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Axiomatizing the Logic of Comparative Probability

John P. Burgess

Abstract Often where an axiomatization of an intensional logic using only finitely many axioms schemes and rules of the simplest kind is unknown, one has a choice between an axiomatization involving an infinite family of axiom schemes and one involving nonstandard "Gabbay-style" rules. The present note adds another example of this phenomenon, pertaining to the logic comparative probability ("p is no more likely than q"). Peter Gärdenfors has produced an axiomatization involving an infinite family of schemes, and here an alternative using a "Gabbay-style" rule is offered. Both axiomatizations depend on the Kraft-Pratt-Seidenberg theorem from measurement theory.

1 Choice Conjecture

In axiomatizing nonclassical extensions of classical sentential logic one tries to make do, if one can, with adding to classical sentential logic a finite number of axiom schemes of the simplest kind and a finite number of inference rules of the simplest kind. The simplest kind of axiom scheme in effect states of a particular formula P that for any substitution of formulas for atoms the result of its application to P is to count as an axiom. The simplest kind of one-premise inference rule in effect states of a particular pair of formulas P and Q that for any substitution of formulas for atoms, if the result of its application to P is a theorem, then the result of its application to Q is to count as a theorem; similarly for many-premise rules. Such are the schemes and rules of all the best-known modal and tense logics, for instance.

Sometimes it is difficult to find such simple schemes and rules (though it is usually even more difficult to prove that none exist). In that case one may resort to less simple schemes or less simple rules. There is no generally recognized rigorous definition of "next simplest kind" of scheme. (One fact that makes a rigorous definition difficult is that, if the logic in question is axiomatizable at all, which is to say, if the set of formulas wanted as theorems is recursively enumerable, then by Craig's trick one

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can always get a *primitive recursive* set of schemes of the simplest kind, even if one cannot get a *finite* set. Intuitively, some primitive recursive sets are much simpler than others, but it is difficult to reduce this intuition to a rigorous definition.)

Neither is there any generally recognized definition of "next simplest kind" of rule, and hence there is no fully rigorous enunciation of the *choice conjecture*, the conjecture that schemes of the next simplest kind can always be avoided in favor of rules of the next simplest kind and vice versa. Nonetheless, there are cases where intuitively one does recognize that the schemes or rules in a given axiomatization are only slightly more complex than the simplest kind, including cases where one does have a choice between adopting slightly-more-complex-than-simplest schemes and adopting slightly-more-complex-than-simplest rules.

In tense logic early examples of slightly more complex rules are found in [2] and [3]: there is one example of the embarrassed use of such rules in the former, and many examples of the enthusiastic use of such rules in the latter and its sequels. Accordingly the rules in question have come to be called "Gabbay-style" rules. A typical such rule reads "from $A(p) \rightarrow B$ to infer B, provided the atom p does not occur in B," and intuitively adopting such a rule serves many of the same purposes as would be served by introducing sentential quantification and adopting as an axiom $\exists p A(p)$. Alternatives to the use of such rules in [2], involving instead slightly more complex schemes, were subsequently found to be possible; see [6].

The aim of the present note (a kind of very belated sequel to [1]) is to encourage work toward a rigorous statement and proof or disproof of the choice conjecture by making available a case study, another example (known to the author since the middle 1970s, but not previously published) where one has a choice between slightly more complex than simplest schemes and "Gabbay-style" rules. The nonclassical operator whose logic is to be considered here is that of comparative probability, "p is more probable than q," as in [4]. There an axiomatization involving a scheme of slightly-less-than-simplest kind is proved complete using the results of [5], an early classic of measurement theory. Here an alternative involving a "Gabbay-style" rule, also adapting to the context of nonclassical sentential logics ideas from that classic paper, will be offered.

2 Formal System

The formal language will have the symbols of classical sentential logic, including the constant false $\perp$, and one more two-place connective $\leq$. The operator $\leq$ is understood to bind more tightly than truth functional operators, so that, for instance, $\neg A \leq B$ means $\neg(A \leq B)$ rather than $(\neg A) \leq B$. Formulas not involving the new connective, which is to say, truth-functional compounds of atoms $p, q, r, \ldots$, will be called *classical* formulas; formulas of type $A \leq B$ with A and B classical will be called *comparisons*, and truth-functional compounds of comparisons will be called *probabilistic* formulas. To keep things simple, these last will be the only formulas admitted as axioms and theorems.

As axioms we add to classical logic (that is to say, to the set of all probabilistic formulas that are substitution instances of classical tautologies) all probabilistic formulas of any of the following forms:

(A1) $A \leq B$ where $A \rightarrow B$ is a tautology;
(A2) $A \leq B \wedge B \leq C \rightarrow A \leq C$;

(A3) $A \leq B \vee B \leq A$;
(A4) $A \leq B \rightarrow (A \vee C) \leq (B \vee C)$
where $\neg(A \wedge C)$ and $\neg(B \wedge C)$ are tautologies;
(A5) $(A \vee C) \leq (B \vee C) \rightarrow A \leq B$
where $\neg(A \wedge C)$ and $\neg(B \wedge C)$ are tautologies;
(A6) $\neg((\neg\bot) \leq \bot)$.

Here (A1) is not a scheme of the simplest kind as "simplest kind" was defined above, but it could be replaced by finitely many such schemes in known ways. The replacements would be

$$(\neg(A \rightarrow B)) \leq \bot \rightarrow A \leq B$$
$$(\neg(A \rightarrow B)) \leq \bot \rightarrow ((\neg A) \leq \bot \rightarrow (\neg B) \leq \bot)$$

and finitely many others, each of the form $(\neg C) \leq \bot$ where as the finitely many C involved can be taken the axioms of any of the usual axiomatizations of classical sentential logic. Again (A5) and (A6) are not literally of the simplest kind, but given (A1) or axioms yielding it, (A5) and (A6) can be replaced by the following:

$$(B \wedge C) \leq \bot \wedge A \leq B \rightarrow (A \vee C) \leq (B \vee C)$$
$$(A \wedge C) \leq \bot \wedge (A \vee C) \leq (B \vee C) \rightarrow A \leq B.$$

As rules add to modus ponens (which, given that we are taking all classical tautologies as axioms, will permit inference from any premises to any conclusion that is a classical tautological consequence thereof) the following rule of *polarizability*:

(R1) From $(p \wedge A) \leq (\neg p \wedge A) \wedge (\neg p \wedge A) \leq (p \wedge A) \rightarrow B$
to infer B, provided p does not occur in A or B.

This is not literally a "Gabbay-style" rule in the sense defined above, because of the mention of the absence of p from A as well as from B, but in our context it is equivalent to the following:

From $(p \wedge A) \leq (\neg p \wedge A) \wedge (\neg p \wedge A) \leq (p \wedge A) \rightarrow (A \leq A \rightarrow B)$
to infer $A \leq A \rightarrow B$, provide p does not occur in $A \leq A \rightarrow B$.

As usual a *proof* is a sequence of probabilistic formulas each of which is either an axiom or follows from earlier items in the sequence by application of one of the rules, a *theorem* is any formula appearing in a proof, and a probabilistic formula (respectively, set of such formulas) is *consistent* if its negation (respectively, the negation of the conjunction of its elements) is not a theorem.

If classical formulas A and B are tautologically equivalent, then so are any pair of $C(A)$ and $C(B)$ of classical formulas that are alike except for the former having occurrences of A where the latter has occurrences of B. If $D(A)$ and $D(B)$ is another such pair, then $C(A) \leq D(A)$ and $C(B) \leq D(B)$ are provably equivalent using axioms (A1) and (A2). It follows that if $P(A)$ and $P(B)$ are probabilistic formulas that are alike except for the former having occurrences of A where the latter has occurrences of B, then they are provably equivalent also, a result to be called the *replacement lemma*.

3 Probabilistic Models

A *model* for our logic is a triple (U, π, V) consisting of a Boolean algebra U, a probability measure π on it, and an assignment V of elements of the algebra to atoms.

The valuation extends in the usual way to an assignment, by abuse of language also denoted V, of elements of the algebra to truth-functional compounds of atoms. A formula $A \leq B$ is true in the model if and only if $\pi(V(A)) \leq \pi(V(B))$, and the assignment of truth values extends in the usual way to truth-functional compounds of such formulas.

A formula is *valid* if and only if it is true in all models and a formula (respectively, set of formulas) is *satisfiable* if and only if it is true (respectively, all its elements are true) in some model. The proof of the soundness theorem, stating that all theorems are valid, is tedious but routine, and left to the reader. What remains to be proved is the completeness theorem, stating that all valid formulas are theorems, or equivalently, that any consistent formula (equivalently, any consistent finite set of formulas) is satisfiable.

A (classical or probabilistic) formula involving only the alphabetically first n atoms $p_1, \ldots, p_n$ will be called an n-formula. For purposes of proving completeness we will need to distinguish several special classes of n-formulas and sets thereof, as follows:

Class $\boldsymbol{\alpha(n)}$: Conjunctions of form $(\neg)p_1 \wedge \cdots \wedge (\neg)p_n$. Call the elements of $\alpha(n)$ in alphabetically order $X_1, \ldots, X_N$, where $N = 2^n$.

Class $\boldsymbol{\beta(n)}$: Disjunctions of elements of $\alpha(n)$, with the disjuncts in alphabetically order and no disjunct repeated. The elements of $\beta(n)$ are to be understood as including the "empty disjunction" $\perp$. Every classical n-formula is tautologically equivalent to an element of $\beta(n)$, its *disjunctive normal form.*

Class $\boldsymbol{\gamma(n)}$: Comparisons $A \leq B$ of elements A and B of $\beta(n)$. By the replacement lemma, every comparison that is an n-formula is provably equivalent to an element of $\gamma(n)$.

Class $\boldsymbol{\delta(n)}$: Negations $\neg(A \leq B)$ of elements of $\gamma(n)$.

Class $\boldsymbol{\varepsilon(n)}$: Conjunctions of elements of $\gamma(n)$ and $\delta(n)$.

Class $\boldsymbol{\zeta(n)}$: Disjunctions of elements of $\varepsilon(n)$. Every probabilistic n-formula is equivalent to an element of $\zeta(n)$. Moreover, such a disjunction will be consistent (respectively, satisfiable) if and only if one of its disjuncts is consistent (respectively, satisfiable). Hence to prove completeness it suffices to show (for every n) that every consistent element of $\varepsilon(n)$ is satisfiable.

Class $\boldsymbol{\xi(n)}$: Consistent sets of elements of $\gamma(n)$ and $\delta(n)$. Since a finite set is satisfiable if and only if the conjunction of its elements is, to prove completeness it suffices to show than every element of $\xi(n)$ is satisfiable.

Class $\boldsymbol{\eta(n)}$: Maximal elements of $\xi(n)$. Elements of $\eta(n)$ will contain all formulas of classes $\gamma(n)$ and $\delta(n)$ that are theorems of our logic. These will include the formulas $\perp \leq X_i$ for all i, these being theorems by (A1), as well as $\neg X_1 \vee \cdots \vee X_N \leq \perp$, which is a theorem by (A6) and the replacement lemma. Since any element of $\xi(n)$ can be extended to an element of $\eta(n)$ by the usual argument (*Lindenbaum's lemma*), to prove completeness it suffices to show that any element of $\eta(n)$ is satisfiable.

4 Inequality Systems

Any element of any of the above-named classes has an *algebraic translation*, obtained by replacing each X_i by an indeterminate x_i and $\perp$ by 0 and $\vee$ by $+$. The algebraic translation H^* of an element H of $\eta(n)$ will be a system of inequalities (and negated inequalities) containing $0 \leq x_i$ for all i, as well as $\neg x_1 + \cdots + x_N \leq 0$. The reader can verify that if such an H is satisfiable, then H^* has a solution $x_1 = a_1, \ldots, x_N = a_N$ in real numbers. (Given a model (U, π, V) in which every element of H is true, taking $x_i = \pi(V(X_i))$ gives a solution of H^*.)

The converse also holds. Given such a solution the a_i are each nonnegative and their sum a is positive. We can then obtain a model in which every element of H is true by considering: (i) the Boolean algebra U whose elements are equivalence classes $[A]$ of classical n-formulas under tautological equivalence, with the complementation and meet and join operations corresponding to negation and conjunction and disjunction; (ii) the probability measure π that for any element A of $\beta(n)$ assigns as probability $\pi([A])$ to $[A]$ the sum of the a_i for which X_i is a disjunct of A, divided by the sum a of all the a_i; and (iii) the valuation V assigning each atom p_i its equivalence class $[p_i]$ if $i \leq N$ (and the zero element of the Boolean algebra, which is to say, the equivalence class $[\perp]$, if $i > N$).

These facts and Tarski's theorem on the decidability of elementary algebra together imply the decidability of the class of satisfiable probabilistic formulas (a result already, by a different proof, implicit in [5] and explicit in [6]).

Since no disjunct is repeated in any element $X_i \vee X_j \vee \ldots$ of $\beta(n)$, no indeterminate is repeated in its algebraic translation $x_i + x_j + \ldots$, and this suggests an alternative *set-theoretic translation* $\{x_i, x_j, \ldots\}$, which can be extended from elements $\beta(n)$ to elements of the other classes. In particular, any element H of $\eta(n)$ will have a set-theoretic translation $H\dagger$ describing an ordering of the power set of $I = \{x_1, \ldots, x_N\}$.

This order satisfies the *De Finetti conditions*, which is to say that for all subsets of I we have the following:

(D1) *Transitivity* If $u \leq v$ and $v \leq w$, then $u \leq w$;

(D2) *Comparability* Either $u \leq v$ or $v \leq u$;

(D3) *Additivity* If w is disjoint from u and from v, then $u \leq v$ if and only if $u \cup w \leq v \cup w$;

(D4) *Nontriviality* Not $I \leq \varnothing$.

Our axioms were chosen to insure this result: (D1) follows from axiom (A2), (D2) from axiom (A3) and the maximality of H, and (D3) from axioms (A4) and (A5), and (D4) from axiom (A6).

Any assignment of nonnegative real numbers a_i, not all zero, as to the x_i as "weights" induces an order satisfying the De Finetti conditions, namely, the one in which sets are ordered by the sums of the weights of their elements. Another way of putting the criterion for satisfiability of H stated above is just this, that the order determined by $H\dagger$ must agree with some order induced by an assignment of weights.

Unfortunately, the main counterexample in [5] shows that *not* every order satisfying the De Finetti conditions satisfies the additional condition of agreeing with the order induced by some assignment of weights. For this additional condition to be met, the following is proved in the paper cited to be necessary and sufficient:

(KPS) For any M and any $u_1, \ldots, u_M$ and $v_1, \ldots, v_M$, if each x_i belongs to exactly as many of the u_j as of the v_j and if $u_j \leq v_j$ for $j = 2, \ldots, K$, then $v_1 \leq u_1$.

The approach in [4] proceeds by directly translating the KPS condition into a primitive recursive set of axioms schemes. To finish the proof of completeness for the alternative axiomatization of the present paper, we must show using the polarizability rule that the KPS condition is met by any order given by the translation $H\dagger$ of any maximal consistent set H. Let us call such an order a *translated* order, for short.

5 Applying Polarizability

The De Finetti conditions do guarantee that the KPS condition holds in the special case where each x_i belongs to *at most one* of the u_j. What will be shown here is that in the case of a *translated* order, the KPS condition holds also when each x_i belongs to *at most two* of the u_j; the method of proof then easily generalizes to cover the cases of at most four, at most eight, and so on, as needed to establish completeness.

The proof will, in fact, be given only in a sketch, since it is really no more than an adaptation to the present context of an argument at the end of [5]. Rather than consider any possible violation of the KPS conditions, let us consider just the counterexample from that paper. In that example there is an order meeting the De Finetti conditions but violating the KPS condition because we have

$$\begin{gathered}\{x_3\} \le \{x_1, x_2\}\\ \{x_2, x_4\} \le \{x_1, x_3\}\\ \{x_1, x_5\} \le \{x_2, x_3\}\\ \neg(\{x_4, x_5\} \le \{x_1, x_2, x_3\}).\end{gathered}$$

What will be shown is that such a situation cannot arise in a *translated* order.

The key observation is the following. Let us abbreviate the formula

$$(Y \wedge Z) \le (\neg Y \wedge Z) \wedge (\neg Y \wedge Z) \le (Y \wedge Z)$$

as $Y|Z$, read "Y polarizes Z." Then if we add to a finite consistent set any formula $p|A$ where p does not occur in A or in the set, then the result remains consistent. (Just apply the polarizability rule to $p|A \rightarrow B$, where B is the negation of the conjunction of the elements of the set.)

Hence if H is in $\eta(n)$ and the q_i are new atoms, we may successively add $q_1|X_1, \ldots, q_N|X_N$ and preserve consistency. Then letting Y be the disjunction of all $q_i \wedge X_i$, for each i the formulas $Y \wedge X_i$ and $q_i \wedge X_i$ will be tautologically equivalent, and likewise $\neg Y \wedge X_i$ and $\neg q_i \wedge X_i$, and the replacement lemma will give us $Y|X_i$ for each i. If a contradiction could be deduced from the conjunction of the formulas in H and the formulas $p_{N+1}|X_i$, then by substituting Y for p_{N+1} in the deduction, a contradiction could be derived from the conjunction of the formulas in H and the formulas $Y|X_i$, which is impossible. Therefore, the result of adding the formulas $p_{N+1}|X_i$ to H remains consistent and can be extended to a maximal consistent set H^* in $\eta(n+1)$.

Each element of $\alpha(n+1)$ will be of one of the two forms

$$\begin{gathered}X_i' = X_i \wedge p_{N+1} \quad \text{or}\\ X_i'' = X_i \wedge \neg p_{N+1},\end{gathered}$$

and in translating H^* each indeterminate x_i from H will correspond to a pair of indeterminates x_i' and x_i''. The order displayed above would become the following:

$$\begin{gathered}
\{x_3', x_3''\} \leq \{x_1', x_1'', x_2', x_2''\} \\
\{x_2', x_2'', x_4', x_4''\} \leq \{x_1', x_1'', x_3', x_3''\} \\
\{x_1', x_1'', x_5', x_5''\} \leq \{x_2', x_2'', x_3', x_3''\} \\
\neg(\{x_4', x_4'', x_5', x_5''\} \leq \{x_1', x_1'', x_2', x_2'', x_3', x_3''\})
\end{gathered}$$

while we will also have $\{x_i'\} \leq \{x_i''\}$ and $\{x_i''\} \leq \{x_i'\}$ for all i. But the De Finnetti conditions then imply

$$\{x_3', x_3''\} \leq \{x_1', x_1'', x_2', x_2''\} \text{ if and only if } \{x_3^{***}\} \leq \{x_1^*, x_2^{**}\} \tag{1}$$

where each of $*$ and $**$ and $***$ may be either of $'$ or $''$, as well as the analogues for the other three inequalities. To see this it suffices to consider the case where all three of $*$ and $**$ and $***$ are $'$, which is to say, the case

$$\{x_3', x_3''\} \leq \{x_1', x_1'', x_2', x_2''\} \text{ if and only if } \{x_3'\} \leq \{x_1', x_2'\}. \tag{2}$$

To give the argument for the implication from right to left, leaving the other direction to the reader, we have $\{x_3''\} \leq \{x_3'\}$, and the right-hand side of (2) is $\{x_3'\} \leq \{x_1', x_2'\}$. Since we have $\{x'\} \leq \{x_1''\}$, additivity gives $\{x_1', x_2'\} \leq \{x_1'', x_2'\}$, and similarly we get $\{x_1'', x_2'\} \leq \{x_1'', x_2''\}$. So by transitivity we get $\{x_3''\} \leq \{x_1'', x_2''\}$. Again by additivity we get $\{x_3', x_3''\} \leq \{x_1', x_2', x_3''\}$ as well as $\{x_1', x_2', x_3''\} \leq \{x_1', x_1'', x_2', x_2''\}$. Transitivity then gives the left-hand side of (2).

Application of (1) and its analogues would then give

$$\begin{gathered}
\{x_3'\} \leq \{x_1', x_2'\} \\
\{x_2', x_4'\} \leq \{x_1'', x_3'\} \\
\{x_1', x_5'\} \leq \{x_2'', x_3''\} \\
\neg(\{x_4'', x_5''\} \leq \{x_1'', x_2'', x_3''\}).
\end{gathered}$$

But this would be a failure of the KPS conditions in a case where no indeterminate appears more than *once* among the pertinent u_i, which as has been said is impossible by the De Finetti conditions.

This concludes the sketch of the proof. In comparing the case of a choice between more complex kinds of axiom schemes and a "Gabbay-style" rule presented by [6] *versus* [2] with the case of the similar choice between [3] and the foregoing, no obvious general pattern suggests itself. We thus have no very strong evidence for the choice conjecture, supposing that conjecture even admits of rigorous formulation. But as was stated at the outset, the aim here has been merely to encourage further work.

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Department of Philosophy
Princeton University
Princeton NJ 08544-1006
USA
jburgess@princeton.edu