Notre Dame Journal of Formal Logic
Volume 49, Number 4, 2008

The Sum of Irreducible Fractions with Consecutive Denominators Is Never an Integer in PA^-

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Abstract Two results of elementary number theory, going back to Kürschák and Nagell, stating that the sums $\sum_{i=1}^{k} \frac{m_i}{n+i}$ (with $k \geq 1$, $(m_i, n+i) = 1$, $m_i < n+i$) and $\sum_{i=0}^{k} \frac{1}{m+in}$ (with n, m, k positive integers) are never integers, are shown to hold in PA^-, a very weak arithmetic, whose axiom system has no induction axiom.

A well-known problem in elementary number theory asks one to show that the sum

$$1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

is never an integer for $n \geq 2$. The proof one often finds offered for this fact is based on Chebyshev's theorem (Bertrand's postulate). If one asks for a proof that, more generally, the sum

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+k},$$

with $k \geq 1$, can never be an integer, then the proof based on Chebyshev's theorem needs to be amended. One first notes that, if $k < n$, then the above sum must be less than 1, and thus cannot be an integer, and if $k \geq n$, then one applies the same proof based on Chebyshev's theorem. (This fact seems to have been overlooked by Oblath [8], who wants to use Chebyshev's theorem, but finds himself constrained to use, in case $k < n$, another result, the Sylvester-Schur theorem, as well.) However, Kürschák [5] (see also Pólya and Szegö [9], Problem 251; Sierpinński [10], p. 139; DeTemple [2]; and Farmer [3]) found a much simpler proof which relies on the very simple observation that among any number (≥ 2) of consecutive positive integers there is precisely one which is divisible by the highest power of 2 from among all the

Received January 24, 2008; accepted March 14, 2008; printed September 25, 2008
2000 Mathematics Subject Classification: Primary, 03C62; Secondary, 03B30, 11A05
Keywords: Kaye's PA^-, Kürschák's theorem, Nagell's theorem, weak arithmetic
 10.1215/00294527-2008-021

given numbers. Aside from its didactic use, one may wonder whether Kürschák's proof is not in a very formal way much simpler, that is, whether it does not require simpler methods of proof in the sense of formal logic.

When formalized, arithmetic is usually presented as Peano Arithmetic, which contains an induction axiom schema, stating, loosely speaking, that any set that can be defined by an elementary formula in the language of arithmetic (i.e., in terms of some undefined operation and predicate symbols such as $+$, $\cdot$, 1, 0, $<$), which contains 1, and which contains $n+1$ whenever it contains n, is the set of all numbers. Several weak arithmetics have been studied in which the types of elementary formulas allowed in the definitions of the sets used in induction are restricted by certain syntactic constraints (see D'Aquino [1]). One might think that Kürschák's proof would make it in a weaker formal arithmetic than the one dependent on Chebyshev's theorem. It turns out, in fact, that no amount of induction is needed at all!

To see this, let's first generalize the problem further, along the lines of the generalization in Oblath [7], so that there can be no proof based on Chebyshev's theorem.

Theorem 1 *The sum*

$$\frac{m_0}{n} + \frac{m_1}{n+1} + \cdots + \frac{m_k}{n+k} \tag{1}$$

with $(m_i, n+i) = 1$, $m_i < n+i$, *and* $k \geq 1$, *is never an integer.*

Proof ([5], [2]) Let $a = \max\{\alpha : 2^\alpha | (n+i) \text{ for some } 0 \leq i \leq k\}$. Then 2^a divides exactly one of the numbers $n, n+1, \ldots, n+k$. (If we had $n+i = 2^a(2r+1)$ and $n+j = 2^a(2s+1)$, with $0 \leq i < j \leq k$, then $n+i < 2^a(2r+2) < n+j$, contradicting the maximality assumption on a.) Let $l = \operatorname{lcm}(n, n+1, \ldots, n+k)$. Notice that l must be even, and that only one of the integers $l \cdot \frac{m_i}{n+i}$ is odd (the one with $2^a | (n+i)$). Suppose the sum in (1) is an integer b. Multiplying both (1) and b by l, we obtain on the one hand an odd number, as the sum of (several) even numbers and of one odd number, and on the other the even number lb, which have to be equal. □

Moreover, to make it a theorem of arithmetic, we will do away with the fractions appearing in it, and state it, for all positive $k \in \mathbb{N}$, as φ_k, the following statement (where, for $u \geq 1$, we denote by $\overline{u}$ the term $((\ldots((1+1)+1)+\cdots)+1)$, in which there are u many 1s, and we let $\overline{0} = 0$; the terms $\overline{u}$ will be referred to as *numerals*)

$$(\forall n)(\forall m_0)\ldots(\forall m_k)(\forall p) \bigvee_{i=0}^{k} ((\forall a)(\forall b)\, m_i a \neq (n+\overline{i})b + 1) \vee \bigvee_{i=0}^{k} n + \overline{i} < m_i$$
$$\vee \sum_{i=0}^{k} (m_i \prod_{0 \leq j \leq k, j \neq i} (n + \overline{j})) \neq p \prod_{j=0}^{k} (n + \overline{j}). \tag{2}$$

The first disjunct of this formula states that m_i and $n+\overline{i}$ are not relatively prime (i.e., that $(m_i, n+i) \neq 1$); the second, that one of the conditions $m_i < n+\overline{i}$ does not hold (the case $m_i = n+\overline{i}$ is covered by the first disjunct); and the third one, that the sum in (1) is not equal to p. We have not explicitly stated that $n \neq 0$ and $m_i \neq 0$, as this was not necessary. If $m_i = 0$, then, for all b, we have $(n+\overline{i})b + 1 \neq 0$, so the first disjunct holds. If $n = 0$ and $m_0 \neq 0$, then $n + \overline{0} < m_0$, and the second disjunct holds for $i = 0$.

The arithmetic which we will show that (2) holds in is PA^-, which is expressed in a language containing as undefined operation and predicate symbols only $+$, $\cdot$, 1, 0, and $<$, and whose axioms A1–A15 were presented in Kaye ([4], pp. 16–18). We will repeat them here for the reader's convenience, and we will omit the universal quantifiers for all universal axioms.

A 1 $(x + y) + z = x + (y + z)$

A 2 $x + y = y + x$

A 3 $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

A 4 $x \cdot y = y \cdot x$

A 5 $x \cdot (y + z) = x \cdot y + x \cdot z$

A 6 $x + 0 = x \wedge x \cdot 0 = 0$

A 7 $x \cdot 1 = x$

A 8 $(x < y \wedge y < z) \rightarrow x < z$

A 9 $\neg x < x$

A 10 $x < y \vee x = y \vee y < x$

A 11 $x < y \rightarrow x + z < y + z$

A 12 $(0 < z \wedge x < y) \rightarrow x \cdot z < y \cdot z$

A 13 $(\forall x)(\forall y)(\exists z)\, x < y \rightarrow x + z = y$

A 14 $0 < 1 \wedge (x > 0 \rightarrow (x > 1 \vee x = 1))$

A 15 $x > 0 \vee x = 0$

What is missing from PA^-, and makes it so weak (indeed, the positive cone of every discretely ordered ring is a model of PA^-), is the absence of any form of induction.

The proof that φ_k holds in PA^- will be carried out in an arbitrary model $\mathfrak{M}$ of PA^-. The idea of proof will be to show that all variables that appear in φ_k must be numerals. An essential ingredient of the proof is the following fact, which holds in PA^- (see [4], Lemma 2.7, p. 22), for all positive $k \in \mathbb{N}$,

$$x < \overline{k} \rightarrow x = 0 \vee x = 1 \vee \cdots \vee x = \overline{k-1}, \tag{3}$$

and which allows us to deduce that any element which is bounded from above by a numeral must be a numeral.

Suppose that, for some positive $k \in \mathbb{N}$, φ_k does not hold in $\mathfrak{M}$. Then, for all $i = 0, \ldots, k$, there are m_i, p, a_i and b_i, with $m_i a_i = (n + i)b_i + 1$ and $m_i < n + \overline{i}$, and such that

$$\sum_{i=0}^{k}(m_i \prod_{0 \le j \le k, j \ne i} (n + \overline{j})) = p \prod_{j=0}^{k}(n + \overline{j}). \tag{4}$$

This can be rewritten, by leaving only the first term of the sum on the left-hand side and sending all others to the right-hand side with changed sign (subtraction being meaningful by A13), as $m_0(n + \overline{1}) \ldots (n + \overline{k}) = nq$, where by q we have denoted

$$p \prod_{j=1}^{k}(n + \overline{j}) - (\sum_{i=1}^{k} m_i \prod_{1 \le j \le k, j \ne i} (n + \overline{j})).$$

The product $(n+\overline{1})\ldots(n+\overline{k})$ can also be written as a polynomial in n, whose constant term is $\overline{k!}$, that is, as $nr+\overline{k!}$; thus $m_0(nr+\overline{k!}) = nq$. Given that there are a_0 and b_0 such that $m_0a_0 = nb_0+1$, if we multiply both sides of the equality $m_0(nr+\overline{k!}) = nq$ by a_0 we obtain $(nb_0+1)\overline{k!} = n(a_0q - a_0m_0r)$; thus $\overline{k!} = n(a_0q - a_0m_0r - b_0\overline{k!})$. We know that $\mathfrak{M}$ must contain a copy of $\mathbb{N}$, and it may contain other elements as well, called *nonstandard* numbers. Could n be in $\mathfrak{M}$ but not of the form $\overline{m}$ for some $m \in \mathbb{N}$? If it were such an element of $\mathfrak{M}$, then it would be greater than all $\overline{m}$ with $m \in \mathbb{N}$, and thus so would $n(a_0q - a_0m_0r - b_0\overline{k!})$, unless $a_0q - a_0m_0r - b_0\overline{k!} = 0$, which cannot be the case, as $\overline{k!}$ is not zero. However, $n(a_0q - a_0m_0r - b_0\overline{k!})$ cannot be greater than all $\overline{m}$ with $m \in \mathbb{N}$, for it is equal to such a number, namely, to $\overline{k!}$. Thus n must be an $\overline{m}$ for some $m \in \mathbb{N}$. Given that $m_i < n+\overline{i}$, the m_i must be numerals as well, and thus all variables appearing in (4), except p, must be numerals. Thus p is a numeral as well, for else the right-hand side in (4) would be greater than the left-hand side. This means that in (4) all variables are numerals. However, we know from Kürschák's proof that such an equation, involving only numerals, cannot exist. Thus, for all $k \in \mathbb{N}$, φ_k holds in PA^-.

Another generalization of the original problem, proved by Nagell [6], states that the sum

$$\frac{1}{m} + \frac{1}{m+n} + \frac{1}{m+2n} + \cdots + \frac{1}{m+kn}$$

is never an integer if n, m, k are positive integers. The proof is rather involved and uses both a Kürschák-style argument and Chebyshev's theorem. This statement turns out to be, with $\overline{k}$ instead of k, valid in PA^- as well. To see this, let, for all positive $k \in \mathbb{N}$, ν_k stand for

$$(\forall m)(\forall n)(\forall p)\, m > 0 \wedge n > 0 \rightarrow \sum_{i=0}^{k} \prod_{0\le j\le k, j\ne i} (m+\overline{j}n) \ne p \prod_{0\le j\le k} (m+\overline{j}n), \quad (5)$$

and let $\mathfrak{M}$ be again a model of PA^-. Notice that, if $m > \overline{k}$, then $\neg\nu_k$ cannot hold for any n and p. To see this, suppose that, for some n and p, we had equality in (5). Given that $(m+n)\ldots(m+\overline{k}n)$ is the largest of all the summands on the left-hand side, and there are $\overline{k}$ summands, the sum on the left-hand side is $\le \overline{k}(m+n)\ldots(m+\overline{k}n)$, and thus $< m(m+n)\ldots(m+\overline{k}n)$; thus equality cannot hold in (5). Thus $m \le \overline{k}$, and thus, by (3), m must be standard; that is, it must be $\overline{u}$ for some $0 < u \le k$. It remains to be shown that n must be standard as well. To see this, suppose again that, for some $n > 0$ and p, we have equality in (5). Notice that, since $m(m+2n)\ldots(m+\overline{k}n)$ is the largest product among all $\prod_{0\le j\le k, j\ne i}(m+\overline{j}n)$, for $i = 1, 2, \ldots k$, the sum on the left-hand side of our equality is $\le (m+n)\ldots(m+\overline{k}n) + \overline{k}m(m+2n)\ldots(m+\overline{k}n)$ (with equality if and only if $k=1$). Thus $pm(m+n)\ldots(m+\overline{k}n) \le (m+n+\overline{k}m)(m+2n)\ldots(m+\overline{k}n)$, which implies $pm(m+n) \le m+n+\overline{k}m$. If n were nonstandard, then this inequality were possible only if $pm = 1$, that is, if $p = m = 1$, which is not possible, for in that case the first summand on the left-hand side of (5) is equal to the right-hand side; thus the left-hand side must be larger than the right-hand side, so equality could not have taken place in (5). Now that m, n, k have all been shown to be standard, Nagell's proof implies the truth of our statement, which thus holds in PA^-.

Although the proofs that φ_k and ν_k hold in PA^- can be turned into formal derivations from the axioms of PA^-, their length will depend, in the form presented by us, on k. If there were no proofs with length not depending on and increasing with k, then one may see this as a trade-off for the parsimony in assumptions. Weakening the axiom system would lengthen the proofs.

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Acknowledgments

I thank Andreas Blass for several corrections and suggestions, Justus Diller and an anonymous referee for having pointed out that the proof presented here can be transformed into a laborious syntactic proof.

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