

## Baire Categoricity and $\Sigma_1^0$ -Induction

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**Abstract** We investigate the reverse-mathematical status of a version of the Baire category theorem known as BCT. In a 1993 paper Brown and Simpson showed that BCT is provable in  $\text{RCA}_0$ . We now show that BCT is equivalent to  $\text{RCA}_0$  over  $\text{RCA}_0^*$ .

Consider the following version of the Baire category theorem.

**Definition 1** Let BCT be the statement that, in any complete separable metric space, the intersection of any countable sequence of dense open sets is dense. Thus BCT is essentially the usual statement of the Baire category theorem for complete separable metric spaces.

The purpose of this paper is to prove a new result concerning the reverse-mathematical status of BCT. From Brown and Simpson [1, Theorem 2.1] we already know that BCT is provable in  $\text{RCA}_0$ . Here, of course,  $\text{RCA}_0$  is the usual base theory for reverse mathematics (see [5]), consisting of  $\Delta_1^0$ -comprehension plus  $\Sigma_1^0$ -induction. We now prove that BCT is logically equivalent to  $\text{RCA}_0$  over the weaker base theory  $\text{RCA}_0^*$ . The system  $\text{RCA}_0^*$  was first introduced in Simpson and Smith [6]. Two recent papers making use of  $\text{RCA}_0^*$  are Simpson and Yokoyama [8] and Yokoyama [10].

In addition to BCT itself, we consider the following special case of BCT.

**Definition 2** The *Cantor space* is the space  $\{0, 1\}^{\mathbb{N}}$  of infinite sequences of 0's and 1's. We endow  $\{0, 1\}$  with the discrete topology and  $\{0, 1\}^{\mathbb{N}}$  with the product topology. Let  $\text{BCT}(\{0, 1\}^{\mathbb{N}})$  be the statement that BCT holds for the Cantor space.

More precisely, let  $\{0, 1\}^*$  be the set of finite sequences of 0's and 1's. For  $\sigma \in \{0, 1\}^*$  and  $X \in \{0, 1\}^{\mathbb{N}}$ , we write  $\sigma \subset X$  to mean that  $\sigma$  is an initial segment of  $X$ . We also write  $\llbracket \sigma \rrbracket = \{X \in \{0, 1\}^{\mathbb{N}} \mid \sigma \subset X\}$ . Note that the sets  $\llbracket \sigma \rrbracket$

Received April 17, 2012; accepted June 13, 2012

2010 Mathematics Subject Classification: Primary 03B30; Secondary 03F25, 54E52

Keywords: reverse mathematics, second-order arithmetic, Baire category theorem,  $\text{RCA}_0$ ,  $\text{RCA}_0^*$ ,  $\Sigma_1^0$ -induction

© 2014 by University of Notre Dame 10.1215/00294527-2377887

where  $\sigma \in \{0, 1\}^*$  form a basis for the topology of  $\{0, 1\}^{\mathbb{N}}$ . For  $\sigma, \tau \in \{0, 1\}^*$ , we write  $\sigma \subset \tau$  to mean that  $\sigma$  is a proper initial segment of  $\tau$ . We say that  $D \subseteq \{0, 1\}^*$  is *dense in*  $\{0, 1\}^*$  if for all  $\sigma \in \{0, 1\}^*$  there exists  $\tau \in D$  such that  $\sigma \subset \tau$ . Thus the dense open sets in  $\{0, 1\}^{\mathbb{N}}$  are just the sets of the form  $\llbracket D \rrbracket = \bigcup_{\tau \in D} \llbracket \tau \rrbracket$ , where  $D$  is dense in  $\{0, 1\}^*$ . We say that  $X \in \{0, 1\}^{\mathbb{N}}$  *meets*  $D$  if there exists  $\tau \in D$  such that  $\tau \subset X$ . Within  $\text{RCA}_0^*$  let  $\text{BCT}(\{0, 1\}^{\mathbb{N}})$  be the statement that for all sequences of dense sets  $D_i \subseteq \{0, 1\}^*$ ,  $i \in \mathbb{N}$ , and all  $\sigma \in \{0, 1\}^*$  there exists  $X \in \{0, 1\}^{\mathbb{N}}$  such that  $\sigma \subset X$  and  $X$  meets  $D_i$  for each  $i \in \mathbb{N}$ .

**Theorem 1** *The following are pairwise equivalent over  $\text{RCA}_0^*$ :*

1.  $\text{RCA}_0$ ,
2.  $\text{BCT}$ ,
3.  $\text{BCT}(\{0, 1\}^{\mathbb{N}})$ ,
4. *for all finite sequences of dense sets  $D_i \subseteq \{0, 1\}^*$ ,  $1 \leq i \leq n$ , there exists  $X \in \{0, 1\}^{\mathbb{N}}$  such that  $X$  meets  $D_i$  for each  $i = 1, \dots, n$ .*

**Proof** We reason in  $\text{RCA}_0^*$ . The implication  $1 \Rightarrow 2$  is already known (see [1, Theorem 2.1]). The implications  $2 \Rightarrow 3$  and  $3 \Rightarrow 4$  are obvious.

It remains to prove  $4 \Rightarrow 1$ . For this purpose we use the following lemma from [8]. Within  $\text{RCA}_0^*$  a set  $C \subseteq \mathbb{N}$  is defined to be *infinite* if it is not finite, or equivalently, it is *unbounded*, that is,  $\forall n \exists c (n < c \in C)$ .

**Lemma 1** *Over  $\text{RCA}_0^*$  the following are equivalent:*

1.  $\text{RCA}_0$ ,
2. *each infinite subset of  $\mathbb{N}$  includes arbitrarily large finite subsets.*

**Proof** This is [8, Lemma 3.2]. □

We now prove  $4 \Rightarrow 1$ . Assume 4. By Lemma 1 it suffices to prove that each infinite subset of  $\mathbb{N}$  includes arbitrarily large finite subsets. Given an infinite set  $C \subseteq \mathbb{N}$ , for each  $i \in \mathbb{N}$  let  $D_i$  be the set of strings in  $\{0, 1\}^*$  of the form

$$\sigma \wedge \langle 1 \rangle \wedge \underbrace{\langle 0, \dots, 0 \rangle}_c \wedge \langle 1 \rangle \wedge \underbrace{\langle 0, \dots, 0 \rangle}_i \wedge \langle 1 \rangle, \quad (1)$$

where  $c \in C$  and  $c$  is greater than the length of  $\sigma$ . The sequence of sets  $\langle D_i \mid i \in \mathbb{N} \rangle$  exists by  $\Delta_1^0$ -comprehension. Since  $C$  is infinite, each  $D_i$  is dense in  $\{0, 1\}^*$ . Given  $n \in \mathbb{N}$ , apply 4 to obtain  $X \in \{0, 1\}^{\mathbb{N}}$  such that  $X$  meets  $D_i$  for each  $i = 1, \dots, n$ . By  $\Sigma_1^0$ -bounding (see [6]) plus  $\Delta_1^0$ -comprehension, there exists a finite sequence of strings  $\tau_i$ ,  $1 \leq i \leq n$ , such that  $\tau_i \in D_i$  and  $\tau_i \subset X$  for each  $i = 1, \dots, n$ . Consider the finite sequence  $c_1, \dots, c_n$ , where  $\tau_i$  is as in (1) with  $c = c_i$ . For  $i \neq j$ , we have  $\tau_i \neq \tau_j$ ; hence  $\tau_i \subset \tau_j$  or  $\tau_j \subset \tau_i$ ; hence  $c_i < c_j$  or  $c_j > c_i$ ; hence  $c_i \neq c_j$ . Thus  $\{c_1, \dots, c_n\}$  is a finite subset of  $C$  of cardinality  $n$ . □

**Theorem 2**  *$\text{BCT}$  is not  $\Pi_1^0$ -conservative over  $\text{RCA}_0^*$ .*

**Proof** Recall from [5, Section X.4] and [6] that  $\text{RCA}_0^*$  is  $\text{RCA}_0$  with  $\Sigma_1^0$ -induction weakened to  $\Sigma_0^0$ -induction plus *natural number exponentiation*, that is, the assertion that  $m^n$  exists for all  $m, n \in \mathbb{N}$ . It is known that  $\text{RCA}_0^*$  is  $\Pi_2^0$ -equivalent to elementary function arithmetic (see [6]) and hence is much weaker than  $\text{RCA}_0$ , which is  $\Pi_2^0$ -equivalent to primitive recursive arithmetic (see [5, Section IX.3]). Since primitive recursive arithmetic proves the consistency of elementary function arithmetic (see,

e.g., [4], [5, Theorems II.8.11, IX.3.16]), it follows that  $\text{RCA}_0$  is not  $\Pi_1^0$ -conservative over  $\text{RCA}_0^*$ . This fact together with Theorem 1 gives Theorem 2.  $\square$

### Remarks

1. Beyond  $\text{RCA}_0^*$  one may consider even weaker base theories for reverse mathematics. In this direction there is the following result of Fernandes [2]: BCT is  $\Pi_1^1$ -conservative over  $\Sigma_1^b$ -NIA +  $\nabla_1^b$ -CA. Note that  $\Sigma_1^b$ -NIA +  $\nabla_1^b$ -CA is “feasible”; that is, it does not include natural number exponentiation.
2. Actually, Fernandes [2] showed that BCT is conservative over  $\Sigma_1^b$ -NIA +  $\nabla_1^b$ -CA not only for  $\Pi_1^1$ -sentences but also for sentences of the form  $(\forall X)(\exists \text{ unique } Y)\Phi$ , where  $\Phi$  is arithmetical. And Yamazaki [9] showed that  $\Pi_\infty^0$ -BCT is conservative over  $\text{RCA}_0$  for this same class of sentences, which arose previously in connection with Tanaka’s conjecture (see [7, Theorem 4.18]).
3. Our Theorems 1 and 2 were inspired by Fernandes [2, Proposition 1] and Hirschfeldt, Shore, and Slaman [3, Theorem 4.3].

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