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Semi-Isolation and the Strict Order Property

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Abstract We study semi-isolation as a binary relation on the locus of a complete type and prove that—under some additional assumptions—it induces the strict order property.

0 Introduction

Throughout the paper T is a fixed, complete, first-order theory in a countable language and M is its (infinite) monster model. T is an *Ehrenfeucht theory* if it has finitely many, but more than one, countable models. The class of Ehrenfeucht theories is quite interesting. There are numerous results and a large bibliography in this area (see Baizhanov, Sudoplatov, and Verbovskiy [1] and Sudoplatov [8] for references). The first example was found by Ehrenfeucht in Vaught [11, Section 6]: $T_E = \mathrm{Th}(\mathbb{Q}, <, n)_{n \in \omega}$. It eliminates quantifiers and has three countable models: the prime model, the saturated model, and the model prime over a realization of a nonisolated type. T_E is also a *binary theory*: every formula is equivalent modulo T_E to a Boolean combination of formulas with at most two free variables. Not all Ehrenfeucht theories are binary: nonbinary examples can be found in Peretyat'kin [4] and Woodrow [13]. The motivating question for our work is the following.

Question 1 Is there a binary, Ehrenfeucht theory without the strict order property (SOP)? In particular, is there such a theory with three countable models?

An important relation in any Ehrenfeucht theory is semi-isolation as a binary relation on the locus of a powerful type $p \in S(\emptyset)$ in a model of T (all these notions are defined in Section 1). There the semi-isolation relation is either empty (if p is omitted) or a $\bigvee$-definable quasiorder with no maximal elements. If in addition T has precisely three countable models, then the isomorphism type of any countable model N can be described by combinatorial properties of the quasiorder:

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1. N is prime if and only if $p(N) = \emptyset$;
2. N is prime over a realization of p if and only if there is a minimal (with respect to semi-isolation) element in $p(N)$; in this case, N is prime over any minimal element;
3. N is saturated if and only if $p(N) \neq \emptyset$ has no minimal elements.

We note that in Ehrenfeucht's example the type $\{n < x \mid n \in \omega\}$ determines a complete 1-type p on whose locus, in any countable model, the semi-isolation (defined precisely later and denoted by SI_p) coincides with $\leq$. In particular, semi-isolation is a relatively definable relation on the locus of p. The strict order property in this example is induced by the semi-isolation, and it is natural to examine whether this will happen in any binary Ehrenfeucht theory.

One result in this direction was obtained by Woodrow in [12]. He proved that if a theory in the language of Ehrenfeucht's example eliminates quantifiers and has three countable models, then it is quite similar to the original one; in particular, semi-isolation is a relatively definable ordering on the locus of a powerful type. Ikeda, Pillay, and Tsuboi proved that the same happens in the case of an almost $\aleph_0$-categorical theory with three countable models (see [3, Theorem 7]). Another result in this direction was obtained by Pillay in [5, Theorem 5], who proved that in any Ehrenfeucht theory with few links there exists a definable linear ordering. The ordering relation that he found, when restricted to the locus of a powerful type, is induced by the semi-isolation relation.

In this article we will investigate proper quasiorders of the form $(p(M), \mathrm{SI}_p)$, where $p \in S(\emptyset)$ is a nonisolated type in an arbitrary first-order theory, and prove that, under some additional assumptions, a relatively definable suborder can be found. The additional assumptions have a topological flavor. That is not surprising because SI_p has a natural topological "definition" as a subspace of the compact space $S_{p,p}$ consisting of all complete extensions of $p(x) \cup p(y)$. The semi-isolation SI_p corresponds to the subspace $S^p_{\rightarrow}$ of all types $\mathrm{tp}(a, b)$, where $(a, b) \in \mathrm{SI}_p$. We will decompose $S_{p,p}$ into four parts, adequate for studying definability properties of SI_p (see Definition 1.1 and Remark 1.2). Then we will translate definability properties of semi-isolation into topological (complexity) properties of these parts.

In Section 2 we will prove that certain assumptions on the complexity imply the existence of a proper, relatively definable suborder of SI_p. For example, we will prove in Theorem 2.7 that if the theory T has *closed asymmetric links on* $p(M)$ (meaning that one of the parts, the set $S^p_{\mapsto}$, is nonempty and closed in $S_{p,p}$), then there exists a nontrivial, relatively definable suborder of SI_p. This is one direction in which we generalize Pillay's result: if p is a powerful type of an Ehrenfeucht theory with few links, then $S^p_{\mapsto}$ is finite (hence closed) and nonempty.

In Sections 3 and 4 we concentrate on the existence of antichains in SI_p in the case of the negation of the strict order property (NSOP), that is the case in which there is no formula $\varphi(\bar{x}, \bar{y})$ of given theory and tuples $\bar{a}_i$, $i \in \omega$, such that the following equivalence holds:

$$\vdash \varphi(\bar{a}_i, \bar{y}) \to \varphi(\bar{a}_j, \bar{y}) \Leftrightarrow i \leq j.$$

We do not do much in this direction: assuming that the underlying theory is binary, NSOP, and has three countable models, with lots of effort we prove that there are at least two distinct types of SI_p-incomparable pairs of elements on the locus of a powerful type. This indicates that the answer to Question 1 may be affirmative.

In Section 5 we consider a powerful type p in a binary theory for which SI_p is downwards directed in a specific way (PGPIP; see Definition 5.1). We prove that in the NSOP case the Cantor–Bendixson rank of $S_{p,p}$ is finite, indicating that maybe there is no binary, Ehrenfeucht, NSOP theory with PGPIP at all. So the answer to Question 1 may be negative after all.

1 Preliminaries

Throughout the paper $S_n(A)$ denotes the set of all complete n-types with parameters from A. The topology on $S_n(A)$ is defined in the usual way. If $\varphi(\bar{x})$ is a formula over A in n free variables, then by $[\varphi]$ we will denote the set of all types from $S_n(A)$ containing $\varphi(\bar{x})$. The set $S(A)$ denotes $\bigcup_n S_n(A)$. If $p, q \in S(\emptyset)$, then $S_{p,q}(\emptyset)$ is the subspace of all the extensions of $p(\bar{x}) \cup q(\bar{y})$ in $S_m(\emptyset)$ (where $\bar{x}$ and $\bar{y}$ are disjoint and $m = |\bar{x}| + |\bar{y}|$). Similarly, if $q \in S_n(\emptyset)$, then $S_q(A)$ denotes the set of all completions of $q(\bar{x})$ in $S_n(A)$. For any $\bar{c}$ realizing p there is a canonical homeomorphism between $S_{p,q}(\emptyset)$ and $S_q(\bar{c})$: the one sending $r(\bar{x}, \bar{y})$ to $r(\bar{c}, \bar{y})$.

Next we recall the definition of the Cantor–Bendixson rank. It is defined on the elements of a topological space X by induction: $\mathrm{CB}_X(p) \geq 0$ for all $p \in X$; $\mathrm{CB}_X(p) \geq \alpha$ if and only if for any $\beta < \alpha$, p is an accumulation point of the points of CB_X-rank at least β. We have that $\mathrm{CB}_X(p) = \alpha$ if and only if both $\mathrm{CB}_X(p) \geq \alpha$ and $\mathrm{CB}_X(p) \not\geq \alpha + 1$ hold; if such an ordinal α does not exist, then $\mathrm{CB}_X(p) = \infty$. Isolated points of X are precisely those having rank 0; points of rank 1 are those which are isolated in the subspace of all nonisolated points. For a nonempty $C \subseteq X$ we define $\mathrm{CB}_X(C) = \sup\{\mathrm{CB}_X(p) \mid p \in C\}$; in this way $\mathrm{CB}_X(X)$ is defined and $\mathrm{CB}_X(\{p\}) = \mathrm{CB}_X(p)$ holds. If X is compact and Hausdorff and C is closed in X, then the sup is achieved: $\mathrm{CB}_X(C)$ is the maximum value of $\mathrm{CB}_X(p)$ for $p \in C$; there are finitely many points of maximum rank in C, and the number of such points is the CB_X*-degree* of C. If X is countable and compact, then $\mathrm{CB}_X(X)$ is a countable ordinal and every closed subset has ordinal-valued rank and finite CB_X-degree.

$S_n(A)$ is compact, so CB-rank is defined there on points (complete types) and is well behaved on closed subsets (they correspond to partial types). So whenever p is a partial type in n free variables and parameters from A, then $\mathrm{CB}_n^A(p)$ is the CB-rank of the compact space consisting of all completions of p in $S_n(A)$; usually the meaning of n and A will be clear from the context, so we will simply write $\mathrm{CB}(p)$. Similarly, the CB-degree is defined. Thus the CB-rank and degree are defined on all partial types and, in particular, they are defined on formulas. If T is small (i.e., $|S(\emptyset)| = \aleph_0$), then the CB-rank of any partial type over a finite domain is an ordinal.

$\varphi(M, \bar{a})$ denotes the solution set of $\varphi(\bar{x}, \bar{a})$; if $p(\bar{x})$ is a (partial) type, then by $p(M)$ we denote the set of all its realizations. $D \subseteq M^n$ is definable if it is defined by a formula with parameters; it is A-definable (or definable over A) if the defining formula can be chosen to use only parameters from A. We have that D is type-definable ($\bigvee$-definable) if it is the intersection (union) of $< |M|$ definable sets; if all the sets in the intersection (union) are definable over a fixed set $A \subset M$, then we say that D is *type-definable* ($\bigvee$-definable) over A. In this paper we will consider only countable intersections and unions of sets definable over a finite parameter set. Let $C \subseteq M^n$ be type-definable, and let $C_1 \subseteq C$. Then C_1 is *relatively definable within* C if there is a definable $D \subseteq M$ such that $C_1 = C \cap D$; similarly, relative $\bigvee$-definability is defined.

Semi-isolation was introduced by Pillay in [5]; here we will sketch its basic properties (the reader may find more details in [1]). $\bar{b}$ is *semi-isolated over* $\bar{a}$ (or $\bar{a}$ *semi-isolates* $\bar{b}$) if and only if there is a formula $\varphi(\bar{a}, \bar{x}) \in \mathrm{tp}(\bar{b}/\bar{a})$ such that $\varphi(\bar{a}, \bar{x}) \vdash \mathrm{tp}(\bar{b})$; we will denote that by $\bar{b} \in \mathrm{Sem}(\bar{a})$ or by $\bar{a} \to \bar{b}$. $\varphi(\bar{x}, \bar{y})$ is said to *witness* the semi-isolation; we will also write $\bar{a} \xrightarrow{\varphi} \bar{b}$ ($\bar{a}$ φ-arrows $\bar{b}$). Thus

$$\bar{a} \xrightarrow{\varphi} \bar{b} \quad \text{if and only if} \quad \models \varphi(\bar{a}, \bar{b}) \text{ and } \varphi(\bar{a}, \bar{y}) \vdash \mathrm{tp}_{\bar{y}}(\bar{b}).$$

If $\bar{a} \to \bar{b}$, then there are many formulas witnessing the semi-isolation: if $\varphi(\bar{x}, \bar{y})$ is a witness, then $\varphi(\bar{x}, \bar{y}) \wedge \bar{x} = \bar{x}$ is a witness too. Therefore we can have many distinct named arrows between a fixed pair of tuples.

The reader may note that our definition of $\bar{a} \to \bar{b}$ does not exclude the existence of an arrow in the opposite direction. If, in addition to $\bar{a} \to \bar{b}$, we know that the opposite arrow does not exist (i.e., that $a \notin \mathrm{Sem}(b)$), we will write $\bar{a} \mapsto \bar{b}$. Therefore $\bar{a} \mapsto \bar{b}$ means that both $\bar{a} \to \bar{b}$ and $\bar{a} \notin \mathrm{Sem}(\bar{b})$ hold; $\bar{a} \to \bar{b}$ and $\bar{a} \mapsto \bar{b}$ may be consistent. $\bar{a} \leftarrow\!\!\dashv \bar{b}$ means $\bar{b} \mapsto \bar{a}$. And $a \overset{\varphi}{\mapsto} b$ means that both $a \xrightarrow{\varphi} b$ and $a \mapsto b$ hold, while $\bar{a} \leftrightarrow \bar{b}$ means that both $\bar{a} \to \bar{b}$ and $\bar{b} \to \bar{a}$ hold.

Consider semi-isolation as a binary relation on $M^{<\omega}$. It is trivially reflexive and it is not hard to see that it is transitive:

$$\bar{a} \xrightarrow{\varphi} \bar{b} \text{ and } \bar{b} \xrightarrow{\psi} \bar{c} \text{ together imply } \bar{a} \xrightarrow{\varphi} \bar{c},$$

where $\varphi(\bar{x}, \bar{z})$ is $\exists \bar{y}(\varphi(\bar{x}, \bar{y}) \wedge \psi(\bar{y}, \bar{z}))$. Thus semi-isolation is a quasiorder on $M^{<\omega}$. We note an interesting consequence of transitivity:

$$\bar{a} \mapsto \bar{b} \to \bar{c} \text{ implies } \bar{a} \mapsto \bar{c}.$$

We will be interested mainly in semi-isolation as a binary relation on the locus of a complete type $p \in S(\emptyset)$. Then it is relatively $\bigvee$-definable within the locus: to simplify notation we will consider only 1-types; this is justified by passing to an appropriate sort in M^{eq}. So fix for a while $p \in S_1(\emptyset)$. Define

$$\mathrm{SI}_p = \{(a, b) \in p(M)^2 \mid a \to b\}.$$

For any $(a, b) \in \mathrm{SI}_p$ there exists an L-formula $\varphi(x, y)$ witnessing p-semi-isolation. This implies that SI_p is defined by $\bigvee \varphi(x, y)$ within $p(M)^2$ (here the disjunction is taken over all such φ's), so SI_p is a relatively $\bigvee$-definable subset of $p(M)^2$.

Define

$$\overline{\mathrm{SI}}_p = \{(a, b) \in p(M)^2 \mid a \to b \text{ or } b \to a \text{ holds}\}, \qquad \perp_p = p(M)^2 \setminus \overline{\mathrm{SI}}_p.$$

$(a, b) \in \perp_p$ means that a, b are incomparable in the quasiorder, in which case we will write $a \perp_p b$. The semi-isolation $\overline{\mathrm{SI}}_p$ is relatively $\bigvee$-definable within $p(M)^2$, while $\perp_p$ is type-definable.

We will use the following syntax: $x \notin \mathrm{Sem}_p(y)$ will denote the type consisting of all negated formulas witnessing that y p-semi-isolates x; $x \perp^p y$ will denote the type $x \notin \mathrm{Sem}_p(y) \cup y \notin \mathrm{Sem}_p(x)$. Therefore the type $p(x) \cup p(y) \cup x \perp^p y$ defines the set $\{(a, b) \in p(M)^2 \mid a \perp_p b\}$ whose complement in $p(M)^2$ is $\overline{\mathrm{SI}}_p$.

Each $\varphi(x, y)$ witnessing p-semi-isolation defines a binary relation on $p(M)$, so the quasiorder SI_p may also be viewed as the union of a family of binary relations; this has already been suggested by the arrows notation. The relations defined by arrows correspond naturally to subsets of $S_{p,p}$, and relative definability properties translate into topological properties of these subsets.

Definition 1.1 For a nonisolated $p \in S(\emptyset)$ and $\sigma \in \{\mapsto, \leftarrowtail, \to, \leftarrow, \leftrightarrow, \perp\}$, define

$$S^p_\sigma = \{\mathrm{tp}(ab) \in S_{p,p} \mid a \; \sigma \; b\}.$$

The nonisolation of p in the definition is assumed in order to exclude the trivial case $\mathrm{SI}_p = p(M)^2$, which is not interesting at all.

Remark 1.2 Let $p \in S(\emptyset)$ be nonisolated. We list some observations related to the defined parts of $S_{p,p}$.

(1) $S^p_{\mapsto} \cup S^p_{\leftrightarrow} = S^p_{\to}$ and $S^p_{\leftarrowtail} \cup S^p_{\leftrightarrow} = S^p_{\leftarrow}$. We have that $S_{p,p}$ is the disjoint union

$$S_{p,p} = S^p_{\mapsto} \;\dot\cup\; S^p_{\leftarrowtail} \;\dot\cup\; S^p_{\perp} \;\dot\cup\; S^p_{\leftrightarrow}.$$

(2) The mapping taking $\mathrm{tp}(a,b)$ to $\mathrm{tp}(b,a)$ is a homeomorphism of $S_{p,p}$. It fixes setwise $S^p_\perp$ and $S^p_\leftrightarrow$ and maps $S^p_\mapsto$ onto $S^p_\leftarrowtail$ and $S^p_\to$ onto $S^p_\leftarrow$. In particular, $S^p_\mapsto$ and $S^p_\leftarrowtail$, as well as $S^p_\to$ and $S^p_\leftarrow$ are homeomorphic.

(3) $S^p_\leftrightarrow$ has at least one member (containing $x = y$). We have that $S^p_\leftrightarrow \neq S_{p,p}$ holds; otherwise, there would be a formula $\varphi(x,y)$ witnessing that each of x and y p-semi-isolates the other such that $p(x) \cup p(y) \vdash \varphi(x,y)$. Then, by compactness, there would be $\theta(x) \in p$ such that $\models (\theta(x) \wedge \theta(y)) \Rightarrow \varphi(x,y)$ and, if $a \models p$ and $b \in \theta(M) \smallsetminus p(M)$, we would get $\models \varphi(a,b)$, which is not possible by our choice of $\varphi(x,y)$.

(4) Each of $S^p_\mapsto$, $S^p_\leftarrowtail$, and $S^p_\perp$ may be empty while their union is nonempty (because of $S^p_\leftrightarrow \neq S_{p,p}$). By part (2), $S^p_\mapsto$ and $S^p_\leftarrowtail$ are homeomorphic, so they are either both empty or both nonempty.
   - Consider the theory of an infinite set with infinitely many elements named, and let $p \in S_1(\emptyset)$ be the unique nonalgebraic type. Then $S^p_\mapsto = S^p_\leftarrowtail = \emptyset$, while $S^p_\perp$ is a singleton with a member containing $x \neq y$.
   - Consider the type $p \in S_1(\emptyset)$ containing $\{n < x \mid n \in \omega\}$ in Ehrenfeucht's theory T_E. There $S^p_\mapsto$ and $S^p_\leftarrowtail$ have members containing $x < y$ and $y < x$, respectively, while $S^p_\perp = \emptyset$ because any two elements are comparable.

(5) $S^p_\to$, $S^p_\leftarrow$, and $S^p_\leftrightarrow$ are open in $S_{p,p}$: $S^p_\to$ is open because $S^p_\to = \bigcup_\varphi [\varphi]$, where the union is taken over all formulas $\varphi(x,y)$ witnessing p-semi-isolation; by homeomorphism, $S^p_\leftarrow$ is open too. If $\mathrm{tp}(a,b) \in S^p_\leftrightarrow$, then there is a formula $\varphi(x,y) \in \mathrm{tp}(a,b)$ witnessing $a \leftrightarrow b$ and $S^p_\leftrightarrow$ is the union $\bigcup_\varphi [\varphi]$ taken over all such $\varphi(x,y)$. And so $S^p_\leftrightarrow$ is open in $S_{p,p}$.

(6) $S^p_\perp$ is closed in $S_{p,p}$ because it is the set of all completions of $p(x) \cup p(y) \cup x \perp^p y$.

(7) Since SI_p corresponds to $S^p_\to$, SI_p is relatively definable within $p(M)^2$ if and only if $S^p_\to$ is clopen in $S_{p,p}$. But $S^p_\to$ is always open, so SI_p is relatively definable if and only if $S^p_\to$ is closed in $S_{p,p}$.

(8) $\overline{\mathrm{SI}}_p$ corresponds to $S^p_\to \cup S^p_\leftarrow$, which is open. Therefore relative definability of $\overline{\mathrm{SI}}_p$ within $p(M)^2$ is equivalent to any of the following conditions:
   - $S^p_\to \cup S^p_\leftarrow$ is clopen in $S_{p,p}$;
   - $S^p_\to \cup S^p_\leftarrow$ is closed in $S_{p,p}$;
   - $S^p_\perp$ is clopen in $S_{p,p}$ (because it is the relative complement of $S^p_\to \cup S^p_\leftarrow$).

(9) We have $\mathrm{cl}(S^p_{\mapsto}) \subseteq S^p_{\mapsto} \cup S^p_{\perp}$ (where cl denotes the topological closure in $S_{p,p}$). Since $S^p_{\leftarrow}$ is open and disjoint from $S^p_{\mapsto}$, we have $\mathrm{cl}(S^p_{\mapsto}) \subseteq S_{p,p} \smallsetminus S^p_{\leftarrow} = S^p_{\mapsto} \cup S^p_{\perp}$. In particular, if $S^p_{\mapsto}$ is not closed, then it has an accumulation point in $S^p_{\perp}$ and $S^p_{\perp} \neq \emptyset$.

Definition 1.3 A nonisolated type $p \in S(\emptyset)$ is *symmetric* if and only if SI_p is a symmetric binary relation on $p(M)$. Otherwise, p is *asymmetric*.

Since semi-isolation is transitive, it follows that p is asymmetric if and only if $(p(M), \mathrm{SI}_p)$ is a proper quasiorder (with infinite strictly increasing chains). Asymmetric types may exist even in an ω-stable theory, so their existence, in general, does not imply the strict order property (examples of that kind can be found in Sudoplatov [7], [8] and Tanović [10]).

Remark 1.4 It is well known that the symmetry of semi-isolation implies the symmetry of isolation. We will sketch the proof of this fact.

(1) If $\mathrm{tp}(a/b)$ is isolated and $b \in \mathrm{Sem}(a)$, then $\mathrm{tp}(b/a)$ is isolated too. To prove this fact, choose $\varphi(x, b) \in \mathrm{tp}(a/b)$ witnessing the isolation and choose $\psi(a, y) \in \mathrm{tp}(b/a)$ witnessing the semi-isolation. Then $\psi(a, y) \wedge \varphi(a, y) \vdash \mathrm{tp}(b/a)$. If b' satisfies this formula, then $\models \psi(a, b')$ implies $\mathrm{tp}(b') = \mathrm{tp}(b)$. Combining with $\models \varphi(a, b')$ (and $\varphi(x, b) \vdash \mathrm{tp}(a/b)$), we derive $\mathrm{tp}(ab') = \mathrm{tp}(ab)$; $\mathrm{tp}(b/a)$ is isolated.
(2) Suppose that $\mathrm{tp}(a/b)$ is isolated and that $\mathrm{tp}(b/a)$ is nonisolated. Then $b \to a$ and, by part (1), $b \notin \mathrm{Sem}(a)$. This shows that the asymmetry of isolation on a pair of elements implies the asymmetry of semi-isolation on the same pair. In particular, if $p \in S(\emptyset)$ and there are $a, b \models p$ such that $\mathrm{tp}(a/b)$ is isolated and $\mathrm{tp}(b/a)$ is nonisolated, then p is asymmetric.
(3) Suppose that $\mathrm{tp}(a/b)$ is isolated. By part (1) we have

$$\mathrm{tp}(b/a) \text{ is nonisolated iff } b \notin \mathrm{Sem}(a) \text{ iff } b \mapsto a.$$

We will use a version of Remark 1.4 localized to p: if semi-isolation is symmetric on $p(M)$, then isolation is symmetric on $p(M)$ too. The following example shows that the converse is not true: symmetry of isolation on $p(M)$ does not necessarily imply the symmetry of semi-isolation on $p(M)$.

Example 1.5 Let $T = \mathrm{Th}(\omega, <)$. Here there is a unique nonalgebraic 1-type $p(x)$ over $\emptyset$ (the type of an infinite element). Any infinite element has an immediate successor and a predecessor, so $x \pm n$ are well-defined functions and

$$\mathrm{SI}_p = \bigcup_{n \in \omega} \{(x, y) \in p(M)^2 \mid x - n < y\}$$

(note that $x + n < y$ implies $x < y$). We have that p is asymmetric. Take a, b realizing p such that $a + n < b$ holds for all integers n; then $a \mapsto b$. On the other hand, isolation on $p(M)$ is symmetric because it is witnessed by a formula of the form $x = y \pm n$ for some n.

Note that SI_p is not relatively definable within $p(M)^2$ because the union is strictly increasing. On the other hand, $\overline{\mathrm{SI}}_p = p(M)^2$ is obviously relatively definable within $p(M)^2$. Therefore there are asymmetric types for which $\overline{\mathrm{SI}}_p$ is relatively definable, while SI_p is not relatively definable within the locus.

Recall that a nonisolated type $p \in S(\emptyset)$ is called *powerful* if the model prime over a realization of p is weakly saturated (realizes all finitary types over $\emptyset$). Benda in [2] proved that powerful types exist in any Ehrenfeucht theory. Consider all the (isomorphism types of) countable models atomic over a finite subset, and order them by elementary embeddability. Then there is a maximal element (since there are finitely many isomorphism types); the maximal models are precisely those that are weakly saturated.

Remark 1.6 We note some well-known facts about powerful types. We sketch their proofs for the reader's convenience.

(1) Any powerful type is asymmetric. Let $p(x)$ be powerful, and let $a \models p$. Since p is nonisolated, we can find a' realizing a nonisolated extension of p in $S(a)$. Further, because $\mathrm{tp}(aa')$ is realized in any maximal model, there is $b \models p$ such that $\mathrm{tp}(aa'/b)$ is isolated. Note that $\mathrm{tp}(a'/ab)$ is isolated. If $\mathrm{tp}(b/a)$ were isolated, then by transitivity of isolation, $\mathrm{tp}(a'b/a)$ would be isolated too. The latter implies isolation of $\mathrm{tp}(a'/a)$, which is a contradiction. Therefore $\mathrm{tp}(b/a)$ is nonisolated while $\mathrm{tp}(a/b)$ is isolated, so isolation is asymmetric on $p(M)$. By Remark 1.4(2), we conclude that p is asymmetric.

(2) Let p be powerful. Then the proof of part (1) shows that for any $a \models p$ there exists $b \models p$ such that $b \mapsto a$.

(3) Semi-isolation is a downwards-directed quasiorder on the locus of a powerful type. If a, b realize p, then by maximality there is d realizing p such that $\mathrm{tp}(ab/d)$ is isolated. In particular, $\mathrm{tp}(a/d)$ and $\mathrm{tp}(b/d)$ are isolated, by $\varphi(d, x)$ and $\psi(d, y)$, say, and we have $d \overset{\varphi}{\to} a$ and $d \overset{\psi}{\to} b$. We have that d is a lower bound for a and b.

By a *p-principal formula* we mean an L-formula $\varphi(x, y)$ such that for some (any) a realizing p,

$$\varphi(a, x) \text{ isolates an extension of } p \text{ in } S_1(a) \text{ and } a \overset{\varphi}{\mapsto} b \text{ holds for all } b \in \varphi(a, M).$$

By Remark 1.4(3), the condition $a \overset{\varphi}{\mapsto} b$ can be replaced by "$\mathrm{tp}(a/b)$ is nonisolated."

Remark 1.7 Suppose that p is powerful. We strengthen the conclusion of Remark 1.6(3): for all $a, b \in p(M)$ there is $d \in p(M)$ and p-principal formulas φ and ψ such that both $d \overset{\varphi}{\mapsto} a$ and $d \overset{\psi}{\mapsto} b$ hold. To prove it, first choose $c_a, c_b \models p$ satisfying $c_a \mapsto a$ and $c_b \mapsto b$ (here we use Remark 1.6(2)). Then choose $d \models p$ such that $\mathrm{tp}(c_a c_b ab/d)$ is isolated. Then $\mathrm{tp}(c_a/d)$ is isolated, by $\varphi(d, x)$, say. Further, $d \to c_a \mapsto a$ implies $d \mapsto a$ and $d \overset{\varphi}{\mapsto} a$. Similarly, $d \overset{\psi}{\mapsto} b$ for a suitably chosen ψ.

Recall that a theory T is *binary* if every formula is equivalent modulo T to a Boolean combination of formulas with at most two free variables. Binary theories are a special case of Δ-*based* theories (see Saffe, Palyutin, and Starchenko [6]). There Δ is a fixed set of formulas (without parameters), and every formula without parameters is equivalent to a Boolean combination of formulas from Δ. As noted in [6], this means precisely that any complete type $p \in S(\emptyset)$ is Δ-based, that is, that p is forced by the set of formulas $\varphi^\delta \in p$, where $\varphi \in \Delta$ and $\delta \in \{0, 1\}$. In particular, a theory is binary if and only if any complete type is forced by the union of its 2-subtypes.

2 Definability of Semi-Isolation

In this section we study definability properties of semi-isolation on the locus of an asymmetric type $p \in S(\emptyset)$. We know that SI_p is $\bigvee$-definable within $p(M)^2$. We will prove that certain additional assumptions on the topological complexity of $S_{p,p}$ imply the strict order property. The ordering relation found will always be a subset of SI_p, as formalized in the next definition.

Definition 2.1 Suppose that $p \in S(\emptyset)$ and that $(p(M), \leq)$ is a quasiorder with infinite strictly increasing chains. We will say that $\leq$ is a *p-order* if

(1) $\leq$ is a relatively definable subset of $p(M)^2$, and
(2) $a \leq b$ implies $(a, b) \in \mathrm{SI}_p$.

The next proposition shows that a p-order is the restriction of a definable quasiorder to $p(M)$; the domain of such a quasiorder can be chosen to be definable and unbounded (contains no maximal elements).

Proposition 2.2 *Suppose that $p \in S(\emptyset)$, $(p(M), \leq)$ is a p-order, and that $\varphi(x, y)$ relatively defines $\leq$ within $p(M)^2$. Then there exists $\theta(x) \in p$ such that the formula $\theta(x) \wedge \theta(y) \wedge \varphi(x, y)$ witnesses p-semi-isolation and defines an unbounded quasiorder on $\theta(M)$.*

Proof Denote by $\tau(x, y, z)$ the formula $\varphi(x, x) \wedge (\varphi(x, y) \wedge \varphi(y, z) \Rightarrow \varphi(x, z))$.
The first condition from the definition of a p-order implies

$$p(x) \cup p(y) \cup p(z) \vdash \tau(x, y, z). \tag{2.1}$$

The second can be expressed by

$$p(x) \cup p(y) \cup \{\varphi(x, y)\} \vdash \bigvee_{i \in I} \varphi_i(x, y), \tag{2.2}$$

where the disjunction is taken over all formulas witnessing p-semi-isolation. By compactness there exists a finite $I_0 \subset I$ such that (2.2) holds with I_0 in place of I. Then

$$p(x) \cup p(y) \cup \{\varphi(x, y)\} \vdash \varphi(x, y), \tag{2.3}$$

where $\varphi(x, y)$ is the formula $\bigvee_{i \in I_0} \varphi_i(x, y)$. Note that $\varphi(x, y)$ witnesses p-semi-isolation. Now we apply compactness simultaneously to (2.1) and (2.3): there exists a formula $\theta_0(x)$ such that

$$\theta_0(x) \wedge \theta_0(y) \wedge \theta_0(z) \vdash \tau(x, y, z) \quad \text{and} \quad \theta_0(x) \wedge \theta_0(y) \wedge \varphi(x, y) \vdash \varphi(x, y). \tag{2.4}$$

The first relation here implies that $\varphi(x, y)$ defines a quasi-order $\leq_\varphi$ on $\theta_0(M)$; its restriction to $p(M)$ is $\leq$. The second implies that $\theta_0(x) \wedge \theta_0(y) \wedge \varphi(x, y)$ witnesses p-semi-isolation. Now we show that there is no $\leq_\varphi$-maximal element in $\theta_0(M)$ above $a \in p(M)$. We have that $a \leq_\varphi b$ implies $b \in p(M)$ and, because $\leq$ is a p-order, there exists a strictly $\leq$-increasing chain above b. Thus b is not $\leq$-maximal. But $\leq$ is a restriction of $\leq_\varphi$, so b is not $\leq_\varphi$-maximal.

Let $\theta(x)$ be the conjunction of $\theta_0(x)$ and the formula saying that there is no $\leq_\varphi$-maximal element above x. Clearly, $\theta(x) \wedge \theta(y) \wedge \varphi(x, y)$ witnesses p-semi-isolation and defines the restriction of $\leq_\varphi$ on $\theta(M)$. To finish the proof it remains to show that the restricted quasiorder is unbounded; this holds because $\theta(M)$ is $\leq_\varphi$-closed upwards in $\theta_0(M)$ and $\theta_0(M)$ is unbounded. □

As an immediate corollary we obtain the following.

Corollary 2.3 *If $p(x) \in S(\emptyset)$ is asymmetric and SI_p is a relatively definable subset of $p(M)^2$, then there is $\theta(x) \in p$ and a definable, unbounded quasiorder on $\theta(M)$ whose restriction to $p(M)$ is SI_p. In particular, T has the strict order property.*

This fact is well known and can be found in different forms in [1], [3], [5], and Tanović [9]. An example of an asymmetric type with relatively definable semi-isolation is the unique nonisolated 1-type in Ehrenfeucht's example. A similar situation appears in any almost $\aleph_0$-categorical theory: recall that T is *almost $\aleph_0$-categorical* (see [3]) if $p_1(x_1) \cup p_2(x_2) \cup \cdots \cup p_n(x_n)$ has only finitely many completions $r(x_1, \ldots, x_n) \in S(\emptyset)$ for all n and all complete types $p_i(x_i) \in S(\emptyset)$. For any p in such a theory, SI_p is relatively definable within $p(M)^2$: $S_{p,p}$ is finite, so all its relevant parts are clopen, and by Remark 1.2(7), SI_p is relatively definable; alternatively, there are only finitely many inequivalent formulas witnessing p-semi-isolation, so their disjunction relatively defines SI_p within $p(M)^2$.

Corollary 2.4 *If $p(x) \in S(\emptyset)$ is asymmetric and $S_{p,p}$ is finite, then there is $\theta(x) \in p$ and a definable, unbounded quasiorder on $\theta(M)$ whose restriction to $p(M)$ is SI_p. In particular, T has the strict order property.*

Example 2.5 Let $T = (\mathbb{Q}, <, c_n, d_n)$, where (c_n) is an increasing and (d_n) is a decreasing sequence such that both converge to $\sqrt{2}$. We have that T is an Ehrenfeucht theory having six countable models. Let p be the 1-type representing $\sqrt{2}$. Then the locus of p is convex and linearly ordered by $<$. However, p is symmetric and SI_p is the identity relation. Thus there is no p-order there!

Therefore, the locus of a symmetric type may be properly ordered and the asymmetry of semi-isolation is not an exclusive reason for the presence of the strict order property. However, we believe that in this example the reason for the absence of p-orders lies in nonpowerfulness of p.

Question 2 Suppose that p is a powerful type in an Ehrenfeucht theory. Does the existence of a nontrivial, relatively definable, partial order on $p(M)$ always imply the existence of a p-order?

It is easy to realize that relative definability of SI_p implies relative definability of $\overline{\mathrm{SI}}_p$ within $p(M)^2$. The converse is, in general, not true as Example 1.5 shows. There the asymmetric type $p \in S_1(\emptyset)$ is such that $\overline{\mathrm{SI}}_p$ is relatively definable within $p(M)^2$, while SI_p is not.

We will prove in Corollary 2.8 below that relative definability of $\overline{\mathrm{SI}}_p$ for asymmetric p implies the existence of a p-order. Actually, the order found in the proof will have an additional property which will witness that semi-isolation is *partially definable* on $p(M)$. This notion was introduced in [10], and here we give an equivalent definition which relies on the notion of a p-order.

Definition 2.6 We say that semi-isolation is *partially definable on p* if there is a definable quasi-order $\leq$ such that for all $a \in p(M)$,

(i) the restriction of $\leq$ to $p(M)$ is a p-order, and
(ii) $a \overset{\leq}{\mapsto} b \to b'$ and $b' \in p(M)$ imply $a \overset{\leq}{\mapsto} b'$.

Clearly, partial definability of semi-isolation implies that T has the strict order property.

Question 3 Does the existence of a p-order imply partial definability of semi-isolation on p?

Theorem 2.7 *Suppose that $p \in S(\emptyset)$ is asymmetric and that $S^p_{\mapsto}$ is closed in $S_{p,p}$. Then semi-isolation is partially definable on $p(M)$. In particular, T has the strict order property.*

Proof Suppose that $S^p_{\mapsto}$ is closed in $S_{p,p}$. Then it is compact. For each $q(x,y) \in S^p_{\mapsto}$, choose a formula $\varphi_q(x,y) \in q(x,y)$ witnessing p-semi-isolation. Then $S^p_{\mapsto} \subseteq \bigcup\{[\varphi_q] \mid q \in S^p_{\mapsto}\}$. Since $S^p_{\mapsto}$ is compact, there is a finite subcover. Let $\varphi(x,y)$ be the disjunction of all the φ_q's from the subcover. Then φ witnesses p-semi-isolation and $S^p_{\mapsto} \subseteq [\varphi] \subseteq S^p_{\rightarrow}$ holds. Let $x \leq y$ be

$$x = y \vee \big(\varphi(x,y) \wedge (\forall t)\big(\varphi(y,t) \Rightarrow \varphi(x,t)\big)\big).$$

Clearly, $\leq$ defines a quasiorder on M; it also witnesses p-semi-isolation.

Claim 1 *If $a \mapsto b$ realize p, then $\varphi(b,M) \subsetneq \varphi(a,M)$ and $a < b$.*

Proof Suppose that $d \in \varphi(b,M)$. Then $a \mapsto b \rightarrow d$ implies $a \mapsto d$ and $\mathrm{tp}(ad) \in S^p_{\mapsto} \subseteq [\varphi]$. Thus $d \in \varphi(a,M)$ and $\varphi(b,M) \subsetneq \varphi(a,M)$ holds. Similarly, $a \mapsto b$ implies $\mathrm{tp}(ab) \in S^p_{\mapsto} \subseteq [\varphi]$, so $\models \varphi(a,b)$. Finally, $\models \varphi(a,b)$ and $\varphi(b,M) \subsetneq \varphi(a,M)$ imply $a < b$. □

Since p is asymmetric, no element of p is maximal in the semi-isolation quasiorder. Then, by the claim, no realization of p is $\leq$-maximal. We conclude that $\leq$ defines a p-order on $p(M)$, proving condition (i) from the definition of partial semi-isolation. To prove (ii), suppose that $a \overset{\leq}{\mapsto} b \rightarrow c$ holds. Then $a \mapsto c$ and the claim implies $a < c$. Therefore $a \overset{\leq}{\mapsto} c$ holds, proving (ii). The symbol $\leq$ partially defines semi-isolation on p. □

Corollary 2.8 *Suppose that $p(x) \in S(\emptyset)$ is asymmetric and that $\overline{\mathrm{SI}}_p$ is a relatively definable subset of $p(M)^2$. Then semi-isolation is partially definable on $p(M)$. In particular, T has the strict order property.*

Proof Suppose that $\overline{\mathrm{SI}}_p$ is relatively definable within $p(M)^2$. We will show that $S^p_{\mapsto}$ is closed in $S_{p,p}$. By Remark 1.2(8) $S^p_{\rightarrow} \cup S^p_{\leftarrow}$ is closed; clearly it contains $S^p_{\mapsto}$, so $\mathrm{cl}(S^p_{\mapsto}) \subseteq S^p_{\rightarrow} \cup S^p_{\leftarrow}$. On the other hand, by Remark 1.2(9) we have $\mathrm{cl}(S^p_{\mapsto}) \subseteq S^p_{\mapsto} \cup S^p_{\perp}$. Therefore

$$\mathrm{cl}(S^p_{\mapsto}) \subseteq (S^p_{\rightarrow} \cup S^p_{\leftarrow}) \cap (S^p_{\mapsto} \cup S^p_{\perp}) = S^p_{\mapsto}\ .$$

Therefore $S^p_{\mapsto}$ is closed in $S_{p,p}$, and the conclusion follows by Theorem 2.7. □

Corollary 2.9 (T is NSOP) *If $p \in S(\emptyset)$ is asymmetric, then $S^p_{\mapsto}$ (is infinite and) has an accumulation point in $S^p_{\perp}$. In particular, $S^p_{\perp} \neq \emptyset$ and $p(x) \cup p(y) \cup x \perp^p y$ is consistent.*

Proof By Remark 1.2(9) we have $\mathrm{cl}(S^p_{\mapsto}) \subseteq S^p_{\mapsto} \cup S^p_{\perp}$. The NSOP assumption combined with Theorem 2.7 implies that $S^p_{\mapsto}$ is not closed in $S_{p,p}$, so there exists $q \in \mathrm{cl}(S^p_{\mapsto}) \setminus S^p_{\mapsto}$. Then q is an accumulation point of $S^p_{\mapsto}$ and $q \in S^p_{\perp}$. In particular, $S^p_{\perp} \neq \emptyset$, so $p(x) \cup p(y) \cup x \perp^p y$ is consistent. □

Theories with few links were introduced by Benda in [2]: T has *few links* if whenever $p(\bar{x})$ and $q(\bar{y})$ are complete types, then there are only finitely many complete types $r(\bar{x}, \bar{y}) \supset p(\bar{x}) \cup q(\bar{y})$ such that $r(\bar{c}, \bar{y})$ is nonisolated in $S(\bar{c})$ for all $\bar{c}$ realizing $p(\bar{x})$. Pillay in [5, Theorem 5] proved that any Ehrenfeucht theory with few links has the strict order property. He noted that his proof uses only the assumption when $p = q$ is a powerful type. Indeed, it is not hard to realize that the few-links assumption implies that $S^p_{\mapsto}$ is finite for any $p \in S(\emptyset)$: If $\bar{a}, \bar{b} \models p$ and $\bar{a} \mapsto \bar{b}$, then $\mathrm{tp}(\bar{a}/\bar{b})$ is nonisolated; there are only finitely many possibilities for $\mathrm{tp}(\bar{a}/\bar{b})$, so $S^p_{\mapsto}$ is finite. In particular, $S^p_{\mapsto}$ is closed in $S_{p,p}$, and we have the following.

Corollary 2.10 *Any theory with few links and an asymmetric type has the strict order property.*

In the same article, Pillay [5, Section 6] commented on the few-links assumption: "This condition is admittedly rather artificial, but it enables some proofs to go through." An easy consequence of the few-links assumption is that $\mathrm{CB}(S_{p,p}) \leq 1$ holds for all $p \in S(\emptyset)$ (simply because $S_{p,p}$ cannot have infinitely many accumulation points). So $\mathrm{CB}(S_{p,p}) = 1$ seems to be a more natural condition. There are such Ehrenfeucht theories, the first example having been found by Woodrow in [13].

Question 4 Is there a powerful type p in an NSOP theory satisfying $\mathrm{CB}(S_{p,p}) = 1$?

In this article, we do not give much evidence towards answering this question.

Corollary 2.11 (T is small, NSOP) *Suppose that $p \in S(\emptyset)$ is asymmetric (not necessarily powerful) and that $\mathrm{CB}(S_{p,p}) = 1$ holds. Then*

(1) $|S^p_{\mapsto}| \geq \aleph_0$ and $|S^p_{\perp}| \geq 1$, and
(2) there are infinitely many pairwise inequivalent p-principal formulas.

Proof Condition (1) follows from Corollary 2.9. To prove (2), note that $\mathrm{CB}(S_{p,p}) = 1$ implies that there are infinitely many members of $S^p_{\mapsto}$ isolated in $S_{p,p}$. If $\mathrm{tp}(ab) \in S^p_{\mapsto}$ is such a type, then $\mathrm{tp}(b/a)$ is isolated and contains a p-principal formula. □

3 Incomparability

In this section, we start dealing with the SI_p-incomparability of realizations of an asymmetric type. By Corollary 2.9, it is an interesting relation especially in NSOP theories. The next theorem deals with the case when $\overline{\mathrm{SI}}_p$ has relatively definable intersection with the product of two relatively definable subsets of $p(M)$. We will prove that there is a pair of incomparable elements $(a, b) \in D_1 \times D_2$. The intended combinatorial description of this situation is formalized in Proposition 4.3: if we have two large, unbounded, relatively definable subsets of $p(M)$, then some pair of their elements is incomparable.

Theorem 3.1 *Suppose that $p \in S_1(\emptyset)$ is nonisolated and that $D_1, D_2 \subset M$ are $\bar{e}$-definable subsets of M such that the following conditions are satisfied.*

(1) $\overline{\mathrm{SI}}_p \cap (D_1 \times D_2) \neq \emptyset$ is relatively $\bar{e}$-definable within $D_1 \times D_2$.
(2) For all $a \in D_1 \cap p(M)$ there is $b \in D_2 \cap p(M)$ such that $a \mapsto b$.
(3) For all $b \in D_2 \cap p(M)$ there is $a \in D_1 \cap p(M)$ such that $b \to a$.

Then there is an $\bar{e}$-definable quasiorder on M such that no element of $D_1 \cap p(M)$ is below a maximal one of D_1. In particular, T has the strict order property.

Proof Suppose that D_i is defined by $D_i(x, \bar{e})$ and that relative definability is witnessed by $\theta(x, y, \bar{e})$. So we have

$$p(x) \cup p(y) \cup \{D_1(x, \bar{e}), D_2(y, \bar{e}), \theta(x, y, \bar{e})\} \vdash y \in \mathrm{Sem}_p(x) \vee x \in \mathrm{Sem}_p(y).$$

The right-hand side is a long disjunction, so by compactness there is an L-formula $\varphi(x, y)$ witnessing $y \in \mathrm{Sem}_p(x)$ and there is an L-formula $\psi^*(x, y)$ witnessing $x \in \mathrm{Sem}_p(y)$ such that

$$p(x) \cup p(y) \cup \{D_1(x, \bar{e}), D_2(y, \bar{e}), \theta(x, y, \bar{e})\} \vdash \varphi(x, y) \vee \psi^*(y, x).$$

Let $\psi(x, y) := \psi^*(y, x)$. Then for any pair $(a, b) \in D_1 \times D_2$ of realizations of p, we have

$$\text{either } \models \neg\theta(a, b, \bar{e}) \text{ or: at least one of } a \overset{\varphi}{\to} b \text{ and } b \overset{\psi}{\to} a \text{ holds.} \tag{3.1}$$

(The first disjunction here is exclusive because $\theta(x, y, \bar{e})$ relatively defines $\overline{\mathrm{SI}}_p \cap D_1 \times D_2$.) Further, we express assumption (3) by

$$p(x) \cup \{D_2(x, \bar{e})\} \vdash \bigvee_{\psi'(x,y)} \exists y \big(D_1(y, \bar{e}) \wedge \psi'(x, y)\big), \tag{3.2}$$

where the disjunction is taken over all $\psi'(x, y)$ witnessing p-semi-isolation. By compactness, for some $\psi'(x, y)$ we have

$$\text{for all } b \in D_2 \cap p(M) \text{ there is } c \in D_1 \cap p(M) \text{ such that } b \overset{\psi'}{\to} c \text{ holds.} \tag{3.3}$$

After replacing both ψ and ψ' by their disjunction, we may assume that $\psi = \psi'$. Let $\varphi(x, y, \bar{e})$ be $\exists z (D_2(z, \bar{e}) \wedge \varphi(x, z) \wedge \psi(z, y))$. Then $\varphi(a, y, \bar{e}) \vdash p(y)$ for any a realizing p.

Claim 1 *For any $a \in D_1 \cap p(M)$, there exists $c \in D_1$ satisfying $a \mapsto c$ and $\models \varphi(a, c, \bar{e})$.*

Proof Let $a \in D_1 \cap p(M)$. By (3.2) there is $b \in D_2 \cap p(M)$, and by (3.3) there is $c \in D_1 \cap p(M)$ such that $a \mapsto b \overset{\psi}{\to} c$ holds. Then $(a, b) \in \overline{\mathrm{SI}}_p$ implies $\models \theta(a, b, \bar{e})$, and $a \notin \mathrm{Sem}_p(b)$ implies that $b \overset{\psi}{\to} a$ does not hold. By (3.1) we derive $a \overset{\varphi}{\mapsto} b$. Thus $a \overset{\varphi}{\mapsto} b \overset{\psi}{\to} c$, and so $\models \varphi(a, c, \bar{e})$. □

Define $a' \leq b'$ iff $\varphi(b', M, \bar{e}) \cap D_1 \subseteq \varphi(a', M, \bar{e}) \cap D_1$. Clearly, $\leq$ is a definable quasiorder on M. We will show that no element of $D_1 \cap p(M)$ is below a maximal one of D_1.

Claim 2 *If $a, c \in D_1 \cap p(M)$ and $a \mapsto c$, then $a \leq c$.*

Proof Suppose that $d \in \varphi(c, M, \bar{e}) \cap D_1$, and let $b \in D_2$ be such that $c \overset{\varphi}{\to} b \overset{\psi}{\to} d$. Then $a \mapsto c \to b$ implies $a \mapsto b$, so $b \overset{\psi}{\to} a$ does not hold; also, $(a, b) \in \overline{\mathrm{SI}}_p$ implies $\models \theta(a, b, \bar{e})$. By (3.1) we conclude that $a \overset{\varphi}{\mapsto} b$ holds, and then $a \overset{\varphi}{\mapsto} b \overset{\psi}{\to} d$ implies $\varphi(a, d, \bar{e})$. Thus $d \in \varphi(a, M, \bar{e})$. This shows that $\varphi(c, M, \bar{e}) \cap D_1 \subseteq \varphi(a, M, \bar{e}) \cap D_1$; that is, $a \leq c$. □

Now, let $a_1 \in D_1 \cap p(M)$. By Claim 1 there is $c_1 \in D_1$ such that $a_1 \mapsto c_1$ and $\models \varphi(a_1, c_1, \bar{e})$. By Claim 2 we have $a_1 \leq c_1$. Repeating the same procedure with c_1, we find $a_2 \in D_1$ satisfying $c_1 \mapsto a_2$, $\models \varphi(c_1, a_2, \bar{e})$, and $c_1 \leq a_2$. In particular, $a_1 \leq a_2$; that is, $\varphi(a_2, M, \bar{e}) \cap D_1 \subseteq \varphi(a_1, M, \bar{e}) \cap D_1$. Then $c_1 \notin \varphi(a_2, M, \bar{e})$; otherwise $\models \varphi(a_2, c_1, \bar{e})$ would witness $a_2 \rightarrow c_1$, which is in contradiction with $c_1 \mapsto a_2$. Thus $c_1 \in \varphi(a_1, M, \bar{e}) \setminus \varphi(a_2, M, \bar{e})$ and $a_1 < a_2$. Continuing in this way we get an infinite strictly increasing chain of elements of $D_1 \cap p(M)$. □

4 Semi-Isolation on Minimal Powerful Types

Throughout this section we will assume that T (is small and) has a powerful type. We will say that $p \in S(\emptyset)$ is a *minimal powerful* type if it is powerful and there is a formula $\theta(x) \in p$ such that p is the unique powerful type containing θ. Minimal powerful types exist in any Ehrenfeucht theory—take a powerful type of minimal CB-rank. To simplify notation, unless otherwise stated we will assume that $p \in S_1(\emptyset)$ is powerful.

We will be interested in sets definable over a single parameter; we do not a priori assume that the parameter realizes even a nonisolated type. We will say that $D = \varphi(d, M)$ is a *p-set* if $D \cap p(M) \neq \emptyset$ and there exists $b \in D \cap p(M)$ such that at least one of the following two conditions holds:

1. b does not semi-isolate d;
2. $\mathrm{tp}(d)$ is not powerful.

The intended intuitive description of a p-set is that $D \cap p(M)$ is large and unbounded; this is formalized in Lemma 4.2 below.

Remark 4.1 Suppose that p is a powerful type.

(1) If $\mathrm{tp}(d)$ is not powerful, then the second condition from the definition of a p-set is satisfied, so $D = \varphi(d, M)$ is a p-set if and only if it contains a realization of p.

(2) Suppose that p is a minimal powerful type and that $\theta(x) \in p$ witnesses the minimality. Let $d \in \theta(M) \smallsetminus p(M)$. Then, by part (1), $D = \varphi(d, M)$ is a p-set whenever it contains a realization of p.

(3) Suppose that $d \models p$ and that $\varphi(x, y)$ witnesses the asymmetry of p-semi-isolation; there are $a, b \in p(M)$ such that $a \overset{\varphi}{\mapsto} b$. Then b witnesses that the first condition from the definition holds for $D = \varphi(a, M)$, so $\varphi(a, M)$ is a p-set. In particular, $\psi(a, M)$ is a p-set for any p-principal formula $\psi(x, y)$ and $a \models p$.

(4) Suppose that p is a minimal powerful type and that the minimality is witnessed by $\theta(x) \in p(x)$. If $\varphi(x, y)$ is a p-principal formula, then for all $d \in \theta(M)$, $D = \varphi(d, M)$ is a p-set if and only if it contains a realization of p. For $d \in p(M)$ this follows from part (3), and for $d \notin p(M)$ from part (1).

Lemma 4.2 *Suppose that $\theta(x) \in p(x)$ witnesses that $p \in S_1(\emptyset)$ is a minimal powerful type, $d \in \theta(M)$, and that $D = \varphi(d, M)$ is a p-set. Then $D \cap p(M)$ does not have an SI_p-upper bound.*

Proof Suppose on the contrary that $a \in p(M)$ is an upper bound for $D \cap p(M)$. Then $c \to a$ holds for all $c \in D \cap p(M)$:

$$p(x) \cup \{\varphi(d, x)\} \vdash \bigvee_{\psi} \psi(x, a).$$

By compactness there are $\theta_0(x) \in p(x)$ (wlog implying $\theta(x)$) and $\psi(x, y)$ witnessing p-semi-isolation such that $\models (\theta_0(x) \wedge \varphi(d, x)) \Rightarrow \psi(x, a)$. Define

$$\sigma(y, z) := \forall t\big((\theta_0(t) \wedge \varphi(y, t)) \Rightarrow \psi(t, z)\big).$$

Then $\models \sigma(d, a)$ holds, and according to the definition we have two cases.

Case 1 There exists $b \in D \cap p(M)$ such that b does not semi-isolate d.

In this case, we have

$$\models \varphi(d, b) \wedge \theta(d) \wedge \exists z \sigma(d, z). \tag{4.1}$$

Since b does not semi-isolate d, any formula from $\mathrm{tp}(d/b)$ is consistent with infinitely many types from $S_1(\emptyset)$, so there exists $d' \in M$ which does not realize p and satisfies (4.1) in place of d. Note that $\models \theta(d')$ and the minimality of p together imply that $\mathrm{tp}(d')$ is not powerful. Let a' be such that

$$\models \varphi(d', b) \wedge \theta(d') \wedge \sigma(d', a').$$

We *claim* that $\sigma(d', z) \vdash p(z)$ holds. Assume $\models \sigma(d', c)$. Then from $b \in \theta_0(M) \cap \varphi(d', M)$ and the definition of σ, we get $\models \psi(b, c)$. Since ψ witnesses p-semi-isolation, the claim follows.

T is small, so there is an isolated type in $S_1(d')$ containing $\sigma(d', t)$ (it is an extension of p). Thus d' isolates an extension of p, and because p is powerful, $\mathrm{tp}(d')$ has to be powerful too. This is a contradiction.

Case 2 $\mathrm{tp}(d)$ is not powerful.

Since D is a p-set, there exists $b' \in \varphi(d, M) \cap p(M)$. Assuming $\models \sigma(d, c')$ and arguing as in the first case, we derive $b' \xrightarrow{\psi} c'$, so $\sigma(d, z) \vdash p(z)$. Again, we can find an isolated extension of p in $S_1(d)$ and conclude that $\mathrm{tp}(d)$ is powerful. This is a contradiction. □

Next we show that SI_p-incomparability appears quite often on the locus of a minimal powerful type in an NSOP theory.

Proposition 4.3 (T is NSOP) *Suppose that $\theta(x) \in p(x)$ witnesses that p is a minimal powerful type, $d_i \in \theta(M)$, and that each $D_i = \varphi_i(d_i, M)$ is a p-set for $i = 1, 2$. Then there are $a \in D_1, b \in D_2$ realizing p such that $a \perp_p b$.*

Proof Otherwise, for all $a \in D_1, b \in D_2$ realizing p we have $(a, b) \in \overline{\mathrm{SI}}_p$, so

$$\text{at least one of } a \to b \text{ and } b \to a \text{ holds.} \tag{4.2}$$

In particular, $\overline{\mathrm{SI}}_p \cap (D_1 \times D_2)$ is relatively $d_1 d_2$-definable within $p(M)^2$, and the first assumption of Theorem 3.1 is satisfied. We will prove that the other two are satisfied too.

Suppose that the second condition fails, and witness the failure by $a \in D_1 \cap p(M)$. Then, by (4.2), $b \to a$ would hold for all $b \in D_2 \cap p(M)$, so a would be an upper bound for $D_2 \cap p(M)$; this is in contradiction with Lemma 4.2. Therefore the second and, similarly, the third condition are fulfilled. By Theorem 3.1, T has the strict order property. This is a contradiction. □

Thus SI_p is in some sense a "wide" quasiorder. Because p is powerful, it is also directed downwards. It is interesting to know whether it has to be directed upwards.

Question 5 Must SI_p be directed upwards on the locus of a minimal powerful type in an NSOP theory?

We have proved in Corollary 2.9 that $S^p_\perp$ has at least one element, and here, under much stronger assumptions, we will prove that $|S^p_\perp| \geq 2$.

Proposition 4.4 *Suppose that T is a binary NSOP theory with three countable models and that $p \in S_1(\emptyset)$ has* CB*-rank* 1*. Then $q(x,y) = p(x) \cup p(y) \cup x \perp_p y$ has at least two completions in $S_2(\emptyset)$.*

Proof In a theory with three countable models there is a unique isomorphism type of a "middle model," that is, a countable model prime over a realization of a nonisolated type. The middle model is weakly saturated because every finitary type is realized in some finitely generated model. Thus any nonisolated type is powerful and, in particular, p is powerful. Let $\theta(x) \in p$ be a formula of CB-rank 1 and CB-degree 1. Then p is the unique nonisolated type containing $\theta(x)$, and p is a minimal powerful type.

p is asymmetric, so by Corollary 2.9, $q(x,y)$ is consistent. Now suppose that the conclusion of the proposition fails: $q(x,y)$ has a unique completion $q'(x,y) \in S_2(\emptyset)$. Choose $a\,b \models q'$; then $a \perp_p b$ holds. By Corollary 2.9, q' is an accumulation point of $S^p_{\mapsto}$, so each of $\mathrm{tp}(ab)$, $\mathrm{tp}(a/b)$, and $\mathrm{tp}(b/a)$ is nonisolated. By the three model assumption, we know that the model prime over ab is also prime over a realization d of p (because any two models prime over a realization of a nonisolated type are isomorphic). Note that both $\mathrm{tp}(ab/d)$ and $\mathrm{tp}(d/ab)$ are isolated. Hence there is a formula $\tau(x,y,z) \in \mathrm{tp}(dab)$ such that $\tau(d,y,z)$ isolates $\mathrm{tp}_{yz}(ab/d)$ and $\tau(x,a,b)$ isolates $\mathrm{tp}_x(d/ab)$. Now we use the assumption that T is binary: there are formulas φ', ψ', σ such that

$$\models \big(\varphi'(x,y) \wedge \psi'(x,z) \wedge \sigma(y,z)\big) \leftrightarrow \tau(x,y,z).$$

The assumed isolation properties of τ imply

$$\varphi'(x,a) \wedge \psi'(x,b) \wedge \sigma(a,b) \vdash p(x); \tag{4.3}$$

$$\varphi'(d,y) \wedge \psi'(d,z) \wedge \sigma(y,z) \vdash \mathrm{tp}(ab/d). \tag{4.4}$$

Let $\mathrm{tp}(a/d)$ be isolated by $\varphi(d,y) \in \mathrm{tp}(a/d)$, and let $\mathrm{tp}(b/d)$ be isolated by $\psi(d,z) \in \mathrm{tp}(b/d)$. Without loss of generality, assume that they are chosen so that $\models (\varphi(x,y) \Rightarrow \varphi'(x,y)) \wedge (\psi(x,y) \Rightarrow \psi'(x,y))$. Then by (4.3) and (4.4):

$$\varphi(x,a) \wedge \psi(x,b) \wedge \sigma(a,b) \vdash p(x); \tag{4.5}$$

$$\varphi(d,y) \wedge \psi(d,z) \wedge \sigma(y,z) \vdash \mathrm{tp}(ab/d). \tag{4.6}$$

Now consider the formula $(\exists x)(\theta(x) \wedge \varphi(x,y) \wedge \psi(x,z) \wedge \sigma(y,z))$ which is in $\mathrm{tp}_{yz}(ab) = q'(y,z)$. Since $S^p_\perp = \{q'\}$, by Corollary 2.9, q' is an accumulation point of $S^p_{\mapsto}$, so there are $a'b'$ satisfying this formula such that $\mathrm{tp}(a'b') \in S^p_{\mapsto}$; hence $(a',b') \in \mathrm{SI}_p$. Then for some d' we have

$$\models \theta(d') \wedge \varphi(d',a') \wedge \psi(d',b') \wedge \sigma(a',b'). \tag{4.7}$$

d' does not realize p; otherwise (4.6) would imply $a'b' \models q'$, which is in contradiction with $(a',b') \in \mathrm{SI}_p$. Thus $d' \in \theta(M) \smallsetminus p(M)$, so by Remark 4.1(2),

$D_1 = \varphi(d', M)$ and $D_2 = \psi(d', M)$ are p-sets. By Proposition 4.3, there are $a'' \in D_1$ and $b'' \in D_2$ realizing p such that $a'' \perp_p b''$ holds. The uniqueness of q' implies $a''b'' \models q'$ and $\models \sigma(a'', b'')$. Thus

$$\models \varphi(d', a'') \wedge \psi(d', b'') \wedge \sigma(a'', b'').$$

By (4.5) and $\mathrm{tp}(ab) = \mathrm{tp}(a''b'') = q'$, we get $d' \models p$. This is a contradiction. □

5 PGPIP for Binary Theories

Throughout this section we will assume that T is a small, binary theory and that p is a powerful 1-type. We have already noted in Remark 1.6 that SI_p is directed downwards. In Remark 1.7 we noted a stronger form: for any pair of elements $a, b \in p(M)$ there exists $d \in p(M)$ and p-principal formulas φ, ψ such that both $d \xrightarrow{\varphi} a$ and $d \xrightarrow{\psi} b$ hold. In all the basic examples φ and ψ can be chosen from a finite (fixed in advance) set. This property is labeled in [8] as the global pairwise intersection property (GPIP) for p. Precisely, it means that there is a formula $\varphi(x, y)$ which is a disjunction of p-principal formulas and such that $(p(M), \varphi(M^2))$ is an acyclic digraph satisfying

$$\text{for all } a, b \in p(M) \text{ there exists } d \models p \text{ such that} \models \varphi(d, a) \wedge \varphi(d, b). \tag{5.1}$$

Here we introduce a somewhat stronger property.

Definition 5.1 p has PGPIP if there is a formula $\varphi(x, y)$ which is a disjunction of p-principal formulas and is such that $(p(M)^2, \varphi(M))$ is an acyclic digraph, and for all $a, b \in p(M)$ there exists $d \models p$ satisfying

$$\mathrm{tp}(ab/d) \text{ is isolated and } \models \varphi(d, a) \wedge \varphi(d, b). \tag{5.2}$$

We leave it to the reader to check that nonisolated 1-types from Ehrenfeucht's and Peretyatkin's (see [4]) examples have PGPIP.

Theorem 5.2 (T is binary, NSOP) *Suppose that $\varphi(x, y) = \bigvee_{i=1}^{n} \varphi_i(x, y)$, where each $\varphi_i(x, y)$ is p-principal, witnesses PGPIP for p. Then $n \geq 2$ and* $\mathrm{CB}(S_{p,p}(\emptyset)) < n^2$.

Proof Fix d realizing p. For each pair $i, j \leq n$, define

$$D_{(i,j)} = \{(a, b) \in p(M)^2 \mid \mathrm{tp}(ab/d) \text{ is isolated and } \models \varphi_i(d, a) \wedge \varphi_j(d, b)\},$$
$$C_{(i,j)} = \{\mathrm{tp}(ab/d) \mid (a, b) \in D_{(i,j)}\} \qquad S_{(i,j)} = \{\mathrm{tp}(ab) \mid (a, b) \in D_{(i,j)}\}.$$

Note that PGPIP implies that $\bigcup_{(i,j)} S_{(i,j)} = S_{p,p}(\emptyset)$ holds; in particular, if $n = 1$, then $S_{(1,1)} = S_{p,p}(\emptyset)$.

Claim 1 *For every $q(x, y) \in S_{(i,j)}$ there is $\theta_q(x, y) \in q$ which has a unique extension in $C_{(i,j)}$.*

Proof Let $(a, b) \in D_{(i,j)}$ realize q. Then $\mathrm{tp}(ab/d)$ is isolated and, because T is binary and φ_i's are p-principal, there is a formula $\theta_q(x, y) \in q(x, y)$ such that

$$\varphi_i(d, x) \wedge \varphi_j(d, y) \wedge \theta_q(x, y) \vdash \mathrm{tp}(ab/d).$$

Since any extension of $\theta_q(x, y)$ in $C_{(i,j)}$ contains the formula on the left-hand side, we conclude that the extension is unique. □

Now, we claim that each $S_{(i,j)}$ is a discrete subset of $S_{p,p}(\emptyset)$. Suppose on the contrary that $q(x, y) \in S_{(i,j)}$ is an accumulation point of $S_{(i,j)}$. Then θ_q is contained in some $q' \in S_{(i,j)}$ which is distinct from q. Thus θ_q has at least two extensions in $C_{(i,j)}$: the one extending q and the one extending q'. This is a contradiction.

The first part of our theorem follows: if $n = 1$, then $S_{(1,1)} = S_{p,p}(\emptyset)$ is discrete and, because it is compact, it has to be finite. Then by Corollary 2.4, T has the strict order property. This is a contradiction. Therefore $n \geq 2$.

The second part follows from the following topological fact: a compact space which is a union of m discrete subsets has CB-rank smaller than m (easily proved by induction). In our situation $S_{p,p}(\emptyset) = \bigcup_{(i,j)} S_{(i,j)}$ is a union of n^2 discrete subsets, so $\mathrm{CB}(S_{p,p}(\emptyset)) < n^2$. $\square$

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