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Logical Consequence and First-Order Soundness and Completeness: A Bottom Up Approach

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Abstract What is the philosophical significance of the soundness and completeness theorems for first-order logic? In the first section of this paper I raise this question, which is closely tied to current debate over the nature of logical consequence. Following many contemporary authors' dissatisfaction with the view that these theorems ground deductive validity in model-theoretic validity, I turn to measurement theory as a source for an alternative view. For this purpose I present in the second section several of the key ideas of measurement theory, and in the third and central section of the paper I use these ideas in an account of the relation between model theory, formal deduction, and our logical intuitions.

1 Logical Consequence and First-Order Soundness and Completeness

The soundness and completeness theorems for first-order logic prove the existence of two converse inclusion relations: of the standard first-order deductive proof relation within first-order model-theoretic validity, and vice versa. As such their mathematical content is straightforward and not a matter of interpretation or dispute. However, when it comes to the extra-formal content that formal logicians and philosophers read into these theorems things are different. The interpretation of first-order soundness and completeness theorems is closely related to the philosophical analysis of the concept of logical consequence: any position with respect to the latter issue (the nature of logical consequence) will have implications to the former. Therefore, the ongoing controversy concerning logical consequence implies disagreement with respect to the significance of first-order soundness and completeness as well.

As noted by Etchemendy [4], Shapiro [17], and others, there is a received view on these matters that is given explicit expression in many logic textbooks and implicit expression in standard terminology. According to this received view our intuitive notion of logical consequence was successfully captured formally through

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Tarski's 1936 model-theoretic analysis. This conception has obvious consequences to the way first-order soundness and completeness theorems are viewed: if model-theoretic validity captures most directly and completely our intuitions with respect to logical consequence, then the degree to which deductive systems (based on first-order languages) coincide with these intuitions can be assessed by their relation to the Tarskian notion. The names commonly assigned to these theorems express very clearly this view. Thus the inclusion of the deductive consequence relation within model-theoretic consequence is called soundness, because it supposedly confirms the validity of formal proofs by showing that they are all model-theoretically valid. Similarly, the inverse inclusion is called completeness, because it shows that the deductive system is complete with respect to intuition—as expressed by model-theoretic validity. And even the very terms semantic and syntactic consequence that are assigned, respectively, to the model-theoretic and deductive formal relations, imply these hierarchical relations: logical consequence has to do with meaning (even if only with formal aspects of meaning), and therefore a formalism that is labeled semantic is immediately granted predominance in capturing this notion.

This received view has been subject to many attacks and criticisms, as well as to staunch defense and friendly suggestions for change and improvement. Here are several of the main trajectories of the debate over the Tarskian notion of consequence, together with their implications to the standing and interpretation of first-order soundness and completeness theorems. (No claim is made here that the list below exhausts the extant positions on the question of logical consequence, or that it provides a full exposition of the positions that appear in it.)

1.1 Proof-theoretic accounts of consequence The major opposition to the Tarskian account of consequence (and later semantic accounts in its mold) has traditionally come from proof-theoretic, so-called constructivist conceptions of logic (Prawitz [15], Martin-Löf [13]). Such conceptions relegate proof rather than truth the central role in an account of logical consequence, tying this notion to human epistemic and cognitive capabilities. The obvious consequence of such a perspective is that formal deduction systems are conceived as the closest in form and spirit to the philosophical analysis of the intuitive notion, and therefore it is not required nor possible to justify these systems through their association with the Tarskian formalism which is truth-theoretic in its orientation.

Many of those who belong to this camp would probably simply show disinterest in classical first-order logic and the theorems associated with it, preferring alternative formal systems (such as intuitionistic systems) that are in tune with their philosophical convictions. Those who do give attention to first-order deductive and model-theoretic systems and the relations thereof will reject the view implicit in current terminology. From their perspective the soundness theorem certainly does not justify the deduction rules: if anything, it consists in a completeness theorem, showing that the model-theoretic consequence relation does not miss any of the more intuitively and philosophically robust pronouncements of the deductive system. Similarly, what is usually known as the completeness theorem will be termed according to this opposite conception a soundness theorem, ensuring that all model-theoretically valid inferences do not go astray, that is, that they do not characterize an argument as valid without it being so characterized by the more trustworthy deductive system (Shapiro [17]).

An obvious result of this exercise of identity-switching is that the terms involved are value-laden, and therefore are sensitive to philosophical outlook and possibly also to context. (That is, it cannot be ruled out that in one context the deductive system in question could be viewed as closer to intuition than the model theory assigned to it, and that therefore the assessments of soundness and completeness would be along the lines just mentioned, and in another context things will be the other way around and standard usage of the terms "soundness" and "completeness" would be in place.) Alternatively, it can be explicitly decided that "soundness" and "completeness" always indicate the by-now accepted relations between deduction and model-theoretic systems. In this case, however, care needs to be taken not to read into them any superfluous extra-formal content.

1.2 Etchemendy's criticism of Tarskian model-theoretic semantics In a much debated book Etchemendy [4] accepts the general semantic outlook on logical consequence, according to which the consequence of a valid argument must be true in all possible cases in which the premises of the argument are true. However, by Etchemendy's lights Tarski's formalism (both as expressed in Tarski's original work and as used today) does not adequately capture this outlook. Tarskian semantics are interpretational, in that in order to decide if an argument is valid or not they enumerate different ways in which the nonlogical terms appearing in it can be interpreted. The notion of logical consequence (the intuitive, philosophically defensible notion), on the other hand, is representational: it keeps the meaning of the terms (both logical and not) fixed, and (implicitly) quantifies over possible ways the world might have been. The interpretational mechanism does not coincide with representational intuition, and therefore it fails both intentionally (as a formal rendering of the correct analysis of the notion) and extensionally (i.e., in the classification of valid arguments and logical truths).

It is beyond the scope of this paper to pronounce judgment on the historical question whether Tarski's original work was indeed interpretational (in particular, whether Tarski had in mind a single domain of quantification or varied such domains). Much has been written about this question (Sher [19], Mancosu [12]). As for contemporary model-theoretic systems, the discussion in Section 3 below will bear upon their status vis-à-vis Etchemendy's distinction. For now let it only be noticed that if Etchemendy's position is accepted, then the soundness and completeness theorems for first-order logic change their orientation in a way similar to that described in the previous clause, albeit for different reasons. If Tarskian semantics is misguided, then certainly we should trust it less than we do standard first-order deductive systems. It follows that the completeness theorem for first-order logic should be thought of as proving the soundness of Tarski's suspect system, by including its pronouncements within those of the more trustworthy deductive system. Similarly, the soundness theorem turns out to be a proof of the completeness of the model-theoretic system with respect to the deductive one. In both cases, the deviant interpretation of the theorems does not arise from wholehearted trust in deduction. Rather, along with Tarski and unlike constructivist logicians, Etchemendy holds that truth-theoretic semantics should foot the bill for the notion of logical consequence. It is just that extant interpretational semantics does not do so adequately, and therefore we are left for the time being to rely on first-order deductive systems.

1.3 Widening the bounds of logic Yet another varied and important group of thinkers upholds the general model-theoretic framework initiated by Tarski [23], but rejects the view that first-order logic should be given special status as the real, or *core* logic. Tarski himself is noncommittal on this issue in his 1936 paper, as indicated by his remark that he is unable to defend the choice of logical constants made in the paper. In later writings Tarski [24] does present a substantive conception of the logical terms, saying that they are those that are insensitive to permutations over the domain of discourse, and Sher [20] follows this direction and suggests an extension of logic to all terms that satisfy this requirement. Those who promote second-order logic as a legitimate (and much stronger) alternative to first-order systems may be placed in this camp as well (Shapiro [18]), as they expand the bounds of logic on the basis of semantic considerations. And even more radical suggestions to change or replace classical first-order logic can be enumerated here, as long as they are model-theoretic rather than deductive in their orientation. For example, Hintikka and Sandu's [10] IF logics break away from classical logic in many respects; in particular, their semantics is game- rather than model-theoretic. However, they may be said to belong to the domain delineated here due to their all-out emphasis on semantic considerations.

Conceptions of this kind adhere to (a generalized version of) what was labeled above as the received view of the relations between formal proof systems and formal semantics. Hence they provide grounds for continued interest in soundness theorems—albeit such theorems that relate their semantics of choice to adequate proof systems (and not the soundness theorem for first-order logic, which loses its pride of place). And as for completeness theorems, in many of these contexts it can be shown that no axiomatizable deductive system is complete with respect to the chosen semantics. This state of affairs demotes the calibration of the semantics with some deductive system from its status as an objective that should be aimed at and pursued and leaves deduction behind as a necessarily crippled tool through which to follow the lead of the semantic formalism.

1.4 Shapiro's "eclectic attitude" to logical consequence In a recent article Shapiro [17] presents deductive and model-theoretic formal systems as representing distinct strains in our pretheoretic intuitions about logical consequence. According to his view the deductive framework expresses the intuition that logic is epistemically grounded, and the model-theoretic framework represents intuitions that ground logic in modality. Neither of these two strains of intuition (that are not necessarily the only ones) is prior to (or more basic than) the other. Each of the two kinds of formalism just mentioned should be judged not on the basis of its correspondence with the other, but rather solely on its success in capturing the intuitive notion it arises from. Thus Shapiro suggests a pair of *theses* with respect to classical formal deductive systems, namely, that they are correct and adequate with respect to the epistemic intuition they represent, and a similar pair concerning the relation between model theory and modal intuition. (The term thesis, as opposed to theorem, indicates that the relations in question are between an intuitive notion and a formal one, as is the case of Turing's celebrated thesis concerning computability.) Correctness and adequacy are the analogues of soundness and completeness, and the change in terminology should help distinguish these relations, that obtain (or not) between an intuitive notion and a formal one, from the standard relations between two formally defined notions.

According to Shapiro's outlook, then, the philosophical burden that is typically assigned to the soundness and completeness theorems is shifted to the shoulders of the above-defined theses. This is not to say, though, that the theorems are devoid of interest and function. Rather, they are viewed as calibrating between two self-standing strains of intuition, albeit through the mediation of the formal renderings of these two strains. As such they are claimed by Shapiro to help prove the two less obvious of the four theses enumerated above (on the basis of the two other ones, that are argued for on intuitive grounds).

We saw above four views (or types thereof) of logical consequence and its relation to first-order soundness and completeness. What is common to all these views is that they are top-down, in the following sense: they begin with a conception of logical consequence and derive from it certain consequences with respect to the role and meaning of the said theorems. This is certainly a reasonable course of thought, but its weakness is that it leaves untapped what could turn out to be a useful resource—the particulars of the statement, proof, and use of the theorems. These are not given rigorous attention by the above-mentioned accounts, but I argue that they might be of utility in our thinking about the issues involved. Therefore, following the lead of Floyd [5] I propose to start pursuing in this paper an alternative, bottom-up approach. That is, I intend to make several observations regarding first the soundness theorem and then the completeness theorem for first-order logic and derive from these observations some consequences vis-à-vis the philosophical content of these theorems. These consequences, in turn, will give rise to an alternative picture of the relation between model theory, proof theory, and the intuitive notion of logical consequences.

Before this is done, however, there will be an interlude: in the next section I give a brief exposition of several key notions and ideas in the theory of measurement. The rationale behind this detour is as follows. The soundness and completeness theorems can be said to consist in mappings between two structured domains: the set of sentences in a given first-order language structured through the deductive consequence relation and the same set structured through model-theoretic consequence. (Alternatively, the second domain can be construed as consisting of sets of models, related to one another set theoretically.) As argued above, the interpretation of these mappings implicit in the terms "soundness" and "completeness" should not be presupposed, and hence possible alternatives to it should be looked for. One prominent such alternative can be found in the representational theory of measurement. As will be elaborated below, in this theory measurement is construed as a mapping between an empirical domain and a numeric one. Not surprisingly, such measurement theoretic mappings are conceived of very differently from the logical mappings considered above. However, it will be argued in Section 3 below that there is some conceptual affinity between mappings in the two domains, which can shed some light on the function of the theorems we are concerned with here.

2 Measurement Theoretic Representation

Measurement theory is concerned with the foundations of measurement: What does numeric measurement—for example, of length, mass or temperature—consist in? What is required from a certain property to be measurable? What is the relation between physical objects (or states) and numbers that is claimed to hold by measurement statements? Although the practice of measurement has been in existence for thousands of years, the study of these questions began only toward the end of the nineteenth century, notably by Helmholtz [8]. During the first half of the twentieth century measurement was given further attention and treatment, for example, by Campbell [3] and Stevens [21]. These earlier efforts were later integrated by Suppes and his collaborators and presented in a mature form in the *Foundations of Measurement* [11] as the Representational Theory of Measurement (RTM), which is currently the most widely accepted view of measurement.

According to RTM measurement consists in a homomorphism, that is, as structure-preserving mapping, from what is called an *empirical relational structure* to a numerical structure. The empirical structure is described and characterized without any initial appeal to the numbers or to numeric properties whatsoever. It consists in a class of objects, and several operations and relations defined on these objects. For example, in length measurement the empirical relational structure involves qualitative length-comparison and the concatenation operation as two primitives, relating physical objects to each other and induced by empirically legitimate operations. (Length comparison is affected by placing objects next to each other and observing which extends the other, and concatenation is realized in the way that is relevant to length-measurement, that is, by placing the objects one next to the other in the right way.) Such a structure can be shown to have various formal properties: for example, in the case of length measurement the comparison relation involved can be seen to consist in an ordering, the concatenation operation to be commutative and positive with respect to the ordering relation, and so on. (The sense in which the relations/operations and their formal properties can be said to be empirical is debatable. We shall not enter this debate here.)

In some cases these formal properties of the empirical relational structure are sufficient to prove what is called in measurement theory a *representation theorem*. Such a theorem establishes the existence of a structure-preserving mapping from the empirical structure to the numbers (typically the real numbers)—that is, a function that assigns a number to each object and numerical relations and operations to their empirical counterparts in such a way that a given empirical relation holds between any given array of objects if and only if its numerical counterpart holds between the numbers assigned by the mapping to these objects. (We can indeed talk here of a theorem in the strict mathematical sense because only the formal aspects of the empirical structure are taken into account in the proof of a representation theorem; their physical concreteness is abstracted away.) The existence of such a mapping is a necessary and sufficient condition for the applicability of measurement to a given physical domain—it is what measurement amounts to.

The above-described representation theorem is usually accompanied in any given context by what is called a *uniqueness theorem*, a mathematical characterization of all the possible mappings from the empirical domain to the numbers. (The existence of one such mapping is already established by the representation theorem, but usually there is more than one.) A switch from one such mapping to another consists in scale change, for example, from measurement of length in inches to the metric system. Uniqueness theorems are of interest partly because in any specific case of measurement they can help us distinguish between numerical properties and relations between measures that do represent physical facts in the empirical domain and such properties and relations that do not. For example, the fact that 20 is two times 10 does represent a physical relation in mass measurement (i.e., that a body assigned 20 units of a given scale has two times the mass of a body assigned 10 units), and indeed this relation between mass-measures is preserved by any switch to a different mass scale. In temperature measurement, on the other hand, things are not this way, as attested by the fact that changes in temperature scale (e.g., from Celsius to Fahrenheit) do not preserve absolute size relations between measures.

A key intuitive aspect of measurement is that it allows us to use our acquaintance with the domain of the numbers in order to reason about various physical aspects of the world. RTM captures this aspect of measurement well: the homomorphic mapping from the empirical domain to the numbers, the existence of which is proved by a representation theorem, allows us to start with certain objects (or the properties thereof), reason about their numeric measures, and apply the results of this reasoning back to the physical world. Swoyer [22] calls this *surrogative reasoning*.

Note how distinct requirements from the homomorphic mapping give rise to distinct aspects of surrogative reasoning (Swoyer [22], Mundy [14]). According to RTM the homomorphism from the physical domain to the numeric one needs to satisfy a bidirectional requirement (“if-and-only-if”):

(i) whenever an empirical relation holds its numeric counterpart must hold as well (between the representing numeric measures), and conversely,
(ii) whenever a numerical relation holds its physical counterpart does too.

(As follows from the discussion of uniqueness in the previous paragraphs, clause (ii) applies only to those numeric relations that are meaningful in the given context.) The second requirement, that goes backward (against the direction of the mapping function), ensures that the measurement function yields *sound* surrogative reasoning: whatever the numeric representations tell us about the represented domain is correct. The first requirement entails that the representation is *complete*: whenever a physical relation obtains between objects in the domain (a relation that is part of the empirical relational structure, of course) it will not be missed—its numeric counterpart will obtain between the numeric representations as well. Also, note that complete representation of a given relation is equivalent to sound representation of its negation: if whenever a physical relation R obtains among objects its numeric counterpart R' obtains among the numeric measures of these objects, then ipso facto whenever $\sim R'$ obtains among numeric measures $\sim R$ will obtain among the objects these measures are assigned to.

These observations complete this quick survey of several key ideas in the theory of measurement. I believe that even before (and also after) it is attempted below to apply some of these ideas to the domain of logic there is an important lesson to be gained from the mere juxtaposition of logical soundness and completeness theorems on the one hand and measurement-theoretic representation theorems on the other hand. Such juxtaposition shows that there are radically different nonformal types of content that can be ascribed to mappings from one domain to another. (So far there were reviewed here two kinds of such content, but of course there can be others.) Therefore, a pluralistic approach to such mappings is called for: we should be willing to consider various possible interpretations of every such mapping and accept those (not necessarily a single one, but rather possibly several) that seem to be most in accord with intuition and practice.

3 First-Order Soundness and Completeness Revisited

Let us return, then, to the discussion of logic and take another look at first-order soundness, and then at first-order completeness. I believe that an obvious observation is that the soundness theorem for standard axiomatic and natural deduction proof systems of first-order logic does not deserve its name. As noted by Shapiro [17], the term "soundness" tells us that the theorem is supposed to give us assurance in the validity of the inference rules and/or axioms of the said systems. However, it seems unreasonable to say so: in the proof of the theorem for each of these systems the very inference rules and axioms whose soundness is at stake are applied (in the metalanguage) to the set-theoretic models that are supposed to ground this soundness. As Girard [6] bluntly puts it, Tarskian semantics (and the soundness theorem as part of it) consists in a pleonasm.

Note that this is not to say that every so-called soundness theorem—that is, every theorem showing that a class of syntactically valid inferences is included in a class of model-theoretically valid inferences—does not deserve its name. There can be cases—even for first-order proof systems—where our intuitive confidence in the deduction system (i.e., the degree to which it is in accord with preformal intuition) is weaker than our confidence in the model-theoretic semantics (i.e., the degree to which *it* captures our intuition). In such cases a soundness proof would relegate (at least some) of our assurance in the latter to the former. Is there a clear-cut distinction between cases of the two kinds (that is, of "real" proofs of soundness as opposed to "bogus" such proofs)? There need not be. Each specific case should be decided on its own, depending on the intuitive appeal of the two sides involved, on the proof itself, and possibly on other factors as well. My claim here is only that the proof of the soundness theorem for standard proof systems of first-order logic disqualifies it from counting as a real such theorem. If it is of interest at all it is not because it confers soundness on the proof system.

Also, note that this is certainly not to say that we should jump to the conclusion that we should in fact think of the soundness theorem as a completeness theorem, showing that the suspect model-theoretic system covers all the inferences that are sanctioned as valid by the more intuitively robust proof system, First, as noted by Shapiro [17], we may have strong intuitions about the intuitive soundness of proof system(s) in question, but this is not to say that we have similarly strong confidence in their completeness; arguably we don't. And second (and more generally), a key general point of this paper is that soundness and completeness (in the traditional logical sense) are not the only two options. Thus ruling out one of these options does not entail the other. By the same coin, the fact that the soundness theorem is not convincing as a completeness theorem does not imply that it is a soundness theorem after all. I argue it is neither.

As further (albeit circumstantial) support for rejecting the content of the soundness theorem suggested by its name, consider Tarski's [23] famous paper in which he introduces the notion of model-theoretic validity. It is clear that proving the soundness of proof-theoretic validity is *not* among Tarski's objectives in this paper: his complaint against proof-theoretic validity is not that it is not intuitively sound.

So what is the content that can be reasonably assigned to the soundness theorem? Does the theorem have any intuitive, methodological, or philosophical significance? I think we should reject the definite article that appears in the first question (and may be implicit in the second): there need not be a single label that should replace the extant one and point to the sentence's role. Rather, there could be an array of such roles. Here I want to argue for two, which are related to each other, but there could be more.

3.1 The soundness theorem as an extension theorem As noted above, in his 1936 paper Tarski does not motivate the introduction of model-theoretic validity by saying that the intuitive soundness of deductive validity needs further support. Rather, his complaint against deduction systems is that at least in some cases they do not seem to capture *all* valid inferences. This complaint goes hand in hand with his claim that the model-theoretic machinery expresses in a better way our intuition about validity: because the model-theoretic formalism is in closer accord with intuition it can, at least in some cases, capture formally inferences that are intuitively valid but are not so characterized deductively. (By 1936 Tarski would have known that this is not the case for first-order logic. Nevertheless, he can (and does) still hold the general claim with respect to the relation between deductive and model-theoretic systems.)

But before it is examined in each specific case whether model-theoretic validity does indeed extend deductive validity it must be verified that the former (model-theoretic) notion does not miss any valid arguments that the latter (deductive) notion diagnoses as such; otherwise we would not have a case of possible extension, but rather failure of overlap (or proper inclusion of model-theoretic validity in proof-theoretic validity). This task is realized by so-called soundness theorems, and, in particular, by the soundness theorem for first-order logic: it shows that every deductively valid argument is indeed model-theoretically valid as well. Thus I argue that we should conceive of the theorem in this way: it provides the required assurance that the model theory extends proof theory.

One advantage of this interpretation is that it legitimizes the triviality of the soundness proof. The standard interpretation of the theorem, according to which it proves the soundness of the deductive system on the basis of the model theory, equates this triviality with circularity: if the inference rules whose soundness needs support are merely rehearsed in the construction of possible model-theoretic interpretations, then how can these rules be said to have gained any intuitive support? However, if it is acknowledged that the theorem involves no pursuit of support, but rather mere verification of conservation (no loss of valid arguments), then the theorem's straightforwardness becomes unproblematic.

Another related advantage of the suggested outlook is this: it does not represent our intuitive confidence in the deductive system as weaker than (or dependent on) our confidence in the model theory, and this without falling into either of two pitfalls. One pitfall is the simplistic claim that things are the other way around, that is, that the proof system is the benchmark according to which the model theory should be judged. We saw earlier that this view is avoided by a refusal to construe the soundness theorem as a completeness theorem. The extension-interpretation goes even further in this direction—that is, this interpretation reads the theorem as leaving the door open for the possibility that model-theoretic validity extends proof-theoretic validity, and that therefore the latter is not exhaustive. The other pitfall, that Shapiro [17] falls into, is the radical claim that the deductive system and the model-theoretic one express completely independent strains of intuition with respect to logical consequence (see Section 1 above). It is my view that this claim goes too far, and the soundness theorem (according to its construal suggested here) supports this view. The theorem shows in a straightforward manner that model-theoretic validity encompasses deductive validity, and therefore it is unreasonable to say that the former formalism expresses an intuition that is wholly independent from the one expressed by the latter. Rather, as will be argued below, the connections between the formal systems and the intuitions behind them are more complex than either complete independence on the one hand, or one-sided intuitive dependence of one system upon the other system on the other hand.

3.2 The soundness theorem as a representation theorem for consistency The proof of the soundness theorem for standard first-order deductive systems is indeed almost trivial, but it yields valuable and useful results. One of them (shared by other soundness theorems) is this: the theorem can be used in the proof that a certain set of sentences is deductively consistent. The familiar line of reasoning is that if a model is found where all members of the set are true, then by the soundness theorem they cannot lead to contradiction—otherwise, there would be a deductively valid argument that is not model-theoretically valid. Thus the soundness theorem allows us to use model theory in order to address questions of nonprovability in the deductive system, which are not afforded a straightforward answer by the system itself. (Note that it is certainly not claimed here that there exists a procedure (an algorithm) such that whenever an argument is invalid the procedure can produce a model that attests to this fact. The undecidability of first-order logic tells us that this is impossible.) An example from relatively recent literature is Kripke's new model-theoretic proof of Gödel's incompleteness theorem (Putnam [16]): Kripke produces a nonstandard model of PA in which a sentence S (that is true in the standard model) is false, and reasons (via soundness) that this sentence is not provable from PA.

Note that this application of the soundness theorem does not depend in any way on its construal as a soundness theorem, in the standard sense: it is not required that we view deductive validity as being justified by its remaining within the bounds of model-theoretic validity for the use of model existence in the proof of deductive consistency. Rather, it is only required for such use that we appreciate that (i) model construction, as a proof of model-theoretic invalidity, is in some cases a feasible task, and that (ii) according to the mapping established by the soundness theorem every model-theoretically invalid argument is a syntactically invalid one. These two observations together make it useful and legitimate to use the model-theoretic construction in order to reason about deductive relations.

I therefore argue that the soundness theorem has a function of a partial measurement-theoretic representation theorem. As noted in the previous section, whenever each case of a relation R holding in a domain A is mapped into a case of a relation R' holding in a domain B, this can be viewed as a complete (but not necessarily sound) representation of R by R', or a sound (but not necessarily complete) representation of $\sim R$ by $(\sim R)'$. In our case the domains A and B are equal to each other—they comprise the sentences of the first-order language in question. The relations R and R' are (respectively) deductive and model-theoretic validity. And the application of the soundness theorem considered above shows that it can be usefully construed as ensuring the sound representation of $\sim R$, that is, deductive consistency (invalidity), by $(\sim R)'$, that is, model-theoretic invalidity (existence of a countermodel). Thus the term “soundness” receives a surprising new interpretation and vindication: the theorem is a soundness theorem not because it ascribes soundness to deductive validity in virtue of its inclusion within model-theoretic validity, but rather because it ascribes representational soundness to model-theoretic invalidity because of its inclusion in deductive invalidity.

The assimilation of the logical case with the measurement theoretic case can be used to stress again that this interpretation rejects the construal of the soundness theorem as founding deductive validity on the basis of model-theoretic validity. Recall that measurement-theoretic representation consists in a structure-preserving mapping between an empirical domain and an abstract, mathematical (numeric) domain. The former is certainly not viewed as being founded by the theorem on the latter. Rather, the empirical domain is self-standing and of prior interest, and the mathematical domain is only used to reason about it. Similarly, the interpretation suggested here of the soundness theorem as one half of a measurement-theoretic representation theorem allows us to view deduction as having self-standing and prior interest and construe the model theory in this case as a vehicle through which we can reason about it.

Note that it is not my claim that this measurement-theoretic relation between model theory and deduction exhausts the interplay between these formalisms and the intuitions that they capture. It is not suggested here that this relation should just replace the one expressed by the standard name of the theorem. Rather, I argue (as already stated above) that the relation between the formalisms may involve various distinct factors and aspects, having to do both with philosophical outlook and mathematical practice. The foregoing discussion rejected one such aspect (the traditionally received one) and suggested two others (conservation and representation) instead, without commitment that they are exhaustive. Indeed, in the discussion of the completeness theorem below further dimensions of this interplay between the formalisms will be suggested.

This qualification having been made, it should be noted that the interpretation of the soundness theorem as (one half of) a representation theorem for syntactic consistency is in accord with various other cases where the association of semantic values with syntactic entities is made for the same purpose—proof of consistency. One example is Hilbert and Bernay’s 1918 proof of the consistency of the propositional calculus (Zach [25]), using assignments of 0 and 1 to formulas and showing that

(i) all axioms get 0 under every assignment,
(ii) all inference rules preserve this property, and
(iii) a formula and its negation cannot be both assigned 0 and therefore cannot be both provable.

The mechanism is identical to that of standard contemporary soundness proofs for propositional logic. However, this mechanism is not viewed by Hilbert and Bernays as grounding syntactic consistency in a more basic truth-theoretic apparatus, but rather as a means for capturing and proving syntactic relations that are in no need of further grounding in any way. A different, much newer example, is the apparatus of Boolean valued models (Bell [2]). These are assigned as interpretations for standard first-order languages, but certainly without any claim that they stand in closer proximity to our logical intuitions than typical deductive rules for these languages. (For one thing, the elements of such models are viewed as names, and identity statements involving these names (intuitively meaning that they refer to the same object) are assigned the elements of a Boolean algebra, as generalized truth values.) Rather, these models were invented as an alternative presentation of the *forcing* proof-method used in set theory, and as such they too play the function of consistency representation, via soundness: the fact that there exists such a model for a given theory is a proof for the (more intuitively accessible and significant) fact that this theory is syntactically consistent.

The upshot of placing the soundness theorem of first-order logic within the context of these two examples is this. First, there cannot be drawn a clear-cut distinction between so-called merely instrumental semantic assignments on the one hand, whose objective is to represent aspects of self-standing and possibly also intuitively prior syntax, and semantic assignments that are "real," in the sense that they are supposed to consist in interpretations of the syntax and provide grounding for its axioms and/or inference rules. Similar semantic formalisms can be interpreted and used in opposing ways in this respect. And second, the suggested interpretation of a first-order soundness theorem as one half (the soundness half) of a representation theorem of deductive consistency is in accord with actual mathematical usage of similar semantic formalisms.

We turn now to the completeness theorem for first-order logic, first proved by Gödel in 1929. My goal, again, will be to suggest an interpretation of this theorem that is not bound by a prior philosophical outlook on the intuitive weight of the two formalisms related by it. Rather, as noted in Section 1, the analysis undertaken here goes in the opposite direction.

As opposed to the claim made above, that the soundness theorem does not deserve its name (in its usual construal), I suggest that the completeness theorem does wear its title rightly, at least in a basic, extensional sense. As stated in Section 3.1 of the foregoing discussion, in the soundness theorem it is shown that model-theoretic validity includes deductive validity, and thus the question is left open whether there is indeed a relation of proper inclusion between the two notions. The completeness theorem gives a negative answer to this question: the deductive notion is shown to be complete with respect to the model-theoretic notion. Thus, we have here an initial construal of the completeness theorem as indeed proving the completeness of deductive validity with respect to model-theoretic validity. However, this construal is justified by relating the completeness theorem to the soundness theorem, not by the presupposition that the model-theoretic framework is more intuitively fundamental. (This presupposition is not ruled out by this interpretation, though. See below.)

Following this first modest step, I suggest two further interpretations of the theorem, echoing those discussed vis-à-vis soundness but presented in reverse order.

3.3 The completeness theorem as the second half of a representation theorem for consistency In view of the argument presented in Section 3.2 of the discussion of the soundness theorem above, it is clear that the completeness theorem for first-order logic can be viewed as the completeness half of a measurement-theoretic representation theorem of deductive consistency. If, indeed, it is acknowledged that one role (possibly among others) played by the model-theoretic formalism is of representing certain aspects of the deductive system, then the foregoing analysis of the soundness theorem can be applied to the completeness theorem as well. According to this analysis the theorem is not viewed as grounding the intuitive completeness of the deductive system by showing that it is coextensive with model-theoretic validity. Rather, what the theorem proves is the representational completeness of model-theoretic invalidity—that is, the existence of a countermodel—with respect to deductive consistency. The soundness theorem proved the soundness of this representation—that is, that whenever there is a model for a theory it is consistent—and the completeness theorem proves the completeness of the representation: whenever there is a deductively consistent theory there is a model that attests to this fact.

Note that this interpretation is distinct from, but *not* at odds with, the construal of completeness suggested in the previous paragraph. The theorem proves that deductive validity is coextensive with model-theoretic validity, and thereby also allows for model-theoretic invalidity to completely represent deductive consistency.

Next, I argue that the suggested measurement-theoretic construal of the completeness theorem is in tune with actual mathematical and logical practice—otherwise, such a construal remains an artificial exercise. For one thing, consider the way the theorem is stated when it is proved, both by Gödel [7] and later by Henkin [9]. In both cases the theorem is stated in the very form considered above, namely, as the claim that every deductively consistent set of sentences of a first-order language has a model. The fact that it is under this formulation that the theorem is proved is indicative of its significance. We usually do not start with a formal argument that we know to be model-theoretically valid and use the theorem to find (or as an assurance that we *can* find) a deduction that proves it. Rather, we often start with a theory that is known (or hypothesized) to be deductively consistent and use the theorem as indicating that there is indeed a model that attests to this fact. An example of such a consideration can be found following the proof of Gödel's 1931 incompleteness theorem: together with the earlier completeness theorem the later result shows that there are nonstandard models of arithmetic, where the (indirectly) self-referring G is false.

As was the case in the discussion of soundness, the claim made here with respect to first-order completeness is not to be understood as applying to completeness theorems in general: it is not argued here that any such theorem can be usefully viewed as the completeness half of a representation theorem of deductive consistency by model-theoretic invalidity. In some cases the main interest of such a theorem may be indeed the assurance it yields that model-theoretically valid arguments can deductively (and therefore oftentimes also algorithmically) be reproduced. Rather, the point made here is of a local nature, namely, that in the case of first-order logic this is a central way in which the theorem is used and (albeit implicitly) thought of. Some of the ramifications of this will be considered in Section 3.4 below.

In the same vein, it certainly needs to be acknowledged that the symmetry that is entailed by the conjunction of the soundness and the completeness theorems, together with the requirements for representation as presented in Section 2, open the door for various other claims to the effect that (in)validity in one system represents (in)validity in the other. It is not argued here that only the representational claims considered above have formal basis, and that the others do not. Rather, the objectives of the introduction of measurement-theoretic conceptualization to the discussion of first-order soundness and completeness were

(i) to show how talk of representation can fruitfully replace (or be added to) talk of grounding, and
(ii) to argue that in the cases considered above the formal possibility of introducing representational concepts is in accord with mathematical practice and preformal intuition.

An agreement on the first, more general of these two theses can be accompanied with disagreement on the specific way it was applied here, and/or with different applications of it elsewhere.

3.4 Completeness as expressive completeness In the foregoing discussion of the soundness theorem it was argued that reading this theorem as grounding the validity of the deductive rules in the model-theoretic semantics is misguided: the construction of models appeals to analogues of these rules in the metalanguage, and therefore cannot convincingly be said to intuitively underlie them. This rejection of the standard hierarchical view of the two systems (i.e., that model theory is intuitively more fundamental than deduction) carries over to the discussion of completeness. If the simplistic view that model theory is intuitively more basic than deduction is rejected, then we cannot read the completeness theorem as simply proving the adequate scope of the deduction rules by showing that these rules are coextensive with the self-standing and more fundamental model theory.

On the other hand, as was shown above, the proof of the soundness theorem does not support Shapiro's [17] claim that the two formalisms express two independent strains of intuition as regards the question what logical consequence is. Thus I believe we need a conception of the relation between model theory and deduction that avoids the Scylla of complete, one-sided dependence on the one hand, and the Charybdis of complete independence of the two sides on the other. In what follows I present an outline of such a conception.

First-order predicate logic can be rightly described as a logic of object-talk and predication-talk. In the atomic formulas of first-order formal languages predicates are applied to names of objects (or relations are applied to series thereof), and complex formulas are yielded from atomic ones through the application of the truth-functional sentential connectives and the quantifiers. The latter too are object-oriented—they allow us to apply predicates (either simple or complex) to all or some objects in the given domain of discourse, through the mediation of variables. The fact that first-order logic is object- (and predication-) oriented is of course given expression in the rules of typical first-order deductive systems. In a natural deduction system, for example, these rules tell us how we can introduce and eliminate quantifiers from formulas, thus capturing formally some of our intuitions concerning what moves are allowed in reasoning about objects and their properties. By the same coin, model theory captures in a different fashion the fact that first-order languages are object- and predication-oriented. The constants and predicate symbols are, respectively, associated with objects and extensions in any given model-theoretic interpretation, and the invariable interpretation of the quantifiers expresses their object-oriented meaning.

But how are these two formal representations of object-oriented talk and conceptualization related to each other? Sher [20] answers this question in the following way, that gives philosophical priority to Tarskian model-theoretic semantics. According to her view, model theory captures through set-theoretical means the fact that our world is made up of objects that satisfy (or do not satisfy) various properties. This so-called formal structure of the world is presented by Sher as independent of any linguistic description of it. The success of model theory is in that it allows us to use mathematical means to represent the various combinatorial arrangements (of objects and properties) that this basic structure makes possible, and thus sift out the logical concepts and the logically valid arguments. The logical concepts are those second-order concepts (such as nonemptiness) that for any model are insensitive to permutations of the objects in the model—this ensures that they appeal only to general formal structure that is common to all models, and, supposedly, characterizes the world. The logical arguments are those arguments in which logical concepts (as defined above) are treated as logical constants (their meaning remains fixed), and which satisfy the regular model-theoretic requirement for validity (i.e., the consequence of the argument is true in all interpretations in which the premises are true). On the basis of this analysis, the roots of which can be found in Tarski's later writings, Sher rejects the claim (made by Tarski himself in his 1936 paper) that the choice of logical constants in first-order languages is arbitrary. She shows that these constants satisfy the above-stated requirement and argues that the bounds of logic should include further concepts (such as finiteness) that satisfy this requirement as well.

I believe that Sher's analysis of the way model theory captures the formal underpinnings of objects talk is correct and valuable. However, I also believe that we can enjoy the fruit of this analysis without accepting Sher's presupposition that the formal structure captured by model theory applies to the world as it is, and that therefore model theory is more fundamental than deduction in accounting for our logical intuition. Instead, I suggest that the structure captured by the model-theoretic apparatus should be viewed as representing and elaborating object *talk*: not the most general formal characteristics of the world as it is, but rather the most general formal characteristics of the way we talk (and think) of the world as being comprised of objects that satisfy (or do not satisfy) various properties.

Such a change of perspective is of course motivated by a general philosophical outlook according to which language consists in Nuerath's famous boat—the framework that our physics, metaphysics, epistemology, and logic cannot escape or transcend. It is certainly not my aim to defend this outlook here: it is at the center of the linguistic turn in analytic philosophy, and as such has been supported and elaborated by some of the main figures of this tradition (and, of course, attacked by other prominent members of it). Instead, I want to show how there can be derived from this outlook a conception of the relation between first-order model theory, deductive systems, and our pretheoretic logical intuition that is both plausible and in accord with the observations made above as regards the soundness and completeness theorems. If such a conception is seen to be of interest and value it can serve as yet another source of support of the said general language-oriented outlook.

If model theory is viewed as capturing the structure (or, rather, the possible structures) imputed on the world by object talk, then both first-order deductive systems and model-theoretic systems are placed "on the same trajectory," so to speak: systems of both types represent formal aspects of the way we talk (and think) of the world in terms of objects and properties. Deductive systems do so by formally capturing inferential moves involving predication and quantification that we hold to be valid. In this they may be described as staying relatively close to our natural language inferential practices. Model theory goes a step (or several steps) farther: it abstracts away from this or that set of inference rules and aims to formally represent all the possible combinations that are allowed for by our scheme of objectification and predication. However, in doing so it should not be viewed as probing underneath language, and therefore going deeper than the deductive systems that represent linguistic inferential practices. Rather, model theory represents in a more general (and possibly extended) way the formal underpinnings of deduction.

Consider how this perspective is supported by various aspects of the foregoing discussion of the soundness and completeness theorem for first-order logic. First, it is in accord with the characterization of the soundness theorem as an extension theorem. Model theory is not needed to provide intuitive grounding for formal deductive systems—these are intimately tied to our intuition as regards what inferential moves are valid, and do not need (and cannot get) support from model theory, that simply rehearses them. Rather, the theorem's philosophical value is in that it shows that the more abstract (and possibly more general) representation of object talk is an extension of the more concrete, deductive representation, and hence does not violate its pronouncements: whatever is deductively valid is also model-theoretically valid.

Second, the suggested outlook coheres with the claim that the soundness theorem (which, as has just been said, does not prove soundness in the usual sense) is not a completeness theorem as well—completeness of the model theory with respect to the deductive system. The reason is that although the deductive system is not viewed as in need of support from model theory, still it is viewed as possibly being inferior to model theory in exhausting our object-oriented notion of validity. This is because model theory gets at this validity in a more abstract and general way. Thus the soundness theorem opens the door for the possibility that model theory goes beyond deduction, that is, that it is able to characterize as valid inferences that are not so characterized deductively. (Recall that this was Tarski's proclaimed objective in defining model-theoretic validity: he did not want to ground deduction, but rather to go beyond its bounds.) However, note again that such extension is possible not because model theory describes the world from without the prism of deduction or language, but rather because it elaborates more generally and abstractly the formal mechanisms that underlie deduction (and presumably natural language as well).

Third, this view yields an interpretation of the completeness theorem that is in accord with its title, yet infuses new content into this title, that is in tune with the foregoing discussion. The theorem does indeed prove that the deductive system is complete with respect to model theory, but this is of interest and importance not because model theory invokes more fundamental intuitions with respect to logical validity. Rather, model theory is shown by the soundness theorem to possibly extend deductive validity, and the completeness theorem proves that in the first-order case this possibility is not realized. Therefore, the concrete deductive mechanism is proved to be equivalent to the abstract model-theoretic one in expressing object-oriented validity. But this equivalence is of interest and importance, again, not because the two formalisms are ordered hierarchically in their philosophical strength. Rather, one is more abstract and therefore seemingly more general, and the completeness theorem shows that in the first-order case it does not, in fact, extend the bounds of validity.

Fourth, the suggested framework coheres with the representational content ascribed to both the soundness theorem and the completeness theorem. A formalism of a higher level of abstraction can indeed be used to represent aspects of a less abstract formalization of a given concept. This is as opposed to the view of model theory as being intuitively more fundamental, in which case its representational uses (in the measurement-theoretic sense) are more difficult to account for.

Fifth, note how the position presented here allows for a new perspective on the history of formal logic from Frege's *Begriffsschrift* onward. (In this remark and the following one I go beyond the foregoing discussion of the soundness and completeness theorems.) The standard construal of model-theoretic validity as more fundamental than deduction makes the earlier development of deductive systems look like a historical accident: formal logic went the wrong (deductive) way, and then turned to the right (semantic) tracks. The view advocated for here changes this pronouncement. According to this view the model-theoretic route is not distinct and more fundamental, but rather a more abstract continuation of the deductive outlook. This way the historical development of formal logic becomes more rational and coherent. Now admittedly history need not necessarily be rational and coherent—even not the history of formal logic—but certainly it is not a disadvantage of a philosophical position that it helps the intellectual developments in the domain it applies to make more sense.

Sixth, the suggested construal of Sher's work allows us to enjoy its above-described fruit without sharing Sher's view that the formal structure captured by model theory (including the added structure suggested by Sher that goes beyond first-order logic) is Logic, with a capital L. If the structure captured by set theoretic model theory is not found *out there*, in the language-independent world, but is rather extracted and abstracted from language, then it has no priority over other types of structure that may be extracted and abstracted from language, possibly through other mathematical means. One can (and should, I think) accept the centrality of object talk in our language and thought, and therefore accept the centrality of set-theoretically analyzed structure in our intuitive notion of logic, but accepting these claims can (and should, I think) go hand in hand with a view that the bounds of logic are vague and contextual, and may reasonably include other kinds of structure. Thus the outlook presented here leaves the doors of logic open for the multitude of formal systems that are currently called logics. This so-called logical pluralism (Beall and Restall [1]), that respects actual pluralistic usage of the term "logic," seems to me as yet another advantage of this position.

I claim to have thus fulfilled the promise of a bottom-up approach to the issue of logical consequence. On the basis of a series of observations as regards the content and proof of the soundness and completeness theorems for first-order logic there was presented here a conception of the intuitive, philosophical, and historical interplay among model theory, deductive systems, and pretheoretic logical intuition. As already noted above, the general framework underlying this conception can be applied to the discussion of other logical systems and the soundness/completeness theorems that can (or cannot) be proved for them, although the results of such an application need not be identical to those reached here.

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