

Many Normal Measures

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Abstract We characterize the situation of having at least $(2^\kappa)^+$ -many normal ultrafilters on a measurable cardinal κ . We also show that if κ is a compact cardinal, then κ carries $(2^\kappa)^+$ -many κ -complete ultrafilters, each of which extends the club filter on κ .

0 Introduction

The number of normal measures on large cardinals has been investigated extensively in recent years. We know (by Kunen, in [7]) that it is consistent to have a measurable cardinal which carries just one normal measure on it. The opposite direction is taken by Kunen and Paris [9], who proved the consistency of having 2^{2^κ} -many normal measures on a measurable cardinal κ . Since then, many papers have dealt with all the values in between (see [12] under some assumptions on the Mitchell order, Apter, Cummings, and Hamkins [1] for having κ^+ -many normal measures on the measurable cardinal κ without further assumptions, Baldwin [2], and recently Friedman and Magidor [3], which covers for the measurable cardinal κ all the possibilities between 2 and κ^{++} in a uniform manner). However, a measurable cardinal may carry just one normal measure. On the other hand, if κ is supercompact, then there are many normal ultrafilters on it (see Magidor [10]).

We characterize, in this paper, the existence of many normal measures on a measurable cardinal. By “many” we mean at least $(2^\kappa)^+$, which is an upper bound if the generalized continuum hypothesis holds.

The idea is pretty simple. By a basic result of Scott (from Keisler and Tarski [6]), we have a canonical way to normalize a κ -complete ultrafilter on κ . If we start with two distinct κ -complete ultrafilters, we may get the same normal ultrafilter after normalizing both of them. But if we have many different κ -complete ultrafilters, and some property of the function which represents κ in the ultrapower, then we will be able to show that κ carries many different normal ultrafilters.

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We indicate that these results apply to every measurable cardinal, but if one assumes that κ is compact, then many κ -complete ultrafilters on κ are available. Moreover, we can show that many of them include the club filter on κ . This fact does not suffice for normality, since we need an extra property on these ultrafilters, but it is a meaningful step toward proving that there are many normal ultrafilters on every compact cardinal κ .

Some comments about notation. We shall use $\kappa, \lambda, \mu, \tau$ for infinite cardinals, typically $\tau = (2^\kappa)^+$. We use $\alpha, \beta, \gamma, \delta, \zeta, \eta, \xi$ for ordinals. D will be a filter, and U will be an ultrafilter. For an uncountable regular cardinal κ we denote the club filter on κ by D_κ . A filter is κ -complete if it is closed under intersections of fewer than κ -many members from it. A filter is normal if it is closed under diagonal intersections of length κ . We assume, from now on, that all the filters and ultrafilters are nonprincipal. For a regular cardinal κ and $A, B \subseteq \kappa$, A and B are almost disjoint if $|A \cap B| < \kappa$.

If κ is measurable and U is a κ -complete ultrafilter on κ , then we can form the ultrapower \mathbf{V}^κ/U . Being well founded, there is a transitive model M which is isomorphic to \mathbf{V}^κ/U . We denote the elementary embedding of \mathbf{V} into \mathbf{V}^κ/U by J , so the picture is $J : \mathbf{V} \hookrightarrow \mathbf{V}^\kappa/U \cong M$, and we shall not distinguish between \mathbf{V}^κ/U and M .

Recall that κ is the first ordinal moved by J , and we call it the *critical point*. Dealing with many ultrafilters, we denote the elementary embedding which U generates by J_U , and if $\delta = J_U(\kappa)$, then we say that δ is the *critical value* of J_U .

If U is a κ -complete ultrafilter on κ , then $\kappa \in M$ and corresponds to an equivalence class in \mathbf{V}^κ/U . One can choose a representative for this class, that is, a function $f : \kappa \rightarrow \kappa$ such that $[f]_U = \kappa$. We shall use the following.

Lemma 0.1 *κ -complete ultrafilters and the identity function.*

Let κ be a measurable cardinal; let U be a κ -complete ultrafilter on κ . Let $g : \kappa \rightarrow \kappa$ be the identity (i.e., $g(\alpha) = \alpha$ for every $\alpha < \kappa$). We have the following:

- (a) $[g]_U \geq \kappa$ in \mathbf{V}^κ/U ;
- (b) U is normal if and only if $[g]_U = \kappa$.

For a regular cardinal κ , $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ is almost disjoint if $A, B \in \mathcal{A} \Rightarrow |A \cap B| < \kappa$. We quote the following theorem from [8], to be used in Theorem 2.2 below. The proof appears in [8, p. 48].

Theorem 0.2 *If $\kappa \geq \aleph_0$ and $2^{<\kappa} = \kappa$, then there is an almost disjoint family $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ with $|\mathcal{A}| = 2^\kappa$.*

The paper contains two additional sections. In the first one we deal with measurable cardinals, and we give a general characterization for having at least $(2^\kappa)^+$ -many normal measures on the measurable cardinal κ . In the second we deal with compact cardinals, with the goal of showing the plausibility of having many normal measures on every compact cardinal.

1 Many Normal Measures on Measurable Cardinals

Let us start with the following lemma, which is due to Scott [6]. Originally, he used it for showing that if κ is an uncountable measurable cardinal (in the sense that κ carries a κ -complete ultrafilter on it), then there is also a normal ultrafilter on κ .

Proposition 1.1 (Normalizing an ultrafilter) *Suppose that κ is an uncountable measurable cardinal, and suppose that U is a κ -complete ultrafilter on κ . Assume that $f : \kappa \rightarrow \kappa$ satisfies $[f]_U = \kappa$.*

Set $U^ = \{B \subseteq \kappa : f^{-1}(B) \in U\}$. Then U^* is a normal ultrafilter on κ .*

Starting with two distinct ultrafilters U_α, U_β , we can normalize and get U_α^*, U_β^* . But even if $U_\alpha \neq U_\beta$ it may happen that $U_\alpha^* = U_\beta^*$. This setting happens, for instance, in $\mathbf{L}[U]$ (see [7]). It is proved there that there exists a measurable cardinal κ which carries just one normal measure U_* . Take two distinct κ -complete ultrafilters U_α and U_β on κ , and normalize them. As U_* is the only normal measure on κ , one must conclude that $U_\alpha \neq U_\beta$ yet $U_\alpha^* = U_\beta^* = U_*$.

Moreover, it is known that in the model $\mathbf{L}[U]$ there are κ^+ -many nonprincipal κ -complete ultrafilters on κ (see Jech [4, Corollary 19.22]). It means that the Scott process of normalizing projects all of them onto the same normal ultrafilter. The key point which enables us to prove the converse is based on the following.

Definition 1.2 (Bounded functions) Let U be a κ -complete ultrafilter on κ , let $\theta \leq \kappa, \theta \geq 1$ be a (finite or infinite) cardinal, and let $f : \kappa \rightarrow \kappa$. Then,

- (**ℵ**) f is θ -bounded if $|f^{-1}(\beta)| < \theta$ for every $\beta < \kappa$;
- (**⊂**) f is U -bounded if $|f(A) \cap f(\kappa \setminus A)| < \kappa$ for every $A \in U$;
- (**⊃**) f is (U, U^*) -bounded if $f(A) \cap f(\kappa \setminus A) \notin U^*$ for every $A \in U$ (U^* is the normalization of U according to Proposition 1.1);
- (**⊄**) f is stationarily bounded if $f^{-1}(\beta)$ is not a stationary set for every $\beta < \kappa$.

We observe that parts (**ℵ**) and (**⊂**) of the definition are incompatible, but part (**⊃**) gives a weaker assumption on f than part (**⊂**). For this, see Remark 1.4 below. We shall use part (**⊄**) in Section 2 when we deal with compact cardinals. The following is the first characterization that we can prove.

Theorem 1.3 (Normal ultrafilters and U -boundedness) *Let κ be a measurable cardinal, $\tau = (2^\kappa)^+$.*

The following conditions are equivalent:

- (α) κ carries (at least) τ -many normal ultrafilters;
- (β) κ carries τ -many κ -complete ultrafilters ($U_\alpha : \alpha < \tau$), such that for every $\alpha < \tau$ one can choose a U_α -bounded $f_\alpha : \kappa \rightarrow \kappa$ with $[f_\alpha]_{U_\alpha} = \kappa$.

Moreover, one can replace the (strong) requirement of U_α -boundedness by the (weaker) assumption of (U_α, U_α^) -boundedness.*

Proof (α) \Rightarrow (β) is immediate, since every one-to-one function is U -bounded (and the identity is one-to-one). We shall prove (β) \Rightarrow (α). Let f_α be a U_α -bounded function such that $[f_\alpha]_{U_\alpha} = \kappa$, for every $\alpha < \tau$.

Since $\tau = \text{cf}(\tau) > 2^\kappa$, there exists $f : \kappa \rightarrow \kappa$ such that $f = f_\alpha$ for τ -many (and without loss of generality, every) $\alpha < \tau$. Use Proposition 1.1 to normalize each U_α and to get a normal ultrafilter U_α^* for every $\alpha < \tau$. We claim now that $\alpha < \beta < \tau \Rightarrow U_\alpha^* \neq U_\beta^*$.

Recall that $U_\alpha \neq U_\beta$ and that being ultrafilters one can choose $A \in U_\alpha$ such that $\kappa \setminus A \in U_\beta$. Denote $f(A)$ by B_0 and $f(\kappa \setminus A)$ by B_1 . The assumption on f_α (which equals f) implies $|B_0 \cap B_1| < \kappa$. Now $B_0 \in U_\alpha^*$ (since $f^{-1}(B_0) \supseteq A$) and $B_1 \in U_\beta^*$ (since $f^{-1}(B_1) \supseteq \kappa \setminus A$), but $B_1 \notin U_\alpha^*$ (since $B_0 \in U_\alpha^*$ and $|B_0 \cap B_1| < \kappa$). It

means that $U_\alpha^* \neq U_\beta^*$, and we are done. The same argument also gives the same conclusion under the weaker assumption of (U_α, U_α^*) -boundedness, so the proof is complete. \square

Remark 1.4 We defined (in Definition 1.2 above) the notions of θ -boundedness and U -boundedness. These notions are really distinct, as a representing function might be θ -bounded (even for $\theta = 3$) but not U -bounded (e.g., if there is an $A \in U$ such that $|A| = |\kappa \setminus A| = \kappa$ and for every $\beta < \kappa$ there exist unique $\alpha_0 \in A, \alpha_1 \in \kappa \setminus A$ so that $f(\alpha_0) = f(\alpha_1) = \beta$). Similarly, a representing function might be U -bounded but not κ -bounded (e.g., if there is some A of cardinality $\kappa, A \notin U$, and $f(A) = \{\beta\}$ for some $\beta < \kappa$).

It means that parts **(\aleph)** and **(\beth)** in Definition 1.2 are incompatible. Clearly, part **(\beth)** is weaker than **(\aleph)**, as U^* is nonprincipal and κ -complete and hence contains no set of size less than κ .

Nevertheless, we have another characterization (similar to the previous one), based on θ -bounded functions.

Theorem 1.5 (Normal ultrafilters and θ -boundedness) *Let κ be a measurable cardinal, let $\tau = (2^\kappa)^+$, and let $\theta_\alpha < \kappa$ for every $\alpha < \tau$.*

The following conditions are equivalent:

- (α) κ carries (at least) τ -many normal ultrafilters;
- (β) κ carries τ -many κ -complete ultrafilters $(U_\alpha : \alpha < \tau)$, such that for every $\alpha < \tau$ one can choose a θ_α^+ -bounded $f_\alpha : \kappa \rightarrow \kappa$ with $[f_\alpha]_{U_\alpha} = \kappa$.

Proof Since $\tau = \text{cf}(\tau) > \kappa$, the correspondence $\alpha \mapsto \theta_\alpha$ yields an unbounded subset $T \subseteq \tau$ and some $\theta < \kappa$ so that $\alpha \in T \Rightarrow \theta_\alpha \equiv \theta$, so without loss of generality $\theta_\alpha \equiv \theta$ for every $\alpha < \tau$. As in Theorem 1.3, we need only the direction of $(\beta) \Rightarrow (\alpha)$. Fix, for a while, an ordinal $\alpha < \tau$. Denote $f_\alpha^{-1}(\beta)$ by A_β , for every $\beta < \kappa$. Enumerate the members of A_β by $\{a_\gamma^\beta : \gamma < \theta\}$, for each $\beta < \kappa$ (if $|A_\beta| < \theta$ use repetitions, and if $A_\beta = \emptyset$ ignore it). For every $\gamma < \theta$, set $C_\gamma = \{a_\gamma^\beta : \beta < \kappa\}$.

It follows that $\kappa = \bigcup\{C_\gamma : \gamma < \theta\}, \kappa \in U_\alpha$, and U_α is κ -complete, so there is an ordinal $\gamma < \theta$ such that $C_\gamma \in U_\alpha$. By the construction, $f_\alpha \upharpoonright C_\gamma$ is one-to-one. Our purpose is to show that there exists a one-to-one function on κ (not just on C_γ) $h_\alpha \in [f_\alpha]_{U_\alpha}$.

Decompose C_γ into two sets, say, $C_\gamma = C_0^\gamma \cup C_1^\gamma$, such that $C_0^\gamma \cap C_1^\gamma = \emptyset$ and $|C_0^\gamma| = |C_1^\gamma| = \kappa$. Without loss of generality, $C_0^\gamma \in U_\alpha$ and $C_1^\gamma \notin U_\alpha$. Let $B = \kappa \setminus \{f_\alpha(\xi) : \xi \in C_0^\gamma\}$. Notice that $|B| = \kappa$ since $f_\alpha \upharpoonright C_1^\gamma \subseteq B$, and f_α is one-to-one on C_1^γ .

Choose a one-to-one map $g_\alpha : (\kappa \setminus C_\gamma) \cup C_1^\gamma \rightarrow B$, and define $h_\alpha = f_\alpha \upharpoonright C_0^\gamma \cup g_\alpha$. It follows that h_α is a one-to-one function from κ into κ , and $h_\alpha \in [f_\alpha]_{U_\alpha}$, so $[h_\alpha]_{U_\alpha} = \kappa$.

Repeat this process for every $\alpha < \tau$, and use the fact that $\tau = \text{cf}(\tau) > 2^\kappa$ to get a fixed one-to-one function h and τ -many ultrafilters with h as their representing function of κ . Since h is one-to-one, we have $h(A) \cap h(\kappa \setminus A) = \emptyset$ for every $A \subseteq \kappa$. Hence h is U_α -bounded for every $\alpha < \tau$. Now employ Theorem 1.3 to conclude that there are (at least) τ -many normal ultrafilters on κ . \square

Remark 1.6 The above theorems focus on the case of $\tau = (2^\kappa)^+$, but τ can be replaced, verbatim, by any $\lambda \geq \text{cf}(\lambda) \geq \tau$.

2 Compact Cardinals and the Club Filter

Suppose that κ is a compact cardinal. An old open problem (which appears in [4] and in Kanamori [5]) is whether κ carries more than one normal measure. Let us try to explain the background behind this problem.

By a fundamental paper of Magidor [11], the first compact cardinal has an identity crisis. Magidor proved that on one hand it is consistent that the first compact cardinal is also the first measurable cardinal. On the other hand, it is consistent that the first compact cardinal is the first supercompact cardinal. So from this point of view, the compact cardinal is flexible. But we can judge the compact cardinals from a different point of view.

As noted above, a measurable cardinal may carry just one normal measure. A supercompact cardinal carries many normal measures (see [10]). The problem of how many normal measures exist on a compact cardinal can be viewed as another aspect of the identity crisis. Most of the papers which decrease the number of normal measures on a measurable cardinal, mentioned in the introduction, make use of the universe $\mathbf{L}[U]$ (when U is a normal ultrafilter on κ). This includes the pivotal result of Kunen (which gives the consistency of having just one normal measure on a measurable cardinal). We know that there are no compact cardinals in $\mathbf{L}[U]$ (see Vopěnka and Hrbáček [14]). Yet it is an open problem whether every compact cardinal carries at least two normal measures.

The first claim below shows that if U is a κ -complete ultrafilter which extends D_κ , then a simple property of the representing function of κ in the ultraproduct suffices for implying the normality of U . This claim does not assume compactness. The main theorem of this section is that many ultrafilters extending D_κ are available for every compact cardinal.

Claim 2.1 (Stationary boundedness and normality) *Suppose that κ is an uncountable measurable cardinal, and suppose that U is a κ -complete ultrafilter on κ so that $U \supseteq D_\kappa$. Assume that there is a stationarily bounded function $f : \kappa \rightarrow \kappa$ so that $[f]_U = \kappa$. Then U is normal.*

Moreover, if we call β a good ordinal when $f^{-1}(\beta)$ is not stationary, then the conclusion holds when there is some $\beta_ < \kappa$ so that $\beta \in [\beta_*, \kappa) \Rightarrow \beta$ is a good ordinal and $[f]_U = \kappa$.*

Proof Let g be the identity function on κ . Define $S_0 = \{\alpha < \kappa : f(\alpha) \geq g(\alpha)\}$ and $S_1 = \{\alpha < \kappa : f(\alpha) < g(\alpha)\}$. If $S_1 \in U$, then S_1 is stationary (since $U \supseteq D_\kappa$). By Fodor’s lemma there exists a stationary subset $S \subseteq S_1$ such that f is constant on S , contrary to the assumptions of the claim.

Hence $S_1 \notin U$, so $S_0 \in U$. It follows that $g \leq_U f$. On the other hand, $[f]_U = \kappa$ and $[g]_U \geq \kappa$ (due to Lemma 0.1(a)), so $f \leq_U g$. Together, $f \equiv_U g$, and by Lemma 0.1(b) we infer that U is normal.

For the additional part of the claim, suppose that there is just one bad ordinal β . It means that $f^{-1}(\beta)$ is stationary, but notice that this set does not belong to U (otherwise, $[f]_U = \beta$). Hence there is $h \in [f]_U$ such that h coincides with f on $\kappa \setminus f^{-1}(\beta)$, and $h \upharpoonright f^{-1}(\beta)$ is one-to-one. Since $h \upharpoonright \kappa \setminus f^{-1}(\beta) \equiv f \upharpoonright \kappa \setminus f^{-1}(\beta)$ we get a stationarily bounded function. By the κ -completeness of U , this process can be carried out for fewer than κ -many bad ordinals, so we are done. \square

Recall that a family \mathcal{C} of subsets of κ is κ -independent if given disjoint subcollections $C_0, C_1 \subseteq \mathcal{C}$ each of size less than κ , $\bigcap\{x : x \in C_0\} \cap \bigcap\{\kappa \setminus x : x \in C_1\} \neq \emptyset$. It is known that if κ is inaccessible, then there exists a κ -independent family of size 2^κ (the proof goes back to Hausdorff; see [8, p. 288]; in order to get κ -independence one has to replace ω there by κ).

A κ -independent family on κ of size 2^κ yields 2^{2^κ} -many distinct filters. Indeed, let \mathcal{C} be such a family. Let f be a function from \mathcal{C} into $\{0, 1\}$. Set $F = \{x : x \in \mathcal{C}, f(x) = 0\} \cup \{\kappa \setminus x : x \in \mathcal{C}, f(x) = 1\}$. Insofar as \mathcal{C} is κ -independent, F is a base for a κ -complete filter. Clearly, $f \neq g$ create two distinct filters. Inasmuch as there are 2^{2^κ} -many functions from \mathcal{C} into $\{0, 1\}$, we have 2^{2^κ} -many distinct κ -complete filters.

One of the defining properties of compact cardinals is the extension property of κ -complete filters into κ -complete ultrafilters. Basically, one can create 2^{2^κ} -many κ -complete ultrafilters on a compact cardinal in the traditional way. Namely, first create a κ -independent family of size 2^κ , which yields 2^{2^κ} -many different κ -complete filters. Extending each filter into a κ -complete ultrafilter gives the desired result.

It is not clear if the above method can be employed in order to create many κ -complete ultrafilters on a compact cardinal κ , each of which extends the club filter D_κ . Nonetheless, the following theorem suggests a way to define such ultrafilters.

Theorem 2.2 (Many club ultrafilters on a compact cardinal) *Suppose that κ is a compact cardinal, and let τ be $(2^\kappa)^+$. Then there exist τ -many distinct ultrafilters $(U_\alpha : \alpha < \tau)$ such that $D_\kappa \subseteq U_\alpha$ for every $\alpha < \tau$.*

Proof We shall prove that for every ordinal $\delta < (2^\kappa)^+$ there is an ultrafilter U_δ so that $D_\kappa \subseteq U_\delta$ and $J_{U_\delta}(\kappa) > \delta$. By induction on τ this yields the desired collection of ultrafilters. Let us start with a disjoint partition of κ into κ -many stationary sets, say, $\langle S_\alpha : \alpha < \kappa \rangle$. We employ here the celebrated theorem of Solovay from [13]. In fact, Solovay proves much more. He shows that every stationary subset of κ can be partitioned into κ -many disjoint stationary subsets.

Let $\langle t_\beta : \beta < \kappa \rangle$ be an enumeration of $[\kappa]^{<\kappa}$ in such a way that each t_β appears stationarily many times. (One can use the sequence $\langle S_\alpha : \alpha < \kappa \rangle$ to scatter the members of $[\kappa]^{<\kappa}$ on these stationary sets.) Finally, choose an almost disjoint family $\mathcal{A} = \{A_\gamma : \gamma < 2^\kappa\}$ in $[\kappa]^\kappa$. Such a family exists, due to Theorem 0.2, upon noticing that κ is compact; hence $2^{<\kappa} = \kappa$.

For every $\gamma < 2^\kappa$ we choose a function $f_\gamma : A_\gamma \rightarrow \kappa \setminus \{0\}$ such that $\beta < \kappa \Rightarrow |f_\gamma^{-1}(\beta)| = \kappa$. For each $A_\gamma \in \mathcal{A}$ we define a function $g_{A_\gamma} : \kappa \rightarrow \kappa$ as follows. For every $\alpha < \kappa$ we check if $t_\alpha \cap A_\gamma$ is a singleton $\{x\}$. If so, then we set $g_{A_\gamma}(\alpha) = f_\gamma(x)$. In all other cases we define $g_{A_\gamma}(\alpha) = 0$.

Set $G = \{g_{A_\gamma} : \gamma < 2^\kappa\}$. It is easily verified that $A \neq B \Rightarrow g_A \neq g_B$. Indeed, assume $A, B \in \mathcal{A}$ and $A \neq B$. By the fact that A, B are almost disjoint, we can choose some $x \in \kappa$ such that $x \in A$ but $x \notin B$. Set $t = \{x\}$; so $t \in [\kappa]^{<\kappa}$. Pick an ordinal $\alpha < \kappa$ so that $t = t_\alpha = \{x\}$. It follows that $t_\alpha \cap A = \{x\}$, hence $g_A(\alpha) = f_\gamma(x)$ (where γ is the index of A in the enumeration of \mathcal{A}). Recall that the range of f_γ is $\kappa \setminus \{0\}$, and conclude that $g_A(\alpha) > 0$.

On the other hand, $t_\alpha \cap B = \emptyset$, as $x \notin B$. In particular, $t_\alpha \cap B$ is not a singleton. By the definition of g_B we have $g_B(\alpha) = 0$. Consequently, $g_B(\alpha) \neq g_A(\alpha)$, so the functions g_B and g_A are not the same; that is, $g_A \neq g_B$.

By the above paragraphs, $|G| = 2^\kappa$. Let $\langle g_\varepsilon : \varepsilon \leq \delta \rangle$ be an enumeration of G . We emphasize that the enumeration is taken along the ordinal $\delta + 1$ (whose cardinality is 2^κ), as we shall need this in the sequel. For every pair $\alpha < \beta \leq \delta$ we define

$$X_{\alpha,\beta} = \{\xi < \kappa : g_\alpha(\xi) < g_\beta(\xi)\}.$$

Notice that $X_{\alpha,\beta}$ includes a stationary subset of κ for every $\alpha < \beta \leq \delta$. Indeed, the functions g_α, g_β correspond to some $A, B \in \mathcal{A}$. Choose any $x \in B \setminus A$ (recall that \mathcal{A} is a collection of almost disjoint sets), and we know that $\{x\} \in [\kappa]^{<\kappa}$. By the enumeration of $[\kappa]^{<\kappa}$ there is a stationary $S \subseteq \kappa$ so that $\xi \in S \Rightarrow t_\xi = \{x\}$. Consequently, $\xi \in S \Rightarrow g_\alpha(\xi) = 0 < f_\varepsilon(x) = g_\beta(\xi)$ (where $B = A_\varepsilon$ in the enumeration of \mathcal{A}), so $S \subseteq X_{\alpha,\beta}$.

Moreover, G is endowed with the following property. If $G' \subseteq G, |G'| < \kappa$ and $0 < \zeta_A < \kappa$ for every $g_A \in G'$, then there exists a stationary set $S \subseteq \kappa$ such that $g_A(\xi) = \zeta_A$ for every $g_A \in G'$ and every $\xi \in S$.

To prove this, choose a member $x_A \in A$ for every $g_A \in G'$ such that $g_B \in G' \wedge g_B \neq g_A \Rightarrow x_A \notin B$ and $f_{y_A}(x_A) = \zeta_A$ (where unless otherwise specified, $A = A_{y_A}$ in the enumeration of G). Let us try to justify the existence of such x_A 's. The cardinality of the set $\bigcup\{B \cap A : g_B \in G' \wedge g_B \neq g_A\}$ is less than κ (as it results as a union of less than κ -many sets, each of size fewer than κ , and κ is regular).

Since $\zeta_A > 0$, the set $f_{y_A}^{-1}(\zeta_A)$ is of size κ and included in A . So choose $x_A \in A$ such that $x_A \notin \bigcup\{B \cap A : g_B \in G' \wedge g_B \neq g_A\}$ and $x_A \in f_{y_A}^{-1}(\zeta_A)$. It follows that $f_{y_A}(x_A) = \zeta_A$ and $x_A \notin B$ whenever $B \neq A$ and $g_B \in G'$.

With the x_A 's at hand, set $s = \{x_A : g_A \in G'\}$. Clearly, $s \in [\kappa]^{<\kappa}$, so there exists a stationary set S such that $\xi \in S \Rightarrow s = t_\xi$. It follows that $s \cap A = \{x_A\}$ for every $g_A \in G'$. Now take any ordinal $\xi \in S$ and any $A \in \mathcal{A}$ such that $g_A \in G'$. We have $t_\xi \cap A = s \cap A = \{x_A\}$; hence $g_A(\xi) = f_{y_A}(x_A) = \zeta_A$, and we are done.

Consequently, the intersection of fewer than κ -many sets of the form $X_{\alpha,\beta}$ includes a stationary set. To prove this, assume that F is a collection of θ -many sets of the form $X_{\alpha,\beta}$, for some $\theta < \kappa$. Let F_{ord} be the set of all ordinals mentioned in the index of $X_{\alpha,\beta}$ for some $X_{\alpha,\beta} \in F$. Clearly, $|F_{\text{ord}}| \leq \theta < \kappa$.

For every pair (α, β) such that $\alpha < \beta$ and $X_{\alpha,\beta} \in F$, choose $\zeta_\alpha, \zeta_\beta$ so that $0 < \zeta_\alpha < \zeta_\beta < \kappa$. To this end, enumerate the members of F_{ord} in an increasing order, and choose $\zeta_\beta > \sup\{\zeta_\alpha : \alpha < \beta\}$ by induction, using the fact that $\theta < \kappa = \text{cf}(\kappa)$.

By the above considerations, we can choose x_α for every $\alpha \in F_{\text{ord}}$ such that $x_\alpha \in A_\alpha$ (where A_α is the set in \mathcal{A} which corresponds to g_α), $x_\alpha \notin A_\beta$ for every $\beta \in F_{\text{ord}} \setminus \{\alpha\}$, and $f_{\varepsilon(\alpha)}(x_\alpha) = \zeta_\alpha$ (where $\varepsilon(\alpha)$ is the index of A_α in the enumeration of \mathcal{A}). We denote the set $\{x_\alpha : \alpha \in F_{\text{ord}}\}$ by s . Since $s \in [\kappa]^{<\kappa}$, there exists a stationary set $S \subseteq \kappa$ such that $\xi \in S \Rightarrow t_\xi = s$. We claim that $S \subseteq X_{\alpha,\beta}$ for every $X_{\alpha,\beta} \in F$.

For this, pick any ordinal $\xi \in S$ and any set $X_{\alpha,\beta} \in F$. We have chosen $\zeta_\alpha < \zeta_\beta$, and $x_\alpha \in A_\alpha \setminus A_\beta$ as well as $x_\beta \in A_\beta \setminus A_\alpha$, such that $f_{\varepsilon(\alpha)}(x_\alpha) = \zeta_\alpha$ and $f_{\varepsilon(\beta)}(x_\beta) = \zeta_\beta$. (Again, $\varepsilon(\alpha)$ is the index of A_α and $\varepsilon(\beta)$ is the index of A_β in the enumeration of \mathcal{A} .) It means that $g_\beta(\xi) = \zeta_\beta$. By the same token, $g_\alpha(\xi) = \zeta_\alpha < \zeta_\beta$, so $\xi \in X_{\alpha,\beta}$ by its very definition. As $\xi \in S$ was arbitrary, we conclude that $S \subseteq X_{\alpha,\beta}$, and as $X_{\alpha,\beta}$ was arbitrary, we are done.

It follows from the discussion above that the collection $F = \{X_{\alpha,\beta} : \alpha < \beta \leq \delta\}$ is a κ -complete filter base. Moreover, $F \cup D_\kappa$ is still a κ -complete filter base. Let D be the filter generated by $F \cup D_\kappa$, and let U_δ be any κ -complete ultrafilter which extends D .

By the construction, $\alpha < \beta \leq \delta \Rightarrow X_{\alpha,\beta} \in U_\delta$, so the sequence $\langle g_\alpha : \alpha \leq \delta \rangle$ establishes an increasing sequence of functions according to $<_{U_\delta}$. We conclude that $J_{U_\delta}(\kappa) \geq \delta + 1$ as required. \square

Discussion 2.3 We call U a *club ultrafilter* if $D_\kappa \subseteq U$. Let $(U_\alpha : \alpha < \tau)$ be a family of club ultrafilters on a compact cardinal κ , stipulating $\tau = (2^\kappa)^+$. Since the number of functions from κ into κ is just 2^κ we may assume, without loss of generality, that the representing function f_α of κ in $\mathbf{V}^\kappa/U_\alpha$ is the same f for every $\alpha < \tau$.

This means that $f^{-1}(\beta)$ is not a club set for every $\beta < \kappa$. Indeed, if $f^{-1}(\beta)$ is a club for some ordinal $\beta < \kappa$, then $f^{-1}(\beta) \in D_\kappa \subseteq U_\alpha$ (for every U_α). Choose one of the U_α 's, and conclude that f represents the ordinal β in $\mathbf{V}^\kappa/U_\alpha$, which is impossible as f represents κ itself.

Moreover, for every stationary subset $S \subseteq \kappa$, it suffices to find one ordinal $\alpha < \tau$ such that $S \in U_\alpha$ to conclude that $S \neq f^{-1}(\beta)$ for every $\beta < \kappa$. The reason is just the same. If $S = f^{-1}(\beta)$ for some $\beta < \kappa$ and $S \in U_\alpha$ for some α , then f represents β in $\mathbf{V}^\kappa/U_\alpha$, contrary to the choice of f .

So if we could catch all the stationary subsets of κ (in the sense that for every S there is some α such that $S \in U_\alpha$), then we could infer that f is stationarily bounded. As f represents κ in U_α for every α , this gives the conclusion that each U_α is a normal ultrafilter, by virtue of Claim 2.1.

In light of Discussion 2.3, Theorem 2.2 gives rise to the possibility that the existence of τ -many club ultrafilters (for $\tau = (2^\kappa)^+$) is a sufficient condition for having τ -many normal ultrafilters on κ , with no need to assume compactness. We conclude with the following problem.

Question 2.4 Let κ be a measurable cardinal, $\tau = (2^\kappa)^+$.

Assume there are τ -many club ultrafilters on κ .

Does this imply that κ carries τ -many normal ultrafilters?

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