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Normal Numbers and Limit Computable Cantor Series

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Abstract Given any oracle, A, we construct a basic sequence Q, computable in the jump of A, such that no A-computable real is Q-distribution-normal. A corollary to this is that there is a Δ^0_{n+1} basic sequence with respect to which no Δ^0_n real is distribution-normal. As a special case, there is a limit computable sequence relative to which no computable real is distribution-normal.

1 Introduction

The effective theory of the reals has been an active area of research for many years. Out of this field have come a number of effective formalizations of the intuitive concept of randomness, for example, Martin-Löf randomness. There are, however, a number of classical formalizations of randomness which derive from ergodic theory. In the present article, we explore one of these classical notions, but in an effective context.

Given $b \in \omega$, $b \geq 2$, a real number x is said to be *b-normal* if the numbers $x, bx, b^2x, \ldots$ are uniformly distributed modulo 1. That is, for each interval $I \subseteq [0,1]$ of length ε, one has

$$\lim_{n\to\infty} \frac{|\{k \in \omega : (0 \leq k < n) \wedge (b^k x (\mathrm{mod}\ 1) \in I)\}|}{n} = \varepsilon.$$

Historically, number theorists have developed several methods for algorithmically producing b-normal numbers. One of the best known of these methods is the Champernowne construction [6]. Let $b^{<\omega}$ denote the set of finite sequences of elements of the set $\{0, 1, \ldots, b-1\}$. If $p_i \in b^{<\omega}$ is the base-b expansion of $i \in \omega$, then the real number with base-b expansion

$$0.p_0\ p_1\ p_2 \ldots$$

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is b-normal. For instance, the nonnegative integers are 0, 1, 10, 11, 100, ... in base 2, and the real with binary expansion

$$0.0\ 1\ 10\ 11\ 100\ 101\ 110\ 111\ldots$$

is 2-normal. In essence, the Champernowne construction shows that, for each b, there is a computable real number which is b-normal.

One may generalize the notion of base-b expansions of real numbers to that of so-called *Cantor series expansions* [5]. Given a sequence $Q = (q_n)_{n\in\omega}$ of positive integers, with each $q_n \geq 2$, and a real number $x \in (0, 1)$, there exist integers $a_0, a_1, \ldots$ such that $0 \leq a_n < q_n$, for each n, and

$$x = \sum_{n=0}^{\infty} \frac{a_n}{q_0 q_1 \cdots q_n}.$$

This expansion is known as the *Cantor series expansion of* x, with respect to Q. The sequence Q is known as a *basic sequence*, that is, a sequence of bases. Over the years, there has been some study of Cantor series expansions under different assumptions on the basic sequence $(q_n)_{n\in\omega}$ (see, e.g., Erdös and Rényi [7], [8]).

There is a corresponding generalization of b-normality in the context of Cantor series. Specifically, if $Q = (q_n)_{n\in\omega}$ is a sequence of positive integers, with each $q_n \geq 2$, then $x \in (0, 1)$ is said to be *Q-distribution-normal* if and only if the sequence $x, q_0 x, q_0 q_1 x, q_0 q_1 q_2 x, \ldots$ is uniformly distributed modulo 1. Thus, b-normality is equivalent to Q-distribution-normality for $Q = (q_n)_{n\in\omega}$, with each $q_n = b$.

It is an active area of research in modern number theory to try to find constructions analogous to the Champernowne construction in the context of Cantor series and other expansions of real numbers (e.g., continued fractions, Lüroth expansions, etc.). Examples of these lines of inquiry can be found in Altomare and Mance [2], Adler, Keane, and Smorodinsky [1], Madritsch [10], and Madritsch, Thuswaldner, and Tichy [11]. There has also been work on relating the various classical notions of normality with recursion-theoretic and descriptive set-theoretic measures of complexity (see, e.g., Ki and Linton [9], Becher, Heiber, and Slaman [3], [4]).

To obtain algorithmic constructions of normal numbers in the context of Cantor series, one often places conditions on the sequence $(q_n)_{n\in\omega}$ that guarantee rapid divergence to infinity, for example, that $\sum_n 1/q_n < \infty$.

In the present article, we provide a group of results which serve as a counterpoint to such attempts to algorithmically produce normal numbers. The following theorem is our main result.

Theorem 1.1 *There is a Δ^0_2 basic sequence Q (consisting of powers of 2) such that no computable real number is Q-distribution-normal.*

2 Preliminaries

As we are presenting Theorem 1.1 in the context of basic sequences consisting of powers of 2 (although it could just as easily be done with an arbitrary b), we introduce some notation for working with binary expansions of real numbers in [0, 1].

Notation

(1) Let 2^ω denote the set of infinite binary sequences.
(2) If $\alpha \in 2^\omega$ and $n \in \omega$, we define $\alpha \restriction n = (\alpha(0), \alpha(1), \ldots, \alpha(n-1))$.
(3) If $\alpha \in 2^\omega$, let x_α denote the real number $\sum_{n\in\omega} \frac{\alpha(n)}{2^{n+1}}$.
(4) If $n \in \omega$ and $\alpha \in 2^\omega$, we will write $n\alpha$ for $(\alpha(n), \alpha(n+1), \ldots)$; that is, $n\alpha$ is the n-bit left shift of α.

Suppose that $Q = (q_n)_{n\in\omega}$ with each $q_n = 2^{s_n}$, for some integers $s_n \geq 1$. If $\alpha \in 2^\omega$ and α does not end with an infinite string of 1's, then, for each n and $p = s_0 + \cdots + s_n$, we have $q_0 \cdot \ldots \cdot q_n x_\alpha (\mathrm{mod} 1) = x_{p\alpha}$.

The following is our key computability-theoretic definition.

Definition 2.1 We say that $x \in [0, 1]$ is Δ^0_n if and only if there is an $\alpha \in 2^\omega$ such that $\{n \in \omega : \alpha(n) = 1\}$ is a Δ^0_n subset of ω and $x = x_\alpha$.

Recall that a subset $A \subseteq \omega$ is Δ^0_n if and only if A is computable in $0^{(n)}$ (the n-fold jump of $\emptyset$). Our definition of Δ^0_n for $x \in [0, 1]$ is equivalent to the standard definition of Δ^0_n (see Nies [12, Section 1.8]).

Next, we require an enumeration that includes all computable reals. To avoid the extra complexity inherent in dealing with partial functions, we define a slightly modified universal Turing machine.

Definition 2.2 Let $\{\varphi_e\}_{e\in\omega}$ be the standard enumeration of all binary-valued partial computable functions. For $e, s \in \omega$, let $\varphi_{e,s}$ be φ_e computed up to s computation stages. We define an array of computable functions, $\{\varphi^*_{e,s}\}_{e,s\in\omega}$, as follows:

$$\varphi^*_{e,s}(n) = \begin{cases} \varphi_{e,s}(n) & \text{if } \varphi_{e,s}(n) \text{ halts,} \\ 0 & \text{otherwise.} \end{cases}$$

We define φ^*_e to be the pointwise limit of $\varphi^*_{e,s}$ as s tends to ∞.

The enumeration, $\{\varphi^*_e\}_{e\in\omega}$, includes all computable reals, although it is obviously not a computable enumeration, and also enumerates some noncomputable reals. We will freely identify each $\varphi^*_{e,s}$ with the infinite sequence it codes.

Following the notation introduced above, we let $n\varphi^*_{e,s}$ denote the n-bit left shift of the infinite sequence determined by $\varphi^*_{e,s}$; that is, if $\varphi^*_{e,s}$ codes the sequence α, then $n\varphi^*_{e,s}$ codes the sequence $(\alpha(n), \alpha(n+1), \ldots) \in 2^\omega$.

Note that every computable real in $[0, 1]$ is of the form $x_{\varphi^*_e}$ for some $e \in \omega$.

3 Diagonalizing against Reals

To prove Theorem 1.1, we will construct a strictly increasing Δ^0_2 function $f : \omega \to \omega$ such that $Q = (q_n)_{n\in\omega}$, with $q_n = 2^{f(n+1)-f(n)}$, is a basic sequence with the property that no computable real is Q-distribution-normal.

As the desired function is to be Δ^0_2, we will construct it as the limit of a computable sequence of finite partial functions, $\{f_s\}_{s\in\omega}$. For an arbitrary s, the function f_s is constructed in $s + 1$ stages. We present the construction of f_s.

Stage 0: We define $f_s(0) = 0$ and end the stage. The domain of f_s is currently $[0, 1) = [0, 3^0)$.

Stage $t + 1$: We define

$$A_k = \{p \in (f_s(3^t - 1), \infty) : \varphi^*_{t,s+1}(p) = k\}.$$

Either $|A_0| \geq 2(3^t)$ or $|A_1| \geq 2(3^t)$. Let $k \in \{0,1\}$ be chosen such that there exist $p_1 < \cdots < p_{2(3^t)}$ in A_k, with $p_{2(3^t)}$ as small as possible. Set $f_s(3^t + i) = p_{i+1}$ for $0 \leq i \leq 2(3^t) - 1$ and end the stage. The domain of f_s is currently $[0, 3^{t+1})$.

By the pigeonhole principle, the interval $(f_s(3^t - 1), 4(3^t) + f_s(3^t - 1))$ must either contain at least $2(3^t)$-many p such that $\varphi^*_{t,s+1}(p) = 0$ or $2(3^t)$-many p such that $\varphi^*_{t,s+1}(p) = 1$. It follows that

$$f_s(3^{t+1} - 1) \leq 4(3^t) + f_s(3^t - 1),$$

for each $t \leq s$. Hence,

$$f_s(3^{t+1}) \leq 0 + 4 + 12 + \cdots + 4(3^t) = 2(3^{t+1} - 1) \tag{1}$$

for all s and t, with $t \leq s$. Note that this upper bound is independent of s.

Now that we have defined f_s for $s \in \omega$, we define $f(x) = \lim_{s\to\infty} f_s(x)$. To verify that we have constructed a function with the desired properties, we must prove two claims. First, we must prove that f is well defined; in other words, for every $p \in \omega$, there exists $m \in \omega$ such that for all $s \geq m$, $f_s(p) = f_m(p)$. We fix $p \in \omega$ and suppose that $i \in \omega$ is such that $p < 3^i$. Pick $m \in \omega$ such that if $s \geq m$, then

$$\varphi^*_e \restriction \max\{f_a(3^i) : a \in \omega\} = \varphi^*_{e,s} \restriction \max\{f_a(3^i) : a \in \omega\},$$

for all $e \leq i$. Note that the maxima above are finite by (1). Clearly $f_s(p) = f_m(p)$ for all $s \geq m$, since $f_s(p)$ depends only on the values of $\varphi^*_e(\ell)$, for $e \leq i$ and

$$\ell \leq \max\{f_a(3^i) : a \in \omega\} < \infty.$$

Thus, f is well defined, and therefore, Δ^0_2.

Let $q_n = 2^{f(n+1)-f(n)}$, and let $Q = (q_n)_{n\in\omega}$. The second claim we must verify is that no real number of the form $x_{\varphi^*_e}$ is Q-distribution-normal. Fix a computable sequence α, let e be such that $\alpha = \varphi^*_e$, and let $i_0 < i_1 < i_2 \ldots$ be a sequence of natural numbers such that $\varphi^*_{i_k} = \alpha$ for all $k \in \omega$. We consider a single value of k. From the definition of f_s it is clear that either

$$\frac{|\{p \leq 3^{i_k} : x_{f(p)\alpha} \leq \frac{1}{2}\}|}{3^{i_k}} \geq 2/3 \qquad \text{or} \qquad \frac{|\{p \leq 3^{i_k} : x_{f(p)\alpha} \geq \frac{1}{2}\}|}{3^{i_k}} \geq 2/3.$$

Since this is true for all $k \in \omega$ and $\varphi^*_{i_k} = \varphi^*_e$, we conclude that

$$\lim_{n\to\infty} \frac{|\{p \leq n : x_{f(p)\alpha} \leq \frac{1}{2}\}|}{n}$$

either does not exist or is not $\frac{1}{2}$. Hence $\alpha = x_{\varphi^*_e}$ is not Q-distribution-normal. As every computable real occurs in the sequence $\{x_{\varphi^*_e}\}_{e\in\omega}$, we have proved the desired result.

4 Generalizations

Relativizing the proof of Theorem 1.1 to an arbitrary oracle, we obtain the following theorem.

Corollary 4.1 *Let A be any subset of the natural numbers. There is a basic sequence Q, limit computable in A, such that no A-computable real is Q-distribution-normal.*

By the relativized version of Shoenfield's limit lemma, a set is limit computable in A if and only if it is computable in A', the jump of A. As a consequence, we obtain a direct generalization of Theorem 1.1 for all the "Δ-classes" of the arithmetical hierarchy.

Corollary 4.2 *There is a Δ^0_{n+1} basic sequence Q such that no Δ^0_n real is Q-distribution-normal.*

Proof Setting $A = 0^{(n)}$, Corollary 4.1 guarantees the existence of a basic sequence Q which is limit computable in $0^{(n)}$ and such that no real computable in $0^{(n)}$ is Q-distribution-normal. If Q is such a sequence, then Q is computable in $0^{(n+1)}$. Equivalently, Q is Δ^0_{n+1}. □

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