Notre Dame Journal of Formal Logic
Volume 51, Number 4, 2010

Metalogic of Intuitionistic Propositional Calculus

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Abstract With each superintuitionistic propositional logic L with a disjunction property we associate a set of modal logics the assertoric fragment of which is L. Each formula of these modal logics is interdeducible with a formula representing a set of rules admissible in L. The smallest of these logics contains only formulas representing derivable in L rules while the greatest one contains formulas corresponding to all admissible in L rules. The algebraic semantic for these logics is described.

1 Preliminaries

Let $\mathcal{L}_0$ be a set of formulas built in a usual way from propositional variables and connectives $\wedge$, $\vee$, $\rightarrow$, $\perp$ and let $\neg A = (A \rightarrow \perp)$. By *IPC* we will denote the intuitionistic propositional calculus with modus ponens and substitution as the only inference rules and by *IPL* we denote intuitionistic propositional logic. If I is a propositional calculus LI denotes the logic defined by calculus I. As usual, a rule

$$\frac{A_1, \ldots, A_n}{B} \tag{1}$$

where $A_1, A_2, \ldots, A_n, B \in \mathcal{L}_0$ is called admissible[1] in *IPL* if for every substitution σ of formulas for propositional variables

$$\text{if } \sigma(A_1), \ldots, \sigma(A_n) \in IPL \text{ then } \sigma(B) \in IPL.$$

Our goal is to study a logic that formalizes reasoning about rules and, specifically, admissible (in *IPL*) rules. Such logic was introduced in [5; 6]. In [12] Iemhoff and Metcalfe construct (based on hypersequents) a Gentzen-style system that allows to prove or disprove admissibility. The Hilbert-style system that is described in this paper is a formalization of reasoning in metalanguage, the system that allows to express properties of rules, rather than a proof system. It is geared toward considering,

Received August 22, 2009; accepted July 9, 2010; printed September 21, 2010
2010 Mathematics Subject Classification: Primary, 03B55, 03F45; Secondary, 06D20
Keywords: intuitionistic logic, modal logic, admissible rule, Heyting algebra, monadic algebra, intermediate logic
 10.1215/00294527-2010-031

for example, different closure operators Cn such that $Cn(\varnothing) = IPL$ rather than the set of all admissible in IPL rules. As it follows from [18; 8; 17; 10] the constructed logic is decidable, but in this paper we are not concerned with deciding procedures or their complexity (see [13]). Because the logic that we are introducing is enforcing, in a way, the disjunction property and in this case multi-conclusion rules (see, for instance, [14]) are equivalent to regular (single-conclusion) rules, we are not considering the multi-conclusion rules.

Let us extend the language by adding a new unary operator $\Box$ and define this operator on Lindenbaum algebra $\mathfrak{F}$ of IPL (a free Heyting algebra of countable rank) as follows:

$$\Box x = \begin{cases} 1, \text{ if } x = \mathbf{1} \\ 0, \text{ otherwise.} \end{cases} \tag{$\Box$}$$

We will call this algebra $\mathfrak{F}^{\Box}$. It is obvious that identity

$$\Box(A_1 \wedge A_2 \wedge \cdots \wedge A_n) \to \Box B = \mathbf{1} \tag{2}$$

on $\mathfrak{F}^{\Box}$ means exactly the same as admissibility of rule (1) in IPL. In particular, if A and B are assertoric formulas (meaning they do not contain operator $\Box$), formula

$$\Box A \to \Box B$$

represents the rule

$$\frac{A}{B}.$$

Later we will demonstrate that in logic $L(\mathfrak{F}^{\Box})$ of algebra $\mathfrak{F}^{\Box}$ (and even in a much weaker logic that will be introduced in the next section), each formula $\Box\alpha$ is equivalent to some conjunction of formulas, representing rules:

$$(\Box A_1 \to \Box B_1) \wedge (\Box A_2 \to \Box B_2) \wedge \cdots \wedge (\Box A_n \to \Box B_n),$$

where $A_i, B_i \in \mathcal{L}_0$ $(i = 1, \ldots, n)$. In other words, $L(\mathfrak{F}^{\Box})$ is a logic of all admissible rules of intuitionistic logic.

From this point on we will be using capital Latin letters $A, B, C, \ldots$ (maybe with indices) to denote assertoric propositional formulas (without occurrences of $\Box$) and use small Greek letters $\alpha, \beta, \gamma, \ldots$ (maybe with indices) to denote formulas in the extended language.

2 Metaintuitionistic Logic

In this section we will construct a modal logic, the purpose of which is to formalize reasoning about inference rules of intuitionistic logic. Let us call $I_0^{\Box}$ an extension of IPC by adding to the language a new symbol $\Box$ and the following axioms.

Ax.1	$\Box p \to p$
Ax.2	$\Box p \to \Box\Box p$
Ax.3	$\Box(p \to q) \to (\Box p \to \Box q)$
Ax.4	$\neg\Box\neg\Box p \to \Box p$
Ax.5	$\Box(p \vee q) \to (\Box p \vee \Box q)$

and additional rule

(R1) $\quad \dfrac{\alpha}{\Box\alpha}.$

It is easy to see that Ax.1–Ax.5 and the rule (R1) hold on $\mathfrak{F}^{\square}$ and, therefore, $LI_0 \subseteq L(\mathfrak{F}^{\square})$. Since $\mathfrak{F}^{\square}$ is a model for both logics $LI_0^{\square}$ and $L(\mathfrak{F}^{\square})$, their assertoric fragment is the same, namely, LI. That is, both of these logics are conservative extensions of the intuitionistic propositional logic. Later we will see that these logics are different, and even more, $L(\mathfrak{F}^{\square})$ is the greatest extension of LI_0 among conservative extensions of LI.

Let us remark that Ax.5 represents Gödel's disjunction theorem; Ax.3 means (on $\mathfrak{F}^{\square}$) that every derivable rule is admissible.

Example 2.1 As mentioned above, the formula of type $\square A \to \square B$ represents a rule. For instance, the following formula

$$\square(\neg p \to (q \vee r)) \to \square((\neg p \to q) \vee (\neg p \to r)) \tag{HF}$$

represents a well-known Harrop's rule [9]:

$$\frac{\neg A \to (B \vee C)}{(\neg A \to B) \vee (\neg A \to C)} . \tag{HR}$$

Since rule (HR) is admissible in intuitionistic logic, formula (HF) is valid on $\mathfrak{F}^{\square}$ and, hence, belongs to $L(\mathfrak{F}^{\square})$.

Remark 2.2 $L(\mathfrak{F}^{\square})$ is a logic of provability in IPC. Nevertheless, it is completely different from the logic of provability in Heyting arithmetic (HA) that was used in various papers (see, for instance, [10]) in order to describe rules admissible in arithmetic as well as in IPC. The difference between two approaches becomes evident if we consider formula $\square(\square\alpha \to \alpha) \to \square\alpha$, which represents the well-known Löb principle. This formula does not hold in $L(\mathfrak{F}^{\square})$ (one can check it by substituting α with $\mathbf{0}$). A very important difference between $L(\mathfrak{F}^{\square})$ and logic of provability in HA is that in $L(\mathfrak{F}^{\square})$ every formula is equivalent to a formula without iterative modalities. Moreover, as it will be demonstrated later, each formula $\square\alpha$ is equivalent to a formula that represents a conjunction of rules.

As usual, $\alpha \sim \beta =_{\text{def}} (\alpha \to \beta) \wedge (\beta \to \alpha)$. First of all, let us establish some equivalences that will be used later. Among other things these equivalences show the "Boolean nature" of a new operator.

Proposition 2.3 *The following formulas are provable in $I_0^{\square}$:*

$$\square\alpha \sim \square\square\alpha \tag{3}$$

$$\square\alpha \sim \neg\square\neg\square\alpha \tag{4}$$

$$\square\alpha \sim \neg\neg\square\alpha \tag{5}$$

$$\neg\square\alpha \sim \square\neg\square\alpha \tag{6}$$

$$\square(\alpha \vee \beta) \sim (\square\alpha \vee \square\beta) \tag{7}$$

$$\square(\alpha \wedge \beta) \sim (\square\alpha \wedge \square\beta) \tag{8}$$

$$\square(\square\alpha \to \square\beta) \sim (\square\alpha \to \square\beta). \tag{9}$$

Proof

(3) Equivalence (3) is a simple consequence from Ax.1 and Ax.2.

(4) In addition to Ax.4 we need only to prove $\Box\alpha\to\neg\Box\neg\Box\alpha$.

$$\text{Ax.1}: \quad \Box\neg\Box\alpha\to\neg\Box\alpha \tag{10}$$
$$(p\to\neg q)\ \sim\ (q\to\neg p): \quad \Box\alpha\to\neg\Box\neg\Box\alpha \tag{11}$$

(5) It is sufficient to prove $\neg\neg\Box\alpha\to\Box\alpha$.

$$\neg\neg\neg p\ \sim\ \neg p: \quad \neg\neg\neg\Box\neg\Box\alpha\to\neg\Box\neg\Box\alpha \tag{12}$$
$$(12), \text{Ax.4}: \quad \neg\neg\neg\Box\neg\Box\alpha\to\Box\alpha \tag{13}$$
$$(4): \quad \neg\neg\Box\alpha\to\Box\alpha \tag{14}$$

(6) It is sufficient to prove $\neg\Box\alpha\to\Box\neg\Box\alpha$.

$$\text{Ax.4}: \quad \neg\Box\neg\Box\alpha\to\Box\alpha \tag{15}$$
$$(5): \quad \neg\Box\neg\Box\alpha\to\neg\neg\Box\alpha \tag{16}$$
$$(p\to\neg q)\ \sim\ (q\to\neg p): \quad \neg\neg\neg\Box\alpha\to\neg\neg\Box\neg\Box\alpha \tag{17}$$
$$\neg\neg\neg p\ \sim\ \neg p: \quad \neg\Box\alpha\to\neg\neg\Box\neg\Box\alpha \tag{18}$$
$$(5): \quad \neg\Box\alpha\to\Box\neg\Box\alpha \tag{19}$$

(7) It is sufficient to prove $(\Box\alpha\vee\Box\beta)\to\Box(\alpha\vee\beta)$.

$$p\to(p\vee q): \quad \alpha\to(\alpha\vee\beta) \tag{20}$$
$$(\text{R1}): \quad \Box(\alpha\to(\alpha\vee\beta)) \tag{21}$$
$$\text{Ax.3}: \quad (\Box\alpha\to\Box(\alpha\vee\beta)) \tag{22}$$
$$\text{In the same way as above}: \quad (\Box\beta\to\Box(\alpha\vee\beta)) \tag{23}$$
$$(22), (23): \quad (\Box\alpha\vee\Box\beta)\to\Box(\alpha\vee\beta) \tag{24}$$

(8) Let us prove first $\Box(\alpha\wedge\beta)\to(\Box\alpha\wedge\Box\beta)$.

$$(p\wedge q)\to p: \quad (\alpha\wedge\beta)\to\alpha \tag{25}$$
$$(\text{R1}): \quad \Box((\alpha\wedge\beta)\to\alpha) \tag{26}$$
$$\text{Ax.3}: \quad \Box(\alpha\wedge\beta)\to\Box\alpha \tag{27}$$
$$\text{In the same way as above}: \quad \Box(\alpha\wedge\beta)\to\Box\beta \tag{28}$$
$$(27), (28): \quad \Box(\alpha\wedge\beta)\to(\Box\alpha\wedge\Box\beta) \tag{29}$$

Now let us prove $(\Box\alpha\wedge\Box\beta)\to\Box(\alpha\wedge\beta)$.

$$(p\to(q\to(p\wedge q))), (\text{R1}): \quad \Box(\alpha\to(\beta\to(\alpha\wedge\beta))) \tag{30}$$
$$\text{Twice Ax.3}: \quad (\Box\alpha\to(\Box\beta\to\Box(\alpha\wedge\beta))) \tag{31}$$
$$(31): \quad (\Box\alpha\wedge\Box\beta)\to\Box(\alpha\wedge\beta) \tag{32}$$

(9) It is sufficient to prove $(\Box\alpha\to\Box\beta)\to\Box(\Box\alpha\to\Box\beta)$

$$(5): \quad (\Box\alpha\to\Box\beta)\sim(\Box\alpha\to\neg\neg\Box\beta) \tag{33}$$
$$(p\to\neg q)\sim\neg\neg(\neg p\vee\neg q): \quad \sim\neg\neg(\neg\Box\alpha\vee\neg\neg\Box\beta) \tag{34}$$
$$(5): \quad \sim\neg\neg(\neg\Box\alpha\vee\Box\beta) \tag{35}$$
$$(6): \quad \sim\neg\neg(\Box\neg\Box\alpha\vee\Box\beta) \tag{36}$$
$$(3): \quad \sim\neg\neg(\Box\neg\Box\alpha\vee\Box\Box\beta) \tag{37}$$
$$(7): \quad \sim\neg\neg\Box(\neg\Box\alpha\vee\Box\beta) \tag{38}$$
$$(5): \quad \sim\Box(\neg\Box\alpha\vee\Box\beta) \tag{39}$$

$$\begin{array}{rll} \text{From the above}: & (\Box\alpha\to\Box\beta)\to\Box(\neg\Box\alpha\vee\Box\beta) & (40)\\ (\neg p\vee q)\to(p\to q): & (\neg\Box\alpha\vee\Box\beta)\to(\Box\alpha\to\Box\beta) & (41)\\ (\mathrm{R1}): & \Box((\neg\Box\alpha\vee\Box\beta)\to(\Box\alpha\to\Box\beta)) & (42)\\ \mathrm{Ax.3}: & (\Box(\neg\Box\alpha\vee\Box\beta)\to\Box(\Box\alpha\to\Box\beta)) & (43)\\ (40), (43): & (\Box\alpha\to\Box\beta)\to\Box(\Box\alpha\to\Box\beta) & (44) \end{array}$$

□

Proposition 2.4 *Formula* $\Box\alpha\vee\neg\Box\alpha$ *is provable in* $I_0^{\Box}$.

Proof

$$\begin{array}{rll} (5): & \neg\neg\Box\alpha\to\Box\alpha & (45)\\ \text{Substitution in } (45): & \neg\neg\Box(\Box\alpha\vee\neg\Box\alpha)\to\Box(\Box\alpha\vee\neg\Box\alpha) & (46)\\ (7): & \neg\neg(\Box\Box\alpha\vee\Box\neg\Box\alpha)\to\Box(\Box\alpha\vee\neg\Box\alpha) & (47)\\ (3): & \neg\neg(\Box\alpha\vee\Box\neg\Box\alpha)\to\Box(\Box\alpha\vee\neg\Box\alpha) & (48)\\ (6): & \neg\neg(\Box\alpha\vee\neg\Box\alpha)\to\Box(\Box\alpha\vee\neg\Box\alpha) & (49)\\ (7): & \neg\neg(\Box\alpha\vee\neg\Box\alpha)\to(\Box\Box\alpha\vee\Box\neg\Box\alpha) & (50)\\ (3): & \neg\neg(\Box\alpha\vee\neg\Box\alpha)\to(\Box\alpha\vee\Box\neg\Box\alpha) & (51)\\ (6): & \neg\neg(\Box\alpha\vee\neg\Box\alpha)\to(\Box\alpha\vee\neg\Box\alpha) & (52)\\ \neg\neg(p\vee\neg p): & \Box\alpha\vee\neg\Box\alpha & (53) \end{array}$$

□

Proposition 2.5 (Deduction theorem) *If* Γ *is a set of formulas and* $\Gamma,\alpha\vdash_{\Box}^{-}\beta$, *then* $\Gamma\vdash_{\Box}^{-}(\Box\alpha\to\beta)$, *where* $\vdash_{\Box}^{-}$ *denotes derivability in* $I_0^{\Box}$ *without substitution.*

Proof This theorem can be proved by induction on length of inference (using (R1), Ax.3, and (3) in the case when (R1) was used at the last step of inference). □

Remark 2.6 As it will be demonstrated later (Corollary 3.7), the following rule is admissible in $I_0^{\Box}$:

$$\frac{\Box\alpha\to\Box\beta}{\Box(\alpha\to\beta)}\,. \tag{PR}$$

The above rule means that LI_0 is the logic of all rules derivable in intuitionistic propositional calculus without substitution while logic $L(\mathfrak{F}^{\Box})$ is a logic of all admissible in I rules.

Remark 2.7 Evidently, the logic of $I_0^{\Box}$ is much stronger than weak S5 (see [3]) or L_4 (see [16]). Since formula $(\Box p\vee\Box\neg p)\to(p\to\Box p)$ is deducible in $I_0^{\Box}$, if we add formula $(p\vee\neg p)$ to $I_0^{\Box}$ as a new axiom, the formula $(p\sim\Box p)$ becomes deducible, that is, modality collapses.

3 Meta-Heyting Algebras

In this section we will construct an algebraic semantic for calculus $I_0^{\Box}$.

Definition 3.1 Algebra $\mathfrak{A}=\langle A;\wedge,\vee,\to,\mathbf{0},\Box\rangle$ will be called *meta-Heyting algebra (mtha)* if $\langle A;\wedge,\vee,\to,\mathbf{0}\rangle$ is Heyting algebra, $\mathbf{1}$ is the greatest element of Heyting algebra, and the following identities hold.

MAx.1	$\Box x \to x = \mathbf{1}$
MAx.2	$\Box x \to \Box\Box x = \mathbf{1}$
MAx.3	$\Box(x \to y) \to (\Box x \to \Box y) = \mathbf{1}$
MAx.4	$\neg\Box\neg\Box x \to \Box x = \mathbf{1}$
MAx.5	$\Box(x \vee y) \to (\Box x \vee \Box y) = \mathbf{1}$
MAx.6	$\Box\mathbf{1} = \mathbf{1}$.

Remark 3.2 Equivalences (3), (6)–(9) mean that $\{\Box\mathfrak{a} | \mathfrak{a} \in \mathfrak{A}\}$ is a subalgebra of $\mathfrak{A}$ and from Proposition 2.4 it follows that this subalgebra is a Boolean algebra. From a general standpoint it means that our reasonings about rules in metaintuitionistic logic are classical.

Let $\mathcal{M}$ be the variety of all meta-Heyting algebras. Obviously $\mathcal{M}$ is a subvariety of variety of monadic Heyting algebras (see [2] for definition). It is evident that all axioms and rules of $I_0^{\Box}$ are valid on algebras from $\mathcal{M}$ and therefore all formulas from LI_0 are valid on all algebras from $\mathcal{M}$. On the other hand, any inference of any identity from identities that define the variety $\mathcal{M}$ can be converted in proof of correspondent formula in $I_0^{\Box}$.

Example 3.3 (Extension) Let us show that formula (HF) is not provable in $I_0^{\Box}$. In order to do so it is sufficient to construct a mtha on which (HF) is not valid. Let us consider the mtha (see Fig. 1), where $\Box$ is defined by $(\Box)$: $\Box\mathbf{1} = \mathbf{1}$ and $\Box\mathfrak{a} = \mathbf{0}$ if $\mathfrak{a} \neq \mathbf{1}$. As we can see, formula (HF),

$$\Box(\neg p \to (q \vee r)) \to \Box((\neg p \to q) \to (\neg p \to r)),$$

is not valid on this mtha, which means that $LI_0 \neq L(\mathfrak{F}^{\Box})$.

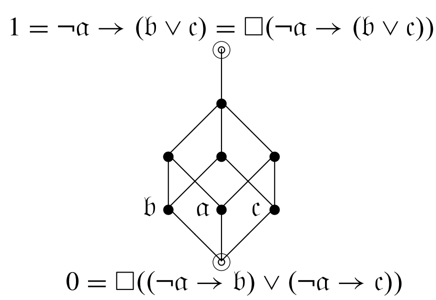

Figure 1

Let $\mathfrak{A}$ be a meta-Heyting algebra and $\nabla \subseteq \mathfrak{A}$. As usual ∇ is said to be a *filter* (of $\mathfrak{A}$) if the following conditions hold:

1. if $\mathfrak{a} \in \nabla$ and $\mathfrak{b} \in \nabla$, then $\mathfrak{a} \wedge \mathfrak{b} \in \nabla$;
2. if $\mathfrak{a} \in \nabla$ and $\mathfrak{a} \leq \mathfrak{b}$, then $\mathfrak{b} \in \nabla$;
3. if $\mathfrak{a} \in \nabla$, then $\Box\mathfrak{a} \in \nabla$.

As in the case of Heyting or topo-Boolean algebras there is a 1-1-correspondence between filters and congruences of algebra $\mathfrak{A}$.

Definition 3.4 Nontrivial algebra $\mathfrak{A}$ is called *simple* if $\mathfrak{A}$ has exactly 2 filters, $\{\mathbf{1}\}$ and itself.

Theorem 3.5 ([20], [5]) *Let $\mathfrak{A}$ be a nontrivial meta-Heyting algebra. The following conditions are equivalent:*

1. *$\mathfrak{A}$ is simple algebra;*
2. *$\mathfrak{A}$ is subdirectly irreducible in $\mathcal{M}$;*
3. *the following two conditions are valid:*
   (a) *if $\mathfrak{a} < \mathbf{1}$ and $\mathfrak{b} < \mathbf{1}$, then $\mathfrak{a} \vee \mathfrak{b} < \mathbf{1}$ (i.e., Heyting algebra $\mathfrak{A}$ is Gödelean or well-connected);*
   (b) *($\Box$) holds on $\mathfrak{A}$.*

Proof $(1 \Rightarrow 2)$ It is evident that every simple algebra is subdirectly irreducible.

$(2 \Rightarrow 3)$ Let $\mathfrak{A}$ be a subdirectly irreducible algebra. Let us establish first that 3(b) holds. From MAx.6 it follows that if $\mathfrak{a} = \mathbf{1}$, then $\Box\mathfrak{a} = \mathbf{1}$. Let now $\mathfrak{a} \in \mathfrak{A}$ and $\mathfrak{a} \neq \mathbf{1}$. We need to demonstrate that $\Box\mathfrak{a} = \mathbf{0}$. Let us note that from MAx.1

$$\Box\mathfrak{a} \neq \mathbf{1}. \tag{54}$$

Proof by contradiction. Assume to the contrary that $\Box\mathfrak{a} \neq \mathbf{0}$. Then

$$\neg\Box\mathfrak{a} \neq \mathbf{1}, \tag{55}$$

but, according to Proposition 2.4, $\Box\mathfrak{a} \vee \neg\Box\mathfrak{a} = \mathbf{1}$ and from (6) it follows that

$$\Box\mathfrak{a} \vee \Box\neg\Box\mathfrak{a} = \mathbf{1}. \tag{56}$$

Let us consider the following two filters:

$$\nabla_1 = \{\mathfrak{b} | \Box\mathfrak{a} \leq \mathfrak{b}, \mathfrak{b} \in \mathfrak{A}\} \text{ and } \nabla_2 = \{\mathfrak{b} | \Box\neg\Box\mathfrak{a} \leq \mathfrak{b}, \mathfrak{b} \in \mathfrak{A}\}.$$

(54) and (55) mean that filters ∇_1 and ∇_2 are proper while (56) means that $\nabla_1 \cap \nabla_2 = \{\mathbf{1}\}$; that is, algebra $\mathfrak{A}$ is subdirectly reducible. This contradiction completes the proof of 3(b).

Let us prove 3(a) by contradiction. Let us assume that there are such $\mathfrak{a}, \mathfrak{b} \in \mathfrak{A}$ and $\mathfrak{a} \neq \mathbf{1}$ and $\mathfrak{b} \neq \mathbf{1}$, but $\mathfrak{a} \vee \mathfrak{b} = \mathbf{1}$. Then $\Box(\mathfrak{a} \vee \mathfrak{b}) = \mathbf{1}$ and, according to (7), $\Box\mathfrak{a} \vee \Box\mathfrak{b} = \mathbf{1}$. On the other hand, from 3(b) it follows that, since $\mathfrak{a} \neq \mathbf{1}$ and $\mathfrak{b} \neq \mathbf{1}$, then $\Box\mathfrak{a} = \Box\mathfrak{b} = \mathbf{0}$; that is, $\Box\mathfrak{a} \vee \Box\mathfrak{b} = \mathbf{0}$. This contradiction proves the assumption wrong and completes the proof of 3(a).

$(3 \Rightarrow 1)$ From 3(b) it follows that every proper filter, since this filter contains distinct from $\mathbf{1}$ elements, coincides with whole algebra, which means that algebra is simple. $\Box$

Remark 3.6 Esakia brought to the author's attention that Theorem 3.5 is a consequence from the Theorem 5.1 from [20].

As it follows from Theorem 3.5, in order to check whether an identity is valid in $\mathcal{M}$ it is enough to check that this identity is valid on all Gödelean Heyting algebras with $\Box$ defined by 3(b). Since on this type of algebras $\Box\mathfrak{a}$ can have only two values $\mathbf{0}$ or $\mathbf{1}$ the task of verifying identity becomes much simpler. For instance, let us check that the following identity is valid in $\mathcal{M}$.

$$\mathfrak{a} \to \Box\mathfrak{b} = \neg\mathfrak{a} \vee \Box\mathfrak{b}. \tag{57}$$

Proof From Theorem 3.5 it follows that on each subdirectly irreducible mtha $\Box\mathfrak{b}$ is either $\mathbf{1}$ or $\mathbf{0}$. In both cases (57) is true. $\Box$

In the same way one can check that the following identities hold on $\mathcal{M}$:

$$\mathfrak{a} \to (\mathfrak{b} \vee \Box\mathfrak{c}) = (\mathfrak{a} \to \mathfrak{b}) \vee \Box\mathfrak{c}; \tag{58}$$

$$\neg\Box\mathfrak{a} \vee \neg\Box\mathfrak{b} = \neg(\Box\mathfrak{a} \wedge \Box\mathfrak{b}) = \neg\Box(\mathfrak{a} \wedge \mathfrak{b}); \tag{59}$$

$$\neg\Box\mathfrak{a} \wedge \neg\Box\mathfrak{b} = \neg(\Box\mathfrak{a} \vee \Box\mathfrak{b}) = \neg\Box(\mathfrak{a} \vee \mathfrak{b}). \tag{60}$$

Corollary 3.7 *A rule*

$$\frac{\Box A \to \Box B}{\Box(A \to B)} \tag{61}$$

is admissible but not derivable in $I_0^{\Box}$.

Proof Let us assume that for some substitution σ formula $\Box(\sigma(A) \to \sigma(B))$ is refutable. Then there exists such a subdirectly irreducible Heyting algebra $\mathfrak{A}$ and such a valuation ν in $\mathfrak{A}$ that $\nu(\sigma(A)) = \mathbf{1}$ while $\nu(\sigma(B)) \neq \mathbf{1}$. Converting Heyting algebra $\mathfrak{A}$ into meta-Heyting algebra by defining $\Box$ by ($\Box$), we can refute formula $\Box A \to \Box B$.

The above rule is not derivable, because formula

$$(\Box A \to \Box B) \to \Box(A \to B)$$

is refutable on a simple 3-element meta-Heyting algebra $\{\mathbf{0}, \mathfrak{a}, \mathbf{1}\}$ by valuation ν, where $\nu(A) = \mathbf{1}$, $\nu(B) = \mathfrak{a}$. $\Box$

Proposition 3.8 *If $A[p_1, \ldots, p_n]$ is an assertoric formula, then $A[p_1, \ldots, p_n]$ is deducible in classic calculus if and only if $A[\Box p_1, \ldots, \Box p_n]$ is deducible in $I_0^{\Box}$.*

Proof This follows simply from the fact that according to Theorem 3.5 on any subdirectly irreducible meta-Heyting algebra, $\Box p_i$ can have only one of two values: $\mathbf{0}$ or $\mathbf{1}$. $\Box$

Lemma 3.9 *Let α be a formula and let $\Box\beta$ be a subformula of α (we will denote this by $\alpha[\Box\beta]$). Then*

(a) $\alpha[\Box\beta] \sim ((\alpha[\bot] \wedge \neg\Box\beta) \vee (\alpha[\top] \wedge \Box\beta))$;

(b) $\alpha[\Box\beta] \sim ((\alpha[\top] \vee \neg\Box\beta) \wedge (\alpha[\bot] \vee \Box\beta))$, *where* $\bot = p \wedge p$ *and* $\top = p \to p$.

Proof In order to prove the lemma it is enough to check that on all the subdirectly irreducible algebras the above formulas hold. According to Theorem 3.5, on every subdirectly irreducible algebra formula $\Box\beta$ can have only two values, $\mathbf{0}$ or $\mathbf{1}$, and it is evident that in any case both equivalences hold. $\Box$

Lemma 3.9 shows that each formula α is equivalent in $I_0^{\Box}$ to a formula without iterated boxes.

Theorem 3.10 (Normalization [5]) *For each formula α there exist assertoric formulas $A_{1,i}, B_{1,i}, C_{1,i} (i = 1, \ldots, n)$ and formulas $A_{2,j}, B_{2,j}, C_{2,j} (j = 1, 2, \ldots, m)$ such that α is equivalent in $I_0^{\Box}$ to formulas*

(a) $\bigwedge_{i=1}^{n} \left(A_{1,i} \vee \Box B_{1,i} \vee \neg\Box C_{1,i}\right)$,

(b) $\bigvee_{j=1}^{m} \left(A_{2,j} \wedge \Box B_{2,j} \wedge \neg\Box C_{2,j}\right)$,

where each of inner disjunctive or conjunctive members may or may not be present.

Proof We will prove this theorem using induction by number n of occurrences of $\Box$ in formula α.

Basis If $n = 0$, then there is no occurrence of $\Box$ in α, which means that α is an assertoric formula.

Step Let us assume that for all the formulas containing less then n occurrences of $\Box$, the statement is true and let α contain n occurrences of $\Box$. Assuming that $n > 0$, there exists such an assertoric formula D that $\Box D$ is a subformula of α. If $\alpha = \Box D$ or $\alpha = \neg\Box D$ the theorem is true. Let $\alpha = \alpha[\Box D]$. Applying Lemma 3.9 we get

$$\text{(a)} \qquad \alpha \sim ((\alpha[\bot] \wedge \neg\Box D) \vee (\alpha[\top] \wedge \Box D)), \tag{62}$$

$$\text{(b)} \qquad \alpha \sim ((\alpha[\top] \vee \Box D) \wedge (\alpha[\bot] \vee \neg\Box D)). \tag{63}$$

Let us denote α_0 and α_1 formulas $\alpha[\bot]$ and $\alpha[\top]$, respectively. Since formulas α_0 and α_1 contain less than n occurrences of $\Box$, we can apply to them our assumption.

$$\alpha_0 \sim \bigwedge_{i=1}^{n}(A_{0i} \vee \Box B_{0i} \vee \neg\Box C_{0i})$$
$$\alpha_1 \sim \bigwedge_{j=1}^{m}(A_{1j} \vee \Box B_{1j} \vee \neg\Box C_{1j})$$

Hence, from (63),

$$\alpha \sim \left(\bigwedge_{i=1}^{n}(A_{0i} \vee \Box B_{0i} \vee \neg\Box C_{0i}) \vee \Box D\right) \wedge \left(\bigwedge_{j=1}^{m}(A_{1j} \vee \Box B_{1j} \vee \neg\Box C_{1j}) \vee \neg\Box D\right). \tag{64}$$

Since disjunction is distributive relative conjunction, that is,

$$((p \wedge q) \vee r)) \sim ((p \vee r) \wedge ((q \vee r)),$$

from (64) it follows that

$$\alpha \sim \left(\bigwedge_{i=1}^{n}(A_{0i} \vee \Box B_{0i} \vee \neg\Box C_{0i} \vee \Box D)\right) \wedge \bigwedge_{j=1}^{m}(A_{1j} \vee \Box B_{1j} \vee \neg\Box C_{1j} \vee \neg\Box D).$$

According to (7) and (59),

$$(\Box B_{0i} \vee \Box D) \sim \Box(B_{0i} \vee D),$$
$$(\neg\Box C_{1j} \vee \neg\Box D) \sim \neg\Box(C_{1j} \wedge D),$$

which means that α is equivalent to a formula of form (a).

The proof that α is equivalent to some formula of form (b) is similar by applying inductive assumption to (62) and using formulas $(p \wedge (q \vee r)) \sim ((p \wedge q) \vee (p \wedge r))$, (8), and (60). $\Box$

Corollary 3.11 ([5]) *In $I_0^{\Box}$ (or any of its extensions) for each formula $\Box\alpha$ there exist such assertoric formulas $A_1, \ldots, A_n, B_1, \ldots, B_n$ that*

$$\Box\alpha \sim \bigwedge_{i=1}^{n}(\Box A_i \to \Box B_i).$$

Proof From Theorem 3.10 formula $\Box\alpha$ is equivalent to

$$\Box\bigwedge_{i=1}^{n}(A_i \vee \Box B_i \vee \neg\Box C_i).$$

$$\text{from (7) and (8)}: \quad \bigwedge_{i=1}^{n}(\Box A_i \vee \Box\Box B_i \vee \Box\neg\Box C_i) \tag{65}$$
$$\text{from (3) and (6)}: \quad \bigwedge_{i=1}^{n}(\Box A_i \vee \Box B_i \vee \neg\Box C_i) \tag{66}$$
$$\text{from (7)}: \quad \bigwedge_{i=1}^{n}(\Box(A_i \vee B_i) \vee \neg\Box C_i) \tag{67}$$
$$\text{from (58)}: \quad \bigwedge_{i=1}^{n}(\Box C_i \rightarrow \Box(A_i \vee B_i)) \tag{68}$$

□

Theorem 3.12 *Variety $\mathcal{M}$ (and calculus $I_0^{\Box}$ or its logic) is finitely approximable.*

Proof In order to prove the theorem we need to demonstrate that if formula α is not valid in $\mathcal{M}$, then there exists such a finite algebra $\mathfrak{A} \in \mathcal{M}$ that α is not valid on $\mathfrak{A}$. It is evident that if α is not valid, then $\Box\alpha$ is not valid too. According to Corollary 3.11, $\Box\alpha$ is equivalent to a formula of type $\bigwedge_{i=1}^{n}(\Box A_i \rightarrow \Box B_i)$, where A_i, B_i are assertoric formulas and they do not contain $\Box$. This means that there exists such $i \in \{1, \ldots, n\}$ that $(\Box A_i \rightarrow \Box B_i)$ is not valid in $\mathcal{M}$. But since formula $(\Box A_i \rightarrow \Box B_i)$ is derivable from formula $(A_i \rightarrow B_i)$ (by simply using (R1), Ax.3, and modus ponens), formula $(A_i \rightarrow B_i)$ cannot be valid in $\mathcal{M}$. The latter means that $(A_i \rightarrow B_i)$ is not provable in intuitionistic propositional calculus which is finitely approximable. Let $\bar{\mathfrak{A}}$ be a finite Heyting algebra such that $(A_i \rightarrow B_i)$ is not valid on $\bar{\mathfrak{A}}$. Let φ be a valuation and

$$\varphi(A_i) = \mathfrak{a} \in \bar{\mathfrak{A}},\ \varphi(B_i) = \mathfrak{b} \in \bar{\mathfrak{A}},\ \text{and}\ \mathfrak{a} \nleq \mathfrak{b}.$$

Let us denote $\nabla(\mathfrak{a})$ a filter in $\bar{\mathfrak{A}}$ generated by $\mathfrak{a}$. Since $\mathfrak{a} \nleq \mathfrak{b}$, $\mathfrak{b} \notin \nabla(\mathfrak{a})$. Let now $\nabla \supseteq \nabla(\mathfrak{a})$ be a maximal filter that does not contain $\mathfrak{b}$ (maximal means that if we add any new element to ∇ and generate a filter this filter will contain $\mathfrak{b}$). At least one such maximal filter exists, because $\bar{\mathfrak{A}}$ is finite. Let us consider $\bar{\mathfrak{A}}/\nabla$ and let $\psi : \bar{\mathfrak{A}} \rightarrow \bar{\mathfrak{A}}/\nabla$ be a natural homomorphism. It is evident that

$$\psi(\varphi(A_i)) = \mathbf{1} \text{ and } \psi(\varphi(B_i)) \neq \mathbf{1}.$$

In order to complete the proof it is enough to show that $\bar{\mathfrak{A}}/\nabla$ is a Gödelean algebra and by defining $\Box$ by ($\Box$) to convert it into finite meta-Heyting algebra . But from the fact that ∇ is maximal it follows that $\bar{\mathfrak{A}}/\nabla$ is indeed a subdirectly irreducible Heyting algebra (any proper filter of $\bar{\mathfrak{A}}/\nabla$ must contain $\psi(\mathfrak{b})$) and as such $\bar{\mathfrak{A}}/\nabla$ is a Gödelean algebra. If we now define $\Box$ by ($\Box$) , we can see that $\psi(\varphi(\Box A_i \rightarrow \Box B_i)) \neq \mathbf{1}$, and this completes the proof of the theorem. □

Remark 3.13 In contrast, logic $L(\mathfrak{F}^{\Box})$ is not finitely approximable. This is a consequence from Theorem 2 ([4]) : the formula

$$\Box((A \rightarrow B) \rightarrow (C \vee D)) \rightarrow \\ \Box(((A \rightarrow B) \rightarrow A) \vee ((A \rightarrow B) \rightarrow C) \vee ((A \rightarrow B) \rightarrow D))$$

that corresponds to Mints's rule is valid in $L(\mathfrak{F}^{\Box})$ and, therefore, it is valid on all models of $L(\mathfrak{F}^{\Box})$. But finite subdirectly irreducible Heyting algebras on which Mints's rule is valid are concatenations of 2- and 4-element Boolean algebras. On all such algebras the following formula holds:

$$((p \rightarrow q) \vee (q \rightarrow r) \vee ((q \rightarrow r) \rightarrow r) \vee (r \rightarrow (p \rightarrow q))),$$

and this formula is not derivable in intuitionistic propositional calculus.

4 Extensions of Logic $L(\mathfrak{F}^{\Box})$

Let $\mathcal{ML}$ be a set of all extensions of logic of metaintuitionistic calculus $I_0^{\Box}$. Evidently $\mathcal{ML}$ is a lattice relative to the closed union and set join.

Proposition 4.1 *The following statements are true:*

(a) *$\mathcal{ML}$ is Heyting algebra (and, therefore, is distributive);*
(b) *the set of all finitely axiomatizable logics from $\mathcal{ML}$ conforms a sublattice.*

Proof The proof of this proposition is a replica of the proof of Theorems 1 and 2 from [15] (in the proof of Lemma 1.3 from [15] the formula that is used is evidently provable in intuitionistic calculus). □

In a natural way one can establish dual isomorphism between $\mathcal{ML}$ (as a lattice) and the lattice of all the subvarieties of $\mathcal{M}$. But we will be more concerned with the relations between logics from $\mathcal{ML}$ and their assertoric fragments.

Let $\mathcal{L} \in \mathcal{ML}$. The set of all assertoric formulas from $\mathcal{L}$ will be denoted by $Asr(\mathcal{L})$ and called assertoric fragment of $\mathcal{L}$. Essentially Asr is a mapping

$$Asr : \mathcal{ML} \rightarrow \mathcal{L},$$

where $\mathcal{L}$ is a lattice of all superintuitionistic logics.

Proposition 4.2 *Asr is an epimorphism of lower semilattices.*

Proof First of all, let us prove that for every superintuitionistic logic $\mathcal{L}'$ there exists such logic $\mathcal{L} \in \mathcal{ML}$ that

$$Asr(\mathcal{L}) = \mathcal{L}'.$$

Let $\mathcal{L}$ be obtained from $\mathcal{L}'$ by adding axioms Ax.1–Ax.5 and closure using rules modus ponens, (R1), and substitution. Obviously, $\mathcal{L}' \subseteq Asr(\mathcal{L})$. Let now $A \notin \mathcal{L}'$. We need to prove that $A \notin Asr(\mathcal{L})$ or, since A is an assertoric formula, to prove that $A \notin \mathcal{L}$.

Since $A \notin \mathcal{L}'$ there exists a subdirectly irreducible Heyting algebra $\mathfrak{A}$ such that A is not valid on $\mathfrak{A}$ while all the formulas from $\mathcal{L}'$ are valid on $\mathfrak{A}$. Let us transform $\mathfrak{A}$ into meta-Heyting algebra by defining □ according to (□). All six axioms of MAx.1–MAx.6 hold (axiom 5 holds because every subdirectly irreducible Heyting algebra is a Gödelean algebra). Evidently, all the formulas from $\mathcal{L}$ are valid on $\mathfrak{A}$ while formula A is not valid on $\mathfrak{A}$. This observation completes the proof that Asr is a mapping onto $\mathcal{L}$. The fact that $Asr(\mathcal{L})$ preserves intersection is evident. □

Remark 4.3 Asr does not preserve union. There exist superintuitionistic logics that are not closed under all rules admissible in I (see, for instance, [11], Lemma 4.2). Let $\mathcal{L}$ be obtained from such a logic by extending the language by □ and adding Ax.1–Ax.5 and closing by inference rules. The union $\mathcal{L}$ and $L(\mathfrak{F}^{\Box})$ will have the assertoric fragment greater than assertoric fragment of $\mathcal{L}$.

It is obvious that every logic $\mathcal{L} \in \mathcal{ML}$ has just one assertoric fragment while the reverse statement is not true.

Proposition 4.4 *There is a continuum of logics whose assertoric fragment is intuitionistic logic.*

Proof In [4] there was introduced a set of rules that are independent and admissible in intuitionistic logic. Let us convert these rules into formulas:

$$\Phi = \left\{\alpha_k = (\Box p^{2k+5} \rightarrow \Box p^{2k+4}) | k = 1, 2, \ldots\right\}$$

(for the definition of degrees, see, for instance, [4]).

In order to prove Proposition 4.4 we need to demonstrate that

(a) each formula α_k is valid on $\mathfrak{F}^{\Box}$ (and, therefore, belongs to $L(\mathfrak{F}^{\Box})$);
(b) the set of formulas Φ is independent (no one formula from Φ is derivable from the rest).

If statement (a) is true then we can add to LI_0 any formulas from Φ and the new logic will have an intuitionistic logic as its assertoric fragment simply because $L(\mathfrak{F}^{\Box})$ is among the models of this newly constructed logic. If statement (b) is valid, then we can construct a continuum of different logics by adding arbitrary subsets of Φ as new axioms and all these extensions will be different because Φ is independent.

Proof of (a) and (b) can be obtained from [4] if we repeat the proof of the Theorem 4 from [4] for metaintuitionistic formulas instead of rules. $\Box$

If $\mathcal{L}'$ is a superintuitionistic logic let $[\mathcal{L}'] = \left\{\mathcal{L} \in \mathcal{ML} | Asr(\mathcal{L}) = \mathcal{L}'\right\}$. Proposition 4.2 states that for any superintuitionistic logic $\mathcal{L}'$ the set $[\mathcal{L}']$ is not empty and the proof of Proposition 4.2 shows that there is the smallest element in $[\mathcal{L}']$ which we will denote by $\Box\mathcal{L}'$.

If Γ is a list of formulas, by $\mathcal{L} + \Gamma$ we will denote the smallest logic from $\mathcal{ML}$ that includes $\mathcal{L} \cup \Gamma$.

Proposition 4.5 *Let $\mathcal{L}'$ be a superintuitionistic logic with disjunction property, $\mathfrak{L}$ be Lindenbaum algebra of $\mathcal{L}'$, and $\mathfrak{L}^{\Box}$ be $\mathfrak{L}$ with $\Box$ defined on it by ($\Box$).*[2] *If formula α is not valid on $\mathfrak{L}^{\Box}$ then*

$$\mathcal{L}' \subset Asr(\Box\mathcal{L}' + \alpha);$$

that is, adding such a formula α to $\Box\mathcal{L}'$ will generate a not conservative extension of $\mathcal{L}'$. In other words, adding a formula that corresponds to not admissible in assertoric fragment rule will result in a nonconservative extension.

Proof Our goal is to demonstrate that there exists such an assertoric formula B that $B \in \Box\mathcal{L}' + \alpha$ even though $B \notin \Box\mathcal{L}'$. Since $\Box\mathcal{L}'$ is closed under rule (R1), $\Box\alpha \in (\Box\mathcal{L}' + \alpha)$ and we can use Corollary 3.7 and consider $\Box\alpha$ being a formula of form

$$\bigwedge_{i=1}^{n} (\Box A_i \rightarrow \Box B_i), \tag{69}$$

and all formulas $(\Box A_i \rightarrow \Box B_i)$ are deducible in $\Box\mathcal{L}' + \alpha$. Since $\Box\alpha$ is not valid on $\mathfrak{L}^{\Box}$ then for some i formula $(\Box A_i \rightarrow \Box B_i)$ is not valid on $\mathfrak{L}^{\Box}$ and there exists such a substitution σ (valuation on $\mathfrak{L}$) that $\sigma(A_i) = \mathbf{1}$ while $\sigma(B_i) \neq \mathbf{1}$. So, using this substitution, from $(\Box A_i \rightarrow \Box B_i)$ we can deduce $(\Box\sigma(A_i) \rightarrow \Box\sigma(B_i))$. Let us remark that $\sigma(A_i)$ is deducible in $\mathcal{L}'$ (simply because it is equal to $\mathbf{1}$ on $\mathfrak{L}$) and by applying (R1) we can deduce $\Box\sigma(A_i)$. Now by modus ponens from $(\Box\sigma(A_i) \rightarrow \Box\sigma(B_i))$ and $\Box\sigma(A_i)$ we can obtain $\Box\sigma(B_i)$ and then $\sigma(B_i)$. But $\sigma(B_i)$ is not valid on $\mathfrak{L}$ and therefore $\sigma(B_i) \notin \mathcal{L}'$, or $\sigma(B_i) \notin \Box\mathcal{L}'$ for this matter. $\Box$

Corollary 4.6 *For each superintuitionistic logic $\mathcal{L}'$ with disjunction property, the set $[\mathcal{L}']$ contains the greatest element, namely, the logic of $\mathfrak{L}^{\Box}$, and $[\mathcal{L}']$ is a lattice relative to the set intersection and closed union.*

Theorem 4.7 ([5; 2]) *$\mathcal{M}$ is a congruence distributive variety.*

Proof Let $\mathfrak{A}$ be a meta-Heyting algebra. Instead of congruences we will consider the lattice of filters:

if $\nabla_1, \nabla_2 \in \mathfrak{A}$ then

$$\nabla_1 \wedge \nabla_2 = \nabla_1 \cap \nabla_2$$
$$\nabla_1 \vee \nabla_2 = \bigcap \{\nabla \mid \nabla \supseteq \nabla_1 \cup \nabla_2\}.$$

Let us prove first of all that

$$\nabla_1 \vee \nabla_2 = \{\mathfrak{a}\wedge\mathfrak{b} \mid \mathfrak{a} \in \nabla_1, \mathfrak{b} \in \nabla_2\}.$$

From the properties of $\wedge$ and $\Box$ it is evident that $\{\mathfrak{a}\wedge\mathfrak{b}|\mathfrak{a} \in \nabla_1, \mathfrak{b} \in \nabla_2\}$ is indeed a filter: if $\mathfrak{a}\wedge\mathfrak{b} \leq \mathfrak{c}$, then $(\mathfrak{a}\vee\mathfrak{c})\wedge(\mathfrak{b}\vee\mathfrak{c}) = \mathfrak{c}$ and, since $\mathfrak{a} \in \nabla_1$ and $\mathfrak{b} \in \nabla_2$, $\mathfrak{c} \in \nabla$. Obviously, $\Box(\mathfrak{a}\wedge\mathfrak{b}) = (\Box\mathfrak{a}\wedge\Box\mathfrak{b}) \in \nabla$.

On the one hand, it is evident that

$$\{\mathfrak{a}\wedge\mathfrak{b}|\mathfrak{a} \in \nabla_1, \mathfrak{b} \in \nabla_2\} \supseteq \nabla_1 \cup \nabla_2 \supseteq \nabla_1 \vee \nabla_2.$$

On the other hand, if $\nabla \supseteq \nabla_1 \cup \nabla_2$, then from the definition of filter

$$\nabla \supseteq \{\mathfrak{a}\wedge\mathfrak{b}|\mathfrak{a} \in \nabla_1, \mathfrak{b} \in \nabla_2\},$$

which means that

$$\nabla_1 \vee \nabla_2 \supseteq \{\mathfrak{a}\wedge\mathfrak{b}|\mathfrak{a} \in \nabla_1, \mathfrak{b} \in \nabla_2\}.$$

Now we can prove that for any filters $\nabla_1, \nabla_2, \nabla_3$,

$$\nabla_1 \wedge (\nabla_2 \vee \nabla_3) = (\nabla_1 \wedge \nabla_2) \vee (\nabla_1 \wedge \nabla_3).$$

Let us first demonstrate that

$$\nabla_1 \wedge (\nabla_2 \vee \nabla_3) \subseteq (\nabla_1 \wedge \nabla_2) \vee (\nabla_1 \wedge \nabla_3). \tag{70}$$

Let $\mathfrak{a} \in \nabla_1 \wedge (\nabla_2 \vee \nabla_3)$, then $\mathfrak{a} \in \nabla_1$ and $\mathfrak{a} \in \nabla_2 \vee \nabla_3$; that is, there are such $\mathfrak{b} \in \nabla_2$, $\mathfrak{c} \in \nabla_3$ that $\mathfrak{a} = \mathfrak{b}\wedge\mathfrak{c}$. From the definition of filter it follows that $\mathfrak{a}\vee\mathfrak{b} \in \nabla_1\wedge\nabla_2$ and $\mathfrak{a}\vee\mathfrak{c} \in \nabla_1\wedge\nabla_3$; hence, $(\mathfrak{a}\vee\mathfrak{b})\wedge(\mathfrak{a}\vee\mathfrak{c}) \in (\nabla_1\wedge\nabla_2)\vee(\nabla_1\wedge\nabla_3)$. But $(\mathfrak{a}\vee\mathfrak{b})\wedge(\mathfrak{a}\vee\mathfrak{c}) = \mathfrak{a}\vee(\mathfrak{b}\wedge\mathfrak{c}) = \mathfrak{a}$. So, $\mathfrak{a} \in (\nabla_1 \wedge \nabla_2) \vee (\nabla_1 \wedge \nabla_3)$ and this completes a proof of (70).

In order to complete the proof of the theorem it is enough to demonstrate that

$$\nabla_1 \wedge (\nabla_2 \vee \nabla_3) \supseteq (\nabla_1 \wedge \nabla_2) \vee (\nabla_1 \wedge \nabla_3). \tag{71}$$

Let $\mathfrak{a} \in (\nabla_1 \wedge \nabla_2) \vee (\nabla_1 \wedge \nabla_3)$. Then $\mathfrak{a} = \mathfrak{b}\wedge\mathfrak{c}$, where

$$\mathfrak{b} \in \nabla_1 \wedge \nabla_2 \subseteq \nabla_1$$
$$\mathfrak{c} \in \nabla_1 \wedge \nabla_3 \subseteq \nabla_1.$$

By definition of filter $\mathfrak{a} = \mathfrak{b}\wedge\mathfrak{c} \in \nabla_1$. In order to complete the proof we simply need to check that $\mathfrak{a} \in \nabla_2\vee\nabla_3$. But $\mathfrak{b} \in \nabla_2$ and $\mathfrak{c} \in \nabla_3$ and, therefore, $\mathfrak{a} = \mathfrak{b}\wedge\mathfrak{c} \in \nabla_2 \vee \nabla_3$. $\Box$

As usual we will call logic *tabular* if it can be defined by a finite algebra.

Theorem 4.8 *Tabular logics from $\mathcal{ML}$ are finitely axiomatizable* [5].

It follows immediately from Theorem 4.7 and Baker's theorem [1].

Theorem 4.9 *Neither logic from $\mathcal{ML}$, except contradictory, possesses a disjunction property.*

Proof Let us remark that the following formula is valid in all logics from $\mathcal{ML}$.

$$(\Box p \to \Box q) \lor (\Box q \to \Box p). \tag{72}$$

First, let us prove (72). From Proposition 2.4, the following formula is valid:

$$(\Box p \lor \neg\Box p) \lor (\Box q \lor \neg\Box q).$$

This formula if obviously equivalent to the following,

$$(\neg\Box p \lor \Box q) \lor (\neg\Box q \lor \Box p),$$

and from (58),

$$\begin{aligned}(\neg\Box p \lor \Box q) &\sim \Box p \to \Box q,\\ (\neg\Box q \lor \Box p) &\sim \Box q \to \Box p.\end{aligned}$$

So, from (72), if logic $\mathcal{L} \in \mathcal{ML}$ has disjunction property, then formula $\Box p \to \Box q$ or $\Box q \to \Box p$ is valid in $\mathcal{L}$. But if $(\Box p \to \Box q) \in \mathcal{L}$, then substituting p with $\top$ we see that $\Box q \in \mathcal{L}$, which means that any formula is valid in $\mathcal{L}$; that is, $\mathcal{L}$ is the contradictory logic. $\Box$

Remark 4.10 From the standpoint of admissibility, (72) simply means that for any two propositional formulas A, B there is no such substitution that will simultaneously refute admissibility of rules A/B and B/A.

Proposition 4.11 *If formula $\Box\alpha \to \Box\beta$ is valid in logic $\mathcal{L} \in \mathcal{ML}$, then formula $\Box(\alpha \lor \gamma) \to \Box(\beta \lor \gamma)$ is valid in $\mathcal{L}$.*

Proof If $\Box\alpha \to \Box\beta$ is valid, then the formulas $\Box\alpha \to (\Box\beta \lor \Box\gamma)$ and $\Box\gamma \to (\Box\beta \lor \Box\gamma)$ are valid. From the latter two formulas the formula $(\Box\alpha \lor \Box\gamma) \to (\Box\beta \lor \Box\gamma)$ is deducible. And applications of equivalence (7) will complete the proof. $\Box$

Corollary 4.12 *If variable p does not have occurrences in formulas α and β, then the following formulas are mutually deducible:*

$$(\Box\alpha \to \Box\beta) \textit{ and } ((\Box\alpha \lor p) \to (\Box\beta \lor p)).$$

5 Metaintuitionistic Logic and Admissibility

As was mentioned above, metaintuitionistic logic $LI_0^{\Box}$ was initially constructed as an attempt to formalize reasoning about rules admissible in intuitionistic propositional calculus. In this section we will focus on segment $[LI]$, the greatest logic of which is logic $L(\mathfrak{F}^{\Box})$ that is a set of formulas representing all admissible in intuitionistic logic rules.

We already saw in Proposition 4.4 that there exists a set of independent in $LI_0^{\Box}$ formulas that are valid in $L(\mathfrak{F}^{\Box})$. In [6] the different set of formulas that are valid in

$L(\mathfrak{F}^{\Box})$ but not in $LI_0^{\Box}$ were introduced:

$$V_n^{\Box} = \Box(A_n \to (r\vee s)) \to \Box\Big(\bigvee_{i=1}^{n}(A_n\to p_i)\vee(A_n\to r)\vee(A_n\to s)\Big), \tag{73}$$

where $A_n = \bigwedge_{i=1}^{n}(p_i\to q_i), n = 1, 2, \ldots$. My conjecture at that time was that $\left\{V_n^{\Box} | n = 1, 2, \ldots\right\}$ is a basis in $L(\mathfrak{F}^{\Box})$.

Remark 5.1 Later the rules V_n corresponding to $V_n^{\Box}$ for Heyting arithmetics were independently introduced by Visser and nowadays they are known as Visser's rules (or principles, see [10], for instance). Visser and de Jongh also conjectured that Visser's rules form a basis for admissible rules. Using technique developed in [8] Rosière [17] and Iemhoff [10] proved that conjecture is indeed correct.

If $I_n^{\Box}$ is a calculus obtained from $I_0^{\Box}$ by adding a new axiom $V_n^{\Box}$, then the logics of this calculus form a chain

$$LI_0^{\Box} \subseteq LI_1^{\Box} \subseteq LI_2^{\Box} \subseteq \ldots \tag{74}$$

and from [17; 10] it follows that every formula valid in $L(\mathfrak{F}^{\Box})$ is deducible from the formulas $V_n^{\Box}$. Since every proof is a finite sequence of formulas,

$$L(\mathfrak{F}^{\Box}) = \bigcup_{i=0}^{\infty} LI_i^{\Box}. \tag{75}$$

In [7] Gabbay and de Jongh constructed such a sequence of intermediate logics $D_n, n = 1, 2, \ldots$ that each logic D_n has disjunction property and $\bigcap_n D_n$ is intuitionistic logic. Let $\mathfrak{F}_n$ be a Lindenbaum algebra of D_n. Since $\mathfrak{F}_n$ is evidently a Gödelian algebra we can convert it into meta-Heyting algebra $\mathfrak{F}_n^{\Box}$ by defining additional operator using ($\Box$). The proof of Lemma 4.2 from [11] can be easily converted into proof of the following.

Proposition 5.2 ([11]) *For each $n = 1, 2, \ldots$ formula $V_n^{\Box}$ holds on $\mathfrak{F}_n^{\Box}$ while formula $V_{n+1}^{\Box}$ does not hold on $\mathfrak{F}_n^{\Box}$.*

The evident corollary from the above proposition is

$$LI_0^{\Box} \subsetneqq LI_1^{\Box} \subsetneqq LI_2^{\Box} \subsetneqq \cdots \subsetneqq L(\mathfrak{F}^{\Box}). \tag{76}$$

Proposition 5.3 ([18]) *Logic $L(\mathfrak{F}^{\Box})$ is not finitely axiomatizable by adding a finite set of new axioms to $LI_0^{\Box}$.*

Proof Proof by contradiction. Assume the contrary: there is a finite set of formulas that constitute the basis of $L(\mathfrak{F}^{\Box})$. In this case, from (75) it follows that there exists such $LI_n^{\Box}$ that all these formulas belong to it, which contradicts to (76). This contradiction completes the proof. $\Box$

Remark 5.4 The absence of the finite basis for admissible in intuitionistic propositional logic rules was first proved by Rybakov [18]. The above statement is stronger than Rybakov's one since the means of inference in $L(\mathfrak{F}^{\Box})$ are stronger than ones for admissible rules (for instance, the Gödel disjunction theorem can be used here). The above proposition means that $L(\mathfrak{F}^{\Box})$ does not have a finite axiomatization if modus ponens, substitution, and (R1) are used as the only inference rules. But the following question remains: is there the finite set of admissible in $L(\mathfrak{F}^{\Box})$ rules that for some integer n $V_n^{\Box}$ can be deduced from $V_{n-1}^{\Box}$? In other words, the following problem remains open.

Problem 5.5 Is $L(\mathfrak{F}^{\Box})$ finitely axiomatizable by adding to $LI_0^{\Box}$ the finite set of new axioms and rules?

6 On Semantic of Metaintuitionistic Logic

Above, we constructed the algebraic semantic for logic LI and its extensions. In this section we will focus on semantic of logic $L(\mathfrak{F}^{\Box})$ from the provability standpoint. By $I^{\Box}$ we will denote a calculus obtained from the calculus $I_0^{\Box}$ by adding formulas $\left\{V_n^{\Box}|n = 1, 2, \dots\right\}$ as axioms. And we use $\vdash$ to denote deducibility in I, and we will use $\vdash_{\Box}$ to denote deducibility in $I^{\Box}$. So, if A is an assertoric formula, then

$$\vdash A \text{ iff } \vdash_{\Box} \Box A \text{ iff } \Box A \in L(\mathfrak{F}^{\Box}). \tag{77}$$

It seems that (77) means that $\Box A$ represents provability of formula A in I; that is, $\Box$ simply means the same as $\vdash$. On the other hand, semantic of $\neg\Box A$ is a bit more complex. Let us consider formula $A = (p \vee \neg p)$. In this case, $\nvdash A$, and we would expect that this is represented in metalogic by $\neg\Box A$. But $\vdash_{\Box} \neg\Box A$ simply is not true (it is enough to substitute p with $\bot$, for instance). Since I contains substitution as an inference rule, if $\vdash A$, then for each substitution σ, $\vdash \sigma(A)$ is also true. But if $\nvdash A$ there very well may exist such a substitution σ that $\vdash \sigma(A)$ is true. In other words, $\vdash_{\Box} \neg\Box A$ is much stronger, than just $\nvdash A$, and if $\vdash_{\Box} \neg\Box A$ then for each substitution σ formula $\sigma(A)$ is not provable.

Let us denote by α^- the assertoric formula obtained from α by omitting all $\Box$.

Theorem 6.1 *For each logic $\mathcal{L} \in \mathcal{ML}$ if $\neg\Box\alpha \in \mathcal{L}$, then $\neg\alpha^-$ is provable in classic logic and $\Box\neg\alpha^- \in \mathcal{L}$.*

Proof If $\neg\Box\alpha \in \mathcal{L}$, then formula $\neg\Box\alpha$ is valid on all meta-Heyting algebras from a corresponding variety. Particularly, this formula is valid on algebra $\mathfrak{B}$—two element Heyting (Boolean) algebra where $\Box$ is defined by ($\Box$). Let us point out that the identity $\Box x = x$ is valid on $\mathfrak{B}$. Therefore, if $\neg\Box\alpha$ is valid on $\mathfrak{B}$, the formula $\neg\alpha^-$ is also valid. Formula $\neg\alpha^-$ is an assertoric formula and from the fact that it is valid on 2-element Boolean algebra it follows that $\neg\alpha^-$ is provable in classical logic. By the Glivenko theorem $\neg\alpha^-$ is provable in intuitionistic propositional calculus and, therefore, $\Box\neg\alpha^- \in LI \subseteq \mathcal{L}$. $\Box$

Corollary 6.2 $\vdash_{\Box} \neg\Box A$ *if and only if* $\vdash_{\Box} \Box\neg A$.

Proof From the theorem it follows that if $\vdash_{\Box} \neg\Box A$, then $\vdash \neg A$ and $\vdash_{\Box} \Box\neg A$. The converse statement is a consequence of Ax.3. $\Box$

Notes

1. The basic information regarding admissibility in various nonclassical logics can be found in [19].

2. If logic does not enjoy a disjunction property we cannot use (□) to define □.

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Acknowledgments

I would like to thank A. Muravitsky for his stimulation, fruitful discussions, and comments. I am also thankful to L. Esakia, V. Rybakov, G. Mints, A. Chagrov, and G. Bezhanishvily for interactions which helped me while I was working on this paper.

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