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A Hyperimmune Minimal Degree and an ANR 2-Minimal Degree

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Abstract We develop a new method for constructing hyperimmune minimal degrees and construct an **ANR** degree which is a minimal cover of a minimal degree.

1 Introduction

A nonrecursive Turing degree $\mathbf{a}$ is *minimal* if the interval $(\mathbf{0}, \mathbf{a})$ is empty; that is, there is no degree strictly between $\mathbf{0}$ and $\mathbf{a}$. Relativizing this definition, a degree $\mathbf{a} > \mathbf{b}$ is a *minimal cover* of $\mathbf{b}$ if the interval $(\mathbf{b}, \mathbf{a})$ is empty. Moreover, $\mathbf{a}$ is a *strong minimal cover* of $\mathbf{b}$ if $\mathcal{D}(< \mathbf{a}) = \mathcal{D}(\leq \mathbf{b})$; that is, every degree strictly below $\mathbf{a}$ is below $\mathbf{b}$.

The primary motivation for this paper is a long-standing question of Yates.

Question 1.1 (Yates) *Does every minimal degree have a strong minimal cover?*

A function f *dominates* g if $f(n) \geq g(n)$ for cofinitely many n. A degree $\mathbf{a}$ is *hyperimmune* if there is a function $f \leq_T \mathbf{a}$ which is not dominated by any recursive function; otherwise, $\mathbf{a}$ is *hyperimmune-free*. A function f is *fixed-point-free* if for every e, $\varphi_e \neq \varphi_{f(e)}$. A degree $\mathbf{a}$ is *fixed-point-free* (**FPF**) if there is a function $f \leq_T \mathbf{a}$ which is fixed-point-free.

A recent and remarkable result in the positive direction for Question 1.1 is Lewis's Theorem [8, Theorem 4.3].

Theorem 1.2 (Lewis) *Every hyperimmune-free degree which is not* **FPF** *has a strong minimal cover.*

So in order to give a negative answer to Question 1.1, we have to look for minimal degrees that are either **FPF** or hyperimmune.

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On one hand, it was also an old question whether **FPF** minimal degrees exist. Kumabe gave a positive solution to this question in an unpublished note and Lewis simplified the proof in [6].

On the other hand, the observation might seem easy: it is well known that there is a minimal degree below $\mathbf{0}'$ [10, Theorem 1] and any nonrecursive degree below $\mathbf{0}'$ is hyperimmune [9, Theorem 1.2]. However, this argument doesn't apply to the degrees that are not recursive in $\mathbf{0}'$. Miller and Martin [9, Section 3] then asked whether there is a hyperimmune minimal degree which is not recursive in $\mathbf{0}'$.

Our second motivation is the finite maximal chain problem for $\overline{\mathbf{GL_2}}$ degrees, that is, those degrees $\mathbf{a}$ such that $\mathbf{a}'' > (\mathbf{a} \vee \mathbf{0}')'$. A degree $\mathbf{a}$ has the *finite maximal chain property* if there is a chain $\mathbf{0} = \mathbf{a}_0 < \mathbf{a}_1 < \cdots < \mathbf{a}_n = \mathbf{a}$ where each $\mathbf{a}_{i+1}$ is a minimal cover of $\mathbf{a}_i$. Lerman [7, Section IV.3] asked whether there is a $\overline{\mathbf{GL_2}}$ degree with the finite maximal chain property. Note that $\mathbf{0}$ is $\mathbf{GL_2}$ (i.e., not $\overline{\mathbf{GL_2}}$). Moreover, it is a classical result that all minimal degrees are $\mathbf{GL_2}$ [5, Theorem 1], and so in order to find a $\overline{\mathbf{GL_2}}$ degree with the finite maximal chain property, we at least need a maximal chain of length 3, that is, $\mathbf{0} < \mathbf{a} < \mathbf{b}$ where $\mathbf{a}$ is minimal and $\mathbf{b}$ is a minimal cover of $\mathbf{a}$.

Regardless of the length of such a maximal chain, we need to build a $\overline{\mathbf{GL_2}}$ degree $\mathbf{b}$ minimal over a $\mathbf{GL_2}$ degree $\mathbf{a}$. It is not difficult to prove (see Propositions 6.1) that if $\mathbf{b}$ is a minimal cover of a $\mathbf{GL_2}$ degree $\mathbf{a}$, and if $\mathbf{b}$ is hyperimmune-free relativized to $\mathbf{a}$ or $\mathbf{b} \leq \mathbf{a}'$, then $\mathbf{b}$ is also $\mathbf{GL_2}$. That is to say, in order to find a $\overline{\mathbf{GL_2}}$ degree with the finite maximal chain property, we need a relativized version of "a hyperimmune minimal degree not recursive in $\mathbf{0}'$."

Cooper [2] answered Miller and Martin's question using an indirect argument. He proved that any $\mathbf{d} \geq \mathbf{0}'$ is the jump of a minimal degree and showed that any minimal degree whose jump is $\mathbf{0}''$ is not below $\mathbf{0}'$. In addition, he used Jockusch's result that $\mathbf{a}' \geq \mathbf{0}''$ implies that $\mathbf{a}$ is hyperimmune [4]. Therefore, any minimal degree whose jump is $\mathbf{0}''$ is hyperimmune and not below $\mathbf{0}'$.

The technical difficulty with producing a direct proof is that in the standard minimal degree construction we don't have a method which explicitly forces the minimal degree to be hyperimmune. In this paper, we will provide a new minimal degree construction and use it to directly construct a hyperimmune minimal degree. In addition, our construction can be easily augmented, given any degree $\mathbf{d}$, to construct a hyperimmune minimal degree which is not recursive in $\mathbf{d}$. Note that this can be done by Cooper's argument: one can find a minimal degree whose jump is $\mathbf{d}''$, and apply Jockusch's result as above to show that it is hyperimmune; in addition, such a minimal degree cannot be recursive in $\mathbf{d}$ because otherwise its jump would be below $\mathbf{d}'$.

The third motivation for this paper comes from some recent research on array nonrecursive (**ANR**) degrees. Recall that the modulus function of K, $m_K(n)$ is defined as the least s such that $\varphi_{m,s}(m) \downarrow$ for every $m \leq n, m \in K$.

Definition 1.3 A degree $\mathbf{a}$ is *array nonrecursive* if there is a function $f \leq_T \mathbf{a}$ which is not dominated by the modulus function of K.

ANR degrees share a lot of nice properties with $\overline{\mathbf{GL_2}}$ degrees (see [3] and [1]). So it is natural to ask whether there is an **ANR** degree with the finite maximal chain property. In addition, the following theorems about **ANR** degrees are quite interesting (in the sense of Question 1.1).

Theorem 1.4 (Downey, Jockusch, Stob [3, Theorem 2.1]) *No* **ANR** *degree is minimal.*

Theorem 1.5 (Downey, Jockusch, Stob [3, Theorem 3.4]) *No* **ANR** *degree has a strong minimal cover.*

In some sense, **ANR** degrees are "high" in the Turing degrees and these degrees with strong minimal covers must be "lower" than **ANR** ones. Minimal degrees are, of course, the lowest possible nonrecursive degrees. From this point of view, Question 1.1 is asking whether these "lowest" degrees have strong minimal covers.

For convenience we state the following definition.

Definition 1.6 A degree $\mathbf{a}$ is 1-*minimal* if it is minimal. A degree $\mathbf{a}$ is $(n+1)$-*minimal* if it is a minimal cover of an n-minimal degree.

It is immediate that a degree is n-minimal for some $n \in \omega$ if and only if it has the finite maximal chain property.

Similarly to Question 1.1, one might ask whether every n-minimal degree has a strong minimal cover. The answer is no and actually we will construct an **ANR** degree which is 2-minimal. This result gives an **ANR** degree with the finite chain property and together with Theorem 1.5 implies the following corollary.

Corollary 1.7 *There is a* 2*-minimal degree that does not have a strong minimal cover.*

2 Basic Definitions

Since, to guarantee hyperimmunity or array nonrecursiveness, we need to code in sufficiently large numbers in a minimal degree or a minimal cover construction, we work with approximations in $\omega^{<\omega}$ rather than $2^{<\omega}$.

A *tree* is a (possibly partial) function from $\omega^{<\omega}$ to $\omega^{<\omega}$ with the usual order-preserving properties. In some cases, it will be more convenient to consider trees whose domains or ranges are restricted to binary strings in $2^{<\omega}$. A *node* on a tree is a string in the range of the tree. A *leaf* is a node which does not have successors on the tree. A *path* on a tree is an (induced) image of an infinite string, and we use $[T]$ to denote the set of all paths on T.

We call a tree T *finite* if the domain of T is finite. We usually use σ to denote strings in the domain of a tree T, and τ to denote strings in the range. We use α, β, γ only for infinite strings. All other lowercase Greek letters denote finite strings.

For two strings σ and τ, we use $\sigma * \tau$ for the usual concatenation of σ with τ. For simplicity, in the case that $\tau = \langle i \rangle$, that is, τ is of length 1, we use $\sigma * i$ instead of $\sigma * \langle i \rangle$.

We use *Even* to denote the set of all strings of even length, and for convenience we call a string *even* if it is in *Even*. A node on T is an *even-node* if it is the image of an even string. Note that "an even node" and "an even-node" have different meanings. In this paper, we will not use the former notion.

Given two strings τ_0, τ_1, we write $\tau_0|_e\tau_1$ if they e-split; that is, there is an x such that $\varphi_e^{\tau_0}(x)\downarrow \neq \varphi_e^{\tau_1}(x)\downarrow$. Our convention is that $\varphi_e^{\sigma}(x)\downarrow$ only if for all $y < x$, $\varphi_e^{\sigma}(y)\downarrow$.

3 A Hyperimmune Minimal Degree

Theorem 3.1 (Cooper) *For any degree* $\mathbf{d}$*, there is a hyperimmune minimal degree which is not recursive in* $\mathbf{d}$.

Proof We approximate an infinite string α in ω^ω by a sequence of (partial) recursive trees $\langle T_i \rangle$ all of which map from $2^{<\omega}$ to $\omega^{<\omega}$.

Definition 3.2 A tree T is *special* if it satisfies the following *properties*:

1. $T(\varnothing)\downarrow$.
2. For any string $\sigma \in$ *Even*, if $T(\sigma)\downarrow$, then neither $T(\sigma)$ nor $T(\sigma * 0)$ is a leaf.
3. For any σ, $T(\sigma * 0)\downarrow \Leftrightarrow T(\sigma * 1)\downarrow$.

It is worth noting that although on a special tree some nodes do not branch, the paths on a special tree still contain a copy of the Cantor space, and this allows us to carry out Spector's minimal degree construction in a slightly different way.

In this construction we will require all trees to be special. In order to code in some sufficiently large numbers we want all trees in the construction to have another property as well.

Definition 3.3 A tree T is *recursively unbounded* if there is no recursive function that dominates every path on T.

In particular, for any special tree T considered in the construction, we also provide a recursive function $f(n)$ such that $f(n)$ is even for every n and for every string σ of length $f(n)$, if $T(\sigma)\downarrow$, then $\varphi_n(|T(\sigma*1)|)\downarrow$ if and only if $T(\sigma*1*0)\downarrow$. In addition, if both converge, then for $i = 0, 1$, $T(\sigma*1*i) \supset T(\sigma*1)*(\varphi_n(|T(\sigma*1)|)+i+1)$. We call such a function a *witness function* for T.

It is easy to see that if we have a witness function $f(n)$ for a special tree T, then T is recursively unbounded, and we can take an appropriate full subtree to satisfy one requirement for hyperimmunity. In our construction, a common example of such a function is $f(n) = 2n$.

The following lemma is very easy to prove but it is convenient to make it explicit.

Lemma 3.4 *If T is a special tree and σ is an even string in the domain of T, then $FS(T,\sigma)$, the full subtree of T above σ defined by $FS(T,\sigma)(\sigma') = T(\sigma*\sigma')$, is also special.*

Proof Immediate. □

3.1 Requirements First of all, as in a standard minimal degree construction, we need the following requirements to guarantee that $\deg(\alpha)$ is minimal:

R_e: either φ_e^α is not total, or it is recursive, or $\alpha \le_T \varphi_e^\alpha$.

It is easy to see that hyperimmunity of a degree $\mathbf{a}$ is equivalent to having a function $f \le_T \mathbf{a}$ which is not dominated everywhere by any recursive function; that is, for every recursive function h, there is an x such that $f(x) > h(x)$. We will require α to be such a function.

P_e: either φ_e is not total, or $\exists x(\alpha(x) > \varphi_e(x)\downarrow)$.

To make sure that $\alpha \not\le_T \mathbf{d}$, we fix a set $D \in \mathbf{d}$ and the following requirements.

Q_e: either φ_e^D is not total, or $\exists x(\alpha(x) \neq \varphi_e^D(x)\downarrow)$.

A tree T *forces* a requirement R_e, P_e, or Q_e if the requirement holds for every path $\alpha \in [T]$. Next, we provide basic modules to find a subtree T' of any given special T with a witness function such that T' forces one requirement R_e, P_e, or Q_e. It is then easy to construct a sequence of trees $\langle T_i \rangle$ which approximates a string α satisfying all the requirements.

3.2 Initial tree T_0 We begin our construction with a special tree T_0. First let $T_0(\varnothing) = \varnothing$, $T_0(0) = 0$, $T_0(1) = 1$, $T_0(00) = 00$, and $T_0(01) = 01$. Compute $\varphi_0(1)$. If it does not converge then $T(1) = 1$ is a leaf; if it converges, let $T_0(10) = 1 * (\varphi_0(1) + 1)$ and $T_0(11) = 1 * (\varphi_0(1) + 2)$.

Inductively for any σ of length $2k$, if we have defined $T_0(\sigma) = \tau$, then we let $T_0(\sigma * 0) = \tau * 0$, $T_0(\sigma * 1) = \tau * 1$, $T_0(\sigma * 00) = \tau * 00$, and $T_0(\sigma * 01) = \tau * 01$. Next we compute $\varphi_k(2k+1)$. If it does not converge, $\tau * 1$ is then a leaf; if it converges, let $T_0(\sigma * 10) = \tau * 1 * (\varphi_k(2k+1) + 1)$ and $T_0(\sigma * 11) = \tau * 1 * (\varphi_k(2k+1) + 2)$. Note that T_0 is a special tree and $f(n) = 2n$ is a witness function for it.

3.3 Forcing α to be hyperimmune Suppose we are given a special tree T with a witness function $f(n)$ and we want to force P_e. If φ_e is not total then we are done. If it is total, then we need a subtree T' of T which forces α to be not dominated everywhere by φ_e, that is, $\alpha(x) > \varphi_e(x)$ for some x.

First find any σ of length $f(e)$ in the domain of T. By the definition of a witness function we know that $\varphi_e(|T(\sigma * 1)|)\downarrow$ if and only if $T(\sigma * 1 * 0)\downarrow$. By assumption $\varphi_e(|T(\sigma * 1)|)$ converges, so $T(\sigma * 1 * i) \supset T(\sigma * 1) * (\varphi_n(|T(\sigma * 1)|) + i + 1)$. We can now take the full subtree of T above $\sigma * 1 * 0$ as T'. By Lemma 3.4 it is also a special tree. By the Padding Lemma we can recursively find a witness function f' for T' from f.

3.4 Forcing $\alpha \not\leq_T \mathbf{d}$ Given a special tree T and an index e, we want to find a subtree T' of T which forces $\alpha \neq \varphi_e^D$, if φ_e^D is total. Pick any two different even-nodes τ_0 and τ_1 on T. One of them must be incompatible with φ_e^D if it is total. Without loss of generality we can assume that τ_0 is incompatible with φ_e^D, that is, $\exists x(\tau_0(x)\downarrow \neq \varphi_e^D(x)\downarrow)$; then we take $T' = FS(T, \sigma_0)$ and again by Lemma 3.4 and the Padding Lemma we are done.

3.5 Forcing α to be minimal Now given a special tree T, its witness function f and an index e, we need to find a subtree of T to force R_e; that is, either φ_e^α is not total, or it is recursive, or $\alpha \leq_T \varphi_e^\alpha$. Note that Posner's Lemma is usually used to show that the nonrecursiveness of α is guaranteed by a splitting tree construction. Here this is automatically guaranteed by the requirements, because hyperimmune degrees are not recursive.

As in a standard minimal degree construction, we first try to ask whether there are e-splitting pairs above every node, that is, whether we can construct a splitting subtree of T. However, we need to be careful here. Some nodes on T are leaves, and these nodes cannot be extended on T. For example, we start from the root of T and search for two nodes above it which form an e-splitting pair. If the two nodes we find are leaves, then we cannot extend them at the next step.

A possible way to solve this problem is to ask a similar but more specific question. First let $E = \mathit{Even} \cap \operatorname{dom}(T)$, that is, the collection of even strings in the domain of

T, then ask whether the following is true:

$$\exists \sigma \in E\ \forall \sigma_0, \sigma_1 \supset \sigma, \sigma_0, \sigma_1 \in E[\neg(T(\sigma_0)|_e T(\sigma_1))].$$

Intuitively, property 2 of special trees guarantees that $T(\sigma_0)$ and $T(\sigma_1)$ are not terminal nodes, so we can continue to construct our subtree, if the answer is no.

Now if the answer is yes, then we can find an even σ in the domain of T such that for any even σ_0, σ_1 above σ, $T(\sigma_0)$ and $T(\sigma_1)$ do not form an e-splitting. For any α an infinite path on $FS(T, \sigma)$, if φ_e^α is total, then we can compute it as follows: Given x, we search above σ for the first even σ' such that $\varphi_e^{T(\sigma')}(x) \downarrow$ and output that value. We can always find one because $\varphi_e^\alpha(x) \downarrow$. We cannot find an answer different from $\varphi_e^\alpha(x)$ because of the positive answer to our question.

If the answer is no, then we know that for any even σ in the domain, we can find two even strings σ_0, σ_1 such that $T(\sigma_0)$ and $T(\sigma_1)$ form an e-splitting. So we can easily construct a splitting tree. In addition, we want the splitting tree to be recursively unbounded. We guarantee this by arranging for $f'(n) = 2n$ to be a witness function as in the following construction. First, put $T'(\varnothing) = T(\varnothing)$. By the negative answer to our question we can find two even strings σ_0, σ_1 (above $\varnothing$) such that $T(\sigma_0)$ and $T(\sigma_1)$ e-split. Now find t such that $\varphi_t = \varphi_0$ and $f(t)$ bounds $|\sigma_0|$ and $|\sigma_1|$. Extend σ_0 and σ_1, respectively, to σ_0' and σ_1' with $|\sigma_0'| = |\sigma_1'| = f(t)$. Let $T'(0) = T(\sigma_0' * 0)$ and $T'(1) = T(\sigma_1' * 1)$.

On the 0 side, search above $\sigma_0' * 0$ for even strings $\sigma_{00} \supset \sigma_0' * 0 * 0$ and $\sigma_{01} \supset \sigma_0' * 0 * 1$ such that $T(\sigma_{00})|_e T(\sigma_{01})$ and let $T'(0i) = T(\sigma_{0i})$ for $i = 0, 1$. (We can find such two even strings because of our convention that $\varphi_e^\sigma(x) \downarrow$ only if for all $y < x$, $\varphi_e^\sigma(y) \downarrow$.)

On the 1 side, we wait for $\varphi_0(|T(\sigma_1' * 1)|) = \varphi_t(|T(\sigma_1' * 1)|)$ to converge. If it doesn't, $T'(1)$ is a leaf. If it converges, then we know that $T(\sigma_1' * 1 * i) \supset T(\sigma_1' * 1) * (\varphi_0(|T(\sigma_1' * 1)|) + i + 1)$. Note that $\sigma_1' * 1 * 0$ and $\sigma_1' * 1 * 1$ are even, so we can search above them for even strings $\sigma_{10} \supset \sigma_1' * 1 * 0$, $\sigma_{11} \supset \sigma_1' * 1 * 1$ such that $T(\sigma_{10})|_e T(\sigma_{11})$. Then let $T'(1i) = T(\sigma_{1i})$ for $i = 0, 1$.

Inductively, suppose we have defined $T'(\sigma) = T(\sigma^*)$ for $|\sigma| = 2n$, and σ^* is even. We find two even strings σ_0, σ_1 extending σ^* such that $T(\sigma_0)|_e T(\sigma_1)$, and pick t such that $\varphi_t = \varphi_n$ and $f(t)$ bounds the lengths of σ_0 and σ_1. Then we extend σ_0 and σ_1, respectively, to σ_0' and σ_1' with $|\sigma_0'| = |\sigma_1'| = f(t)$. Now we define $T'(\sigma * i) = T(\sigma_i' * i)$ for $i = 0, 1$. The rest of the construction here is similar to the one in the base case. It is easy to see that T' is special and recursively unbounded witnessed by $f'(n) = 2n$. This finishes the construction and the proof. □

4 Tree Systems

Next we construct an **ANR** 2-minimal degree. We use a tree construction to find a string α of minimal degree. At the same time, we do another tree argument relativized to α to find a string β which is minimal over α.

Usually, such iterations are implemented with uniform trees (see [7, Chapter VI]). Here we provide a different and more general approach. The intuition is to build a "tree of trees" where "tree" refers to the construction of α and "trees" refers to the construction of β.

A *tree system* is a pair of functions (T, S) where T is a partial recursive tree from $\omega^{<\omega}$ to $\omega^{<\omega}$ and S is a recursive function defined on the range of T with values finite trees mapping from a subset of $2^{<\omega}$ to $\omega^{<\omega}$ such that if $\tau \subset \tau'$ are both in the range of T, then $S(\tau')$ is an extension of $S(\tau)$. That is, $S(\tau')$ restricted to the domain of $S(\tau)$ is equal to $S(\tau)$.

In this setting T is called a tree and S is called a system. We use σ to denote strings in the domain of T, τ (and less frequently π) to denote strings in the range of T and the domain of S. We use R to denote finite trees in the range of S. We use μ to denote strings in the domain of such an R and ρ (and less frequently η, ξ, or ζ) to denote strings in the range of such an R. We write $\mathrm{lb}(\rho)$ to denote the last bit of ρ, that is $\rho(|\rho|-1)$.

In our proof we will construct tree systems with the following *Properties*:

1. For any σ, if $T(\sigma)\downarrow$ then for any $i \leq |T(\sigma)|+1$, $T(\sigma * i)\downarrow\supset T(\sigma)*i$, and $T(\sigma * i)\uparrow$ for $i > |T(\sigma)|+1$.
2. For any node τ on T, the maximum length of leaves on $S(\tau)$ is $3|\tau|+1$.
3. For any node τ on T, all leaves of $S(\tau)$ are of length $3i+1$ for some i and all leaves are images of odd strings. We call these leaves of length $3|\tau|+1$ the *top leaves* on $S(\tau)$, and the other leaves on $S(\tau)$ the *terminal leaves*. In addition, if $S(\tau)(\mu)=\rho$ is a terminal leaf, then $\mathrm{lb}(\mu)=\mathrm{lb}(\rho)=1$.
4. For any infinite path $\alpha \in [T]$, $S(\alpha)=\cup_{\tau\subset\alpha} S(\tau)$ is a special tree (see Definition 3.2).
5. For any $\tau = T(\sigma)$ and any even-node ρ on $S(\tau)$, there are $\eta_0, \eta_1 \in S(\tau)$ extending ρ which are top leaves of $S(\tau)$ with $\mathrm{lb}(\eta_0)=0$ and $\mathrm{lb}(\eta_1)=1$.

We call such tree systems *special*. Property 1 assures that we can infinitely often code some information into α. Properties 2 and 3 provide a convenient specific framework. Property 4 is crucial: it allows us to "guess" whether something happens or not as in our hyperimmune minimal degree construction. Property 5 would follow from our construction but it is more convenient to make it explicit.

Before starting our main construction, we provide some technical lemmas about special tree systems.

Lemma 4.1 *Given a special tree system (T, S), if T' is a subtree of T which also satisfies Property 1, and S' is the restriction of S to the range of T', then (T', S') is also a special tree system.*

Proof Immediate. □

We give an analog of the "full subtree" in the usual tree construction.

Definition 4.2 Given a tree system (T, S), $\tau = T(\sigma)$ and $\rho = S(\tau)(\mu)$, the *full subtree system* (T', S') of (T, S) above (σ, μ), is defined as follows: Let T' be the usual full subtree of T above σ. For any π in the range of T', define $S'(\pi)$ to be the full subtree of $S(\pi)$ above μ. We denote this T' by $FSTS(T, S, \sigma, \mu)$.

Intuitively, this is a full subtree of full subtrees. Now we show that with one easy requirement, this full subtree system is special if the original tree system is special.

Lemma 4.3 *Suppose (T, S) is a special tree system, $\tau = T(\sigma)$, and μ is an even string in the domain of $S(\tau)$, then $FSTS(T, S, \sigma, \mu)$ is also a special tree system.*

Proof Properties 1, 3, and 5 are immediate. Property 4 follows from Lemma 3.4 and finally Property 2 follows from Properties 3 and 4. □

5 An ANR 2-Minimal Degree

Theorem 5.1 *There is a* 2*-minimal* **ANR** *degree.*

Proof We construct a sequence of special tree systems (T_i, S_i) with each one a *subtree system* of the previous one; that is, T_{n+1} is a subtree of T_n and for any τ in the range of T_{n+1}, $S_{n+1}(\tau)$ is a subtree of $S_n(\tau)$. In the end, the T_is will approximate an infinite string α. That is to say, α is the only string in all $[T_i]$. In addition, $S_i(\alpha)$ will approximate β in the same way. Our plan is to make $\deg(\beta)$ an **ANR** degree minimal over the minimal degree $\deg(\alpha)$.

We take $\lambda(n, s)$ to be the standard approximation of the modulus function of K with the number of changes in the column $\{\lambda(n, s)\}_{s\in\omega}$ bounded by $n+1$. Notice that in our construction of a hyperimmune minimal degree we could replace the partial recursive φ_n used at level $f(n)$ by any partial recursive function specified uniformly in n. Relativizing this idea, when we construct a minimal cover β of α, we can code in some partial α-recursive function infinitely often.

Now we construct an α which is bounded by $f(n) = n + 1$ and each value $\alpha(n)$ will be regarded as a guess at the true number of changes in the corresponding column $\{\lambda(n, s)\}_{s\in\omega}$. If we assume that infinitely often we can code the true number of changes into α, then α computes a partial recursive function $h(n)$ which infinitely often equals the modulus function of K: for each n, we first read $\alpha(n)$, and then run through the column $\{\lambda(n, s)\}_{s\in\omega}$ looking for $\alpha(n)$ many changes. If we can find such a position we output the value of $\lambda(n, s)$ at that place, and if we cannot, the computation diverges.

If β can infinitely often exceed the values of $h(n)$ where it is equal to $m_K(n)$, then $\deg(\beta)$ is **ANR**. One thing to worry about here is how to make sure that those places where β exceeds h are the positions where h coincides with the modulus function of K. The main difficulty in the proof is to find a framework within which we can carry out this idea.

In our construction we place one more requirement on our tree systems:

6. For any τ on T and ρ a terminal leaf of length $3i + 1$ on $S(\tau)$, we have $\mathrm{lb}(\rho) = 1$ and $|\{s : s < |\tau|, \lambda(i, s) \neq \lambda(i, s + 1)\}| < \tau(i)$. This implies that, for any $\tau' \supset \tau$ on T such that ρ is no longer a leaf on $S(\tau')$, $|\{s : s < |\tau'|, \lambda(i, s) \neq \lambda(i, s + 1)\}| \geq \tau(i)$.

Note that Lemmas 4.1 and 4.3 are still true with Property 6 added.

Again note that α being nonrecursive and β being strictly above α are both automatic: β is **ANR** while α is not **ANR** by Theorem 1.4, so $\deg(\beta)$ is not below $\deg(\alpha)$. In addition, $\deg(\alpha)$ is strictly above **0** because otherwise $\deg(\beta)$ would be an **ANR** minimal degree, contradicting Theorem 1.4.

5.1 Requirements

First of all, we want α to be minimal.

R_e**:** either φ_e^α is not total, or it is recursive, or $\alpha \leq_T \varphi_e^\alpha$.

In addition, we require β to be minimal over α.

P_e**:** either φ_e^β is not total, or it is recursive in α, or $\beta \leq_T \varphi_e^\beta \oplus \alpha$.

We guarantee array nonrecursiveness by the following:

Q_n**:** there exists at least n distinct xs such that $\beta(3x + 2) > m_K(x)$.

Then β computes a function which is not dominated by m_K. Therefore, it is **ANR** by definition.

The definition of forcing here is similar to that in Section 3: we say (T, S) *forces* a requirement if for every $\alpha \in [T]$ and every $\beta \in [S(\alpha)]$, the requirement holds for α and β. Finally, $\alpha \leq_T \beta$ is guaranteed by the initial tree system defined below.

5.2 Initial tree system (T_0, S_0) We define an initial tree system (T_0, S_0) in this subsection. For simplicity we write (T, S) instead of (T_0, S_0).

T is rather simple. It is the restriction of the identity function to a recursive domain defined as follows: $\varnothing$ is in the domain; if σ is in the domain then $\sigma * 0, \sigma * 1, \ldots, \sigma * (|\sigma| + 1)$ are exactly the immediate successors of σ in the domain.

Given τ of length n in the range of T (and so in the domain of S), we define $S(\tau) = R$ as a finite tree as follows: The domain of R is a subset of $2^{2|\tau|+1}$. First let $R(\varnothing) = \varnothing$. Inductively, for any μ of length $\leq 2|\tau|$ and $R(\mu) = \rho$, if μ is even, then put $R(\mu * 0) = \rho * 0$ and $R(\mu * 1) = \rho * 1$.

If $|\mu| = 2k + 1$ and $\mathrm{lb}(\mu) = 0$, we put $R(\mu * 0) = \rho * \tau(k) * 0$ and $R(\mu * 1) = \rho * \tau(k) * 1$. If $\mathrm{lb}(\mu) = 1$, then we search for the $\tau(k)$th change in the column $\{\lambda(k, s)\}_{s \in \omega}$ up to $s = |\tau|$. If we cannot find that many changes, then both $R(\mu * 0)$ and $R(\mu * 1)$ diverge; if we can find such, then let x be the value of $\lambda(k, t)$ where t is the place we see the $\tau(k)$th change, and let $R(\mu * 0) = \rho * \tau(k) * (x + 1)$ and $R(\mu * 1) = \rho * \tau(k) * (x + 2)$.

It is easy to see that R is a finite tree and the maximum length of its leaves is $3|\tau| + 1$. For every $\alpha \in [T]$ and every $\beta \in [S(\alpha)]$, it is immediate that $\forall x(\alpha(x) = \beta(3x + 1))$; hence $\alpha \leq_T \beta$. Intuitively, along any path on the tree $S(\tau)$, position $3i + 1$ codes the (possible) number of changes, position $3i$ guesses whether it can be found (1) or not (0), and position $3i + 2$ codes a sufficiently large number, if position $3i$ guesses that such number of changes happens and it is actually found.

Infinitely often Property 1 can be used to find a full subtree of T (with the induced "subsystem" of S) to code the "correct number of changes along a column" into α. So in β we can code in a number greater than the corresponding value of the modulus function of K.

5.3 Forcing α to be minimal Suppose we are given (T, S) and an index e, and we need to find a subtree system (T', S') of (T, S) to force R_e. By Lemma 4.1 we only need to find such a subtree T' of T which satisfies Property 1 and take S' to be the corresponding restriction.

As usual we ask whether the following holds (for simplicity we always let D be the domain of T):

$$\exists \sigma \in D \forall \sigma_0, \sigma_1 \supset \sigma, \sigma_0, \sigma_1 \in D(\neg(T(\sigma_0)|_e T(\sigma_1))).$$

If the answer is yes, then we take $T' = FS(T, \sigma)$. It is routine to argue that φ_e^α is either not total or recursive for every $\alpha \in [T']$.

If the answer is no, then we do the following construction. Start with $T'(\varnothing) = T(\varnothing) = \tau$. Now, in order to satisfy Property 1, we need to find $|\tau| + 2$ pairwise e-splitting nodes on T, each extending one of $T(i) \supset \tau * i$ for $i = 0, 1, \ldots, |\tau| + 1$, respectively. In the inductive step we need a similar argument; that is, once we have defined $T'(\sigma) = \tau = T(\sigma')$, then we need to find $|\tau| + 2$ pairwise e-splitting nodes on T, each extending one of $T(\sigma' * i) \supset \tau * i$ for $i = 0, 1, \ldots, |\tau| + 1$, respectively. We start with these nodes $T(\sigma' * i)$. Each time we pick one pair of them, and using the splitting property we can extend these two strings on T to make them e-split. We can iterate this process for each pair. Finally, we end up with pairwise e-splitting extensions and let them be $T'(\sigma * i)$ for $i = 0, 1, \dots, |\tau| + 1$. This construction gives us an e-splitting subtree T' of T and it is easy to argue that $\alpha \leq_T \varphi_e^\alpha$.

5.4 Forcing β to be ANR Given (T, S) we want to code a sufficiently large number into β; that is, we need to find a new n such that $\beta(3n + 2) \geq m_K(n)$. By Property 5 we know that there exist $\tau = T(\sigma)$ and $\rho = S(\tau)(\mu)$ a top leaf of $S(\tau)$ with $\mathrm{lb}(\rho) = 1$. Then, by Property 1, we know that $\tau_i = T(\sigma * i) \supset \tau * i$, $i = 0, 1, \dots, |\tau| + 1$ are exactly the successor nodes of τ. Intuitively, they are guessing that the $|\tau|$th column of λ has i many changes, respectively, for each i.

Then we take i to be the actual number of changes through that column; that is, we let $i = |\{s : \lambda(|\tau|, s) \neq \lambda(|\tau|, s + 1)\}|$. That is to say, $T(\sigma * i) = \tau_i \supset \tau * i$ has the correct guess at the number of changes in this column. Also let t be the position in that column where we see the ith change; that is, t is the least such that $\lambda(|\tau|, t') = \lambda(|\tau|, t)$ for all $t' > t$.

Note that by Property 3, $\mu * 0$ is an even string. By the construction of the initial tree and Property 6, we know that for all $T(\sigma^*) = \tau^* \supset \tau_i$ with length $\geq t$, $S(\tau^*)(\mu * 0) \supset \rho * i * |m_K(|\tau|) + 1|$ or $\rho * i * |m_K(|\tau|) + 2|$. Pick any such σ^* and take $(T', S') = FSTS(T, S, \sigma^*, \mu * 0)$. By Lemma 4.3, this tree system is special and it is easy to see that it forces β to have one more position n where $\beta(3n + 2)$ is greater than $m_K(n)$.

5.5 Forcing β to be minimal over α In this subsection we will present the main splitting construction. Suppose we are given a special tree system (T, S) and an index e. We need to find a special subtree system (T', S') which forces P_e; that is, either φ_e^β is not total, or it is recursive in α, or $\beta \leq_T \varphi_e^\beta \oplus \alpha$.

Notice that in the requirement above, if we could force $\beta \leq_T \varphi_e^\beta$, then we would make β a strong minimal cover over α, but this cannot happen because no **ANR** degree can be a strong minimal cover (for example, by [3, Theorem 2.5]).

Now we ask the following key question (note that D is the domain of T and we let $E(\tau)$ be the set of all even strings in the domain of $S(\tau)$):

$$\exists\sigma \in D \exists\mu \in E(T(\sigma)) \forall\sigma' \supset \sigma, \sigma' \in D \forall\mu_0, \mu_1 \supset \mu, \mu_0, \mu_1 \in E(T(\sigma')) \\ [\neg(S(T(\sigma'))(\mu_0)|_e S(T(\sigma'))(\mu_1))].$$

That is, we ask whether there is a node $\tau = T(\sigma)$ and an even-node $\rho = S(\tau)(\mu)$ such that for any $\tau' = T(\sigma')$ extending τ on T, there is no e-splitting pair of even-nodes on $S(\tau')$ extending ρ.

If the answer is yes, then we pick a witness pair (σ, μ), take $(T', S') = FSTS(T, S, \sigma, \mu)$ and claim that if φ_e^β is total, then it is recursive in α. To compute $\varphi_e^\beta(x)$, we simply search on the tree $S'(\alpha)$ for an even-node η which makes $\varphi_e^\eta(x)$ converge. We must find an answer because $\beta \in [S(\alpha)]$ and φ_e^β is total. We cannot find an answer different from $\varphi_e^\beta(x)$ by the positive answer to our key question.

Now if the answer is no, then we know that

$$(\dagger):\ \forall\sigma \in D \forall\mu \in E(T(\sigma)) \exists\sigma' \supset \sigma, \sigma' \in D \exists\mu_0, \mu_1 \supset \mu, \mu_0, \mu_1 \in E(T(\sigma')) \\ [S(T(\sigma'))(\mu_0)|_e S(T(\sigma'))(\mu_1)].$$

We will use this property to construct a "splitting" subtree system (T', S'). Here by "splitting" we mean the following:

$(*)$: For all τ on T', all leaves of $S'(\tau)$ pairwise e-split.

First we argue that with property $(*)$ the requirement is satisfied. From α one can compute $S'(\alpha)$ and β is a path on $S'(\alpha)$. $(*)$ guarantees e-splitting so φ_e^β can pick a unique leaf which is an initial segment of β, on $S'(\tau)$ for every $\tau \subset \alpha$. This proves that $\beta \leq_T \varphi_e^\beta \oplus \alpha$.

To finish the proof we will provide such a splitting subtree system construction; that is, we construct a subtree system (T', S') which is special and has property $(*)$. To find $T'(\varnothing)$, first, by (†), we search above $\varnothing$ for a σ such that there exist two e-splitting even-nodes ρ_0 and ρ_1 on $S(T(\sigma))$. Then by Property 5 we know that on the tree $S(T(\sigma))$, ρ_0 and ρ_1 have extensions η_0 and η_1, respectively, such that $\mathrm{lb}(\eta_0) = 0$, $\mathrm{lb}(\eta_1) = 1$, and both are top leaves of $S(T(\sigma))$. Then define $T'(\varnothing) = T(\sigma)$ and $S'(T'(\varnothing))$ to be the subtree of $S(T(\sigma))$ defined by $S'(T'(\varnothing))(\varnothing) = S(T(\sigma))(\varnothing)$, $S'(T'(\varnothing))(0) = \eta_0$, and $S'(T'(\varnothing))(1) = \eta_1$.

Now suppose we have defined $T'(\sigma') = \tau = T(\sigma)$ and $S'(\tau)$ a subtree of $S(\tau)$. We need to define $T'(\sigma' * i) \supset T(\sigma * i)$ for each $i \in \{0, 1, \ldots, |\tau| + 1\}$ and define the corresponding $S'(T'(\sigma' * i))$ with pairwise splitting leaves. The good thing is that we only need e-splitting for leaves on each $S'(T'(\sigma' * i))$ separately but not for all leaves on every tree altogether. As we know the number i from α in the end, we only need to find e-splitting extensions separately for each i.

Suppose $\rho_0, \rho_1, \ldots, \rho_n$ are all leaves on $S'(\tau)$. Note that by induction, they pairwise e-split. Let $X = \{\rho_0, \rho_1, \ldots, \rho_k\}$ be the set of all top leaves and $Y = \{\rho_{k+1}, \rho_{k+2}, \ldots, \rho_n\}$ be the set of all terminal leaves on $S'(\tau)$. Some of the ρ_js in X are terminal leaves on $S(T(\sigma * i))$ and we can put them aside (into Y) at this time. Now let $Z = \{\rho_0, \rho_1 \ldots, \rho_l\}$ be the set of all leaves on $S'(\tau)$ that are not terminal leaves on $S(T(\sigma * i))$.

The aim of the construction is to find $\tau_i \supset T(\sigma * i) \supset \tau * i$ on T such that for each $\rho_j \in Z$, $\rho_j = S'(\tau)(\mu_j)$ (note that $|\mu_j|$ is always odd by Property 3), one can find on $S(\tau_i)$ six appropriate nodes defined to be $S'(\tau_i)(\mu_j * 0)$, $S'(\tau_i)(\mu_j * 1)$, $S'(\tau_i)(\mu_j * 00)$, $S'(\tau_i)(\mu_j * 01)$, $S'(\tau_i)(\mu_j * 10)$, and $S'(\tau_i)(\mu_j * 11)$. Moreover, the latter four are top leaves of $S(\tau_i)$ and they have the desired e-splitting property. For each $\rho_j \in Z$, $\rho_j = S'(\tau)(\mu_j) = S(T(\sigma))(\mu'_j)$ we know that $\rho'_j = S(T(\sigma * i))(\mu'_j * 0)$ is an even-node on $S(T(\sigma * i))$, hence also an even-node on $S(\pi)$ for any $\pi \supset T(\sigma * i)$.

Now start from ρ'_0. By (†) we know that there is a $\pi_0 \supset T(\sigma * i)$ and two even-nodes ξ_0^1, ξ_0^1 on $S(\pi_0)$ extending ρ'_0 that e-split. Using the same idea, we apply (†) twice to get $\pi_2 \supset \pi_1 \supset \pi_0$ such that both ξ_0^0 and ξ_0^1 have two even-node extensions ξ_0^{00}, ξ_0^{01} and ξ_0^{10}, ξ_0^{11}, respectively.

Now on $S(\pi_2)$, some nodes in Y will become nonterminal and we need to move them into Z. Pick the next node in Z, say ρ_1. ρ'_1 defined above is also an even-node in $S(\pi_2)$ and similarly by applying (†) three times one can get $\pi_5 \supset \pi_4 \supset \pi_3 \supset \pi_2$ and even-nodes ξ_1^0, ξ_1^1, ξ_1^{00}, ξ_1^{01}, and ξ_1^{10} and ξ_1^{11} extending ρ'_1 (to be explicit, the subscript 1 refers to ρ'_1 and superscripts 0, 1, 00, 01, 10, 11 refer to their positions in the relative 2-level tree structure extending ρ'_1).

Again some nodes in Y will appear nonterminal at this level and we need to move them into Z. There are only finitely many leaves on $S'(\tau)$ at the beginning. So we can iterate this process for all $\rho_j \in Z$ and finally end up with $\pi = \pi_{3t-1}$ for some t. On $S(\pi)$ each ρ'_j similarly has six extensions with appropriate e-splitting properties, and these nodes in Y are still terminal leaves. Now define $T'(\sigma * i) = \pi$, and define $R = S'(\pi)$ extending $S'(\tau)$ as follows: For each $\rho_j \in Z$, $\rho_j = S'(\tau)(\mu_j)$, define $S'(\pi)(\mu_j * 0) = \xi_j^0$ and $S'(\pi)(\mu_j * 1) = \xi_j^1$.

By Property 5 we can find a top leaf ζ_j^{mn} on $S(\pi)$ extending ξ_j^{mn} ($m, n \in \{0, 1\}$) such that $\mathrm{lb}(\zeta_j^{mn}) = n$. Then we define $S'(\pi)(\mu_j * m * n) = \zeta_j^{mn}$ for $m, n \in \{0, 1\}$. This finishes the splitting subtree system construction and it is not difficult to see that the subtree system (T', S') we constructed is a special tree system. □

6 Appendix

Here we prove a proposition mentioned in the Introduction.

Proposition 6.1 *Suppose* $\mathbf{a}$ *is* $\mathbf{GL_2}$ *and* $\mathbf{b}$ *is a minimal cover of* $\mathbf{a}$. *Then* $\mathbf{b}$ *is also* $\mathbf{GL_2}$ *if either of the following holds:*

1. $\mathbf{b} < \mathbf{a}'$, *or*
2. $\mathbf{b}$ *is hyperimmune-free relative to* $\mathbf{a}$*; that is, every function recursive in* $\mathbf{b}$ *is dominated by a function recursive in* $\mathbf{a}$.

Proof In the first case, by a relativized version of [5, Corollary 1], we know that if $\mathbf{b} < \mathbf{a}'$ then $\mathbf{b}$ is $\mathbf{L_2}$ relativized to $\mathbf{a}$. That means $\mathbf{b}'' = \mathbf{a}'' = (\mathbf{a} \vee \mathbf{0}')' \leq (\mathbf{b} \vee \mathbf{0}')'$. Therefore, $\mathbf{b}$ is $\mathbf{GL_2}$.

In the second case, it is well known [5, Lemma 1] that $\mathbf{a}$ being $\mathbf{GL_2}$ is equivalent to there being a function g recursive in $\mathbf{a} \vee \mathbf{0}'$ which dominates every function recursive in $\mathbf{a}$. Then this function g also dominates every function recursive in $\mathbf{b}$ and, of course, $g \leq_T \mathbf{a} \vee \mathbf{0}' \leq \mathbf{b} \vee \mathbf{0}'$. So $\mathbf{b}$ is also $\mathbf{GL_2}$. □

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