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A Long Pseudo-Comparison of Premice in $L[x]$

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Abstract A significant open problem in inner model theory is the analysis of $\mathrm{HOD}^{L[x]}$ as a strategy premouse, for a Turing cone of reals x. We describe here an obstacle to such an analysis. Assuming sufficient large cardinals, for a Turing cone of reals x there are proper class 1-small premice M, N, with Woodin cardinals δ, ε, respectively, such that $M|\delta$ and $N|\varepsilon$ are in $L[x]$, $(\delta^+)^M$ and $(\varepsilon^+)^N$ are countable in $L[x]$, and the pseudo-comparison of M with N succeeds, is in $L[x]$, and lasts exactly $\omega_1^{L[x]}$ stages. Moreover, we can take $M = M_1$, the minimal iterable proper class inner model with a Woodin cardinal, and take N to be M_1-like and short-tree-iterable.

1 Introduction

A central program in descriptive inner model theory is the analysis of HOD^W, for transitive models W satisfying $\mathsf{ZF} + \mathsf{AD}^+$ (see Steel [5], [7], Steel and Woodin [8], Sargsyan [4]). For the models W for which it has been successful, the analysis yields a wealth of information regarding HOD^W (including that it is fine structural and satisfies GCH), and in turn about W.

Assume that there are ω many Woodin cardinals with a measurable cardinal above them. A primary example of the previous paragraph is the analysis of $\mathrm{HOD}^{L(\mathbb{R})}$. Work of Steel and Woodin showed that $\mathrm{HOD}^{L(\mathbb{R})}$ is an iterate of M_ω augmented with a fragment of its iteration strategy (where M_n is the minimal iterable proper class inner model with n Woodin cardinals). The addition of the iteration strategy does not add reals, and so the $\mathrm{OD}^{L(\mathbb{R})}$ reals are just $\mathbb{R} \cap M_\omega$. The latter has an analogue for $L[x]$, which has been known for some time: for a cone of reals x, the $\mathrm{OD}^{L[x]}$ reals are just $\mathbb{R} \cap M_1$. Given this, and further analogies between $L(\mathbb{R})$ and $L[x]$ and their respective HODs, it is natural to ask whether the full $\mathrm{HOD}^{L[x]}$ is an iterate of M_1, adjoined with a fragment of its iteration strategy. Woodin has conjectured that

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this is so for a cone of reals x (for a precise statement, see Koellner and Woodin [2, Conjecture 8.23]). Woodin has proved approximations to this conjecture. He analyzed $\mathrm{HOD}^{L[x,G]}$, for a cone of reals x, and $G \subseteq \mathrm{Coll}(\omega, <\kappa)$ a generic filter over $L[x]$, where κ is the least inaccessible of $L[x]$ (see [2, Theorem 8.21] and [8]). However, the conjecture regarding $\mathrm{HOD}^{L[x]}$ is still open.

In this note, we describe a significant obstacle to the analysis of $\mathrm{HOD}^{L[x]}$.

1.1 Background We give a brief summary of some relevant definitions and facts. We assume familiarity with the fundamentals of inner model theory (see [7], Mitchell and Steel [3]). One does not really need to know the analysis of $\mathrm{HOD}^{L[x,G]}$, but familiarity does help in terms of motivation; the system $\mathcal{F}$ described below relates to that analysis. We do rely on some terminology and smaller facts from [8, Section 3], and we recall some in what follows. We say that a premouse N is *pre-M_1-like* if and only if N is proper class, 1-small, and has a (unique) Woodin cardinal, denoted δ^N. A premouse N is *M_1-like* if and only if N is pre-M_1-like and $N \models$"For all $\eta < \delta^N$, I am (η, η)-iterable." Let P, Q be pre-M_1-like. Given a normal iteration tree $\mathcal{T}$ on P, $\mathcal{T}$ is *maximal* if and only if $\mathrm{lh}(\mathcal{T})$ is a limit and $L[M(\mathcal{T})]$ has no Q-structure for $M(\mathcal{T})$ (so $L[M(\mathcal{T})]$ is pre-M_1-like with Woodin cardinal $\delta(\mathcal{T})$). A premouse R is a (*nondropping*) *pseudo-normal iterate* of P if and only if there is a normal tree $\mathcal{T}$ on P such that either $\mathcal{T}$ has successor length and $R = M^{\mathcal{T}}_\infty$, the last model of $\mathcal{T}$ (and $[0,\infty]_{\mathcal{T}}$ does not drop), or $\mathcal{T}$ is maximal and $R = L[M(\mathcal{T})]$. A premouse R is a *pseudo-iterate* of P if and only if there is $n < \omega$ and $(R_0, R_1, \dots, R_n)$ such that $R_0 = P$ and $R_n = R$ and each R_{i+1} is pre-M_1-like and is a pseudo-normal iterate of R_i. A *pseudo-comparison* of (P, Q) is a pair $(\mathcal{T}, \mathcal{U})$ of normal padded iteration trees of equal length, formed according to the usual rules of comparison, such that either $(\mathcal{T}, \mathcal{U})$ is a successful comparison, or either $\mathcal{T}$ or $\mathcal{U}$ is maximal. A (z-)*pseudo-genericity iteration* is defined similarly, formed according to the rules for genericity iterations making a real (z) generic for Woodin's extender algebra. We say that P is *normally short-tree-iterable* if and only if for every normal, nonmaximal iteration tree $\mathcal{T}$ on P of limit length, there is a $\mathcal{T}$-cofinal well-founded branch through $\mathcal{T}$, and every putative normal tree $\mathcal{T}$ on P of length $\alpha+2$ has a well-founded last model (i.e., we never first encounter an ill-founded model at a successor stage). If $P|\delta^P \in \mathrm{HC}^{L[x]}$, then normal short-tree-iterability is absolute between $L[x]$ and V. If P, Q are normally short-tree-iterable, then there is a pseudo-comparison $(\mathcal{T}, \mathcal{U})$ of (P, Q), and if $\mathcal{T}$ has a last model, then $[0,\infty]_{\mathcal{T}}$ does not drop, and likewise for $\mathcal{U}$.

By *Turing determinacy* we mean the statement that every set of Turing degrees either contains or is disjoint from a cone.

1.2 The HOD of $L[x]$ It has been suggested[1] that one might analyze $\mathrm{HOD}^{L[x]}$ using an $\mathrm{OD}^{L[x]}$ directed system $\mathcal{F}$ such that:

- the nodes of $\mathcal{F}$ are pairs (N, s) such that $s \in \mathrm{OR}^{<\omega}$ and N is a normally short-tree-iterable, pre-M_1-like premouse with $N|\delta^N \in \mathrm{HC}^{L[x]}$,
- for $(P,t), (Q,u) \in \mathcal{F}$, we have $(P,t) \le_{\mathcal{F}} (Q,u)$ if and only if $t \subseteq u$ and Q is a pseudo-iterate of P, and
- $(M_1, \emptyset) \in \mathcal{F}$.

If such systems existed, satisfying some further requirements regarding the sets s, strengthening the iterability requirements, and including countable directedness (for each fixed s), then there would have been a reasonable scenario for analyzing $\mathrm{HOD}^{L[x]}$, making use of Neeman's genericity iterations.[2]

The primary difficulty in analyzing $\mathrm{HOD}^{L[x]}$ in this manner is in arranging that $\mathscr{F}$ be directed, even finitely. For this, it seems most obvious to try to arrange that $\mathscr{F}$ be closed under pseudo-comparison of pairs. However, we show here that, given sufficient large cardinals, there is a cone of reals x such that if $\mathscr{F}$ is as above, then $\mathscr{F}$ is *not* closed under pseudo-comparison of pairs.

The proof proceeds by finding a node $(N, \emptyset) \in \mathscr{F}$ such that, letting $(\mathcal{T}, \mathcal{U})$ be the pseudo-comparison of (M_1, N), then $\mathcal{T}, \mathcal{U}$ are in fact pseudo-genericity iterations of M_1, N, respectively, making reals $y, z \in L[x]$ generic, where $\omega_1^{L[y]} = \omega_1^{L[z]} = \omega_1^{L[x]}$. Letting W be the output of the pseudo-comparison, we will have $W|\delta^W \in L[x]$, so $\omega_1^{W[z]} = \omega_1^{L[x]}$, which implies that $\delta^W = \omega_1^{L[x]}$, so $(W, \emptyset) \notin \mathscr{F}$. We now proceed to the details.

2 The Pseudo-Comparison

For a formula φ in the language of set theory (LST), $\zeta \in \mathrm{OR}$, and $x \in \mathbb{R}$, let $A^x_{\varphi,\zeta}$ be the set of all $M \in \mathrm{HC}^{L[x]}$ such that $L[x] \models \varphi(\zeta, M)$, and $L[M]$ is a normally short-tree-iterable pre-M_1-like premouse with $M = L[M]|\delta^{L[M]}$.

Note that φ does not use x as a parameter. So by absoluteness of normal short-tree-iterability (as stated in Section 1.1), $A^x_{\varphi,\zeta}$ is $\mathrm{OD}^{L[x]}$. So $A^x_{\varphi,\zeta}$ is a collection of premice like those involved in the system $\mathscr{F}$, restricted to their Woodin cardinals.

Theorem *Assume Turing determinacy and that $M_1^\#$ exists and is fully iterable. Then for a cone of reals x, for every formula φ in the LST and every $\zeta \in \mathrm{OR}$, if $M_1|\delta^{M_1} \in A^x_{\varphi,\zeta}$, then there is $R \in A^x_{\varphi,\zeta}$ such that the pseudo-comparison of M_1 with $L[R]$ has length $\omega_1^{L[x]}$.*

The theorem easily proves the statement made in the abstract; for example, we can ensure that $N = L[R]$ is M_1-like by incorporating this requirement into φ.

Proof Suppose not. Then we may fix φ such that for a cone of x, the theorem fails for φ, x. Fix z in this cone with $z \geq_T M_1^\#$. Let $\mathcal{W}$ be the z-genericity iteration on M_1 (making z generic for the extender algebra), and let $Q = M^{\mathcal{W}}_\infty$. By standard arguments (see [8]), $Q[z] = L[z]$,

$$\mathrm{lh}(\mathcal{W}) = \omega_1^{L[z]} + 1 = \delta^Q + 1,$$

$Q|\delta^Q = M(\mathcal{W} \upharpoonright \delta^Q)$, and $\mathcal{T} =_{\mathrm{def}} \mathcal{W} \upharpoonright \delta^Q$ is the z-pseudo-genericity iteration of M_1, and $\mathcal{T} \in L[z]$.

Let $\mathbb{B}$ be the extender algebra of Q, and let $\mathbb{P}$ be the finite support ω-fold product of $\mathbb{B}$. For $p \in \mathbb{P}$ and $i < \omega$, let p_i be the ith component of p. Let $G \subseteq \mathbb{P}$ be Q-generic, with $z_0 = z$, where $x =_{\mathrm{def}} \langle z_i \rangle_{i<\omega}$ is the generic sequence of reals. Then

$$Q[G] = Q[x] = L[x]$$

and $x >_T z$. Let $\zeta \in \mathrm{OR}$ witness the failure of the theorem with respect to φ, x. So $M_1|\delta^{M_1} \in A^x_{\varphi,\zeta}$.

By Farah [1, Lemma 3.4] (essentially due to Hjorth), $\mathbb{P}$ is δ^Q-cc in Q, so $\delta^Q \geq \omega_1^{L[x]}$, but $\delta^Q = \omega_1^{L[z]}$, so $\delta^Q = \omega_1^{L[x]}$. So it suffices to see that there is some $R \in A^x_{\varphi,\zeta}$ such that the pseudo-comparison of M_1 with $L[R]$ has length δ^Q.

For $e \in \omega$ and $y \in \mathbb{R}$, let $\Phi_e^y : \omega \to \omega$ be the partial function coded by the eth Turing program using the oracle y. Let $e \in \omega$ be such that Φ_e^z is total and codes $M_1|\delta^{M_1}$. Let $\dot{x}$ be the $\mathbb{P}$-name for x, and for $n < \omega$ let $\dot{z}_n$ be the $\mathbb{P}$-name for z_n. Let $p \in G$ be such that $p \Vdash^Q_{\mathbb{P}} \psi(\dot{z}_0)$, where $\psi(v)$ asserts "Φ_e^v is total and codes a premouse $R \in A^{\dot{x}}_{\check{\varphi},\check{\xi}}$ such that the v-pseudo-genericity iteration of $L[R]$ produces a maximal tree $\mathcal{U}$ of length $\check{\delta^Q}$ with $M(\mathcal{U}) = L[\check{\mathbb{E}}]|\check{\delta^Q}$." In the notation of this formula,

$$p \Vdash^Q_{\mathbb{P}} \text{"}R \notin \check{V}\text{," because } p \Vdash^Q_{\mathbb{P}} \text{"}E_0^{\mathcal{U}} \notin M(\mathcal{U})\text{."}$$

By genericity, we may fix $q \in G$ such that $q \leq p$ and for some $m > 0$, $q_m = q_0$. Note that $q \Vdash^Q_{\mathbb{P}} \psi(\dot{z}_m)$.

Let $\dot{R}_i$ be the $\mathbb{P}$-name for the premouse coded by $\Phi_e^{\dot{z}_i}$ (or for $\emptyset$ if this does not code a premouse). Also, let $\dot{z}'_0, \dot{z}'_1$ be the $(\mathbb{B} \times \mathbb{B})$-names for the two $(\mathbb{B} \times \mathbb{B})$-generic reals (in order), and let $\dot{R}'_i$ be the $(\mathbb{B} \times \mathbb{B})$-name for the premouse coded by $\Phi_e^{\dot{z}'_i}$.

We may fix $r \leq q$, $r \in G$, such that

$$r \Vdash^Q_{\mathbb{P}} \text{"}\dot{R}_0 \neq \dot{R}_m\text{."} \tag{1}$$

For otherwise there is $r \leq q$, $r \in G$, such that $r \Vdash^Q_{\mathbb{P}}$"$\dot{R}_0 = \dot{R}_m$." But since

$$M_1|\delta^{M_1} = \dot{R}_0^G \notin Q,$$

there are $s, t \in \mathbb{B}$, $s, t \leq r_0$, such that

$$(s,t) \Vdash^Q_{\mathbb{B}\times\mathbb{B}} \text{"}\dot{R}'_0 \neq \dot{R}'_1\text{."}$$

Therefore, there are $u, v \in \mathbb{B}$, with $u \leq r_0$ and $v \leq r_m$, such that

$$(u,v) \Vdash^Q_{\mathbb{B}\times\mathbb{B}} \text{"}\dot{R}'_0 \neq \dot{R}'_1\text{."}$$

Let $w \leq r$ be the condition with $w_i = r_i$ for $i \neq 0, m$, and $w_0 = u$ and $w_m = v$. Then

$$w \Vdash^Q_{\mathbb{P}} \text{"}\dot{R}_0 \neq \dot{R}_m\text{,"}$$

which is a contradiction.

So letting $R = \dot{R}_m^G$, we have $R \neq M_1|\delta^{M_1}$ and $R \in A^x_{\varphi,\xi}$ and $Q|\delta^Q = M(\mathcal{U})$, where $\mathcal{U}$ is the z_m^G-pseudo-genericity iteration of $L[R]$, and $\mathrm{lh}(\mathcal{U}) = \delta^Q$. We defined $\mathcal{T}$ earlier. Let $\mathcal{T}^*, \mathcal{U}^*$ be the padded trees equivalent to $\mathcal{T}, \mathcal{U}$ such that for each α, either $E_\alpha^{\mathcal{T}^*} \neq \emptyset$ or $E_\alpha^{\mathcal{U}^*} \neq \emptyset$, and if $E_\alpha^{\mathcal{T}^*} \neq \emptyset \neq E_\alpha^{\mathcal{U}^*}$, then $\mathrm{lh}(E_\alpha^{\mathcal{T}^*}) = \mathrm{lh}(E_\alpha^{\mathcal{U}^*})$. Let $(\mathcal{T}', \mathcal{U}')$ be the pseudo-comparison of $(M_1, L[R])$ (recall that $L[R]$ is normally short-tree-iterable by definition of $A^x_{\varphi,\xi}$).

We claim that $(\mathcal{T}', \mathcal{U}') = (\mathcal{T}^*, \mathcal{U}^*)$; this completes the proof. For this, we prove by induction on α that

$$(\mathcal{T}', \mathcal{U}') \restriction (\alpha + 1) = (\mathcal{T}^*, \mathcal{U}^*) \restriction (\alpha + 1).$$

This is immediate if α is a limit, so suppose that it holds for $\alpha = \beta$; we prove it for $\alpha = \beta + 1$. Let $\lambda = \mathrm{lh}(E_\beta^{\mathcal{T}^*})$ or $\lambda = \mathrm{lh}(E_\beta^{\mathcal{U}^*})$, whichever is defined. Because $M(\mathcal{T}^*) = Q|\delta^Q = M(\mathcal{U}^*)$, the least disagreement between $M_\beta^{\mathcal{T}^*}$ and $M_\beta^{\mathcal{U}^*}$ has index at least λ, so we just need to see that $E_\beta^{\mathcal{T}^*} \neq E_\beta^{\mathcal{U}^*}$.

So suppose that $E^{\mathcal{T}^*}_\beta = E^{\mathcal{U}^*}_\beta$. In particular, both are nonempty. Then there is $s \in G$ such that $s \leq r$ (see (1)) and $s \Vdash^Q_{\mathbb{P}}$"For $i = 0, m$, let $\mathcal{T}_i$ be the $\dot{z}_i$-pseudo-genericity iteration of $L[\dot{R}_i]$. Then $\mathcal{T}_0$ and $\mathcal{T}_m$ use identical nonempty extenders E of index $\check{\lambda}$." Because

$$s \Vdash^Q_{\mathbb{P}} \psi(\dot{z}_0) \;\&\; \psi(\dot{z}_m),$$

also $s \Vdash^Q_{\mathbb{P}}$"Letting E be as above, $E \subseteq L[\check{\mathbb{E}}]|\check{\lambda}$, but $E \notin \check{V}$"; here $E^{\mathcal{T}^*}_\beta \notin Q$ because λ is a cardinal of Q. But since $\mathcal{T}_i^G$ is computed in $Q[z_i^G]$ (for $i = 0, m$), we can argue as before (as in the proof of the existence of r as in line (1)) to reach a contradiction. □

A slightly simpler argument, using $\mathbb{B} \times \mathbb{B}$ instead of $\mathbb{P}$, proves the weakening of the theorem given by dropping the parameter ζ. The author does not see how to prove the full theorem using $\mathbb{B} \times \mathbb{B}$ instead of $\mathbb{P}$. This is because in the argument given, ζ depends on x, and the choice of the conditions p, q depend on ζ.[3]

Notes

1. For example, at the workshop on descriptive inner model theory at the American Institute of Mathematics, June 2014.

2. Woodin's genericity iterations produce trees of length $\omega_1^{L[x]}$. Moreover, it seems that $\mathrm{HC}^{L[x]}$ need not be sufficiently closed under the existence of collapse generics to allow an obvious analysis of $\mathrm{HOD}^{L[x]}$ using Neeman's genericity iterations. (Thanks to John Steel for pointing out an error that appeared in a draft of this article, regarding this point.) But it seems this issue could have been avoided by using the following fact, due to Steel, together with related calculations. Assume (enough) determinacy. Then for a cone of x, $L[x]$ and $L[x, y]$ have the same theories in ordinal parameters whenever y is Cohen over $L[x]$. The proof, which we include with Steel's permission, is as follows. For reals x, let $[x]$ denote the Turing degree of x. Let $F([x])$ be the least (φ, α) such that $\alpha \in \mathrm{OR}$ and $L[x] \models \varphi(\alpha)$ but for some Cohen-generic y over $L[x]$, $L[x, y] \models \neg\varphi(\alpha)$, if such (φ, α) exists, and let $F([x]) = 0$ otherwise. Then $F([x]) = 0$ for a cone of x, because otherwise the proof of Steel [6, Lemma 4.3] leads to a contradiction. There is also an alternate proof using $M_1^\#$ as follows. We may assume that $L[x] = N[G]$, where N is some nondropping iterate of M_1, and G is $(N, \mathbb{P})$-generic where $\mathbb{P} = \mathrm{Coll}(\omega, \delta^N)$, and so $L[x, y] = N[G][H]$, for some $(N[G], \mathbb{P})$-generic H. It follows that $L[x]$ and $L[x, y]$ have the same theories in ordinal parameters. Clearly, both proofs work with much less than full determinacy, but the former proof has the extra virtue of not needing the appeal to $M_1^\#$.

3. So if one tries to run the same argument but with $\mathbb{B} \times \mathbb{B}$ instead of $\mathbb{P}$, one must first choose a generic pair of reals $x = (z_0, z_1)$, thus determining ζ, but then even if we had tried to be selective about z_1, it seems there might not be any $q \in G$ analogous to that found in the proof using $\mathbb{P}$. On the other hand, if there is no parameter ζ involved, we *can* be selective enough about z_1.

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