Times in Tense Logic

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Abstract   Tense logic is often said to possess insufficient expressive resources to serve as a theory of the nature of time. This paper counters this objection by showing how to obtain quantification over times in a tense logic in which all temporal distinctions are ultimately spelled out in terms of the two simple tense operators "it was the case that" and "it will be the case that." This account of times is similar to what is known as "linguistic ersatzism" about possible worlds, but there are noteworthy differences between these two cases. In particular, while linguistic ersatzism would support actualism, the view of times defended here does not support presentism.

1   Introduction

I am interested in the question of whether a simple tense logic with two conceptually primitive tense operators 'P' ("it was the case that") and 'F' ("it will be the case that") can serve as a metaphysical theory of the nature of time.[1]   Unlike rival accounts, such a *tense primitivism* has no ontological commitments of a specifically temporal nature; all its commitments are "ideological" in the sense of Quine [23].

One might wonder how such an account of time could be reconciled with the theory of relativity, and one might also have worries about intensional theories more generally. What I am concerned with here, though, is a purely logical objection to tense primitivism. Just as there is more to modality than can be expressed in terms of the possibility operator '◇' alone, it is often said that tense logic has insufficient expressive resources to serve as a theory of time ([1], [3]). There are temporal claims that resist regimentation in terms of tense operators, but which are easily accounted for by rival theories that make use of explicit quantification over times, such as the temporal substantivalism endorsed by Mellor [16].

In this paper, I show how the tense primitivist can overcome these problems by acquiring the ability to talk about times himself. Of course, without abandoning his

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project of spelling out all temporal distinctions in terms of his primitive tense oper-
ators, the tense primitivist cannot regard times as metaphysically basic entities. He
must take times to be certain abstractions. In this regard, he is in a similar position
as temporal relationists like Russell [27], who take times to be maximal sets of si-
multaneous events. But since the basic building blocks of tense primitivism are tense
operators rather than temporal relations, it needs a different type of abstraction. The
method I want to advocate takes times to be maximal consistent sets of sentences.

2 A Simple Tense Logic

Take a standard language of propositional logic with logical constants '¬' and '→',
and reserve its sentences for making claims about what is presently the case. To
report what is true at other times, add the tense operators 'P' and 'F'. It is also con-
venient to introduce 'H' ("it has always been the case that") as an abbreviation for
'¬P¬' and 'G' ("it will always be the case that") as shorthand for '¬F¬'. Then the
simple system of tense logic Z (*Zeitlogik*) consists of a standard system of proposi-
tional logic plus the following axioms and rules of *temporal generalization*:

$$\mathsf{H}(\varphi \to \psi) \to (\mathsf{H}\varphi \to \mathsf{H}\psi) \qquad \mathsf{G}(\varphi \to \psi) \to (\mathsf{G}\varphi \to \mathsf{G}\psi)$$
$$\text{If } \vdash \varphi \text{ then } \vdash \mathsf{H}\varphi \qquad\qquad \text{If } \vdash \varphi \text{ then } \vdash \mathsf{G}\varphi.$$

One can think of Z as consisting of two copies of the simple modal system K. The
tense operators 'H' and 'G' play the role of the necessity operator, and 'P' and 'F'
that of the possibility operator.

   The model theory for Z is a doubled-up version of the model theory for modal
logic. A model for tense logic is a quintuple $M = \langle T, p, <, >, @ \rangle$ that consists of a
set T of objects called "times," an element p of T chosen as "present time," and an
"earlier than" relation $<$ and a "later than" relation $>$ on T. The latter can be thought
of as "accessibility relations" for 'P' and 'F', respectively. There is also a "true at"
relation @ between sentences and times that is required to satisfy

$$
\begin{array}{llll}
@(\ulcorner\neg\varphi\urcorner, t) & \text{iff} & \text{it is not the case that } @(\varphi, t); \\
@(\ulcorner\varphi \to \psi\urcorner, t) & \text{iff} & \text{either } @(\ulcorner\neg\varphi\urcorner, t) \text{ or } @(\psi, t); \\
@(\ulcorner\mathsf{P}\varphi\urcorner, t) & \text{iff} & @(\varphi, t') \text{ for some } t' \in T \text{ such that } t' < t; & (1) \\
@(\ulcorner\mathsf{F}\varphi\urcorner, t) & \text{iff} & @(\varphi, t') \text{ for some } t' \in T \text{ such that } t' > t.
\end{array}
$$

A sentence φ is true in the model M if and only if it is true at its present time,
$@(\varphi, p)$, and it is valid just in case it is true in all models.

   By plagiarizing the proof of the soundness and completeness theorem for K, one
can then show that the theorems of tense logic coincide with the valid sentences.

Proposition 2.1 (Model Completeness)   *A sentence is a theorem of* Z *if and only if
it is true in all models.*

Just as the completeness theorem for K does not require the accessibility relation in
models of modal logic to possess any special properties, Z does not put any restric-
tions on the earlier-than and later-than relations. It does not even require that one is
the converse of the other, or that both are transitive. It also tells us nothing about the
denseness, completeness, and comparability of these relations, but there are exten-
sions of Z that impose more stringent conditions. For example, let Z^+ be the system
of tense logic obtained by adding the following axioms to Z:

$$\mathsf{PP}\varphi \to \mathsf{P}\varphi \qquad \mathsf{FF}\varphi \to \mathsf{F}\varphi.$$

This requires the earlier-than and later-than relations to be transitive.

Proposition 2.2 (Model Completeness)   *A sentence is a theorem of Z^+ if and only if it is true in all models in which $<$ and $>$ are transitive.*

Similarly, one can ensure that $<$ and $>$ are converses of each other by adopting

$$\varphi \rightarrow HF\varphi \qquad \varphi \rightarrow GP\varphi.$$

There is a whole range of results of this type, which identify structural properties of $<$ and $>$ with characteristic axiom schemata for tense logic ([7], [2]):

Denseness	$P\varphi \rightarrow PP\varphi \quad F\varphi \rightarrow FF\varphi$
Discreteness	$(\varphi \wedge H\varphi) \rightarrow FH\varphi \quad (\varphi \wedge G\varphi) \rightarrow PG\varphi$
Comparability	$(P\varphi \wedge P\psi) \rightarrow (P(\varphi \wedge \psi) \vee P(\varphi \wedge P\psi) \vee P(P\varphi \wedge \psi))$
	$(F\varphi \wedge F\psi) \rightarrow (F(\varphi \wedge \psi) \vee F(\varphi \wedge F\psi) \vee F(F\varphi \wedge \psi))$
Completeness	$(F\varphi \wedge FG\neg\varphi) \rightarrow F(HF\varphi \wedge G\neg\varphi)$
	$(P\varphi \wedge PH\neg\varphi) \rightarrow P(GP\varphi \wedge H\neg\varphi)$
Endpoints	$H\bot \vee PH\bot \quad G\bot \vee FG\bot$
No Endpoints	$\varphi \rightarrow PF\varphi \quad \varphi \rightarrow FP\varphi$

These are promising results, but in other respects the expressive capacity of tense logic appears to be lacking. In propositional logic, there are choices of truth-functional connectives, such as '¬' and '→', that are *definitionally complete* in that all other connectives of the same type can be expressed in terms of them. There is no comparable result in tense logic, where the tense operators 'P' and 'F' are easily seen to be definitionally *in*complete.

One cluster of problems is given by binary tense operators like "since" and "until." In the model theory, these operators can be characterized via

$$
\begin{aligned}
@(\ulcorner S\psi\varphi\urcorner, t) \quad \text{iff} \quad & @(\varphi, t') \text{ for some } t' < t \text{ and} \\
& @(\psi, t'') \text{ for all } t' < t'' < t; \\
@(\ulcorner U\psi\varphi\urcorner, t) \quad \text{iff} \quad & @(\varphi, t') \text{ for some } t' > t \text{ and} \\
& @(\psi, t'') \text{ for all } t' > t'' > t.
\end{aligned}
\tag{2}
$$

Kamp [12] proves that one cannot define these operators in terms of 'P' and 'F' alone. These two monadic operators only care about whether a time is past or future at all, and to account for 'S' and 'U' one needs to be able to tell whether one time is *more* future or past than another.[2]

Similar problems arise for tense operators that are two-dimensional in the sense of Segerberg [28]. One example is the "now" operator, which permits one to evaluate sentences at the present time of a model even if they occur within the scope of other tense operators:

$$@(\ulcorner N\varphi\urcorner, t) \quad \text{iff} \quad @(\varphi, p).$$

Kamp [13] shows that this operator is redundant in propositional tense logic, where its work is already done by the convention that untensed sentences make claims about what is presently the case. But the following example is said to show that the case of a *quantified* tense logic is different:

A child was born that will become ruler of the world.

With the help 'N', this is easily formalized as 'P$\exists x\,(Bx \land \mathsf{NF}Rx)$'. Kamp argues that it cannot be rendered without it.

Both problems have modal counterparts. There are binary modal operators, such as the counterfactual conditional ([15], sec. 5.2), that cannot be analyzed in terms of the simple possibility operator '$\Diamond$' alone because they require a notion of comparative possibility. The modal analogue of "now" is the two-dimensional actuality operator. This operator is known to be redundant in propositional modal logic [9], but is said to be ineliminable in quantified modal logic ([8], [4], [11]).

It is tempting to brush off these problems by replying that 'P' and 'F' are just a poor choice of primitive tense operators. This may be true, but it does not go to the heart of the matter. Similar problems keep cropping up. Vlach [29] presents a sentence that requires the introduction of yet another operator, the two-dimensional "then" operator 'T':

> One day, all persons alive then would be dead.

Van Benthem [1] takes it one step further by giving an example of a sentence that cannot even be formalized in terms of all of 'P', 'F', 'N', and 'T'. There appears to be no finite set of tense operators that is definitionally complete. If that is right then tense logic suffers from a principled limitation in its expressive capacity. The problem is not that tense primitivism says anything *false* about time but that it says *too little*. There is a wide range of temporal claims that the tense primitivist cannot pass judgment on because he is unable to formulate them in his tense logic.

However, the interpretation of these inexpressibility results is not quite as straightforward as is sometimes suggested. When considering the definitional completeness of truth-functional connectives, one is asking whether any such connective can be expressed in term of certain other connectives *of the same type*. Nobody would expect all tense operators to be expressible in terms of the monadic operators 'P' and 'F' alone. Strings of these operators are again monadic operators, and it is trivially true that one cannot define operators of higher adicity in this way. The interesting question is whether one can express all other tense operators in terms of 'P' and 'F' *plus* the resources of a suitably chosen background logic. This means that expressive shortcomings might be due to the background logic rather than to the choice of primitive tense operators.[3]

I want to show that 'P' and 'F' are indeed sufficient to express everything a tense primitivist would want to say about time, provided he is given the resources necessary to abstract times from them. Just as a temporal relationist needs some set theory to construct times as classes of simultaneous events, the tense primitivist needs a stronger background logic to develop an account of times that suits his needs. As long as this additional structure does not expand the *temporal* ideology and ontology of the account, this does not undermine his central thesis that tense operators can capture everything there is to be said about time.

I shall first present a suitable abstraction method for the weak tense logic Z and then extend it to stronger systems.

3 Times as Abstractions

The model theory for Z talks about "times," but this is of little use to the tense primitivist. The aim of model theory is to characterize the set of theorems, and for that purpose the precise nature of the elements of the models is quite irrelevant.

All the logical work is being done by the restrictions that (1) imposes on the true-at relation @. It does not matter what one chooses as elements of T, as long as there are enough of them. Of course, one would also expect there to be an intended model of tense logic that is made up of genuine times (whatever that amounts to), but even in that case (1) would at best allow one to spell out the tense operators in terms of the times in the intended model. What is at issue here is the exact converse. The task for the tense primitivist is to explain what times are in terms of his tense operators.

The account I want to propose is similar to the modal view of *linguistic ersatzism*, which regards possible worlds as maximal consistent sets of sentences of modal logic [26]. A set of sentences $\mathfrak{s}$ is *maximal* if and only if, for every sentence φ, either $\varphi \in \mathfrak{s}$ or $\ulcorner \neg \varphi \urcorner \in \mathfrak{s}$. Given a logical system S, such a set is *S-consistent* if and only if there are no $\alpha_1, \ldots, \alpha_k \in \mathfrak{s}$ such that $\vdash \neg(\alpha_1 \wedge \cdots \wedge \alpha_k)$, where '$\vdash$' denotes derivability in S. It is easy to show that maximal S-consistent sets are closed under derivability in S:

$$\text{If } \Lambda \subseteq \mathfrak{s} \text{ and } \Lambda \vdash \varphi \text{ then } \varphi \in \mathfrak{s}. \tag{3}$$

Any sentence that follows from a subset of a maximal consistent set $\mathfrak{s}$ is therefore contained in $\mathfrak{s}$ as well. Say that a sentence φ is *true in* $\mathfrak{s}$ if and only if $\varphi \in \mathfrak{s}$. Then a maximal *S-consistent* set is just a way of assigning truth-values to sentences that is compatible with the *S*-logical relations between them. The linguistic ersatzer takes *S* to be a suitable system of modal logic and then proposes maximal *S*-consistent sets as candidates for possible worlds.

In the temporal case, a tense primitivist could try to treat times as *possible presents*, which are maximal Z-consistent sets of sentences of tense logic. The technical details of this abstraction are largely the same as for linguistic ersatzism about possible worlds, but there is one crucial difference. While every maximal consistent set of sentences of modal logic qualifies as a possible world, not every maximal consistent set of sentences of tense logic counts as a time. At best, times are those possible presents that did, do, or will happen, and some of them never do. Since it is a contingent fact which possible presents are times, their abstraction requires an additional step. After the construction of possible presents, one still needs to determine which of them are times.

Here is how this can be done. Let an *interpretation* of tense logic be a fixed but arbitrary choice of possible present $\mathfrak{p}$ that is being put forward as the actual present. (The way I am using these terms here, the "interpretations" and "models" of tense logic are different types of entities; Section 4 explains how the two are related.) Since $\mathfrak{p}$ itself is meant to describe how things presently are, it will serve as the present time of the time-series. But since $\mathfrak{p}$ is maximal consistent, it also specifies what all the other times are like. Hidden within the scope of the tense operators in the sentences in $\mathfrak{p}$, one can find a description of the rest of the time-series. To extract this information, let me define earlier-than and later-than relations '$\lhd$' and '$\rhd$' on sets of sentences.

$$\mathfrak{t} \lhd \mathfrak{t}' \text{ iff } \ulcorner P\varphi \urcorner \in \mathfrak{t}' \text{ for all } \varphi \in \mathfrak{t}; \qquad \mathfrak{t} \rhd \mathfrak{t}' \text{ iff } \ulcorner F\varphi \urcorner \in \mathfrak{t}' \text{ for all } \varphi \in \mathfrak{t}.$$

The sets that enter into these relations may be Z-consistent and maximal, but the definition does not assume that they are. The relations '$\lhd$' and '$\rhd$' must not be confused with the '$<$' and '$>$' of the model theory.

Since there are maximal Z-consistent sets of sentences that contain neither $\ulcorner P\varphi \urcorner$ nor $\ulcorner F\varphi \urcorner$ for all φ, there is no guarantee that there are any sets that are earlier or later than a given interpretation, let alone maximal consistent ones. But one can prove

the following important result, which ensures that the tense primitivist always gets as many times as he needs. If a possible present $\mathfrak{p}$ contains a past (or future) tense claim then there is a possible present that is earlier (or later) than $\mathfrak{p}$.

Proposition 3.1   *Let $\mathfrak{p}$ be a possible present and φ a sentence.*

1. *$\ulcorner P\varphi \urcorner \in \mathfrak{p}$ iff there is a possible present $\mathfrak{t}$ such that $\mathfrak{t} \lhd \mathfrak{p}$ and $\varphi \in \mathfrak{t}$.*
2. *$\ulcorner F\varphi \urcorner \in \mathfrak{p}$ iff there is a possible present $\mathfrak{t}$ such that $\mathfrak{t} \rhd \mathfrak{p}$ and $\varphi \in \mathfrak{t}$.*

Proof   Before starting with the proof itself, let me note two useful facts that can be established by copying the corresponding proofs for the modal system K. The first is that the tense operators are *propositional* connectives that always take the same truth values on logically equivalent sentences:

$$\text{If } \vdash \varphi \to \psi \text{ then } \vdash P\varphi \to P\psi \text{ and } \vdash F\varphi \to F\psi. \tag{4}$$

The second useful fact is that both 'P' and 'F' distribute over disjunctions:

$$\vdash P(\varphi \vee \psi) \leftrightarrow (P\varphi \vee P\psi) \qquad \vdash F(\varphi \vee \psi) \leftrightarrow (F\varphi \vee F\psi). \tag{5}$$

One would get an even weaker system than Z by using (4) in place of the temporal generalization rules, but that system turns out to be too weak for current purposes.

Turning now to the proof of the proposition, it clearly suffices to prove the first claim. The second one could then be shown in an exactly parallel manner, by replacing 'P' and '$\lhd$' with 'F' and '$\rhd$', respectively. Moreover, the right-to-left direction of the first biconditional is trivial. Suppose that there is a maximal Z-consistent $\mathfrak{t}$ such that $\mathfrak{t} \lhd \mathfrak{p}$ and $\varphi \in \mathfrak{t}$. By definition of '$\lhd$', it follows that $\ulcorner P\varphi \urcorner \in \mathfrak{p}$.

The nontrivial part of the proof is the left-to-right direction. Suppose that $\ulcorner P\varphi \urcorner \in \mathfrak{p}$. It needs to be shown that there is a set $\mathfrak{t}$ that is maximal, consistent, earlier than $\mathfrak{p}$, and contains φ. To prove this, let $\psi_1, \psi_2, \dots$ be an enumeration of the sentences of the language. (It is important for the proof that the language of tense logic be countable.) Given any finite set of sentences $\mathfrak{s}$, let $\ulcorner \bigwedge \mathfrak{s} \urcorner$ be the conjunction of the elements of $\mathfrak{s}$. Since conjunctions commute and since the tense operators satisfy (4), the order in which one conjoins the elements of $\mathfrak{s}$ does not matter for the proof. Now consider the following sequence of sets:

$$\mathfrak{t}_0 = \{\varphi\} \qquad \mathfrak{t}_{n+1} = \begin{cases} \mathfrak{t}_n \cup \{\psi_n\} & \text{if } \ulcorner P(\bigwedge \mathfrak{t}_n \wedge \psi_n) \urcorner \in \mathfrak{p} \\ \mathfrak{t}_n \cup \{\ulcorner \neg \psi_n \urcorner\} & \text{otherwise.} \end{cases}$$

The union $\mathfrak{t} = \bigcup_{i \in \mathbb{N}} \mathfrak{t}_i$ is then a possible present with the desired properties. By construction, $\mathfrak{t}$ is maximal and it contains φ. It remains to be shown that it is Z-consistent and that it is earlier than $\mathfrak{p}$.

Let me first show that $\mathfrak{t}$ is earlier than $\mathfrak{p}$. Given any $\lambda \in \mathfrak{t}$, it needs to be shown that $\ulcorner P\lambda \urcorner \in \mathfrak{p}$. By construction of $\mathfrak{t}$, there is some n such that $\lambda \in \mathfrak{t}_n$. Since $\vdash \bigwedge \mathfrak{t}_n \to \lambda$ by truth-functional logic, (4) yields

$$\vdash P \bigwedge \mathfrak{t}_n \to P\lambda.$$

In light of (3), it would suffice to show that $\mathfrak{p}$ contains the antecedent of this conditional. In that case, $\mathfrak{p}$ would also have to contain the consequent. There is no general method for determining the value of n for an arbitrary λ, but that does not matter. By induction on n, one can show that $\ulcorner P \bigwedge \mathfrak{t}_n \urcorner \in \mathfrak{p}$ for all n. The base case of $n = 0$ is trivial because $\mathfrak{t}_0 = \{\varphi\}$ and $\ulcorner P\varphi \urcorner \in \mathfrak{p}$ by assumption. For the inductive step, assume that $\ulcorner P \bigwedge \mathfrak{t}_n \urcorner \in \mathfrak{p}$. Then either $\ulcorner P(\bigwedge \mathfrak{t}_n \wedge \psi_n) \urcorner$

is contained in $\mathfrak{p}$, or it is not. If it is, then $\mathfrak{t}_{n+1} = \mathfrak{t}_n \cup \{\psi_n\}$ and it trivially follows that $\ulcorner P \bigwedge \mathfrak{t}_{n+1} \urcorner \in \mathfrak{p}$. Otherwise, $\mathfrak{t}_{n+1} = \mathfrak{t}_n \cup \{\ulcorner \neg \psi_n \urcorner\}$ and it needs to be shown that $\ulcorner P(\bigwedge \mathfrak{t}_n \wedge \neg \psi_n) \urcorner \in \mathfrak{p}$. One can prove this by contradiction. Suppose that both $\ulcorner P(\bigwedge \mathfrak{t}_n \wedge \psi_n) \urcorner \notin \mathfrak{p}$ and $\ulcorner P(\bigwedge \mathfrak{t}_n \wedge \neg \psi_n) \urcorner \notin \mathfrak{p}$. By maximality of $\mathfrak{p}$, this means that both $\ulcorner \neg P(\bigwedge \mathfrak{t}_n \wedge \neg \psi_n) \urcorner \in \mathfrak{p}$ and $\ulcorner \neg P(\bigwedge \mathfrak{t}_n \wedge \psi_n) \urcorner \in \mathfrak{p}$ must be the case, and hence $\ulcorner \neg (P(\bigwedge \mathfrak{t}_n \wedge \neg \psi_n) \vee P(\bigwedge \mathfrak{t}_n \wedge \psi_n)) \urcorner \in \mathfrak{p}$ by (3). With (5), this yields $\ulcorner \neg P((\bigwedge \mathfrak{t}_n \wedge \neg \psi_n) \vee (\bigwedge \mathfrak{t}_n \wedge \psi_n)) \urcorner \in \mathfrak{p}$. Since $\vdash \bigwedge \mathfrak{t}_n \rightarrow ((\bigwedge \mathfrak{t}_n \wedge \neg \psi_n) \vee (\bigwedge \mathfrak{t}_n \wedge \psi_n))$ by truth-functional logic, this yields $\ulcorner \neg P \bigwedge \mathfrak{t}_n \urcorner \in \mathfrak{p}$ by (4) and (3). Given that $\mathfrak{p}$ is Z-consistent, this contradicts the inductive assumption that $\ulcorner P \bigwedge \mathfrak{t}_n \urcorner \in \mathfrak{p}$, and it follows that $\ulcorner P \bigwedge \mathfrak{t}_{n+1} \urcorner \in \mathfrak{p}$. Hence $\ulcorner P \lambda \urcorner \in \mathfrak{p}$ and thus $\mathfrak{t} \lhd \mathfrak{p}$.

To prove that $\mathfrak{t}$ is also Z-consistent, suppose that it is not. Then there are sentences $\alpha_1, \ldots, \alpha_k \in \mathfrak{t}$ such that $\vdash \neg(\alpha_1 \wedge \cdots \wedge \alpha_k)$, which entails $\vdash H \neg(\alpha_1 \wedge \cdots \wedge \alpha_k)$ by temporal generalization. (This is where one needs the temporal generalization rules rather than the weaker principle (4).) The definition of 'H' then yields $\vdash \neg P(\alpha_1 \wedge \cdots \wedge \alpha_k)$ and thus $\ulcorner \neg P(\alpha_1 \wedge \cdots \wedge \alpha_k) \urcorner \in \mathfrak{p}$ by (3). But that cannot be the case. By construction of $\mathfrak{t}$, there must be some $\mathfrak{t}_n$ that contains all the α_i, and for which $\vdash \bigwedge \mathfrak{t}_n \rightarrow (\alpha_1 \wedge \cdots \wedge \alpha_k)$. Hence $\vdash P \bigwedge \mathfrak{t}_n \rightarrow P(\alpha_1 \wedge \cdots \wedge \alpha_k)$ by (4). In establishing the pastness of $\mathfrak{t}$, it was shown that $\ulcorner P \bigwedge \mathfrak{t}_n \urcorner \in \mathfrak{p}$ for all n, which entails $\ulcorner P(\alpha_1 \wedge \cdots \wedge \alpha_k) \urcorner \in \mathfrak{p}$ by (3), which means that $\mathfrak{p}$ is not Z-consistent. Since it was assumed that it is, $\mathfrak{t}$ must be Z-consistent as well. $\qquad \square$

I have defined what it is for a set of sentences to be earlier or later than $\mathfrak{p}$, but I have not yet said what it is for such a set to be a *time*. A seemingly obvious proposal would be to say that a set of sentences is a time if and only if it is maximal, Z-consistent, and either earlier than *or* later than *or* identical with the choice of present time $\mathfrak{p}$. But this would needlessly prejudge the controversial question of whether it is possible for the time-series to branch. If branches are possible then there can be a time that is future relative to some past time, but is itself neither past, present, or future relative to the present time because it is located in some other branch of time. There are stronger systems of tense logic that prohibit branching times, but Z does not. Consider the following partial interpretation:

$$\mathfrak{p} = \{\neg \varphi, \neg \psi, F(\varphi \wedge H \neg \psi \wedge \neg \psi \wedge G \neg \psi), F(\psi \wedge H \neg \varphi \wedge \neg \varphi \wedge G \neg \varphi), \ldots\}.$$

In this case, there is one future time in which φ is true, another one in which ψ is true, but neither of them is future or past relative to the other. Let me therefore admit branches for now and define the *time-series* $\mathfrak{T}_\mathfrak{p}$ to be the minimal closure of the interpretation $\mathfrak{p}$ under the earlier-than and later-than relations.

Definition 3.2 The *time-series* $\mathfrak{T}_\mathfrak{p}$ derived from the interpretation $\mathfrak{p}$ is the set of all possible presents $\mathfrak{t}$ for which there is a sequence $\mathfrak{s}_0, \mathfrak{s}_1, \ldots, \mathfrak{s}_n$ of possible presents such that $\mathfrak{s}_0 = \mathfrak{p}$, $\mathfrak{s}_n = \mathfrak{t}$, and either $\mathfrak{s}_{i+1} \lhd \mathfrak{s}_i$ or $\mathfrak{s}_{i+1} \rhd \mathfrak{s}_i$ for all $0 \leq i < n$.

Since $\mathfrak{T}_\mathfrak{p}$ need not be linearly ordered by either $\lhd$ or $\rhd$, it is perhaps a bit misleading to call it a time-*series*, but there is no need to worry about this here. Section 6 discusses various ways of ruling out branching times.

4 Models and Interpretations

The models and interpretations of Z are different entities, but they are intimately related. If one constructs the time-series as in Definition 3.2 then one can use set

membership as the true-at relation by saying that a sentence φ is true at a time $\mathfrak{t}$ in $\mathfrak{T}_\mathfrak{p}$ if and only if $\varphi \in \mathfrak{t}$. In this way, every time-series defines a model.

Proposition 4.1 $\mathfrak{M}_\mathfrak{p} = \langle \mathfrak{T}_\mathfrak{p}, \mathfrak{p}, \lhd, \rhd, \in \rangle$ *is a model.*

Proof   It needs to be shown that, when restricted to $\mathfrak{T}_\mathfrak{p}$, set-membership $\in$ satisfies (1). The first two conditions easily follow from the maximality and Z-consistency of the elements of $\mathfrak{T}_\mathfrak{p}$. For the third clause, consider the claim that $\ulcorner \mathsf{P}\varphi \urcorner \in \mathfrak{t}$. Since $\mathfrak{t}$ itself is a maximal Z-consistent set, Proposition 3.1 entails that there is a maximal Z-consistent set $\mathfrak{t}'$ such that $\mathfrak{t}' \lhd \mathfrak{t}$ and $\varphi \in \mathfrak{t}'$. Moreover, since $\mathfrak{t} \in \mathfrak{T}_\mathfrak{p}$, and since $\mathfrak{T}_\mathfrak{p}$ is closed under $\lhd$, it follows that $\mathfrak{t}' \in \mathfrak{T}_\mathfrak{p}$. The proof of the fourth clause is similar.   □

While every time-series gives rise to a model of tense logic, not every model can be obtained in this way. The times in $\mathfrak{M}_\mathfrak{p}$ are always sets of sentences and there is no such restriction on what one can take as the "times" in a model. There are more models than time-series, but this extra structure is irrelevant as far as the model theory for Z is concerned. Say that two models M and M' are *equivalent*, $M \approx M'$, if and only if the same sentences are true in the two models. Given a model M, let $[M]$ be the class of all models to which M bears the relation $\approx$.

Proposition 4.2   *The map* $\mathfrak{p} \mapsto [\mathfrak{M}_\mathfrak{p}]$ *is a bijection from interpretations to equivalence classes of models.*

Proof   Suppose that $\mathfrak{p}$ and $\mathfrak{q}$ are two different interpretations. Since $\mathfrak{p}$ and $\mathfrak{q}$ are maximal consistent sets of sentences, one can only have $\mathfrak{p} \neq \mathfrak{q}$ if there is a φ such that $\varphi \in \mathfrak{p}$ but $\varphi \notin \mathfrak{q}$. So φ is true in $\mathfrak{M}_\mathfrak{p}$ and false in $\mathfrak{M}_\mathfrak{q}$, which entails $\mathfrak{M}_\mathfrak{p} \not\approx \mathfrak{M}_\mathfrak{q}$ and thus $[\mathfrak{M}_\mathfrak{p}] \neq [\mathfrak{M}_\mathfrak{q}]$. This shows that the map is injective and that different interpretations always get assigned different classes. To prove that the map is also surjective, it needs to be shown that every equivalence class of models is the image of some interpretation. Suppose that $M = \langle T, p, <, >, @ \rangle$ is a model and let $\mathfrak{r} = \{\varphi : @(\varphi, p)\}$. Since @ satisfies the first condition of (1), $\mathfrak{r}$ is maximal. To prove that it is also Z-consistent, suppose that it is not. Then there are $\alpha_1, \ldots, \alpha_k \in \mathfrak{r}$ such that $\vdash \neg(\alpha_1 \wedge \cdots \wedge \alpha_k)$. By the left-to-right direction of Proposition 2.1, this means that $@(\ulcorner \neg(\alpha_1 \wedge \cdots \wedge \alpha_k) \urcorner, p)$ must be true and thus $@(\ulcorner \alpha_1 \wedge \cdots \wedge \alpha_k \urcorner, p)$ false by (1). Since all the α_i are in $\mathfrak{r}$, $@(\alpha_i, p)$ is true for all $1 \leq i \leq k$. Hence $@(\ulcorner \alpha_1 \wedge \cdots \wedge \alpha_k \urcorner, p)$ is true by (1), which is a contradiction. Since $\mathfrak{r}$ is a maximal Z-consistent set, it qualifies as an interpretation. By construction of $\mathfrak{r}$, $\varphi \in \mathfrak{r}$ if and only if $@(\varphi, p)$ for all sentences φ, which yields $\mathfrak{M}_\mathfrak{r} \approx M$ and $[\mathfrak{M}_\mathfrak{r}] = [M]$.   □

With these two propositions at hand, one can now prove a variant of the soundness and completeness theorem that is only concerned with time-series, rather than with all models of tense logic.

Proposition 4.3 (Time-Series Completeness)   *A sentence is a theorem of* Z *if and only if it is true in all time-series.*

Proof   Suppose φ is a theorem of Z. According to Proposition 2.1, φ is then true in all models. Since all time-series are models by Proposition 4.1, it follows that φ is true in all time-series. To prove the right-to-left direction, suppose that φ is true in all time-series. Then it must also be true in all models, since Proposition 4.2 shows that every model is $\approx$-equivalent to some time-series. So there are enough time-series to do the logical work of all the models.   □

5 Is There Enough Time?

What kind of time-series one obtains by this method partially depends on the expressive resources of the underlying language. By choosing a language that possesses more fine-grained predicates, and thus describes the world in more detail, one could get more times than one might obtain otherwise. This much was to be expected from a view that takes times to be sets of sentences. For the tense primitivist, there is no language-independent question of getting the "right" number of times. However, it is important for the plausibility of tense primitivism that the number of times is not unduly limited by their construction.

In the modal case, Quine [24] objects that linguistic ersatzism cannot deliver enough possible worlds to give an acceptable account of possibility. If the sentences of the underlying language are finite sequences of symbols from a countable vocabulary, then there are at most $\beth_1$ many sets of such sentences, and it is easy to show that there are at least $\beth_2$ many different possibilities. Consider a continuum of $\beth_1$ many spatial points. All of these points could be either occupied by matter or not, which yields $\beth_2 = 2^{\beth_1}$ many possible distributions of matter in a continuous space. Hence there are far fewer maximal consistent sets of sentences than there are possibilities. (See [15], p. 90 for further discussion of this argument.)

Since nobody believes there to be more than $\beth_1$ many times, this objection does not carry over to the tense case, but nothing said so far guarantees the existence of time-series of cardinality equal to $\beth_1$ either. Proposition 4.2 only guarantees that each model of size $\beth_1$ has an equivalent model $\mathfrak{M}_{\mathfrak{p}}$, but it does not ensure that there is such a model with cardinality $\beth_1$. It requires an independent argument to establish that one can get a time-series of that size.

Proposition 5.1 *There are time-series with continuum many time points.*

Proof Suppose an object a moves along a smooth curve $\mathscr{C}$ on a continuous spatial manifold without moving twice through the same point. The idea of the proof is to adopt a language that can describe this motion and then to distinguish times by the location of a. Since there are $\beth_1$ many points on $\mathscr{C}$, this yields $\beth_1$ many times. Naïvely, one might try to execute this strategy by adding names for *all* the points on $\mathscr{C}$. But this would require continuum many individual constants, and the construction of times explicitly assumes that the language of tense logic has only countably many sentences. (See, e.g., the construction of the t_n in the proof of Proposition 3.1.) As it turns out, though, one does not need names for all the points on $\mathscr{C}$; it suffices to have names for a countable dense subset.

Let '$<$' stand for the less-than relation on the set of real numbers $\mathbb{R}$, and let f be an order preserving bijection from the real numbers to $\mathscr{C}$ such that the reference object a gets to the point $f(r)$ on $\mathscr{C}$ before it gets to $f(r')$ whenever $r < r'$. If i is the natural injection of the rational numbers $\mathbb{Q}$ into $\mathbb{R}$ then the composite map $f \circ i$ densely injects the rational numbers into $\mathscr{C}$:

$$\mathbb{Q} \xrightarrow{\ i\ } \mathbb{R} \xrightarrow{\ f\ } \mathscr{C}.$$

Call the image of this map the "rational" points in $\mathscr{C}$ (relative to the choice of f). Next, add names $k_0, k_1, \ldots$ for the countably many elements of $\mathbb{Q}$ and introduce a location predicate 'L' via the following schema:

$$L\kappa \text{ iff } a \text{ is located at } f(i(\kappa)).$$

Here κ stands for any of the names of the rational numbers that have just been added to the language. Similar to Dedekind's construction of the reals, one can then identify the points on $\mathscr{C}$ with the set of rational points that a needed to pass in order to get to the point in question. Suppose that c and c' are two different points on $\mathscr{C}$. Let t be a time that contains $\ulcorner PL\kappa \urcorner$ if and only if $i(\kappa) \leq f^{-1}(c)$, that is, if and only if $f(i(\kappa))$ is one of the rational points on $\mathscr{C}$ that a had to pass in order to get to c. Similarly, let t' be a time that contains $\ulcorner PL\kappa \urcorner$ if and only if $i(\kappa) \leq f^{-1}(c')$. Since the rational points are dense in $\mathscr{C}$, there is some rational number $\tilde{\kappa}$ such that a needs to pass $f(i(\tilde{\kappa}))$ to get from c to c'. Since $f^{-1}(c) < i(\tilde{\kappa}) < f^{-1}(c')$, it follows that $\ulcorner PL\tilde{\kappa} \urcorner \in t'$ and $\ulcorner PL\tilde{\kappa} \urcorner \notin t'$. Hence $t \neq t'$, which means that there are at least as many times as there are points on $\mathscr{C}$, of which there are $\beth_1$ many. $\qquad\square$

6 Stronger Systems of Tense Logic

The construction of times can be extended to stronger systems of tense logic. After adding another axiom schema E to Z, one could easily repeat the entire construction of times with the tense logic Z+E in place of Z. But since any maximal Z+E consistent set is also Z consistent, all the Z+E time-series one gets in this way are already included among the Z time-series.

Proposition 6.1   *The following claims are equivalent:*

1. $\mathfrak{T}_{\mathfrak{p}}$ *is a* Z+E *time-series;*
2. $\mathfrak{T}_{\mathfrak{p}}$ *is a* Z *time-series with* Z+E-*consistent* $\mathfrak{p}$;
3. $\mathfrak{T}_{\mathfrak{p}}$ *is a* Z *time-series all of whose times contain all instances of* E.

This yields two alternative methods for identifying the Z+E time-series. One can either regard them as Z time-series with Z+E consistent presents, or as Z time-series in which all instances of E are always true.

Before I come to the proof of this proposition, let me note that it is not obvious that (3) entails (2). If every time in $\mathfrak{T}_{\mathfrak{p}}$ contains all instances of E then also $\mathfrak{p}$ must contain all instances of E, but that by itself does not guarantee that $\mathfrak{p}$ is Z+E consistent. Z+E treats instances of E as axioms to which the rule of temporal generalization can be applied, and not as simple truths. To establish the equivalence of (2) and (3), one needs to show that being omnitemporally true is "as good as" being an axiom.

Lemma 6.2   *If all the times in a* Z *time-series* $\mathfrak{T}_{\mathfrak{p}}$ *contain all instances of* E *then they contain all theorems of* Z+E.

Proof of Lemma 6.2   Let '$\vdash_E$' denote derivability in Z+E. It needs to be shown that, given the assumption that all instances of E are always true, φ is contained in an arbitrary time t whenever $\vdash_E \varphi$. The proof is by induction on the *nesting degree* of Z+E proofs. Say that an *application* of temporal generalization has nesting degree 1 if the subproof of which it forms the last step contains no other applications of this rule. Otherwise, it has nesting degree $n+1$, where n is the maximal nesting degree of all earlier applications of temporal generalization in the subproof. The nesting degree of a *proof* is the highest nesting degree of the temporal generalizations occurring within it, or zero if it does not use these rules at all.

As the base case of the induction on nesting degree, suppose that there is a Z+E proof of φ of nesting degree zero. If the proof does not appeal to the axiom schema E then this is already a Z proof of φ. Otherwise, let $E_1, \ldots, E_h$ be the instances of E

used in the proof. Since the proof proceeds without any use of temporal generaliza-
tion, one can convert it into a Z proof of $\ulcorner(E_1 \wedge \cdots \wedge E_h) \rightarrow \varphi\urcorner$ by treating the E_i as
assumptions for conditional introduction rather than as axioms. Since all times are
assumed to contain all instances of E and since $\mathfrak{t}$ is maximal Z-consistent, (3) entails
$\ulcorner E_1 \wedge \cdots \wedge E_h\urcorner \in \mathfrak{t}$ and thus $\varphi \in \mathfrak{t}$ with the conditional Z proof of φ.

For the inductive step, suppose that a sentence is contained in $\mathfrak{t}$ whenever its proof
in Z+E has nesting degree $\leq n$. Consider a Z+E proof of a sentence φ with nesting
degree $n + 1$. This proof can then be rewritten so that it consists of the following:

(a) m proofs in Z+E with nesting degree n ending with sentences $\sigma_1, \ldots, \sigma_m$,
respectively. By assumption, $m > 0$.
(b) m applications of the temporal generalization rules appended to the subproofs
in (a), resulting in $\ulcorner\mathsf{O}_1\sigma_1\urcorner, \ldots, \ulcorner\mathsf{O}_m\sigma_m\urcorner$, where the O_i are either 'H' or 'G'.
(c) k proofs in Z+E with nesting degree $\leq n$ ending with sentences $\psi_1, \ldots, \psi_k$,
respectively. The number k may be zero.
(d) One Z proof of the conditional $\ulcorner(\mathsf{O}_1\sigma_1 \wedge \cdots \wedge \mathsf{O}_m\sigma_m \wedge \psi_1 \wedge \cdots \wedge \psi_k) \rightarrow \varphi\urcorner$.
(e) The conclusion that $\vdash_E \varphi$.

By (a) and the inductive assumption, all times contain all of $\sigma_1, \ldots, \sigma_m$. By
Proposition 3.1 and the definitions of $\mathfrak{T}_\mathfrak{p}$, 'H', and 'G', this means that also
$\ulcorner\mathsf{O}_1\sigma_1\urcorner, \ldots, \ulcorner\mathsf{O}_m\sigma_m\urcorner$ must be contained in all times. Since all times contain
$\psi_1, \ldots, \psi_k$ by the inductive assumption, (3) ensures that they also contain

$$\mathsf{O}_1\sigma_1 \wedge \cdots \wedge \mathsf{O}_m\sigma_m \wedge \psi_1 \wedge \cdots \wedge \psi_k.$$

With (d) and (3), it follows that φ is contained in all times, thus completing the
inductive proof that $\varphi \in \mathfrak{t}$ whenever $\vdash_E \varphi$. $\qquad\qquad\square$

Proof of Proposition 6.1 $\mathfrak{T}_\mathfrak{p}$ is a Z+E time series just in case it is the smallest set
of maximal Z+E consistent sets of sentences that contains $\mathfrak{p}$ and is closed under $\lhd$
and $\rhd$. To show the equivalence of (1) and (2), it therefore suffices to show that a
maximal Z consistent set $\mathfrak{t}'$ must be Z+E consistent whenever $\mathfrak{t}$ is Z+E consistent
and either $\mathfrak{t}' \lhd \mathfrak{t}$ or $\mathfrak{t}' \rhd \mathfrak{t}$. That way, any Z+E time-series is guaranteed to be a Z
time-series, and any Z time-series constructed from a Z+E consistent present must
be a Z+E series. To prove this, suppose that $\mathfrak{t}' \lhd \mathfrak{t}$ but that $\mathfrak{t}'$ is not Z+E consistent.
Then there are sentences $\alpha_1, \ldots, \alpha_k \in \mathfrak{t}'$ such that $\vdash_E \neg(\alpha_1 \wedge \cdots \wedge \alpha_k)$. This entails
$\vdash_E \mathsf{H}\neg(\alpha_1 \wedge \cdots \wedge \alpha_k)$ by temporal generalization, $\vdash_E \neg\mathsf{P}(\alpha_1 \wedge \cdots \wedge \alpha_k)$ by definition
of 'H', and thus $\ulcorner\neg\mathsf{P}(\alpha_1 \wedge \cdots \wedge \alpha_k)\urcorner \in \mathfrak{t}$ and $\ulcorner\mathsf{P}(\alpha_1 \wedge \cdots \wedge \alpha_k)\urcorner \notin \mathfrak{t}$ because $\mathfrak{t}$ is
maximal Z+E consistent. Since all the α_i are in the Z consistent set $\mathfrak{t}'$, (3) entails
$\ulcorner\alpha_1 \wedge \cdots \wedge \alpha_k\urcorner \in \mathfrak{t}'$ and thus $\ulcorner\mathsf{P}(\alpha_1 \wedge \cdots \wedge \alpha_k)\urcorner \in \mathfrak{t}$ because $\mathfrak{t}' \lhd \mathfrak{t}$. This is a
contradiction, so $\mathfrak{t}'$ must be Z+E consistent. In the same way, one can show that $\mathfrak{t}'$
must be Z+E consistent if $\mathfrak{t}' \rhd \mathfrak{t}$. Hence (1) and (2) are equivalent.

To prove that (2) entails (3), suppose that $\mathfrak{p}$ is Z+E consistent. Then all times in
the Z time-series $\mathfrak{T}_\mathfrak{p}$ are Z+E consistent by the equivalence of (1) and (2) and must
therefore contain all instances of E, which are theorems of Z+E. It remains to be
shown that (3) entails (2). Suppose that the Z time-series $\mathfrak{T}_\mathfrak{p}$ contains all instances
of E. To prove that $\mathfrak{p}$ must then be Z+E consistent, suppose that it is not. Then there
are $\alpha_1, \ldots, \alpha_k \in \mathfrak{p}$ such that $\vdash_E \neg(\alpha_1 \wedge \cdots \wedge \alpha_k)$. With Lemma 6.2, this entails
$\ulcorner\neg(\alpha_1 \wedge \cdots \wedge \alpha_k)\urcorner \in \mathfrak{p}$. Since the α_i are in $\mathfrak{p}$, (3) entails $\ulcorner\alpha_1 \wedge \cdots \wedge \alpha_k\urcorner \in \mathfrak{p}$, which
contradicts the assumption that $\mathfrak{p}$ is Z consistent. Hence $\mathfrak{p}$ is Z+E consistent if all
times in $\mathfrak{T}_\mathfrak{p}$ contain all instances of E. $\qquad\qquad\square$

Section 2 listed axioms one could add to Z to ensure that the relations $<$ and $>$ in the models of tense logic have certain structural properties. With the help of Proposition 6.1, one can replicate these results for time-series. To illustrate how this works, let me give one representative example, a time-series version of Proposition 2.2.

Proposition 6.3 (Time-Series Completeness)   *A sentence is a theorem of* Z^+ *if and only if it is true in all* Z *time-series in which* $\lhd$ *and* $\rhd$ *are transitive.*

Proof   Suppose that φ is a theorem of Z^+. By Proposition 2.2, φ is then true in all models with transitive $<$ and $>$. Since Proposition 4.1 guarantees that the Z time-series are included among the models, it follows that φ is true in all Z time-series with transitive relations $\lhd$ and $\rhd$. To prove the right-to-left direction, one can use a similar strategy as in the proof of Proposition 4.2. Suppose that φ is true in a model $M = \langle T, p, <, >, @ \rangle$ with transitive $<$ and $>$. It needs to be shown that there is a time-series with transitive $\lhd$ and $\rhd$ that is $\approx$-equivalent to M. Let $\mathfrak{r} = \{\psi : @(\psi, p)\}$. By construction, $\mathfrak{r}$ is maximal. To prove that it is also Z^+-consistent, suppose that it is not. Then there are $\alpha_1, \ldots, \alpha_k \in \mathfrak{r}$ such that $\neg(\alpha_1 \wedge \cdots \wedge \alpha_k)$ is a theorem of Z^+. Because $<$ and $>$ are transitive on M, Proposition 2.2 guarantees that every theorem of Z^+ is true in M. Hence $@(\ulcorner \neg(\alpha_1 \wedge \cdots \wedge \alpha_k) \urcorner, p)$ and thus $@(\ulcorner \alpha_1 \wedge \cdots \wedge \alpha_k \urcorner, p)$ false by (1). Since all the α_i are in $\mathfrak{r}$, $@(\alpha_i, p)$ is true for all $1 \leq i \leq k$. With (1), this yields $@(\ulcorner \alpha_1 \wedge \cdots \wedge \alpha_k \urcorner, p)$, which is a contradiction. Hence $\mathfrak{r}$ is maximal Z^+-consistent. Now consider the time-series $\mathfrak{M}_{\mathfrak{r}}$ obtained from this maximal Z^+-consistent set $\mathfrak{r}$. By construction, $\mathfrak{M}_{\mathfrak{r}}$ is equivalent to M. It remains to be shown that $\lhd$ and $\rhd$ are transitive on $\mathfrak{M}_{\mathfrak{r}}$. Since $\mathfrak{r}$ is Z^+-consistent, Proposition 6.1 entails that all instances of

$$\mathsf{PP}\varphi \rightarrow \mathsf{P}\varphi \qquad \mathsf{FF}\varphi \rightarrow \mathsf{F}\varphi$$

are true at all times in $\mathfrak{T}_{\mathfrak{r}}$. Next, suppose that $\mathfrak{t}, \mathfrak{t}', \mathfrak{t}'' \in \mathfrak{T}_{\mathfrak{r}}$ are such that $\mathfrak{t}'' \lhd \mathfrak{t}'$ and $\mathfrak{t}' \lhd \mathfrak{t}$. Let φ be any element of $\mathfrak{t}''$. Then $\ulcorner \mathsf{P}\varphi \urcorner \in \mathfrak{t}'$ because $\mathfrak{t}'' \lhd \mathfrak{t}'$ and $\ulcorner \mathsf{PP}\varphi \urcorner \in \mathfrak{t}$ because $\mathfrak{t}' \lhd \mathfrak{t}$. Since $\ulcorner \mathsf{PP}\varphi \rightarrow \mathsf{P}\varphi \urcorner \in \mathfrak{t}$, (3) entails $\ulcorner \mathsf{P}\varphi \urcorner \in \mathfrak{t}$ and thus $\mathfrak{t}'' \lhd \mathfrak{t}$. Hence $\lhd$ is transitive. The proof for $\rhd$ is analogous. This shows that every model in which $<$ and $>$ are transitive has an $\approx$-equivalent time-series with transitive $\lhd$ and $\rhd$. So a sentence φ can only be true in all time-series with transitive $\lhd$ and $\rhd$ if it is true in all models in which $<$ and $>$ are transitive. By Proposition 2.2, this requires φ to be a theorem of Z^+. □

We can show similar results for the other axioms and properties listed in Section 2. In particular, one could rule out branching time-series by adopting axioms that ensure the comparability of the earlier-than and the later-than relations.

7 Times in the Object Language

So far, all talk about times took place in the metalanguage of tense logic. To bring this down to the object level, let me extend the vocabulary of the language of tense logic by adding the following:

$\mathfrak{a}, \mathfrak{b}, \ldots, \mathfrak{s}$	dates (singular terms for times)
$\mathfrak{t}, \mathfrak{u}, \ldots, \mathfrak{z}$	time variables
$\forall, \exists$	time quantifiers
$\lhd, \rhd$	earlier-than and later-than relations
$\mid$	true-at operator.

The true-at operator '|' is due to Myro [20] and [21]. To form a sentence, it needs to be complemented with a date α on the left and a sentence φ on the right, and expressions of the form $\ulcorner \alpha \mid \varphi \urcorner$ are read as "at time α, φ." The sentences of the extended language then comprise all and only the following:

1. sentences of the underlying tense logic,
2. expressions of the form $\ulcorner \alpha \mid \varphi \urcorner$, $\ulcorner \alpha \lhd \beta \urcorner$, and $\ulcorner \alpha \rhd \beta \urcorner$, where α and β are dates and φ is a sentence of the underlying tense logic,
3. truth-functional compounds of sentences,
4. expressions that can be obtained from a sentence by using time quantifiers and variables to quantify into a date-position.

A model for this extended language is a pair $\langle M, c \rangle$, where $M = \langle T, p, <, >, @ \rangle$ is a model of ordinary tense logic and c is a *calendar* map that assigns the maximal Z consistent set $\{\varphi : @(\varphi, p)\}$ to the date $\mathfrak{p}$ ("the present"). To all other dates, the map assigns an arbitrary element of the time-series $\mathfrak{T}_{\{\varphi:@(\varphi,p)\}}$. A sentence σ of this extended language is said to be true in the model $\langle M, c \rangle$ if and only if

1. σ is a sentence of the original tense logic and $@(\sigma, p)$;
2. σ is of the form $\ulcorner \alpha \mid \varphi \urcorner$ and $\varphi \in c(\alpha)$;
3. σ is of the form $\ulcorner \alpha \lhd \beta \urcorner$ and $\ulcorner \mathsf{P}\varphi \urcorner \in c(\beta)$ for all $\varphi \in c(\alpha)$;
4. σ is of the form $\ulcorner \alpha \rhd \beta \urcorner$ and $\ulcorner \mathsf{F}\varphi \urcorner \in c(\beta)$ for all $\varphi \in c(\alpha)$;
5. σ is of the form $\ulcorner \neg\varphi \urcorner$ and φ is not true in $\langle M, c \rangle$;
6. σ is of the form $\ulcorner \varphi \to \psi \urcorner$ and either $\ulcorner \neg\varphi \urcorner$ or ψ is true in $\langle M, c \rangle$;
7. σ is of the form $\ulcorner \forall \xi \varphi \urcorner$ and $\varphi[\alpha/\xi]$ is true in all α-variants of $\langle M, c \rangle$.

If α is a date then the model $\langle M^*, c^* \rangle$ is an α-*variant* of $\langle M, c \rangle$ if and only if $M = M^*$ and c and c^* agree everywhere with the possible exception of what they assign to α.

Valid sentences of the extended language include all the theorems of Z and all theorems of standard quantificational logic, adopted to time quantifiers and variables. Since the time variables range over maximal consistent sets, the valid sentences also include all instances of the schemata

$$\alpha \mid (\varphi \to \psi) \to (\alpha \mid \varphi \to \alpha \mid \psi) \qquad \neg\alpha \mid \varphi \leftrightarrow \alpha \mid \neg\varphi.$$

For any date α, $\ulcorner \alpha \mid \ \ \urcorner$ is thus a propositional operator that always takes the same truth-values on logically equivalent sentences of the underlying tense logic. Since calendar maps are defined such that they always assign the present time $\{\varphi : @(\varphi, p)\}$ of the time-series $\mathfrak{T}_{\{\varphi:@(\varphi,p)\}}$ to the date $\mathfrak{p}$, any sentence that is true at that particular time is true simpliciter:

$$\mathfrak{p} \mid \varphi \leftrightarrow \varphi.$$

Finally, Proposition 3.1 yields the valid schemata

$$\alpha \mid \mathsf{P}\varphi \leftrightarrow \exists \xi (\xi \lhd \alpha \wedge \xi \mid \varphi) \qquad \alpha \mid \mathsf{F}\varphi \leftrightarrow \exists \xi (\xi \rhd \alpha \wedge \xi \mid \varphi).$$

Since the underlying tense logic Z imposes no restrictions on them, there are no valid schemata for $\lhd$ and $\rhd$ alone, but all of this can easily be extended to stronger systems of tense logic by using the same strategies as in Section 6.

With this stronger background logic in place, the tense primitivist can now easily rebut the objection that his account lacks the expressive resources to serve as a

theory of time. Whenever his opponent uses explicit quantification over times to explain some temporal notion, the tense primitivist would do exactly the same. The only difference is that his temporal variables range over his "ersatz" times, rather than, say, the metaphysically basic time points postulated by a temporal substantivalist. In particular, the tense primitivist can give an account of any tense operator that can be characterized model-theoretically. Since '|' mirrors all relevant features of the metalinguistic device '@' in the object language, all he needs to do is to employ $\ulcorner t \,|\, \varphi \urcorner$ whenever Section 2 used $\ulcorner @(\varphi, t) \urcorner$. This does not yield a sentence-by-sentence translation of every temporal claim into his original tense logic, but this is not something a tense primitivist needs anyway. What is important is only that the additional structure does not expand the temporal ideology or ontology of his account. Given that two maximal consistent sets of sentences of tense logic only differ if they disagree about the truth-value of some sentence of the underlying tense logic, the tense primitivist would still be entitled to his view that 'P' and 'F' capture everything there is about time. Expressive limitations would only be a problem for the tense primitivist if they concerned recognizably temporal matters.

8 "Since" and "Until" Revisited

By employing this strategy, a tense primitivist could now give the following definition of the operators "since" and "until" discussed in Section 2:

$$
\begin{aligned}
\xi \,|\, \mathsf{S}\varphi\psi &\quad\text{iff}\quad \exists \xi'(\xi' \lhd \xi \wedge (\xi' \,|\, \varphi) \wedge \forall \xi''(\xi' \lhd \xi'' \lhd \xi \rightarrow (\xi'' \,|\, \psi))); \\
\xi \,|\, \mathsf{U}\varphi\psi &\quad\text{iff}\quad \exists \xi'(\xi' \rhd \xi \wedge (\xi' \,|\, \varphi) \wedge \forall \xi''(\xi' \rhd \xi'' \rhd \xi \rightarrow (\xi'' \,|\, \psi))).
\end{aligned}
\tag{6}
$$

To clarify how this is compatible with Kamp's ([12], Thm. IV.1) result that 'S' and 'U' cannot be expressed in terms of 'P' and 'F', let me take a closer look at what Kamp's proof actually establishes. Kamp is considering a scenario where 'S' and 'U' are added as additional primitive tense operators to the underlying tense logic. The new operators are governed by (2) and the question is whether they can be eliminated in favor of 'P' and 'P'. What I am proposing here, though, is to leave the underlying tense logic alone and to define 'S' and 'U' via (6) in the extended object language. There are cases where these rival definitions come apart.

Kamp discusses a model $M_K = \langle \mathbb{R}, 0, <, >, @_K \rangle$ of tense logic that is isomorphic to the real numbers. In this model, there are two atomic sentences π_0 and π_1 such that $@_K(\pi_0, t)$ holds if and only if t is an even integer and $@_K(\pi_1, t)$ if and only if t is an odd integer.

	π_1	π_0	π_1	π_0	π_1	π_0	π_1	
$\cdots$	●	○	●	○	●	○	●	$\cdots$
	-3	-2	-1	0	1	2	3	
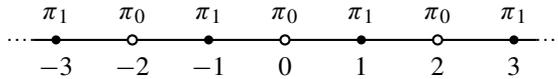
Let η be an arbitrary sentence of the P/F tense logic that contains no atomic sentences *other than* π_0 and π_1. Kamp proves that any such η has the same truth value at all times in the model that are *not* integers. Since $\ulcorner \mathsf{S}\pi_0 \neg \pi_1 \urcorner$ is false at time $-1/2$ and true at time $+1/2$, neither of which is integer, it follows that there is no η that is logically equivalent to this sentence. Similar remarks apply to $\ulcorner \mathsf{U}\pi_0 \neg \pi_1 \urcorner$, which is true at $-1/2$ and false at $+1/2$. Kamp concludes that 'S' and 'U', which are defined via (2), cannot be expressed in terms of 'P' and 'F' alone.

Kamp's proof is based on the assumption that whether or not $\ulcorner \mathsf{S}\pi_0 \neg \pi_1 \urcorner$ and $\ulcorner \mathsf{U}\pi_0 \neg \pi_1 \urcorner$ are true at a given time only depends on the truth-values that are assigned to π_0 and π_1 at the various points of the time-series. If one characterizes 'S' and 'U'

via (6) then this puts the cart before the horse. Until the truth-values of all P/F claims have been specified, there is no fact of the matter what times there are. In definition (6), times are maximally consistent sets of sentences rather than whatever objects play the role of "times" in some model. And for there to be two different times $+1/2$ and $-1/2$ in the time-series $\mathfrak{T}_K = \mathfrak{T}_{\{\varphi : @_K(\varphi, 0)\}}$, there must be some P/F-sentence that is true at the one time but not at the other. Since Kamp has shown that there is no sentence containing only π_0 and π_1 that fits this description, this means that it depends on sentences with *other* atomic constituents whether the time-series $\mathfrak{T}_K$ looks like the real numbers.

If there are sentences that distinguish any two "times" in the model M_K, then the time-series $\mathfrak{T}_K$ is indeed isomorphic to $\mathbb{R}$. In this case, the truth-values of sentences containing 'S' and 'U' do not depend on whether they are defined via the model-theoretic clause (2) or via the definition (6) in the extended object language. Everything that can be said in terms of 'S' and 'U' can be said in the extended P/F-language from Section 7. Both $\ulcorner S\varphi\psi\urcorner$ and $\ulcorner U\varphi\psi\urcorner$ could in principle be spelled out in terms of the tense operators 'P' and 'F' alone, but this would involve atomic sentences other than those occurring in φ and ψ.

The matter is different if there are no P/F-sentences that distinguish between, say, $+1/2$ and $-1/2$. Since times inherit their identity conditions from set theory, there can be no two times in $\mathfrak{T}_K$ at which exactly the same sentences are true. Rather than being isomorphic to $\mathbb{R}$, the time-series $\mathfrak{T}_K$ would look more like a circle in which the same sequence of events keeps repeating itself over and over again, and in which every time bears both $\lhd$ and $\rhd$ to every other time.[4]

This is a case where the two definitions come apart. If one uses (6) to define the binary tense operators then $\ulcorner S\pi_0\neg\pi_1\urcorner$ and $\ulcorner U\pi_0\neg\pi_1\urcorner$ are always true in the time-series $\mathfrak{T}_K$. But if one defines them via (2) then these sentences have different truth-values at the "times" of the model M_K.

If times are maximal consistent sets of sentences then the tense primitivist must reject the possibility of noncyclical eternal recurrence.[5] Other accounts of time can make finer distinctions. Take a temporal substantivalist, who postulates a one-dimensional manifold of metaphysically basic time points that exist independently of what is happening within time. The substantivalist can distinguish two types of eternal recurrence: noncyclical recurrence, in which qualitatively indistinguishable periods of the same type succeed each other in a linear fashion, and cyclical recurrence, with only one such period looping back onto itself. The tense primitivist must reject this distinction, but this is not something a temporal substantivalist can hold against him. The substantivalist can only distinguish cyclical from noncyclical recurrence because he believes that the number of times is independent of what is happening within time, and that is precisely what a tense primitivist denies.

The construction of times is meant to address an issue that arises within the tense primitivist's project. Its purpose is not to convert the adherents of rival views of time. What matters is that the account of times entitles the tense primitivist to the view that 'P' and 'F' capture everything that *he* thinks there is to be said about time. Kamp's proof that "since" and "until" cannot be expressed in terms of 'P' and 'F' either concerns unproblematic cases that the tense primitivist can easily account for,

or it deals with scenarios that he must reject as impossible, anyway, because they involve noncyclical eternal recurrence.

9 Actualism and Presentism

In closing, let me briefly compare this account of times with linguistic ersatzism about possible worlds. The aim of this paper was to defend tense primitivism against the objection that it lacks the expressive resources to serve as a theory of time. In a similar way, one could use linguistic ersatzism to defend *modal primitivism* against the charge that there is more to modality than what can be said in terms of '◇' alone. This is one employment for this account of possible worlds, but linguistic ersatzism is more frequently thought of as a way of being an *actualist*. Unlike the modal primitivist, who takes modal operators as conceptually primitive, the actualist wants to eliminate these operators in favor of actually existing abstract possible worlds.

To use linguistic ersatzism in this way, one needs a slightly different version of the view than the one Roper [26] presents. On Roper's account, possible worlds are maximal consistent sets of sentences of *modal* logic, which means that his worlds contain sentences that themselves contain modal operators. This simplifies the exposition of the proposal, but it yields larger sets of sentences than the proposal actually needs. It would have been sufficient to consider sentences of the underlying propositional logic only. In this case, possible worlds would be more like Carnap's state descriptions. One could then say that an unmodalized sentence is true in a world if and only if it is an element of it, and that $\ulcorner ◇\varphi \urcorner$ is true if and only if φ is true in some world. This allows one to eliminate '◇' in favor of quantification over sets of sentences that do not themselves contain occurrences of modal operators.

Whatever one might think of this view of modality,[6] the tense case is different. The view of times developed here does not permit an elimination of tense operators in favor of quantification over presently existing abstract times. Every maximal consistent set of sentences of tense logic *contains* a maximal consistent set of sentences of the underlying propositional logic, but what matters for the construction of the time-series are the *other* sentences, those with occurrences of the tense operators. The information contained in these sentences permits the construction of the time-series $\mathfrak{T}_\mathfrak{p}$ from an interpretation $\mathfrak{p}$. One cannot eliminate tense operators in favor of quantification over times because the times must themselves contain the very operators one is trying to eliminate. While linguistic ersatzism about possible worlds might support actualism, the construction of times presented here does not support presentism, the view that only present objects exist. But it does support what it was meant to support: tense primitivism.[7]

10 Conclusion

I have shown that a tense logic as simple as Z has sufficient expressive resources to serve as a theory of time. By treating times as maximal Z consistent sets of sentences, a tense primitivist can acquire all the expressive resources he needs, while upholding his central contention that all temporal distinctions are ultimately to be spelled out in terms of a few conceptually primitive tense operators. By itself, this does not show that tense primitivism is to be preferred over rival accounts of time, but it does establish that it cannot be rejected on purely logical grounds, and that is all that I tried to prove here.

Notes

1. This question must not be confused with the question of whether tense logic can serve as a theory of verb tense in natural languages. Nowadays, most authors seem to agree that tense logic fails as a linguistic theory. See [10] for references and discussion.

2. More details of the logic of "since" and "until" can be found in [6].

3. I show in [18] that the two-dimensional "now" and "then" are eliminable in a P/F tense logic that has sufficient quantificational resources. What these operators contribute to tense logic is the ability to rigidify descriptions, and that effect can also be achieved by different means. The ineliminability of "now" and "then" in the cases considered by Kamp and Vlach therefore says more about their impoverished background logics than it does about the ability of 'P' and 'F' to express all temporal claims.

4. The situation is a little bit more complicated than suggested here. If every time is both later and earlier than every other one then nothing would distinguish points on one side of a given time from those on the other side. So there would be no real substance to the claim that the time-series looks like a *circle*. As [5], Sec. 3.1, and [22], Sec. 3.2, note, a cyclical time cannot be described in terms of transitive earlier-than and later-than relations. A tense logic for cyclical time is developed in [25].

5. Special instances of this are changeless worlds, in which nothing ever changes. Changeless worlds are worlds of eternal recurrence with a period of zero length. In such worlds, the primitivist's time-series would consist of a single point, which is a circle of zero circumference.

6. Apart from the cardinality worry mentioned earlier, all the objections to linguistic ersatzism in the philosophical literature are really directed toward linguistic ersatzism as a way of being an actualist. For instance, Lewis [14], Sec. 3.2, complains that linguistic ersatzism does not get rid of all modal notions because it still assumes the notion of consistency. He also objects that it lacks the resources to deal with "alien" properties, which are properties that are only instantiated in worlds other than the actual one.

7. I discuss presentism and actualism in more detail in [17] and [19].

References

[1] van Benthem, J. F. A. K., "Tense logic and standard logic," *Logique et Analyse. Nouvelle Série*, vol. 80 (1977), pp. 395–437. Zbl 0392.03019. MR 0535908. 201, 204

[2] Burgess, J. P., "Basic tense logic," pp. 1–42 in *Handbook of Philosophical Logic 7*, edited by D. Gabbay and F. Guenthner, vol. 165 of *Synthese Library*, Kluwer, Dordrecht, 2002. 203

[3] Cresswell, M. J., *Entities and Indices*, Kluwer, Dordrecht, 1990. 201

[4] Crossley, J. N., and L. Humberstone, "The logic of 'actually'," *Reports on Mathematical Logic*, vol. 8 (1977), pp. 11–29. Zbl 0389.03010. MR 0485193. 204

[5] van Fraassen, B., *An Introduction to the Philosophy of Time and Space*, Random House, New York, 1970. 217

[6] Gabbay, D., M. Finger, and M. Reynolds, "Advanced tense logic," pp. 43–203 in *Handbook of Philosophical Logic 7*, edited by D. Gabbay and F. Guenthner, vol. 165 of *Synthese Library*, Kluwer, Dordrecht, 2002. 217

[7] Gabbay, D. M., "Model theory for tense logics," *Annals of Mathematical Logic*, vol. 8 (1975), pp. 185–236. Zbl 0307.02014. MR 0371608. 203

[8] Hazen, A., "Expressive completeness in modal language," *Journal of Philosophical Logic*, vol. 5 (1976), pp. 25–46. Zbl 0347.02011. MR 0419170. 204

[9] Hazen, A., "The eliminability of the actuality operator in propositional modal logic," *Notre Dame Journal of Formal Logic*, vol. 19 (1978), pp. 617–22. Zbl 0314.02037. MR 508853. 204

[10] Higginbotham, J., *Tense, Indexicality, and Consequence*, edited by J. Butterfield, The Clarendon Press, Oxford, 1999. 217

[11] Hodes, H. T., "Some theorems on the expressive limitations of modal languages," *Journal of Philosophical Logic*, vol. 13 (1984), pp. 13–26. Zbl 0538.03014. MR 737515. 204

[12] Kamp, H., *Tense Logic and the Theory of Linear Orders*, Ph.D. thesis, University of California, Los Angeles, 1968. 203, 214

[13] Kamp, H., "Formal properties of 'now'," *Theoria*, vol. 37 (1971), pp. 227–73. Zbl 0269.02008. MR 0337496. 203

[14] Lewis, D. K., *On the Plurality of Worlds*, Blackwell, Oxford, 1986. 217

[15] Lewis, D. K., *Counterfactuals*, Harvard University Press, Cambridge, 1973. MR 0421986. 204, 209

[16] Mellor, D. H., *Real Time*, Cambridge University Press, Cambridge, 1981. 201

[17] Meyer, U., "The presentist's dilemma," *Philosophical Studies*, vol. 122 (2005), pp. 213–25. 217

[18] Meyer, U., " 'Now' and 'then' in tense logic," *Journal of Philosophical Logic*, vol. 38 (2009), pp. 229–47. 217

[19] Meyer, U., "Worlds and times," *Notre Dame Journal of Formal Logic*, vol. 47 (2006), pp. 25–37. Zbl 1107.03014. MR 2211180. 217

[20] Myro, G., *Existence and Time*, edited by R. Grandy and R. Warner, The Clarendon Press, Oxford, 1986. 213

[21] Myro, G., "Time and essence," pp. 331–41 in *Studies in Essentialism*, edited by P. French, T. Uehling, and H. Wettstein, vol. 11 of *Midwest Studies in Philosophy*, University of Minnesota Press, Minneapolis, 1986. 213

[22] Newton-Smith, W. H., *The Structure of Time*, Routledge, London, 1980. 217

[23] Quine, W. V., "Ontology and ideology," *Philosophical Studies*, vol. 2 (1951), pp. 11–15. 201

[24] Quine, W. V., "Propositional objects," pp. 139–60 in *Ontological Relativity and Other Essays*, Harvard University Press, Cambridge, 1969. 209

[25] Reynolds, M., "Axiomatisation and decidability of *F* and *P* in cyclical time," *Journal of Philosophical Logic*, vol. 23 (1994), pp. 197–224. Zbl 0796.03030. 217

[26] Roper, A., "Towards an eliminative reduction of possible worlds," *Philosophical Quarterly*, vol. 32 (1982), pp. 45–59. 205, 216

[27] Russell, B., "On order in time," *Proceedings of the Cambridge Philosophical Society*, vol. 32 (1936), pp. 216–28. Zbl 0014.19501. 202

[28] Segerberg, K., "Two-dimensional modal logic," *Journal of Philosophical Logic*, vol. 2 (1973), pp. 77–96. Zbl 0259.02013. MR 0424527. 203

[29] Vlach, F., *'Now' and 'Then': A Formal Study in the Logic of Tense Anaphora*, Ph.D. thesis, University of California, Los Angeles, 1973. 204

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