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A Reverse Analysis of the Sylvester-Gallai Theorem

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Abstract Reverse analyses of three proofs of the Sylvester-Gallai theorem lead to three different and incompatible axiom systems. In particular, we show that proofs respecting the purity of the method, using only notions considered to be part of the statement of the theorem to be proved, are not always the simplest, as they may require axioms which proofs using extraneous predicates do not rely upon.

1 Introduction

Sylvester [30] posed in 1893 a question, which resurfaced forty years later as a conjecture by Erdős, to be first proved by Gallai. A comprehensive survey of the proofs for what is now known as the Sylvester-Gallai (SG) theorem can be found in [2]. The theorem can be stated as follows.

> If the points of a finite set S are not all on one line, then there is a line through exactly two of the points.

An enterprise going back to at least Pappus of Alexandria (see [25] for its history), which will be referred to as *reverse analysis*, asks for the axioms needed to prove a given theorem. It has been formulated for modern axiomatics by Hilbert [10]:

> Unter der axiomatischen Erforschung einer mathematischen
> Wahrheit verstehe ich eine Untersuchung, welche nicht dahin zielt, im Zusammenhange mit jener Wahrheit neue oder allgemeinere Sätze zu entdecken, sondern die vielmehr die Stellung jenes Satzes innerhalb des Systems der bekannten Wahrheiten und ihren logischen Zusammenhang in der Weise klarzulegen sucht, daß sich sicher angeben läßt, welche Voraussetzungen zur Begründung jener Wahrheit notwendig und hinreichend sind.

The same concern for the means by which one proves a theorem leads Hilbert [11] to a different problem, namely, that of proving a given statement only with means

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called for by the statement of the problem, which will be referred to as a concern for the *purity of the method*, in his own words:

> In der modernen Mathematik (wird) solche Kritik sehr häufig geübt, woher das Bestreben ist, die Reinheit der Methode zu wahren, d. h. beim Beweise eines Satzes womöglich nur solche Hülfsmittel zu benutzen, die durch den Inhalt des Satzes nahe gelegt sind.

This can be made precise by asking that the proof should proceed inside an axiom system in the same language in which the theorem is stated.

In the case of the SG theorem, the concern for the purity of the method (on which more can found in [1], [8], [9]) was voiced early by Coxeter [5], who repeats it in [7, 12.3]. He deems a proof (a variant of Steinberg's [29] proof) which uses only order axioms to be preferable to one due to Kelly (and published in [5]), which uses metric notions such as perpendicularity and comparison of lengths of segments. Coxeter's [7, p. 181] reaction to this use of "the concept of distance" is, "it is like using a sledge hammer to crack an almond." Arana [1] disagrees with this statement. Coxeter thinks that to understand the concept of a line one must understand the concept of betweenness. However, argues Arana, one may think, in the manner of differential geometry (or, for that matter, in the manner of any geometry in the spirit of Busemann or Alexandrov) that to understand the concept of a line one needs to understand not betweenness, but the notion of distance, as a line may be defined as the shortest path between two points.

Meanwhile, there is a fundamentally different proof of the SG theorem, by Chen [3], which uses assumptions different from both those needed for Kelly's proof and from those needed for the Steinberg-Coxeter proof, a proof validating the distance-geometric understanding of the concept of line.

The aim of this paper is to specify the axiom systems needed for each of the three proofs of the SG theorem, to justify the choice of axioms as natural statements in their own right, independent of the SG theorem, and to show that these axiom systems are incomparable, that is, that each axiom system admits a model which is not a model of any of the two other axiom systems. This shows once more (another similar outcome can be found in the reverse analysis performed in [21]) that the two concerns, for *minimal* and for *pure* axiom systems, lead to different, incompatible results.

To make a statement of first-order logic out of the SG theorem, we have to specify (an upper bound to) the number of points the set S may contain. The language can be chosen to be one-sorted, with variables to be interpreted as *points*, with one ternary predicate L, with $L(abc)$ to be read 'the points a, b, and c are collinear (but not necessarily different)'. If $a \neq b$, a point x with $L(abx)$ is also said to lie on line ab, and we can speak of the point of intersection x of lines ab and cd, whenever $a \neq b$, $c \neq d$, and x is the unique point for which $L(abx) \wedge L(cdx)$. Any axiom system should imply the following basic facts about L, essential for the notion of collinearity (we omit universal quantifiers for all universal sentences):

L 1 $L(aba)$,

L 2 $L(abc) \rightarrow L(acb) \wedge L(bca)$,

L 3 $a \neq b \wedge L(abc) \wedge L(abd) \rightarrow L(acd)$.

The SG theorem for an n-point set S, to be denoted by SG(n), is the statement (it is obvious, given the symmetry in the variables $a_1, \ldots, a_n$ of the antecedent, that the succedent can be sharpened to $\bigwedge_{1\leq i<j<k\leq n} L(p_i p_j p_k)$):

$$\left(\bigwedge_{1\leq i<j\leq n} p_i \neq p_j \wedge \Big(\bigvee_{h\notin\{i,j\}} L(p_i p_j p_h)\Big)\right) \to L(p_1 p_2 p_3). \qquad (1)$$

The statement (1) is in fact the contrapositive of SG for an n-point set S, stating that, if, for any two different points p_i and p_j of S, there is a p_h in S, different from p_i and p_j and collinear with them, then the points p_1, p_2, p_3 (and thus all the p_i) must be collinear.

Notice that SG(n) can be derived from L1–L3 for all $n \leq 6$, so the interesting cases are those with $n \geq 7$. That the case $n = 6$ can be derived from L1–L3 can be seen as follows: If there are three noncollinear points p_1, p_2, and p_3 in S, then there must be an additional point q_{ij} on each line $p_i p_j$ with $1 \leq i < j \leq 3$, and q_{12}, q_{13}, q_{23} must be distinct points, else, by L2 and L3, the points p_1, p_2, and p_3 would have to be collinear. Thus $\{p_1, p_2, p_3, q_{12}, q_{13}, q_{23}\}$ is a set with 6 elements, and thus must coincide with S. There is a line formed by two of its points, p_1 and q_{23}, which contains no other point in S.

2 The Steinberg-Coxeter Proof

For the Steinberg-Coxeter proof of (1), we will understand L as being defined by the definition

$$L(abc) :\Leftrightarrow Z(abc) \vee Z(bca) \vee Z(cab) \vee a = b \vee b = c \vee c = a, \qquad (2)$$

where Z stands for the notion of strict betweenness, with $Z(abc)$ to be read as 'b lies between a and c (and is different from both a and c)'. We will also use the abbreviation λ, with $\lambda(abc) :\Leftrightarrow Z(abc) \vee Z(bca) \vee Z(cab)$, standing for '$a, b, c$ are three distinct collinear points'. We need the following axioms:

Z 1 $Z(abc) \to a \neq c$,

Z 2 $Z(abc) \to Z(cba)$,

Z 3 $Z(abc) \to \neg Z(acb)$,

Z 4 $Z(abc) \wedge Z(acd) \to Z(abd)$,

Z 5 $Z(abc) \wedge Z(adc) \wedge b \neq d \to (Z(abd) \vee Z(adb))$,

Z 6 $Z(abc) \wedge Z(abd) \wedge c \neq d \to (Z(bcd) \vee Z(bdc))$,

Z 7 $Z(abc) \wedge Z(dab) \to \lambda(dac)$,

Z 8 $(\forall a_1 \ldots a_{n-1})\, [a_1 \neq a_2 \wedge \bigwedge_{k=3}^{n-1} L(a_1 a_2 a_k) \to (\exists b)\, \lambda(a_1 a_2 b) \wedge \bigwedge_{i=1}^{n-1} b \neq a_i]$,

Z 9 $(\forall abcde)\, [\neg L(abc) \wedge Z(abd) \wedge Z(aec) \to (\exists f)\, Z(bfc) \wedge Z(dfe)]$,

Z 10 $(\forall abcde)\, [\neg L(abc) \wedge Z(abd) \wedge Z(bec) \to (\exists f)\, Z(afc) \wedge \lambda(def)]$.

Z1, stating that the open interval between a point and itself is empty, is a weaker form of Postulate D of [12]; Z2 and Z3 are Postulate A and C of [12]; Z4 is a variant of Postulate 3 of [12]; Z5 is Postulate 4 of [12], Z6 a variant of Postulate 7 of [12], Z7 a weak form of Postulate 1 (which asks, under the same hypothesis, that not just $\lambda(dac)$, but that $Z(dac)$ should hold[1]) of [12]. Z8 states that, if $a_1, \ldots, a_{n-1}$ are points on a line, with $a_1 \neq a_2$, then there is a point b on the line determined by a_1 and a_2 that is different from all the a_i.

Z10 and Z9 are forms of the Pasch axiom, the former a weak variant of the outer form, the latter the inner form of the Pasch axiom. With $Z(def)$ instead of $\lambda(def)$ in the succedent, Z10 was introduced as axiom XIII, and Z9 as axiom XIV by Peano [24]. In its current form, Z10 was introduced by Veblen [32], who also proved that Z9 follows from Z10, Z2–Z7, and an axiom stating that $(\forall ab)\,[a \neq b \rightarrow (\exists c)\, Z(abc)]$. Given that we do not assume this axiom, and Z8 is too weak a substitute, we have to assume both Z9 and Z10 for the proof to go through. Z10 states that *secant* de (as line) must intersect the *side* ac of $\triangle abc$, and Z9 that the *segment* de must intersect the side bc of $\triangle abc$. Note that one can prove inside our axiom system that the conclusion $\lambda(def)$ in Z10 can be strengthened to $Z(def)$, as shown in [32, Theorem 7, p. 355].

Theorem 2.1 Z1–Z10 $\vdash$ SG(n)*, with* L *defined by* (2).

Proof We repeat the proof from [7, 12.3] inside our axiom system, to emphasize where and why we need all the axioms. Note that, by Z1–Z3, $Z(abc) \rightarrow a \neq b \wedge b \neq c \wedge c \neq a$, a fact we will use throughout without further reference. We will also leave unmentioned the many uses of Z2. Let $p_1, \ldots, p_n$ be such that $\neg L(p_1 p_2 p_3)$ and such that the antecedent of (1) holds. The lines $p_1 p_i$, with $2 \leq i \leq n$ may intersect the line $p_2 p_3$ in at most $n-1$ points r_i. According to Z8, there is a point q with $L(p_2 p_3 q)$, with $q \neq r_i$ for all $2 \leq i \leq n$. The lines $p_j p_k$ with $j \neq k$ meet the line $p_1 q$ in at most $l = (n-1)(n-2)/2 + 1$ points q_j (including p_1 and q). We claim that there exists a point a on the line $p_1 q$ such that

$$\neg Z(p_1 q_i a) \text{ holds for all } q_i. \tag{3}$$

To see this, we first ask whether $\neg Z(p_1 q_i q)$ holds for all q_i. If yes, then we let $a = q$ and are done. If it does not hold, then let i_1 be the first index i for which $Z(p_1 q_i q)$ holds. We now ask whether $\neg Z(p_1 q_i q_{i_1})$ holds for all q_i with $i > i_1$. If yes, we let $a = i_1$ and are done, since we must also have $\neg Z(p_1 q_i q_{i_1})$ for $i < i_1$, given that we know that $\neg Z(p_1 q_i q)$ for $i < i_1$, and that, if we had $Z(p_1 q_i q_{i_1})$, we'd also have $Z(p_1 q_i q)$ (by Z4), which would contradict our definition of i_1). If it does not hold, then we let i_2 be the least $i > i_1$ for which $Z(p_1 q_i q_{i_1})$ holds. This process must stop after a finite number (at most l many) of steps, and in the end we have an i_k such that $\neg Z(p_1 q_i q_{i_k})$ holds for all i, and we let $a = i_k$ and are done. This point a must lie, by the fact that it is a q_i and the definition of the q_i, on a line $p_j p_k$ with $j \neq k$. That line must contain a p_h with $h \notin \{j, k\}$. We know, by our earlier analysis, that there exists $x \in \{p_j, p_k, p_h\}$ such that $\neg Z(ayx)$ for all $y \in \{p_j, p_k, p_h\} \setminus \{x\}$. Without loss of generality, we may assume $x = p_j$. Given that a, p_j, p_k, p_h are all different, we must have $Z(a p_j p_k) \vee Z(p_k a p_j)$ and $Z(a p_j p_h) \vee Z(p_h a p_j)$.

Suppose $Z(a p_j p_k) \wedge Z(a p_j p_h)$. We know, by Z6 and Z7, that we must have one of (i) $Z(a p_k p_h)$ or (ii) $Z(a p_h p_k)$ or (iii) $Z(p_h a p_k)$. However, (iii) cannot hold, for, if it did, then, since $Z(p_k p_j a)$ and $Z(p_k a p_h)$, we would have $Z(p_k p_j p_h)$ (by Z4), and, since $Z(p_k p_j a)$ and $Z(p_k p_j p_h)$, we must have $Z(p_j a p_h)$ or $Z(p_j p_h a)$ (by Z6), both of which contradict $Z(a p_j p_h)$ (by Z3). Suppose (i) holds. On line $p_1 p_k$ there must be a p_m with $m \notin \{1, k\}$. If $Z(p_1 p_m p_k)$, then by Z10, secant $p_h p_m$ must intersect the side $a p_1$ of $\triangle a p_1 p_k$ in some q_s, contradicting (3). If $Z(p_m p_1 p_k)$ (or $Z(p_1 p_k p_m)$), then, by Z9 (or Z10), segment (or secant) $p_m p_j$ must intersect the side ap_1 of $\triangle ap_1p_k$ in some q_s, contradicting (3). If (ii) holds, we follow the same reasoning as for (i) with k and h interchanged.

Suppose $Z(ap_jp_k) \wedge Z(p_hap_j)$. We must have $Z(p_hap_k)$, as we cannot have $Z(ap_hp_k)$ (given that, together with $Z(ap_jp_k)$, $\neg Z(ap_hp_j)$, Z5 would imply $Z(ap_jp_h)$, which, given $Z(p_hap_j)$, would contradict Z3) or $Z(ap_kp_h)$ (given that, with $Z(ap_jp_k)$ we would get, by Z4, $Z(ap_jp_h)$, which in turn, with $Z(p_hap_j)$ would contradict Z3). On line p_1p_k there must be a third p_m. If $Z(p_1p_mp_k)$, then Z9 provides a point q_s of intersection of the secant p_hp_m with the side ap_1 of $\triangle ap_1p_k$, contradicting (3). If $Z(p_1p_kp_m)$ (or $Z(p_kp_1p_m)$), then Z10 (or Z9) provides a point q_s of intersection of the secant p_mp_j with the side ap_1 of $\triangle ap_1p_k$, contradicting (3).

If $Z(ap_jp_h) \wedge Z(p_kap_j)$, then we proceed as above, with h and k interchanged throughout. If $Z(p_kap_j) \wedge Z(p_hap_j)$, then, by Z6, $Z(ap_hp_k) \vee Z(ap_kp_h)$. Suppose $Z(ap_hp_k)$ (the case $Z(ap_kp_h)$ is dealt with by interchanging h and k throughout). On line p_1p_k there must be a third point p_m. If $Z(p_1p_mp_k)$, then the secant p_jp_m intersects, by Z9, side p_1a of $\triangle ap_1p_k$, in a point q_s, contradicting (3). If $Z(p_1p_kp_m)$ (or $Z(p_mp_1p_k)$) then the secant p_mp_h intersects, by Z10 (or by Z9), side p_1a of $\triangle ap_1p_k$, in a point q_s, contradicting (3). □

Given that, as shown in [23], Z10 does not follow from Z1–Z9, it would be of considerable interest to know whether SG can be proved inside {Z1–Z9}.

In the two-dimensional case, there is a weaker axiom system, for *ordered regular incidence planes* from which SG can be derived. It cannot be expressed in terms of *points* and Z, as it is based on the notion of *sides* of a line in a plane, put forward by Sperner in [28], from which Z can be defined, but which cannot, in general, be defined in terms of Z. It can be expressed in a two-sorted language, with variables for *points* (to be represented by lowercase Latin characters) and for *lines* (to be represented by lowercase Gothic characters), with two relation symbols, I, with $I(a\mathfrak{g})$ to be read as 'point a is incident with line $\mathfrak{g}$', and D, with $D(a\mathfrak{g}b)$ to be read as 'the points a and b lie on different sides of line $\mathfrak{g}$'. With $\delta(ab\mathfrak{g}\mathfrak{h}) :\Leftrightarrow [(D(a\mathfrak{g}b) \wedge D(a\mathfrak{h}b)) \vee (\neg D(a\mathfrak{g}b) \wedge \neg D(a\mathfrak{h}b))]$ and ${}^{\epsilon}\delta$ standing for δ if $\epsilon = 1$ and for $\neg\delta$ if $\epsilon = 0$, the axioms are the following (see [13]):

J 1 $(\forall ab)(\exists^{=1}\mathfrak{g})\, a \neq b \rightarrow I(a\mathfrak{g}) \wedge I(b\mathfrak{g})$,

J 2 $(\forall \mathfrak{g})(\exists a_1a_2a_3a_4)\ \bigwedge_{1\leq i<j\leq 4} a_i \neq a_j \wedge \bigwedge_{i=1}^{4} I(a_i\mathfrak{g})$,

J 3 $(\exists abc)(\forall \mathfrak{g})\, \neg(I(a\mathfrak{g}) \wedge I(b\mathfrak{g}) \wedge I(c\mathfrak{g}))$,

J 4 $D(a\mathfrak{g}b) \rightarrow \neg I(a\mathfrak{g})$,

J 5 $D(a\mathfrak{g}b) \rightarrow D(b\mathfrak{g}a)$,

J 6 $\neg I(c\mathfrak{g}) \wedge D(a\mathfrak{g}b) \rightarrow (D(a\mathfrak{g}c) \vee D(b\mathfrak{g}c))$,

J 7 $\neg(D(a\mathfrak{g}b) \wedge D(b\mathfrak{g}c) \wedge D(c\mathfrak{g}a))$,

J 8
$$\begin{aligned}&\Big[\bigwedge_{1\leq i<j\leq 4} a_i \neq a_j \wedge \mathfrak{h}_i \neq \mathfrak{h}_j \wedge \bigwedge_{i=1}^{4} I(a_i\mathfrak{h}_i) \wedge \mathfrak{h}_i \neq \mathfrak{g} \wedge \\ &\quad \Big(\Big(\bigwedge_{i=1}^{4} I(a_i\mathfrak{g})\Big) \vee \Big(\bigwedge_{i=1}^{4} I(o\mathfrak{h}_i)\Big)\Big)\Big] \\ &\quad \rightarrow \Big[\bigvee_{\substack{\epsilon_i\in\{0,1\}\\ \epsilon_1+\epsilon_2+\epsilon_3=2}} {}^{\epsilon_1}\delta(a_3a_4\mathfrak{h}_1\mathfrak{h}_2) \wedge {}^{\epsilon_2}\delta(a_2a_4\mathfrak{h}_1\mathfrak{h}_3) \wedge {}^{\epsilon_3}\delta(a_2a_3\mathfrak{h}_1\mathfrak{h}_4)\Big].\end{aligned}$$

J6 is a weak variant of Pasch's axiom, stating that if a line $\mathfrak{g}$ does not pass through any of the points a, b, and c, and a and b are on different sides of $\mathfrak{g}$, then so are at least one of the pairs $\{a, c\}$ and $\{b, c\}$. J7 is a variant of Pasch's theorem, stating that a line cannot separate all three pairs $\{a, b\}$, $\{b, c\}$, and $\{c, a\}$. One of its special cases, when $a = b = c$, implies that a and b can be on different sides of $\mathfrak{g}$ only if $a \neq b$. That these versions are called "weak" stems from the fact that, if a line $\mathfrak{g}$ separates the points a and b, it no longer means that there is a point on $\mathfrak{g}$ which is between a and b. Indeed, the line $\mathfrak{g}$ and the line determined by a and b may have no point in common (a simple example is provided by the submodel of the ordered affine plane over $\mathbb{Q}$ whose points have coordinates whose denominators are powers of 2, with the plane separation relation inherited from the ordered affine plane over $\mathbb{Q}$). The meaning of J8 is best understood in terms of the notion of *separation* $//$ (with $ab//cd$ to be read as 'the point-pair (a, b) separates the point-pair $(c.d)$'), defined by

$$\begin{aligned} a_1a_2//a_3a_4 \quad :\Leftrightarrow \quad & (\exists \mathfrak{g}\mathfrak{h}\mathfrak{k}) \bigwedge_{i=1}^{4} I(a_i\mathfrak{g}) \wedge \bigwedge_{1\leq i<j\leq 4} a_i \neq a_j \wedge I(a_1\mathfrak{h} \wedge I(a_2\mathfrak{k}) \\ & \wedge \mathfrak{h} \neq \mathfrak{g} \wedge \mathfrak{k} \neq \mathfrak{g} \wedge \neg\delta(a_3a_4\mathfrak{h}\mathfrak{k}). \end{aligned} \tag{4}$$

One part of it (corresponding to the $\bigwedge_{i=1}^{4} I(a_i\mathfrak{g})$ disjunct) states that, if a_1, a_2, a_3, a_4 are four different collinear points, then exactly one of the separation reations $a_1a_2//a_3a_4$, $a_1a_3//a_2a_4$, $a_1a_4//a_2a_3$ holds. Its other part (corresponding to the $\bigwedge_{i=1}^{4} I(o\mathfrak{h}_i)$ disjunct) is the dual statement (in the sense of projective geometry).

Joussen [13] showed that any model $\mathfrak{M}$ of J1–J8 can be embedded in a projective ordered plane $\mathfrak{P}$, whose separation relation $//_{\mathfrak{P}}$ is an extension of the separation relation $//_{\mathfrak{M}}$, defined in $\mathfrak{M}$ terms of $I_{\mathfrak{M}}$ and $D_{\mathfrak{M}}$ by (4).

If a noncollinear SG-configuration (i.e., the negation of SG, which we think of in this context as expressed in terms of points, lines, and I) were to hold in $\mathfrak{M}$, then it would have to hold in the ordered projective plane $\mathfrak{P}$ as well, which cannot be, as Steinberg's proof can be modified to hold in the context of projective ordered planes, as shown in [6, 3.33, p. 30–31] (or, one can remove from the projective plane a line which does not contain any of the points of the SG-configuration, to get an ordered affine plane containing an SG-configuration, which is impossible, as ordered affine planes are models of {Z1–Z10}). Thus, given that, in case there are no three noncollinear points, SG holds trivially, and so J3 is not needed in the proof of SG, we have established the following.

Theorem 2.2 {J1–J2, J4–J8} $\vdash$ SG(n)*, where* SG(n) *is expressed in terms of points, lines, and* I*.*

By defining Z in terms of I and D by

$$Z(abc) :\Leftrightarrow (\exists \mathfrak{g}\mathfrak{h})\, \mathfrak{h} \neq \mathfrak{g} \wedge I(a\mathfrak{g}) \wedge I(b\mathfrak{g}) \wedge I(c\mathfrak{g}) \wedge I(b\mathfrak{h}) \wedge D(a\mathfrak{h}c), \tag{5}$$

one can compare the set of Z-consequences of the axiom system {J1–J2, J4–Z8} to {Z1–Z10}. It turns out that the Z defined by (5) satisfies Z1–Z7 but does not need to satisfy Z8–Z10, so that {J1–J2, J4–J8} cannot be said to be stronger than {Z1–Z10}.

On the other hand, the relation D is not definable in terms of Z on the basis of {Z1–Z10} as there are no "sides" of lines in an arbitrary model of the latter, as its "dimension" may be greater than two, so we cannot even ask whether the axioms J4–J8 hold in {Z1–Z10}.

3 Kelly's Proof

For the axiom system for Kelly's proof we think of axioms that ought to hold not only in any model of absolute geometry (of any dimension), but also in the substructure of a non-Archimedean model of absolute geometry, which consists of the union of the infinitely small neighborhoods of two points at finite, but not infinitely small distance (a universe in which there are only two galaxies, which are so far apart, that, from the vantage point of one galaxy one sees the other galaxy as a galactic nebula of infinitely small diameter.) We will refer to the latter substructure as the "Two Nebulae."

The language in which the axiom system will be expressed contains, beyond L and Z, the quaternary relation J, with $J(abcd)$ to be read as ab is shorter than cd. In this setting, L is *not* an abbreviation (as in (2)); it is one of the primitive notions of our language.[2] To simplify the statement of the axioms, we introduce the following abbreviations: for a, b, c with $\neg L(abc)$, we define

$$au \perp bc :\Leftrightarrow (\forall v)\,[L(bcu) \wedge (L(bcv) \rightarrow \neg J(avau))], \tag{6}$$

which may be read as 'u is a foot of a perpendicular from a to the line bc', given that its definiens states that u is a point on the line determined by bc, with the property that the distance from a to any point v on line bc is not less than that from a to u.

We also define

$$a \sim b :\Leftrightarrow a \neq b \wedge [(\forall c)\,(\neg L(abc) \rightarrow (\exists u)\, au \perp bc)], \tag{7}$$

which may be read as 'a is related to b' (note that $\sim$ is not necessarily a symmetric relation, that is, we may have $a \sim b$ without $b \sim a$). In models of absolute geometry, all points are related to all other points. In the Two Nebulae, only points inside the same nebula are related.

As axioms we have, beside L1–L3, the following statements (addition in the indices in K2 is modulo 3):

K 1 $\bigwedge_{1 \leq i < j \leq 7} a_i \neq a_j \rightarrow \left(\bigvee_{1 \leq i, j \leq 7, i \neq j} a_i \sim a_j\right)$,

K 2 $a \sim b \wedge \bigwedge_{i=1}^{3} a \neq x_i \wedge x_i \neq x_{i+1} \wedge L(ax_ix_{i+1}) \rightarrow \left(\bigvee_{i \neq j} Z(ax_ix_j)\right)$,

K 3 $a \sim b \wedge b \sim c \wedge c \neq a \rightarrow a \sim c$,

K 4 $Z(abc) \rightarrow b \sim a \vee b \sim c$,

K 5 $a \sim b \wedge \neg L(abc) \wedge au \perp bc \rightarrow u \sim a$,

K 6 $J(abcd) \rightarrow \neg J(cdab)$,

K 7 $J(abcd) \wedge J(cdef) \rightarrow J(abef)$,

K 8 $b \sim a \wedge \neg L(abc) \wedge bc \perp ca \rightarrow J(bcab)$,

K 9 $o \sim a \wedge \neg L(aob) \wedge Z(obc) \wedge ao \perp ob \wedge bd \perp ac \rightarrow J(bdao)$.

In all models of absolute geometry, all points are related, so the justification for axioms in which the conclusion refers to the relatedness of two points (such as K1, K3, K4, K5) will come from the Two Nebulae.

K1 states that among seven different points there must be two related ones. It ensures that, under the assumption that there are seven different points, there are related points at all. It is a somewhat weaker statement than the more natural one, that among any three different points there are two related points, which is true in the Two Nebulae, as two of the three points must belong to the same nebula.

K2 states that, if x_1, x_2, and x_3 are three points on a line through a (a point which is related to some other point), then two must lie on the same half-line determined by a (an obvious fact by the pigeonhole principle, if one thinks that a actually divides the line through it into two half-lines).

K3 states a transitivity property of relatedness, which can be understood in the context of the Two Nebulae to state that if a and b belong to the same nebula, and so do b and c, then both a and c belong to the same nebula.

K4 states that if b lies between a and c, then b must be related to one of a or c. In the Two Nebulae, if a and c belong to the same nebula, then any point between them must belong to that nebula as well (given that nebulae are convex); if a and c belong to different nebulae, then b must lie either in the nebula containing a or in that containing c.

In the Two Nebulae, K5 states that, if a is a point outside of line bc, with a and b in the same nebula, then the foot u of the perpendicular from a to bc lies in the same nebula in which a lies (this is easy to see if one notices that the distance from a to u cannot be greater than that from a to b, and since the distance from a to b is infinitely small, so must be the distance from a to u).

K6 and K7 state that J, the *less than* relation, is not symmetric (i.e., that if ab is less than cd, then cd is not less than ab) and that it is transitive.

K8 states that in a right triangle abc, with $bc \perp ca$, the side bc is less than the hypotenuse ab. K8 and K9—which states that, in the figure below, bd is less than ao—may seem unusual as axioms, but one should bear in mind that these two axioms have surfaced independently of Kelly's proof of SG. K8 has been shown in [26] to be weaker than the Pasch axiom in Euclidean Pasch-free geometry. The question regarding the missing link between K8 and the Pasch axiom led in [19] to K9, which, together with K8, turned out to be equivalent to the Pasch axiom in Pasch-free Euclidean geometry.

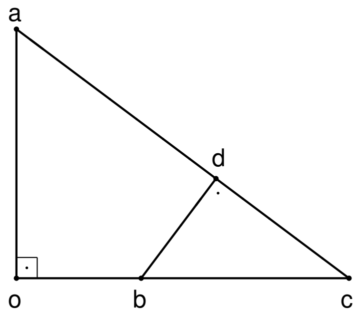

Figure 1 Axiom K9 states that bd is shorter than ao.

Theorem 3.1 K1–K9 $\vdash$ SG*(n)*.

Proof To prove that (1) holds, let $p_1, \ldots, p_n$, with $n \geq 7$, be such that the antecedent of (1) holds, as well as $\neg L(p_1 p_2 p_3)$. By K1, there are p_i and p_j with $p_i \sim p_j$. Given $\neg L(p_1 p_2 p_3)$, one of the lines $p_j p_k$ with $k = 1, 2, 3$ (at least two of the three must actually be lines) does not contain p_i. We denote by k_0 that index

k, and we conclude, by $p_i \sim p_j$ and (7), that there exists a u with $L(p_j p_{k_0} u)$ and $p_i u \perp p_j p_{k_0}$. Let $P := \{p_s u_{srq} : p_s u_{srq} \perp p_r p_q, s \in \{1, \dots, n\}, \neg L(p_s p_r p_q), p_s \sim p_r\}$ (P is, formally speaking, a set of point-pairs, which we denote by xy instead of (x, y)). P is a finite nonempty set (as it contains $p_i u_{ijk_0}$). Given that P is finite, there are p and u with

$$pu \in P \text{ and } \neg J(p'u'pu) \text{ for all } p'u' \in P. \tag{8}$$

That such an element must exist in P can be seen by choosing an arbitrary element $p_1 u_1$ in P and asking whether it satisfies the condition (8). If yes, we are done. If not, then there is an element $p_2 u_2$ of P with $J(p_2 u_2 p_1 u_1)$. If $p_2 u_2$ satisfies (8), then we are done. If not, then there exists an element $p_3 u_3$ of P with $J(p_3 u_3 p_2 u_2)$. We proceed in this manner. The process can last only for finitely many steps, given that P is finite, and that, by K6 and K7, we have $\neg J(p_i u_i p_j u_j)$ for all $i < j$, and thus, we could not proceed after having reached $p_f u_f$, where $f = |P|$.

For the p and the u that satisfy (8), there must exist $1 \leq k, l, m \leq n$ such that $p = p_k$ and $u = u_{klm}$, as well as $p_k \sim p_l$, $\neg L(p_k p_l p_m)$, $p_k u_{klm} \perp p_l p_m$. According to the antecedent of (1), there must be some p_h with $\lambda(p_l p_m p_h)$. Point u_{klm}, which, by K5, satisfies $u_{klm} \sim p_k$, is either equal to one of $\{p_l, p_m, p_h\}$ or it is different from all of them. If u_{klm} is equal to one of them, that is, $u_{klm} = p_i$ with $i \in \{l, m, h\}$, then $p_i \sim p_k$ and there is, by (7), a v with $p_i v \perp p_k p_j$, where j is one of the two indices in $\{l, m, h\} \setminus \{i\}$, and such that $J(p_i v p_k p_i)$ (by K8). Since $p_i v \in P$, this contradicts (8). If u_{klm} is different from the elements of $\{p_l, p_m, p_h\}$, then, by K2, we have $Z(u_{klm} p_i p_j)$, where $i, j \in \{l, m, h\}$. By K4, we have $p_i \sim u_{klm}$ (and, by K3, $p_i \sim p_k$) or $p_i \sim p_j$. In both cases, by (7), there exists a v such that $p_i v \perp p_k p_j$ and $J(p_i v p_k u_{klm})$ (by K9), which contradicts (8). ☐

4 Moszyńska Geometries

Chvátal [4] asked whether SG holds in finite metric spaces, in which the betweenness relation is defined in terms of the metric in the manner of Menger [17]. In all metric spaces, $Z(abc)$ may be defined to hold precisely if a, b, and c are all distinct and the sum of the distances from a to b and from b to c coincides with the distance from a to c. The betweenness relation thus obtained satisfies Z1–Z4, but does not, in general, satisfy Z5, Z6, or Z7. If one were to define the notion of collinearity in the manner of (2), then the line determined by two points would, in general, contain few points. To enrich the number of points on the line determined by two points a and b, Chvátal suggests the following stepwise procedure of constructing the line l_{ab} determined by a and b. First, a, b, as well as all points x with $Z(xab)$ or $Z(axb)$ or $Z(abx)$ are the elements of the first stage in our construction of l_{ab}, to be denoted by l^1_{ab}. Suppose we have finished n stages in the construction of l_{ab}, and the resulting set of points is l^n_{ab}. At the $(n+1)$st stage, all points x with $Z(xuv)$ or $Z(uxv)$ or $Z(uvx)$, where u and v are any two points in l^n_{ab}, are added to l^n_{ab} to form l^{n+1}_{ab}. We define l_{ab} to be the union of all l^n_{ab} for $n \geq 1$. In the case of finite metric spaces, this union is a finite one.

Chvátal's conjecture was settled by Chen [3]. His proof is carried out inside finite metric spaces, and these are structures that carry a lot of information, which the first-order theory of their associated betweenness relation (a theory studied in [16] and [27]) does not capture. For example, the proof requires one to choose among a set of triples (a, b, c) the one for which $\varrho(a, b) + \varrho(b, c) + \varrho(c, a)$ is minimal (here ϱ stands for the metric), something the first-order betweenness theory cannot do, as it does not know the values of the distances between points. Since geometry is a narrative about points only, the real numbers involved, the operation of addition of lengths, as well as the ability to compare two lengths, have to be expressed in elementary terms.

The most austere solution to the problem of expressing such metric-dependent statements inside a first-order theory that expresses betweenness as well has been proposed by Moszyńska [18]. Chen's proof can be rephrased with minor changes in the setting of the equidistance and betweenness spaces considered by Moszyńska, enlarged with one axiom, which allows for the comparison of the lengths of any two segments.

These spaces have many more properties than metric spaces, so in a sense, our proof restricts Chen's theorem to this much more narrow class of spaces for the sake of an elementary axiomatization of the theory inside which we claim a new version of SG to hold.

We first repeat the axiom system from [18], to which we add an axiom, M12, which ensures that any two segments are comparable. The language in which the axiom system is expressed contains two predicates, a quaternary one, $\equiv$, with $ab \equiv cd$ to be read as '*ab* is congruent to *cd*', and the strict betweenness predicate Z (Moszyńska used the nonstrict betweenness predicate B; the differences are insignificant). The axioms are Z1–Z4 and the following (S_k stands for the set of all permutations of the set $\{1, \ldots, k\}$, the numbers k and l appearing in the axioms take on all positive integer values that are $\leq n(n-1)/2$):

M 1 $ab \equiv cd \wedge ef \equiv cd \rightarrow ab \equiv ef$,

M 2 $aa \equiv bb \wedge ab \equiv ba$,

M 3 $ab \equiv cc \rightarrow a = b$,

M 4 $Z(abd) \wedge Z(bcd) \rightarrow Z(abc)$,

M 5 $Z(abd) \wedge Z(bcd) \rightarrow \neg ad \equiv bc$,

M 6 $ab \equiv a'b' \wedge bc \equiv b'c' \wedge ac \equiv a'c' \wedge Z(abc) \rightarrow Z(a'b'c')$,

M 7 $ab \equiv a'b' \wedge ac \equiv a'c' \wedge Z(abc) \wedge Z(a'b'c') \rightarrow bc \equiv b'c'$,

M 8 $\bigwedge_{i=2}^{k}(Z(p_0p_{i-1}p_i) \wedge Z(q_0q_{i-1}q_i)) \wedge \bigwedge_{i=1}^{k}\big(\bigvee_{f \in S_k} p_{i-1}p_i \equiv q_{f(i)-1}q_{f(i)}\big) \rightarrow p_0p_k \equiv q_0q_k$,

M 9 $\bigwedge_{2 \leq i \leq k, 2 \leq j \leq l}\big[Z(p_0p_{i-1}p_i) \wedge Z(q_0q_{j-1}q_j)\big] \wedge p_0 = p'_0 \wedge p_k = p'_l \wedge q_0 = q'_0 \wedge q_l = q'_k \wedge \bigwedge_{1 \leq i \leq k, 1 \leq j \leq l}\big[\bigvee_{f \in S_k, g \in S_l} p_{i-1}p_1 \equiv q'_{f(i)-1}q'_{f(i)} \wedge q_{j-1}q_j \equiv p'_{g(j)-1}p'_{g(j)}\big] \rightarrow p_0p_m \equiv q_0q_n$,

M 10 $(\forall a_1 \ldots a_kb_1 \ldots b_k)\big[\big(\bigwedge_{i=1}^{k} a_i \neq b_i\big) \rightarrow (\exists q_0 \ldots q_k)\big[\bigwedge_{i=2}^{k}\big(\bigvee_{f \in S_k} Z(q_0q_{i-1}q_i)\big) \wedge \bigwedge_{i=1}^{k} q_{i-1}q_i \equiv a_{f(i)}b_{f(i)}\big]\big]$,

M 11 $(\forall abcc')\,[Z(abc) \wedge ac \equiv a'c' \rightarrow (\exists b')\, ab \equiv a'b' \wedge bc \equiv b'c']$,

M 12 $Z(abc) \wedge ab \equiv cb' \wedge cb \equiv ab' \rightarrow (Z(abb') \vee Z(ab'b) \vee b = b')$.

Axioms Z1–Z4, M1–M11 make up the axiom system put forward in [18], with the difference that the restriction that k and l be $\leq n(n-1)/2$ does not occur in [18].

Axiom M5 states that a segment bc properly included in a segment ad cannot be congruent to ad; M6, that if one of two isometric triples (a, b, c) and (a', b', c') is such that b is between a and c, then b' must be between a' and c'. M7 is a form of the Euclidean Common Notion 3, stating that “if equals be subtracted from equals, the remainders are equal.” M8 states that the order in which one adds segments congruent to k given segments is irrelevant, the resulting sum being always “the same.” M9 states that if there is a path of length q_0q_n joining p_0 and p_m, as well as a path of length p_0p_m joining q_0 and q_n, then the lengths q_0q_n and p_0p_m are identical. M10 is a rectifiability axiom, stating that any set of k nondegenerate segments can be placed end to end on a line in some order. M11 states that, if ac and $a'c'$ are two congruent segments, and b is a point between a and c, then there must exist a point b' between a' and c', positioned metrically on $a'c'$ in the same manner b is on ac. M12 states that, if b and b' are two points on the segment ac such that b' is positioned metrically on ca in the same manner b is on ac, then b' must lie on the ray $\overrightarrow{ab}$.

We have added M12 to ensure that any two segments can be compared. Given any two nondegenerate segments ab and cd (with $a \neq b$ and $c \neq d$), there exist, by M10, points q_0, q_1, q_2 such that $Z(q_0q_1q_2)$, $ab \equiv q_0q_1$, and $cd \equiv q_1q_2$. By M11, there exists q' such that $q_2q' \equiv q_0q_1$ and $q_0q' \equiv q_1q_2$, and, by M12, we have (i) $Z(q_0q_1q')$ or (ii) $Z(q_0q'q_1)$ or (iii) $q_1 = q'$. Informally speaking, (i) means $ab < cd$, (ii) means $cd < ab$, and (iii) means $ab \equiv cd$.

That Chen’s [3] proof goes through in this setting can be seen by noticing that the only properties of metric spaces used in the proof are the ability to add and to compare segments, the number of which can never exceed $n(n-1)/2$ if there are n points in the whole space, since the same segment is never used twice. The occurrence of subtractions of lengths of segments on pp. 196–98 of [3] can be all removed, since they all appear when comparing two differences, the general form of them being $\varrho(a, b) - \varrho(c, d) \leq \varrho(a', b') - \varrho(c', d')$. Such comparisons are meaningful inside our setting as well, as they amount to $\varrho(a, b) + \varrho(c', d') \leq \varrho(a', b') + \varrho(c, d)$.

The definition of L, referred to earlier, for which Chen proved Chvátal’s conjecture, depends on the value of n in SG(n), given that the stepwise process which gives rise to the line determined by two different points a and b will have to end in at most $(n-2)$ steps, as it must generate at least one point at every step. With φ defined by

$$\varphi(uvx) :\Leftrightarrow Z(uvx) \vee Z(vxu) \vee Z(xuv) \vee x = u \vee x = v,$$

the definition of L is

$$\begin{aligned} L(abc) \quad :\Leftrightarrow \quad & a = b \vee b = c \vee c = a \vee \left(\exists x_1^1 x_2^1 \ldots x_1^{n-3} x_2^{n-3}\right) \\ & \bigwedge_{i=1}^{2} \varphi\left(abx_i^1\right) \wedge \bigwedge_{j=1}^{n-4} \varphi\left(x_1^j x_2^j x_1^{j+1}\right) \wedge \varphi\left(x_1^j x_2^j x_2^{j+1}\right) \wedge \varphi\left(x_1^{n-3} x_2^{n-3} c\right). \end{aligned} \tag{9}$$

Thus, by [3], we have the following theorem.

Theorem 4.1 {Z1–Z4, M1–M12} $\vdash$ SG*(n), with L defined by* (9).

5 Incompatibility

Having presented the specifics of the three axiom systems inside which SG(n) can be proved, we now ask what we have learned from this example regarding the themes *purity of the method* and *minimality of assumptions*. We have seen that Coxeter preferred Theorem 2.1 to the other two, given that the axioms are expressed solely in terms of Z, whereas the others involve notions going beyond Z. The purity of the method can be considered as preserved only if one thinks of L being a defined relation, which needs Z in its definition. If one thinks that L and Z are unrelated, then the proof of SG(n) in terms of axioms involving Z isn't pure either, as one would expect a proof from axioms expressed in terms of L. Such a proof can be provided from axioms for projective planes (or for projective geometry of arbitrary dimension ≥ 2) in which the coordinate ternary field satisfies the Artin-Schreier-like condition stated in [15], which ensures that the projective plane is orderable. That axiom system would not be minimal, even under all possible axiom systems for projective geometry. As shown in [14], the theory obtained by adding to the axioms for projective planes all the SG(k) for all $k \in \mathbb{N}$ is weaker than the theory of all orderable projective planes (axiomatized using the conditions in [15]).

One point we would like to make with this reverse analysis is that the requirement of *methodological purity* is not stronger than that of *assumptional minimalism*. A proof can respect the former requirement but proceed from a set of assumptions which contains axioms not needed in a different proof, which is not methodologically pure. Regressive analyses may lead to different minimalist axiom systems, some of which may respect the *purity of the method* requirement, but the axiom systems themselves are incomparable.

An easy way to state that the three axiom systems are incompatible is to point out that the language for Moszyńska geometry contains $\equiv$, which does not appear in the languages of the other two, and that the language for K1–K8 contains J, which does not appear in the other two. It remains to show that there are Moszyńska geometries and models of K1–K8 which do not satisfy all the axioms Z1–Z10.

We take a different approach and show that even if $ab \equiv cd$ is defined to be

$$\neg J(abcd) \wedge \neg J(cdab),$$

and if $J(abcd)$ is defined to be

$$(\exists uvw)\, Z(uvw) \wedge cd \equiv uv \wedge ab \equiv uw,$$

which corresponds to the intuitive meaning of J (even though the axioms K1–K8 by no means imply that), the three axiom systems are incomparable.

To see this, consider the following Two Nebulae model of K1–K8, which is neither a model of Z1–Z10, nor a Moszyńska geometry.

Let $L = \mathbb{Q}(t)$, ordered by $\left(\sum_{i=0}^{n} a_i t^i\right)\left(\sum_{j=0}^{m} b_j t^j\right)^{-1} > 0$ if and only if $a_n b_m > 0$. Let K be the real closure of L. The $x \in K$ for which $|x| > n$ for all $n \in \mathbb{N}$ will be called *infinitely large*, and the $x \in K$ for which x^{-1} is infinitely large will be called *infinitely small*. Let $\mathfrak{C}(K)$ be the Cartesian plane with K as coordinate field, with the usual betweenness relation induced by the order of K. Let $\|(x, y)\| = \sqrt{x^2 + y^2}$. Our model $\mathfrak{M}$ has the subset of $\mathfrak{C}(K)$ consisting of $U \cup V$ as universe, where $U := \{(x, y) \in K^2 : \|(x, y)\|$ is infinitely small or is $0\}$ and $V := \{(x, y) \in K^2 : \|(x - 1, y)\|$ is infinitely small or is $0\}$. The betweenness relation B is the restriction of the relation B from $\mathfrak{C}(K)$ to $U \cup V$, which is the union of two infinitely small neighborhoods: that of $(0, 0)$ and that of $(1, 0)$. With the usual interpretations of L and Z, and with $J(\mathbf{abcd})$ to mean $\|\mathbf{a} - \mathbf{b}\| < \|\mathbf{c} - \mathbf{d}\|$, all axioms K1–K8 are satisfied (notice that $\mathbf{a} \sim \mathbf{b}$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are both in U or both in V). However, in this model neither Z9 nor Z10 hold. Let $\epsilon = t^{-1}$, $\mathbf{a} = (0, 0)$, $\mathbf{b} = (1, 0)$, $\mathbf{d} = (1 + \epsilon, 0)$, $\mathbf{c} = (0, \epsilon)$, $\mathbf{e} = \left(0, \frac{\epsilon(\epsilon+1)}{2\epsilon+1}\right)$. We have $Z(\mathbf{abd})$, $Z(\mathbf{aec})$, $\neg L(\mathbf{abc})$, but the segments $\mathbf{de}$ and $\mathbf{bc}$ intersect in $\left(\frac{1}{2}, \frac{\epsilon}{2}\right)$, which is not a point of $\mathfrak{M}$, being neither in U nor in V. Thus Z9 does not hold, and since Z10, would, together with the linear order axioms, which all hold in $\mathfrak{M}$, imply Z9, the outer form of the Pasch axiom cannot hold in $\mathfrak{M}$ either. It is also easy to see that M10 does not hold, as there is no segment whose length is the sum of two segments of length 1.

A Moszyńska geometry, which is neither a model of K1–K8 nor of Z9 or of Z10, is the submodel $\mathfrak{M}'$ of the standard Euclidean plane $\mathfrak{E}(\mathbb{R})$, whose universe U is $\mathbb{R}\times\mathbb{R} \setminus \{(0, 0)\}$. That Z9 does not hold can be seen by taking $\mathbf{a} = (-1, 0)$, $\mathbf{b} = \left(-\frac{1}{3}, -\frac{1}{3}\right)$, $\mathbf{c} = (1, 1)$, $\mathbf{d} = (1, -1)$, $\mathbf{e} = \left(-\frac{1}{3}, \frac{1}{3}\right)$, and noticing that we have $Z(\mathbf{aec})$, $Z(\mathbf{dba})$, but there is no $\mathbf{f}$ in $\mathfrak{M}'$ such that $Z(\mathbf{efd})$ and $Z(\mathbf{bfc})$ as that $\mathbf{f}$ would be $(0, 0)$, which is not in U. In other words, in $\triangle\mathbf{acb}$ secant $\mathbf{ed}$ cuts side $\mathbf{bc}$ in $(0, 0)$, which does not belong to $\mathfrak{M}'$. It is not a model of Z10 either, as Z10 together with all the linear axioms, which $\mathfrak{M}'$ satisfies, would imply Z9. K5 does not hold in this model, for, although $(-1, 1) \sim (1, -1)$, $\neg L((1, 1)(1, -1)(1, 1))$, and $(-1, 1)(1, 1) \perp (1, -1)(1, 1)$, we do not have $(1, 1) \sim (-1, 1)$, as there is no minimal joining segment between the point $(1, 1)$ and the points on the line passing through $(-1, 1)$ and $(1, -1)$.

The question whether predicates to be interpreted as notions of *ordered* geometry are needed in axiom systems from which SG, which is a pure incidence statement, can be derived is still open. It was conjectured in [20] (see also [22]) that SG holds in affine (or projective) planes over fields of characteristic 0, which are not quadratically closed. Even if such a proof were to exist, which would satisfy—just like the proof using the incidence-based characterization of orderable projective planes from [15]—the purity of method criterion, the axiom system used to prove SG would not be weaker than those presented in the three proofs above, but simply incomparable.

Notes

1. That Postulate 1 of [12] cannot be deduced from Z1–Z10 can be seen by taking as model the unit circle, with the distance $\varrho(a, b)$ between two points a and b on it defined as the length of the shorter of the two arcs joining a and b (half the perimeter of the circle, should they be antipodal points), and where $Z(abc)$ holds if and only if a, b, c are three distinct points and $\varrho(a, b) + \varrho(b, c) = \varrho(a, c)$. Postulate 1 of [12] follows from Z1–Z10 and $(\exists abc)\, \neg L(abc)$, as shown in [32, Theorem 9, p. 356–57].

2. We could have opted for a language in which J is the only primitive notion, but we decided to not follow that path, as it would have resulted in a stronger axiom system, one in which Z would be defined in terms of J and L in terms of Z by (2). That J alone suffices (in fact, even J' alone, defined by $J'(abc) :\Leftrightarrow J(abac)$) to axiomatize all of Euclidean geometry coordinatized by Pythagorean ordered fields was shown in [31].

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