume that the recursion has produced the sequence of ordinals $\{\gamma_{\lambda,\delta} \mid \delta < \beta\}$, and construct the set of type definitions for T_λ of TD rank less than β; that is,

$$R_{\lambda,<\beta} = \bigcup_{\delta<\beta} R_{\lambda,\delta} \in L(\alpha, T_\lambda).$$

We first develop an intermediate set of type definitions of TD rank β, denoted $\mathcal{Q}_{\lambda,\beta}$, where

$$\begin{aligned} \mathcal{Q}_{\lambda,\beta} = \big\{ \ulcorner p_{\varphi,\beta} \urcorner(\mathcal{T}_\lambda) \;\big|\; & \ulcorner p_{\varphi,\beta} \urcorner(\mathcal{T}_\lambda) \text{ is a type defn } \ulcorner p_j \urcorner(\mathcal{T}_\lambda) \text{ for some } j \in \mathcal{J}_\lambda \\ & \text{and also asserts } \ulcorner \varphi \urcorner(\mathcal{T}_\lambda) \in \mathcal{L}_\lambda(\mathcal{T}_\lambda) \\ & \text{and } \ulcorner p_{\varphi,\beta} \urcorner(\mathcal{T}_\lambda) \in S\mathcal{T}_\lambda - R_{\lambda,<\beta} \\ & \text{and } \forall \ulcorner q \urcorner(\mathcal{T}_\lambda) \in S\mathcal{T}_\lambda \big[\ulcorner \varphi \urcorner(\mathcal{T}_\lambda) \in \ulcorner q \urcorner(\mathcal{T}_\lambda) \\ & \to \big(\ulcorner q \urcorner(\mathcal{T}_\lambda) \leftrightarrow \ulcorner p_{\varphi,\beta} \urcorner(\mathcal{T}_\lambda) \vee \ulcorner q \urcorner(\mathcal{T}_\lambda) \in R_{\lambda,<\beta} \big) \big] \big\}. \end{aligned}$$

In other words, $\ulcorner p_{\varphi,\beta} \urcorner(\mathcal{T}_\lambda)$ is a definition of the unique type isolated by the formula $\ulcorner \varphi \urcorner(T_\lambda)$ in $ST_\lambda - R_{\lambda,<\beta}(T_\lambda)$. By the small sets lemma and iterated bounding, $\mathcal{Q}_{\lambda,\beta} \in L(\gamma_{\lambda,\beta}, T_\lambda)$ for some $\gamma_{\lambda,\beta} < \omega_1^{T_\lambda}$ for all $T_\lambda \in A_\lambda$.

We now look at a given type definition $\ulcorner p_{\varphi,\beta} \urcorner(\mathcal{T}_\lambda)$ and $\ulcorner \varphi \urcorner(\mathcal{T}_\lambda)$ its isolating-formula definition. We claim that for a given T_λ, there exists a bound $\delta < \lambda$ such that all realized type definitions $\ulcorner q \urcorner(T_\lambda) \in ST_\lambda$ containing $\ulcorner \varphi \urcorner(T_\lambda)$ that have TD rank less than β have an isolating-formula definition below stage δ.

We find it by using Barwise compactness. Let Z be the set of axioms with parameters p, φ, and β all in $L(\omega_1^{T_\lambda}, T_\lambda)$:

1. $q \in ST_\lambda$; $q \neq p$; $\varphi \in q$;

2. ψ does not isolate q in $ST_\lambda - R_{\lambda,<\gamma}$, for all $\psi \in \mathcal{L}_\delta(T_{\delta-})$, all $\delta < \lambda$, and all $\gamma < \beta$;
3. the structure of $L(\omega_1^{T_\lambda}, T_\lambda)$ (so we have an end extension).

Then Z is Σ_1 in $L(\omega_1^{T_\lambda}, T_\lambda)$ thanks to the ordinals we have constructed along the way in the recursion.

If Z is consistent, then φ would have two extensions in ST_λ of rank at least β. But p is the unique such extension, so Z is inconsistent. By Barwise compactness, there is a $z_0 \in L(\omega_1^{T_\lambda}, T_\lambda)$ and $z_0 \subseteq Z$ that is inconsistent. Let $\delta < \omega_1^{T_\lambda}$ bound the ordinals mentioned in z_0 of axiom type (2). Then the axioms (3) and (1) in z_0 imply that there exists an isolating formula in $\mathcal{L}_\delta$ for any such q. Let μ be the Σ_1-function (uniform in T_λ) that takes (p, φ, β) to δ.

We now have enough information to define the set of type definitions of rank β:

$$\begin{aligned}\mathcal{P}_{\lambda,\beta} = \Big\{&\ulcorner p_{\varphi,\beta}\urcorner(\mathcal{T}_\lambda) \in \mathcal{Q}_{\lambda,\beta} \text{ and also asserts}\\ &\ulcorner\varphi\urcorner(\mathcal{T}_\lambda) \text{ is } \textstyle\bigwedge\big(\ulcorner p_{\varphi,\beta}\urcorner(\mathcal{T}_\lambda) \restriction \mathcal{T}_\delta\big) \in \mathcal{L}_{\delta+1}(\mathcal{T}_\delta)\\ &\text{for } \delta = \mu(\ulcorner p_{\varphi,\beta}\urcorner, \ulcorner\varphi\urcorner, \beta)\Big\}.\end{aligned}$$

This definition is effective since we can reconstruct all the T_δ ($\delta < \lambda$) from T_λ and μ is bounded Σ_1.

We claim that for a given T_λ, such a $\ulcorner\varphi\urcorner(T_\lambda)$ (given in $\mathcal{P}_{\lambda,\beta}$) always isolates $\ulcorner p_{\varphi,\beta}\urcorner(T_\lambda)$ in ST_λ. If not, then it splits to say $\ulcorner p\urcorner(T_\lambda)$ ($= \ulcorner p_{\varphi,\beta}\urcorner(T_\lambda)$) and $\ulcorner q\urcorner(T_\lambda) \in ST_\lambda$. We then have that $\ulcorner q\urcorner(T_\lambda)$ is an extension of $\ulcorner\varphi\urcorner(T_\lambda)$ and different to $\ulcorner p\urcorner(T_\lambda)$, so it must have TD rank less than β. Thus $\ulcorner q\urcorner(T_\lambda)$ is itself isolated (in $ST_\lambda - R_{\lambda,<\beta}(T_\lambda)$) by a formula $\ulcorner\psi\urcorner(T_\lambda) \in \mathcal{L}_{\delta+1}(T_\delta)$ by construction (where $\delta = \mu(\ulcorner p\urcorner, \ulcorner\varphi\urcorner, \beta)$). But $(\neg\ulcorner\psi\urcorner(T_\lambda)) \in \ulcorner p\urcorner(T_\lambda) \restriction T_\delta$ otherwise $\ulcorner p\urcorner = \ulcorner q\urcorner$, and $\ulcorner\varphi\urcorner = \bigwedge(\ulcorner p\urcorner(T_\lambda) \restriction T_\delta) \in \ulcorner q\urcorner(\mathcal{T}_\lambda)$ would mean that $\ulcorner\psi\urcorner(T_\lambda)$ is inconsistent with $\ulcorner q\urcorner(T_\lambda)$, which is a contradiction.

For each $\ulcorner p_{\varphi,\beta}\urcorner(\mathcal{T}_\lambda) \in \mathcal{P}_{\lambda,\beta}$, let $\delta = \mu(\ulcorner p_{\varphi,\beta}\urcorner, \ulcorner\varphi\urcorner, \beta)$. For $\gamma \geq \delta$, let $j(\ulcorner\varphi\urcorner(\mathcal{T}_\lambda), \gamma) \in \mathcal{J}_\gamma$ be the least index j in the sense of W_γ such that $\ulcorner\varphi\urcorner(\mathcal{T}_\lambda) \in \ulcorner p_j\urcorner(T_\gamma)$, a member of $\mathcal{P}_\gamma$. Then as in stage 0, the predecessors of $\ulcorner p_{\varphi,\beta}\urcorner(\mathcal{T}_\lambda)$ are

$$f\big(\ulcorner p_{\varphi,\beta}\urcorner(\mathcal{T}_\lambda), \gamma\big) = \begin{cases}\ulcorner p_{j(\ulcorner\varphi\urcorner(\mathcal{T}_\lambda),\gamma)}\urcorner(\mathcal{T}_\gamma) & \text{if } \gamma \geq \delta,\\ f(\ulcorner p_{j(\ulcorner\varphi\urcorner(\mathcal{T}_\lambda),\delta)}\urcorner(\mathcal{T}_\delta), \gamma) & \text{if } \gamma < \delta.\end{cases}$$

To complete stage β, we just need to include the set of types of TD rank β for the following stages in the recursion. Let

$$\begin{aligned}R_{\lambda,\beta} = \big\{x \;\big|\; &x \subseteq T_\lambda \text{ and } x \notin R_{\lambda,<\beta} \text{ and}\\ &\exists\varphi \in \mathcal{L}_\lambda(T_\lambda)\forall q \in ST_\lambda\big[\varphi \in q \to \big(q = x \vee q \in R_{\lambda,<\beta}\big)\big]\\ &\text{and } x \text{ is complete, finitarily consistent, and } \omega\text{-complete}\big\}.\end{aligned}$$

Since the parameters $\mathcal{L}_\lambda(T_\lambda)$, ST_λ, and $R_{\lambda,<\beta}$ are all in $L(\omega_1^{T_\lambda}, T_\lambda)$, then by the small sets lemma, $R_{\lambda,\beta} \in L(\omega_1^{T_\lambda}, T_\lambda)$. And then by iterated bounding, there is a $\gamma < \omega_1^{T_\lambda}$ such that $R_{\lambda,\beta} \in L(\gamma, T_\lambda)$ for all $T_\lambda \in A_\lambda$. Expand $\gamma_{\lambda,\beta}$ to include γ, if necessary.

We now have to show that this recursion is bounded within $L(\alpha, T)$ so that we can construct stage $\lambda + 1$, and continue the effective recovery process. First note that the definition of TD rank is Σ_1 and that the domain of the rank is $ST_\lambda \in L(\alpha, T_\lambda)$. Therefore, by Σ_1-replacement, the range is also bounded in $L(\alpha, T_\lambda)$. Let

$$\gamma_{T_\lambda} = (\text{least } \gamma)\big[\forall q \in ST_\lambda\big(\text{TDR}(q) < \gamma\big)\big].$$

So by iterated bounding, there is a γ_λ that bounds all the γ_{T_λ} and has a uniform Σ_1 definition. We only need to recurse through the TD ranks less than γ_λ. This recursion is Σ_1 at worst with bounded input, and so let $\rho_\lambda < \alpha$ be the least such ordinal such that $L[\rho_\lambda, \mathcal{L}_0; T_\lambda]$ constructs it all.

3 A Bound for Weakly Scattered Theories

3.1 Partial domains We now improve the main bounding theorem in [4] by weakening its effective k-splitting hypothesis assumption. We redefine the effective k-splitting hypothesis so that type definitions have partial domains, and also include our predecessor function from Section 2, so that the predecessor property assumption can be removed. With these changes, the proof remains essentially the same.

Let $\mathcal{L}$ be a countable first-order language, and let $\mathcal{L}_0$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$. Let $T \subseteq \mathcal{L}_0$ be a weakly scattered theory with a model. Finally, let $L(\alpha, T)$ be Σ_1-admissible.

In the effective recovery process, develop our improved set of type definitions $\mathcal{P}_\delta$ at level γ_δ of $L(\alpha, T)$, as given in Section 2. So

$$\mathcal{P}_\delta = \big\{\ulcorner p_j \urcorner(\mathcal{T}_\delta) \;\big|\; j \in \mathcal{J}_\delta\big\}$$

for the improved index set $\mathcal{J}_\delta$.

We define a $\Delta_1^{L(\alpha,T)}$ predicate that determines the domain of a type definition, that is, whether a definition is an actual type for a given theory on a given level of the hierarchy:

$$\text{dom}\big(\ulcorner p_j \urcorner(\mathcal{T}_\delta), T_\delta, \delta\big) \text{ iff } \ulcorner p_j \urcorner(T_\delta) \in ST_\delta \text{ and } T_\delta \text{ is a theory on level } \delta.$$

This definition is clearly $\Delta_1^{L(\alpha,T)}$ from the effective recovery process and Section 2.

We now define a $\Sigma_1^{L(\alpha,T)}$ set of sentences B_α whose models have a node on level α of $\mathcal{RH}(T)$. The set B_α consists of the following sentences.

1. $T \subseteq T_0$ and T_0 is an ω-complete, finitarily consistent theory of $\mathcal{L}_0$.
2. $T_\delta \subseteq T_{\delta+1}$ and $T_{\delta+1}$ is an ω-complete, finitarily consistent theory of $\mathcal{L}_{\delta+1}(T_\delta)$ for all $\delta < \alpha$.
3. $T_\lambda = \bigcup\{T_\delta \mid \delta < \lambda\}$ for all limit ordinals $\lambda < \alpha$.
4. For all $\delta < \alpha$, we have that T_δ has a nonprincipal n-type for some n.

The definition of B_α is the same as [4] because our language is not dependent on the type domains. Note that B_α is $\Delta_1^{L(\alpha,T)}$ because we construct $\mathcal{L}_\delta(T_{\delta-})$ via the ordinal ρ_δ, as defined in the $\Sigma_1^{L(\alpha,T)}$ recursion in Section 2.

Using our improved type definitions and index sets $\mathcal{P}_\delta$ and $\mathcal{J}_\delta$ in Section 2, we say that $\ulcorner p \urcorner$ *is on level* δ if

$$\bigvee_{j \in \mathcal{J}_\delta} \big(\ulcorner p \urcorner = \ulcorner p_j \urcorner(\mathcal{T}_\delta)\big).$$

The crucial definition to be changed is splitting types. We define a *split* $\langle\ulcorner p\urcorner,\ulcorner r\urcorner,\ulcorner r'\urcorner\rangle(\mathcal{T}_{\delta+1})$ *at level* δ to be the sentence

$$\mathrm{dom}(\ulcorner p\urcorner,\mathcal{T}_\delta,\delta)\wedge\mathrm{dom}(\ulcorner r\urcorner,\mathcal{T}_{\delta+1},\delta+1)\wedge\mathrm{dom}(\ulcorner r'\urcorner,\mathcal{T}_{\delta+1},\delta+1)\wedge\ulcorner r\urcorner \neq\ulcorner r'\urcorner\wedge\ulcorner r'\urcorner,\ulcorner r\urcorner\text{ extend }\ulcorner p\urcorner.$$

We then define $\langle\ulcorner p\urcorner,\ulcorner r\urcorner,\ulcorner r'\urcorner\rangle(\mathcal{T}_{\delta+1})$ to be a k*-split* if $\langle\ulcorner p\urcorner,\ulcorner r\urcorner,\ulcorner r'\urcorner\rangle(\mathcal{T}_{\delta+1})$ splits at level δ and $\ulcorner p\urcorner(\mathcal{T}_\delta)$ has arity k. Let K be a set of k-splits. Then we say that

$$K\text{ is }\textit{unbounded}\text{ if }\forall\beta<\alpha\;\exists\delta>\beta\;[K\text{ has a }k\text{-split on level }\delta].$$

Let $\ulcorner p_j\urcorner(\mathcal{T}_\delta)$ be a type definition at stage δ, and let f be the $\Sigma_1^{L(\alpha,T)}$ predecessor function, as given in Section 2. We say that $\ulcorner p_j\urcorner(\mathcal{T}_\delta)$ is K*-unbounded* if the set of all γ such that

$$\exists\langle\ulcorner q\urcorner,\ulcorner r\urcorner,\ulcorner r'\urcorner\rangle\left[\langle\ulcorner q\urcorner,\ulcorner r\urcorner,\ulcorner r'\urcorner\rangle\in K\wedge\ulcorner q\urcorner\text{ is on level }\gamma\wedge f(\ulcorner q\urcorner,\delta)=\ulcorner p_j\urcorner(\mathcal{T}_\delta)\right]$$

is unbounded in α. Note that if K is $\Delta_1^{L(\alpha,T)}$, then since f is $\Sigma_1^{L(\alpha,T)}$ and "$\ulcorner p\urcorner$ is on level γ" is a bounded sentence in $L(\alpha,T)$, we have that K-unboundedness is a $\Pi_2^{L(\alpha,T)}$ property.

We say that the *effective k-splitting hypothesis* holds for T at α if there exists an unbounded $\Delta_1^{L(\alpha,T)}$ set K of k-splits such that B_α and K are consistent if B_α is.

Theorem 3.1 (Improved from [4]) *Let $L(\alpha,T)$ be a countable Σ_2-admissible set. Let T be a weakly scattered theory such that for each $\beta<\alpha$, we have that T has a model of Scott rank at least β. If there exists a k such that the effective k-splitting hypothesis holds for T at α, then T has a countable model $\mathfrak{A}$ such that both*

$$\omega_1^{\mathfrak{A}}=\alpha\qquad\textit{and}\qquad\mathrm{sr}(\mathfrak{A})=\alpha+1.$$

The proof from [4] now goes through with only a trivial change and can be seen in full in [2].

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