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Approximate Similarities and Poincaré Paradox

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Abstract De Cock and Kerre, in considering Poincaré paradox, observed that the intuitive notion of "approximate similarity" cannot be adequately represented by the fuzzy equivalence relations. In this note we argue that the deduction apparatus of fuzzy logic gives adequate tools with which to face the question. Indeed, a first-order theory is proposed whose fuzzy models are plausible candidates for the notion of approximate similarity. A connection between these structures and the point-free metric spaces is also established.

1 Introduction

The so called paradox of Poincaré refers to indistinguishability by emphasizing that, in spite of common intuition, this relation is not transitive (see [19]). In fact, it is possible that we are not able to distinguish d_1 from d_2, d_2 from $d_3, \ldots, d_{m-1}$ from d_m and, nevertheless, that we have no difficulty in distinguishing d_1 from d_m. Now an immediate solution of this paradox would merely conclude that our intuition about this notion is wrong. A different solution is proposed by fuzzy logic in which the paradoxical effect of the transitivity is avoided by assuming that the indistinguishability is a graded property. Indeed, assume that such a notion is represented by a fuzzy $\otimes$-equivalence, that is, a fuzzy relation $\mathrm{eq} : S \times S \to [0, 1]$ such that, for every x, y, z in S,

$$
\begin{array}{ll}
\mathrm{eq}(x, x) = 1 & \text{(reflexive)} \\
\mathrm{eq}(x, y) = \mathrm{eq}(y, x) & \text{(symmetric)} \\
\mathrm{eq}(x, z) \otimes \mathrm{eq}(y, z) \leq \mathrm{eq}(x, y) & \text{($\otimes$-transitive)}
\end{array}
$$

where S is a nonempty set and $\otimes$ a triangular norm. Also assume that $\mathrm{eq}(d_i, d_{i+1}) = \lambda$ where λ is very close to 1 but different from 1. Then from the proposed properties we can conclude only that $\mathrm{eq}(d_1, d_m) \geq \lambda^{(m-1)}$ where $\lambda^{(m-1)}$ denotes the $m - 1$ power of λ with respect to $\otimes$. Such a conclusion is not a paradox at all. In fact if $\otimes$

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is the Łukasiewicz norm and m such that $\lambda^{(m-1)} = 0$, then it asserts only the trivial inequality $\mathrm{eq}(d_1, d_m) \geq 0$ (a more formal argument can be found in Section 4).

Now, De Cock and Kerre in [3] claim that such a solution is not adequate since the hypothesis $\lambda \neq 1$ is not justified. The argumentation of these authors refers to an example such as the following one. Consider the interval $S = [1.50, 2.50]$ of possible heights a man can have and assume that the notion "approximately equal heights" is modeled by a fuzzy equivalence eq such that $\mathrm{eq}(1.50, 1.51) = \mathrm{eq}(1.51, 1.52) = \cdots = \mathrm{eq}(2.49, 2.50) = \lambda$. Then, in accordance with the fact that we cannot distinguish a difference in heights of less than 0.01, we have to assume that $\lambda = 1$. In fact, we have to differentiate a claim as "1.50 is approximately equal to 1.51" which is completely true from a claim such as "1.50 is equal to 1.51" which is only partially true. Moreover, as observed by Bodenhofer [2],

> Even if a measuring device can give seemingly precise numbers, accuracy is limited due to various external influences. It is not even guaranteed that two measurements of the same person give the same result. So how can we justify that two persons whose heights differ only by two millimetres are given a degree of similarity which is strictly less that 1, while two consecutive measurements of the same person may differ in the same range?

On the other hand, if we admit that $\lambda = 1$, then, by the $\otimes$-transitivity and the fact that $1 \otimes 1 = 1$, we can prove that $\mathrm{eq}(1.50, 2.50) = 1$, that is to say that the height of 1.50 is approximately equal to the height of 2.50. This is clearly an absurdity.

Observe that the same considerations apply to the [0, 1]-*valued equalities* defined by Höhle in [14] and [13], that is, the fuzzy relation satisfying the following axioms:

(e1) $\mathrm{eq}(x, y) \leq \mathrm{eq}(x, x)$,
(e2) $\mathrm{eq}(x, y) = \mathrm{eq}(y, x)$,
(e3) $\mathrm{eq}(x, z) \otimes (\mathrm{eq}(z, z) \to \mathrm{eq}(y, z)) \leq \mathrm{eq}(x, y)$

(where $\to$ is the residuum associated with $\otimes$). Indeed, again the paradox is solved by assuming that $\mathrm{eq}(d_i, d_{i+1}) = \lambda \neq 1$, and again in the case $\lambda = 1$ we are forced to conclude that $\mathrm{eq}(d_1, d_m) = 1$. As a matter of fact, as observed in [2], the criticism of De Cock and Kerre applies to all the fuzzy relations eq such that $\mathrm{kernel(eq)} = \{(x, y) \in S \times S : \mathrm{eq}(x, y) = 1\}$ is a transitive relation.

As an alternative, De Cock and Kerre proposed the distance-based notion of "resemblance relation" in which is emphasized the idea that "The closer two objects are to each other, the more they are (approximately) equal" (see Section 2).

Now, even if I completely agree with the criticisms about the hypothesis $\lambda \neq 1$, there is something unsatisfactory in the definition of resemblance relation. Indeed,

(i) there is no reference to the transitivity while, in my opinion, the basic question is to give a formal representation of our intuition suggesting that indistinguishability is transitive in some way;
(ii) there is a strong reference to a pseudo-metric and this precludes an approach within first-order logic formalisms.

In accordance, in this note I propose to face the question by using first-order fuzzy logic and by admitting a "relaxed" transitivity property. The idea is to take into account the capability of each element to be "distinguished" from the remaining ones. This provides a "solution" to Poincaré paradox whose nature is similar to the solution of the Heap paradox proposed by Goguen [8] and others (for example, see Hájek and Novák [12]).

Finally, in the paper we show that, in spite of their logical nature, the proposed notions can be interpreted in a geometrical setting. Indeed we can connect them with the approach to point-free geometry based on the notion of distance between regions and diameter of a region (see Gerla and Volpe [5] and Gerla [6]).

2 The Resemblance Relations

We define the notion of resemblance relation by referring to the simplified definition given by Klawonn in [15].

Definition 2.1 Let (S, d) be a pseudo-metric space, then a fuzzy relation $e : S \times S \to [0, 1]$ is a *resemblance relation with respect to* d provided that

$$\begin{aligned} &e(x, x) = 1, \\ &e(x, y) = e(y, x), \\ &d(x, y) \le d(z, u) \Rightarrow e(x, y) \ge e(z, u). \end{aligned}$$

Given a pseudo-metric space (S, d) and a real number $\varepsilon \ge 0$, a simple (crisp) example of resemblance relation can be obtained by setting e equal to (the characteristic function of) the relation $\equiv$ defined by setting

$$x \equiv y \Leftrightarrow d(x, y) \le \varepsilon.$$

It is apparent that such a relation is not transitive, in general. A more interesting class of graded resemblance relations can be obtained as follows (see Proposition 7 in [3]).

Proposition 2.2 *Consider a pseudo-metric space* (S, d)*, a real number* $\varepsilon \ge 0$ *and set*

$$\begin{array}{ll} e(x, y) = 1 & \text{if } d(x, y) \le \varepsilon, \\ e(x, y) = 0 & \text{if } d(x, y) \ge 1 + \varepsilon, \\ e(x, y) = 1 - (d(x, y) - \varepsilon) & \text{otherwise.} \end{array}$$

Then e is a resemblance relation with respect to d.

We say that e is *the resemblance relation associated with* (S, d) *and* ε. A more synthetic definition of e is given by the following equation,

$$e(x, y) = 0 \vee (1 \wedge (1 - (d(x, y) - \varepsilon)).$$

A characteristic of these fuzzy relations is that they cannot distinguish small differences and that at the same time they are able to detect sufficiently big differences. This shows the existence of a "fuzzy model" of Poincaré's conditions and, therefore, this gives a solution to the paradox from a semantical point of view. Indeed, by reconsidering the example of heights, we can consider the resemblance e obtained by assuming that $S = [1.50, 2.50]$, $d(x, y) = 20 \cdot |x - y|$, and $\varepsilon = 0.2$. In such a case we have

$$e(1.50, 1.51) = e(1.51, 1.52) = \cdots = e(2.49, 2.50) = 1,$$

while

$$e(1.50, 1.52) = 0.8, e(1.50, 1.53) = 0.6, \ldots,$$
$$e(1.50, 1.56) = 0, \ldots, e(1.50, 2.50) = 0.$$

3 Some Basic Notions in Fuzzy Logic

The reader is assumed to be familiar with the basic notions of fuzzy logic (see, for example, Gerla [4] and Novák et al. [17]). In this section we confine ourselves to list some elementary definitions. We consider as set of truth values the interval $[0, 1]$ equipped with a continuous t-norm $\otimes$, that is, an order-preserving associative, commutative operation such that $x \otimes 1 = x$ (as an example, see Klement et al. [16]). From this operation we define a *residuum operation* $\rightarrow$ by setting

$$x \rightarrow y = \mathrm{Sup}\{z \in [0, 1] : x \otimes z \leq y\},$$

a *negation* by setting $\neg x = x \rightarrow 0$ and a *co-norm* $\oplus$ by setting $x \oplus y = \neg(\neg x \otimes \neg y)$. A basic property of the pair $\otimes$ and $\rightarrow$ is that

$$x \otimes z \leq y \Leftrightarrow z \leq x \rightarrow y.$$

Given $x \in [0, 1]$, we denote by $x^{(n)}$ the n-power of x with respect to $\otimes$; that is, we set $x^{(n+1)} = x \otimes x^{(n)}$ and $x^{(0)} = 1$. In this paper we refer to Łukasiewicz triangular norm defined by setting

$$x \otimes y = (x + y - 1) \vee 0.$$

For such a norm $x \rightarrow y = \min\{1, 1 + y - x\}$, $\neg x = 1 - x$, and $x \oplus y = (x + y) \wedge 1$. Also $x^{(n)} = (n \cdot x - n + 1) \vee 0$ and, therefore, $x^{(n)} = 0$ for every n such that $n \geq 1/(1 - x)$.

The algebraic structure $([0, 1], \otimes, \rightarrow)$ defines a first-order multi-valued logic. The languages are the usual first-order languages of classical logic further extended by logical constants $\{\underline{\lambda} : \lambda \in [0, 1]\}$. The conjunction, disjunction, implication are interpreted by $\otimes, \oplus, \rightarrow$, respectively, and the negation by $\neg$ and the logical constant $\underline{\lambda}$ with λ. Moreover, the universal and existential quantifiers are interpreted by the greatest lower bound and the least upper bound operators, respectively. A *fuzzy interpretation* is a pair (S, I) where I is a map associating

(i) any constant c with an element $I(c)$ of S,
(ii) any n-ary operation symbol h with an n-ary operation $I(h) : S^n \rightarrow S$,
(iii) any n-ary relation symbol r with an n-ary fuzzy relation $I(r) : S^n \rightarrow [0, 1]$.

The fuzzy interpretation (S, I) defines a valuation of the formulas in a truth-functional way (see, for example, [17], [4], and [11]). So, given a formula α whose free variables are among $x_1, \dots, x_n$ and $d_1, \dots, d_n$ in S, the truth value $\mathrm{Val}_{(S,I)}(\alpha, d_1, \dots, d_n)$ of α in $d_1, \dots, d_n$ is defined. We denote by $\mathrm{Val}_{(S,I)}(\alpha)$ the truth value of the universal closure of α. Given a formula α whose free variables are among $x_1, x_2, \dots, x_n$, the *extension* of α is the fuzzy relation $|\alpha| : S^n \rightarrow [0, 1]$ defined by the equation $|\alpha|(d_1, \dots, d_n) = \mathrm{Val}_{(S,I)}(\alpha, d_1, \dots, d_n)$.

Definition 3.1 Denote by F the set of closed formulas; then a *fuzzy system of axioms* or *fuzzy theory*, is a fuzzy subset $\tau : F \rightarrow [0, 1]$ of F. A fuzzy interpretation (S, I) is a *fuzzy model of* τ, in brief, $(S, I) \models \tau$, provided that $\mathrm{Val}_{(S,I)}(\alpha) \geq \tau(\alpha)$ for every $\alpha \in F$. The *logical consequence operator* Lc is defined by setting

$$Lc(\tau)(\alpha) = \mathrm{Inf}\{\mathrm{Val}_{(S,I)}(\alpha) : (S, I) \models \tau\}.$$

It is useful to represent a fuzzy theory τ by the set

$$\mathrm{Sign}(\tau) = \{(\alpha, \lambda) : \alpha \in F, \lambda = \tau(\alpha), \lambda \neq 0\}$$

of *signed formulas*. Then $(S, I) \models \tau$ provided that $\mathrm{Val}_{(S,I)}(\alpha) \geq \lambda$ for every (α, λ) in $\mathrm{Sign}(\tau)$. In the case of a crisp theory, we represent τ simply by listing the elements in the support $\{\alpha \in F : \tau(\alpha) \neq 0\}$.

In this paper we are interested mainly in positive clauses, that is, formulas such as $\forall x_1 \ldots \forall x_n((\alpha_1 \wedge \cdots \wedge \alpha_h) \rightarrow \alpha)$ where $\alpha_1, \ldots, \alpha_h$ and α are atomic. A fuzzy interpretation satisfies such a formula provided that, for every $d_1, \ldots, d_n$ in S,

$$|\alpha_1|(d_1, \ldots, d_n) \otimes \cdots \otimes |\alpha_h|(d_1, \ldots, d_n) \leq |\alpha|(d_1, \ldots, d_n).$$

To define a deduction apparatus for fuzzy logic we refer to the formalization proposed by Pavelka in [18] (see also [4], [8], [17]). The idea is that the notion of inference rule has to be extended by specifying how a constraint on the truth value of a conclusion depends on the available constraints on the truth values of the premises. As an example, we can extend the modus ponens rule by assuming that,

IF	you know that α is true at least at degree λ_1
AND	$\alpha \Rightarrow \beta$ is true at least at degree λ_2,
THEN	you can conclude that β is true at least at degree $\lambda_1 \otimes \lambda_2$,

where α and β are formulas and λ_1 and λ_2 are elements in $[0, 1]$. Also, given two formulas α and β, we extend the classical $\wedge$-introduction rule by assuming that

IF	you know that α is true at least at degree λ_1
AND	that β is true at least at degree λ_2,
THEN	you can conclude that $\alpha \wedge \beta$ is true at least at degree $\lambda_1 \otimes \lambda_2$.

Given a fuzzy theory τ, any proof π of α is evaluated by a number $\mathrm{Val}(\pi, \tau) \in [0, 1]$. This number is a constraint (a lower bound) on the truth value of α depending on the information carried on by τ. A fuzzy theory is *contradictory* if there are two proofs π_1 and π_2 of a formula α and its negation $\neg\alpha$, respectively, such that $\mathrm{Val}(\pi_1, \tau) \otimes \mathrm{Val}(\pi_2, \tau) > 0$. We say that τ is *consistent* if it is not contradictory.

Because different proofs give different constraints, we have to consider a constraint $D(\tau)(\alpha)$ obtained by fusing the totality of the constraints furnished by the proofs of α.

Definition 3.2 Given a fuzzy theory τ and a formula α, we set

$$D(\tau)(\alpha) = \mathrm{Sup}\{\mathrm{Val}(\pi, \tau) : \pi \text{ is a proof of } \alpha\}.$$

The fuzzy subset $D(\tau)$ is interpreted as the *fuzzy subset of formulas we can derive from* τ. The operator D is called *deduction operator*.

We call *axiomatizable* a fuzzy logic such that there is a fuzzy deduction apparatus whose deduction operator D coincides with the logical consequence operator Lc. In such a case a fuzzy theory is consistent if and only if it admits a model. The axiomatizability is an important property for a fuzzy logic since, in contrast with the logical consequence operator, the deduction operator is defined by an "effective" procedure. The effectiveness is expressed, for example, by the fact that if the fuzzy set τ of axioms is decidable, then $D(\tau)$ is recursively enumerable; if τ is also complete, then $D(\tau)$ is decidable (see [7]).

In this paper we refer mainly to Łukasiewicz fuzzy logic and to the axiomatization proposed in [17]. To simplify the writing of the proofs, besides the basic fuzzy inference rules proposed in such a book, we also consider some derivable rules.

4 The Paradox and the Fuzzy Equivalence Relations

Consider a language $\mathcal{L}_E$ with a binary relation symbol E. Then a fuzzy interpretation of $\mathcal{L}_E$ is a pair (S, e) where e is a binary fuzzy relation in S. In such a language we reformulate the definition of fuzzy $\otimes$-equivalence relation given in Section 1 in logical terms.

Definition 4.1 A fuzzy relation $e : S \times S \to [0, 1]$ is a fuzzy $\otimes$-*equivalence relation* in S if (S, e) is a fuzzy model of the set of formulas,

$$
\begin{array}{ll}
\forall x E(x, x) & \text{(reflexive)},\\
\forall x \forall y (E(x, y) \Rightarrow E(y, x)) & \text{(symmetric)},\\
\forall z \forall x \forall y (E(x, z) \wedge E(y, z) \Rightarrow E(x, y)) & \text{(transitive)}.
\end{array}
$$

In [20] Valverde shows that the notion of fuzzy $\otimes$-equivalence is strictly related with the one of pseudo-distance. To show this, we have to refer to *Archimedean* t-norms, that is, those norms such that, for any $x, y \in (0, 1)$ an integer n exists such that $x^{(n)} < y$. The Archimedean t-norms can be obtained in a very simple way via additive generators.

Definition 4.2 We call *additive generator* any continuous strictly decreasing map $h : [0, 1] \to [0, \infty]$ such that $h(1) = 0$. Also, we denote by $h^{[-1]} : [0, \infty] \to [0, 1]$ the map defined by setting

$$h^{[-1]}(x) = h^{-1}(x \wedge h(0)).$$

Proposition 4.3 *A binary operation $\otimes$ is a continuous Archimedean t-norm if and only if there exists an additive generator h such that, for all $x, y \in [0, 1]$,*

$$x \otimes y = h^{[-1]}(h(x) + h(y)).$$

As an example, if $h(x) = -\log(x)$, then $\otimes$ is the usual product.

Definition 4.4 We call *Łukasiewicz generator* the map $l : [0, 1] \to [0, \infty]$ defined by setting $l(x) = 1 - x$.

Observe that $l^{[-1]}(x) = 1-(x \wedge 1) = (1-x) \vee 0$ and that l generates the Łukasiewicz t-norm defined by

$$x \otimes y = (x + y - 1) \vee 0.$$

We call *Łukasiewicz fuzzy logic* the fuzzy logic based on such a t-norm.

Theorem 4.5 *Let h be an additive generator, $\otimes$ the related t-norm and d a pseudo-metric d in a set S. Then we obtain a $\otimes$-equivalence* eq *in S by setting*

$$\mathrm{eq}(x, y) = h^{[-1]}(d(x, y)). \tag{4.1}$$

Conversely, let eq *be a $\otimes$-equivalence, then we obtain a pseudo-distance d by setting*

$$d(x, y) = h(\mathrm{eq}(x, y)). \tag{4.2}$$

Trivially, given a pseudo-metric d, the fuzzy equivalence eq associated with d by (4.1) is also a resemblance relation with respect to d.

Assume that in $\mathcal{L}_E$ there is a sequence $c_1, c_2, \ldots$ of constants. Then the following theorem gives a solution of Poincaré paradox once we admit that the formulas $E(c_1, c_2), E(c_2, c_3), \ldots$ are axioms at a degree $\lambda \neq 1$

Theorem 4.6 *Consider Łukasiewicz fuzzy logic and consider the fuzzy theory obtained by adding to the axioms for the fuzzy $\otimes$-equivalence relations the following axioms,*

$(E(c_1, c_2), \lambda)$,
$(E(c_2, c_3), \lambda)$,
$\vdots$
$(\neg E(c_1, c_m), 1)$,

where $\lambda \neq 1$ and m is a fixed number such that $\lambda^{(m-1)} = 0$. Then such a theory admits a fuzzy model and therefore it is consistent. Also such a logic enables us to give a formal representation of Poincaré argument preserving its intuitive content but avoiding its paradoxical character.

Proof Let S be the set of natural numbers and define a distance d in S by setting $d(x, y) = |x - y| \cdot (1 - \lambda)$ if $|x - y| \cdot (1 - \lambda) \leq 1$ and $d(x, y) = 1$ otherwise. Also, set $\mathrm{eq}(x, y) = 1 - d(x, y)$. Then eq is a $\otimes$-equivalence with respect to the Łukasiewicz triangular norm $\otimes$. Moreover, $\mathrm{eq}(n, n + 1) = 1 - 1 + \lambda = \lambda$ and, since $\lambda^{(m-1)} = 0$ entails $(m - 1) \cdot (1 - \lambda) \geq 1$, $\mathrm{eq}(1, m) = 0$. This proves that (S, eq) is a model of the considered fuzzy theory.

Also, in fuzzy logic we can formalize Poincaré argument as follows:

Step 1

Since	$E(c_1, c_2)$	[at degree λ]
and	$E(c_2, c_3)$	[at degree λ]
we can state		
	$E(c_1, c_2) \wedge E(c_2, c_3)$.	[at degree $\lambda \otimes \lambda$]
Therefore, since		
	$E(c_1, c_2) \wedge E(c_2, c_3) \Rightarrow E(c_1, c_3)$	[at degree 1]
we can state		
	$E(c_1, c_3)$.	[at degree $\lambda \otimes \lambda$]

Step 2

Since	$E(c_1, c_3)$	[at degree $\lambda^{(2)}$]
and	$E(c_3, c_4)$	[at degree λ]
we can state		
	$E(c_1, c_3) \wedge E(c_3, c_4)$.	[at degree $\lambda^{(3)}$]
Therefore, since		
	$E(c_1, c_3) \wedge E(c_3, c_4) \Rightarrow E(c_1, c_4)$	[at degree 1]
we can state		
	$E(c_1, c_4)$.	[at degree $\lambda^{(3)}$]

$\vdots$

Step $m-1$

Since	$E(c_1, c_{m-1})$	[at degree $\lambda^{(m-2)}$]
and	$E(c_{m-1}, c_m)$	[at degree λ]
we can state		
	$E(c_1, c_m) \wedge E(c_m, c_{m+1}).$	[at degree $\lambda^{(m-1)}$]
Therefore, since		
	$E(c_1, c_m) \wedge E(c_m, c_{m+1}) \Rightarrow E(c_1, c_{m+1})$	[at degree 1]
we can state		
	$E(c_1, c_{m+1}).$	[at degree $\lambda^{(m-1)}$].

Thus, such a proof entails that the conclusion $E(c_1, c_{m+1})$ is true at least at degree $\lambda^{(m-1)} = 0$ (no information). This is not paradoxical. □

5 Approximate ⊗-Similarity Structures

As argued in Section 1, the assumption that the formulas $E(c_n, c_{n+1})$ are axioms at degree $\lambda \neq 1$ is questionable. Then, to avoid the paradox in the case $\lambda = 1$ we have to consider a new class of fuzzy relations in which the transitivity property is in some way relaxed. At first observe that if a fuzzy relation is not transitive, then we can define the following interesting notions.

Definition 5.1 Given a fuzzy relation e, the *discernibility measure* is the extension $\mathrm{dis} : S \to [0, 1]$ of the formula

$$\mathrm{Dis}(z) \equiv \forall x \forall y (E(x, z) \wedge E(y, z) \Rightarrow E(x, y)).$$

The formula $\mathrm{Dis}(z)$ says that things equal to z are also equal to each other. In other words, it says that z is adequate for comparison. By the way, such a property is the first common notion in Book 1 of Euclid's Elements. Observe that I am not sure that it is correct to interpret $\mathrm{dis}(z)$ as a measure of the degree of discernibility of z from the remaining elements or not. Surely, if e is symmetric, $\mathrm{dis}(z)$ is a measure of the behavior of z with respect to the transitivity. Now, whereas dis is a *local* measure of transitivity, in a sense, we can also define a *global* measure of transitivity as follows.

Definition 5.2 We call *transitivity degree* of a fuzzy relation e the valuation $\mathrm{trans}(e)$ of the formula

$$\forall z \forall x \forall y (E(x, z) \wedge E(y, z) \Rightarrow E(x, y)).$$

Equivalently, $\mathrm{trans}(e)$ is the valuation of the formula

$$\forall z(\mathrm{Dis}(z)),$$

that is, of the claim *every element in S is discernible*. Obviously,

$$\mathrm{dis}(z) = \mathrm{Inf}\{e(x, z) \otimes e(y, z) \to e(x, y) : x, y \in S\} \tag{5.1}$$

and

$$\mathrm{trans}(e) = \mathrm{Inf}\{\mathrm{dis}(z) : z \in S\} \tag{5.2}$$

Notice that the notion of "transitivity degree" was proposed by Gottwald in [9] and [10] (see also Behounek and Cintula [1]). As an example, let e be the crisp resemblance relation defined in Section 2 in the case (S, d) is the usual metric space in the real line and $\varepsilon > 0$. Then it is clear that $\mathrm{dis}(z) = 0$ for every real number z and therefore that $\mathrm{trans}(e) = 0$. Instead, if we consider only the positive real line, then

$\mathrm{dis}(0) = 1$ while again $\mathrm{trans}(e) = 0$. In the graded cases both $\mathrm{trans}(e)$ and $\mathrm{dis}(z)$ depend strongly on the triangular norm, obviously. As an example, let e be the graded resemblance relation defined in Section 2. Then, $\mathrm{dis}(1.50) = \mathrm{dis}(2.50) = 1$ and

$$\mathrm{dis}(1.51) \leq e(1.50, 1.51) \otimes e(1.51, 1.56) \\ \rightarrow e(1.50, 1.56) = 1 \otimes 0.2 \rightarrow 0 = 0.2 \rightarrow 0.$$

So, in the case $\otimes$ is either the minimum or the usual product, we have $\mathrm{dis}(1.51) = 0$ and the function dis ranges continuously from 0 to 1. In such a case $\mathrm{trans}(e) = 0$. Instead, in the case $\otimes$ is the Łukasiewicz t-norm, $\mathrm{dis}(1.51) = 0.8$ and dis ranges continuously from 0.8 to 1. In such a case $\mathrm{trans}(e) = 0.8$.

It is easy to prove that the formula

$$\forall x \forall y \forall z (E(x, z) \wedge E(y, z) \wedge \mathrm{Dis}(z) \Rightarrow E(x, y)) \tag{5.3}$$

is true, that is, that

$$e(x, z) \otimes e(z, y) \otimes \mathrm{dis}(z) \leq e(x, y). \tag{5.4}$$

Indeed, recall the basic property of the pair $\otimes, \rightarrow$ and observe that, for any x, y, z in S,

$$e(x, z) \otimes e(y, z) \rightarrow e(x, y) \geq \mathrm{dis}(z). \tag{5.5}$$

Implication (5.3) suggests the following definition.

Definition 5.3 Consider a language $\mathcal{L}_{E,P}$ with two relation symbols E and P. Then an *approximate $\otimes$-similarity structure*, in brief, an *approximate similarity*, is a fuzzy model of the system of axioms,

A1 $\forall x E(x, x)$,
A2 $\forall x \forall y (E(x, y) \Rightarrow E(y, x))$,
A3 $\forall x \forall y \forall z (E(x, z) \wedge E(y, z) \wedge P(z) \Rightarrow E(x, y))$.

As usual, we denote by (S, e, p) a fuzzy interpretation of $\mathcal{L}_{E,P}$ where $e = I(E)$ and $p = I(P)$. The proof of the following proposition is obvious.

Proposition 5.4 *A fuzzy interpretation (S, e, p) is an approximate $\otimes$-similarity structure if and only if*

(i) $e(x, x) = 1$,
(ii) $e(x, y) = e(y, x)$,
(iii) $e(x, z) \otimes e(y, z) \otimes p(z) \leq e(x, y)$.

We can obtain the usual theory of fuzzy $\otimes$-equivalences by adding the axiom $\forall z P(z)$, that is, the condition $p(z) = 1$ for any $z \in S$. The following proposition relates the approximate similarities with the discernibility measure.

Proposition 5.5 *Let e be a reflexive and symmetric fuzzy relation in a nonempty set S and let p be a fuzzy subset of S. Then the structure (S, e, p) is an approximate similarity if and only if $p \subseteq \mathrm{dis}$. In other words, the approximate similarities are the models of the system of axioms*

A1 $\forall x E(x, x)$,
A2 $\forall x \forall y (E(x, y) \Rightarrow (E(y, x))$,
A3′ $\forall z (P(z) \Rightarrow \mathrm{Dis}(z))$.

Proof If $p \subseteq \mathrm{dis}$, then

$$e(x, z) \otimes e(y, z) \otimes p(z) \leq e(x, z) \otimes e(y, z) \otimes \mathrm{dis}(z) \leq e(x, y).$$

Conversely, if $e(x, z) \otimes e(y, z) \otimes p(z) \leq e(x, y)$, then

$$p(z) \leq e(x, z) \otimes e(y, z) \rightarrow e(x, y) \leq \mathrm{dis}(z).$$

$\square$

Observe that the equivalence between the systems A1–A2–A3 and A1–A2–A3′ can be proved also in a syntactical way by the deduction apparatus of Hájek's basic fuzzy logic.

Implication (5.3) shows that, given a reflexive and symmetric fuzzy relation e, the structure (S, e, dis) is the "best" approximate similarity structure we can define from e. In spite of that, I prefer not to limit the theory to the case $p = \mathrm{dis}$. Indeed, while $\mathrm{dis}(z)$ depends on the behavior of z with respect to the remaining elements in S, $p(z)$ can represent also an intrinsic property of z (as an example "to be precise", to be "sharply defined" and so on). We require only that such a property entails the discernibility property.

The following proposition, whose proof is immediate, shows that in the case p is a constant function, we can give another system of axioms for the approximate similarities in which the predicate P is not involved.

Proposition 5.6 *The class of $\otimes$-similarities in which p is a fuzzy subset constantly equal to ε coincides with the class of models of the axioms* A1 *and* A2 *and the signed formula*

A3″ $(\forall x \forall y \forall z(E(x, z) \wedge E(y, z) \Rightarrow E(x, y)), \varepsilon)$,

or, equivalently, the signed formula

A3‴ $(\forall z \mathrm{Dis}(z), \varepsilon)$.

6 Examples of Approximate Similarities

The following proposition gives a class of examples of approximate $\otimes$-similarities whose geometrical meaning will be evident in Section 10.

Proposition 6.1 *Let $\otimes$ be the Łukasiewicz t-norm, (S, eq) a fuzzy $\otimes$-equivalence, and $p : S \rightarrow [0, 1]$ be a fuzzy subset of S. Moreover, set $m(x, y) = l((p(x) + p(y))/2)$ and*

$$e(x, y) = \mathrm{eq}(x, y) \oplus m(x, y).$$

Then (S, e, p) is an approximate $\otimes$-similarity.

Proof It is evident that e is reflexive and symmetric. To prove that

$$e(x, z) \otimes e(y, z) \otimes p(z) \leq e(x, y)$$

or, equivalently, that

$$e(x, z) + e(y, z) + p(z) - 2 \leq e(x, y),$$

it is not restrictive to assume that $e(x, y) \neq 1$ and therefore that $e(x, y) = \text{eq}(x, y) + 1 - p(x)/2 - p(y)/2$. Then

$$\begin{aligned} e(x, z) + e(y, z) + p(z) - 2 \leq (\text{eq}(x, z) + 1 - p(x)/2 + 1 - p(z)/2) + \\ (\text{eq}(y, z) + 1 - p(y)/2 - p(z)/2) + p(z) - 2 = \\ \text{eq}(x, z) + \text{eq}(y, z) + 1 - p(x)/2 - p(y)/2 \leq \\ \text{eq}(x, y) + 1 - p(x)/2 - p(y)/2 = e(x, y). \end{aligned}$$

□

We call an approximate ⊗-similarity obtained in such a way *the approximate ⊗-similarity associated with* (S, eq) *and* p. The following proposition shows that in the case p is a constant function, these approximate similarities coincide with the resemblance relations defined in Proposition 2.2.

Proposition 6.2 *Let ⊗ be the Łukasiewicz t-norm and* (S, e, p) *be the approximate ⊗-similarity associated with the fuzzy equivalence* eq *and with the fuzzy set* p *constantly equal to* ε. *Then e coincides with the resemblance relation associated with the pseudo-metric* $d(x, y) = l(\text{eq}(x, y))$ *and* ε. *Conversely, let e be the resemblance relation associated with the pseudo-metric space* (S, d) *and* $\varepsilon \in [0, 1]$. *Then e coincides with the approximate similarity associated with the fuzzy equivalence* $\text{eq}(x, y) = l(d(x, y))$ *and the fuzzy set p constantly equal to* ε.

Proof Let e be the ⊗-similarity obtained from the fuzzy ⊗-equivalence (S, eq) and the fuzzy subset p constantly equal to ε and set $d(x, y) = l(\text{eq}(x, y))$. Then

$$e(x, y) = (\text{eq}(x, y) + \varepsilon) \wedge 1 = (1 - d(x, y) + \varepsilon) \wedge 1,$$

and, therefore, $e(x, y) = 1$ if $d(x, y) \leq \varepsilon$, and $e(x, y) = 1 - (d(x, y) - \varepsilon)$ otherwise. Then, in account of the fact that $e(x, y) < 1 + \varepsilon$, this shows that e is the resemblance relation defined from d and ε. In a similar way one proves the second part of the proposition. □

Proposition 6.3 *In the approximate ⊗-similarities defined in Proposition 6.2 we have* $p \neq \text{dis}$, *in general.*

Proof To give an example in which $\text{dis} \neq p$, assume that $S = [0, 1]$, $\text{eq}(x, y) = l(|x - y|)$ and $\varepsilon \neq 1$. Then $e(x, y) = 1$ if $|x - y| \leq \varepsilon$ and $e(x, y) = 1 - |x - y| + \varepsilon$ otherwise. We claim that

$$e(x, y) \geq e(x, 0) \otimes e(y, 0)$$

for any x, y in S and therefore that $\text{dis}(0) = 1 \neq p(0)$. Indeed, it is not restrictive to assume that $x \geq y$ and $e(x, y) \neq 1$ and therefore that $e(x, y) = 1 - x + y + \varepsilon$. Consider the case $e(x, 0) = 1$. Then $x \leq \varepsilon$ and, therefore,

$$1 - x + y + \varepsilon \geq 1 - \varepsilon + y + \varepsilon = 1 + y \geq e(y, 0) = e(x, 0) \otimes e(0, y).$$

Consider the case $e(x, 0) \neq 1$. Then $e(x, 0) = 1 - x + \varepsilon$ and, therefore,

$$1 - x + y + \varepsilon = e(x, 0) + y \geq e(x, 0) \geq e(x, 0) \otimes e(0, y).$$

Consider the just exposed example in the case $\varepsilon = 1/2$. Then while $\text{dis}(0) = \text{dis}(1) = 1$, we have that $\text{dis}(1/2) = 1/2$. □

The structures defined in such a section can be embedded into a unique structure. Given a nonempty set S, we call *fuzzy point* a pair (x, λ) such that $x \in S$ and $\lambda \in [0, 1]$. We denote by x^λ the fuzzy point (x, λ) and by $FP(S)$ the set $S \times [0, 1]$ of all the fuzzy points in S. We can interpret a fuzzy point x^λ as an *event* x together a *discriminability degree* of the observation instrument in x. The proof of the following proposition is trivial.

Proposition 6.4 *Let $\otimes$ be the Łukasiewicz t-norm and (S, eq) be a fuzzy $\otimes$-equivalence. Moreover, define e' by setting*

$$e'(x^\lambda, y^\mu) = \mathrm{eq}(x, y) \oplus l((\lambda + \mu)/2)$$

and define p' by setting $p'(y^\mu) = \mu$. Then $(FP(S), e', p')$ is an approximate $\otimes$-similarity structure. Let (S, e, p) be defined as in Proposition 6.1. Then the map $h : S \to S'$ associating any $x \in S$ with $x^{p(x)}$ is an embedding of (S, e, p) into (S_f, e', p').

7 Approximate ⊗-Similarities and the Paradox

The kernel of an approximate $\otimes$-similarity e is not an equivalence relation, in general. Indeed, given an element $z \in S$ such that $p(z) \neq 1$, from $e(x, z) = 1$ and $e(z, y) = 1$ we can only derive that $e(x, y) \geq p(z)$. This suggests that these relations are good candidates to face the Poincaré paradox in spite of the fact that $E(c_n, c_{n+1})$ is accepted as an axiom at degree 1.

Theorem 7.1 *Consider in Łukasiewicz fuzzy logic the fuzzy theory obtained by adding to the axioms for the approximate $\otimes$-similarities the axioms*

$(E(c_1, c_2), 1)$
$(E(c_2, c_3), 1)$
$\vdots$
$(\neg E(c_1, c_m), 1)$
$(P(c_1), \varepsilon)$
$(P(c_2), \varepsilon)$
$\vdots$

where ε is different from 0 and 1 and m is such that $\varepsilon^{(m-2)} = 0$. Then such a theory admits a fuzzy model and therefore it is consistent. Moreover, such a logic enables us to give a formal representation of the Poincaré argument preserving its intuitive content but avoiding its paradoxical character.

Proof Let S be the set of positive natural numbers and set $e(x, y) = \varepsilon^{(|x-y|-1)}$ if $x \neq y$ and $e(x, y) = 1$ if $x = y$. Then it is evident that e is symmetric and reflexive. We claim that

$$e(x, z) \otimes e(y, z) \otimes \varepsilon \leq e(x, y).$$

In fact, in all the cases $x = z$, $y = z$, and $x = y$ such an inequality is immediate. Otherwise, since $|x - z| + |y - z| - 1 \geq |x - y| - 1$,

$$e(x, z) \otimes e(y, z) \otimes \varepsilon = \varepsilon^{(|x-z|-1)} \otimes \varepsilon^{(|y-z|-1)} \otimes \varepsilon = \varepsilon^{(|x-z|+|y-z|-1)} \leq \varepsilon^{(|x-y|-1)} = e(x, y).$$

As a consequence, if we define p by setting $p(x) = \varepsilon$ for every $x \in S$, we obtain a $\otimes$-similarity structure (S, e, p). Moreover, since $e(n, n+1) = \varepsilon^{(0)} = 1$ and $e(1, m) = \varepsilon^{(m-2)} = 0$, such a structure is a model of the proposed fuzzy theory.

To formalize the Poincaré argument we can consider the following proof:

Step 1

Since	$E(c_1, c_2)$	[at degree 1]
and	$E(c_2, c_3)$	[at degree 1]
and	$P(c_2)$	[at degree ε]
we can state		
	$E(c_1, c_2) \wedge E(c_2, c_3) \wedge P(c_2)$.	[at degree $1 \otimes 1 \otimes \varepsilon$]
Therefore, since		
	$E(c_1, c_2) \wedge E(c_2, c_3) \wedge P(c_2) \Rightarrow E(c_1, c_3)$	[at degree 1]
we can state		
	$E(c_1, c_3)$.	[at degree $\varepsilon^{(1)}$]

Step 2

Since	$E(c_1, c_3)$	[at degree $\varepsilon^{(1)}$]
and	$E(c_3, c_4)$	[at degree 1]
and	$P(c_3)$	[at degree ε]
we can state	$E(c_1, c_3) \wedge E(c_3, c_4) \wedge P(c_3)$.	[at degree $\varepsilon^{(2)}$]
Therefore, since		
	$E(c_1, c_3) \wedge E(c_3, c_4) \wedge P(c_3) \Rightarrow E(c_1, c_4)$	[at degree 1]
we can state		
	$E(c_1, c_4)$.	[at degree $\varepsilon^{(2)}$]

$\vdots$

Step m-1

Since	$E(c_1, c_{m-1})$	[at degree $\varepsilon^{(m-3)}$]
and	$E(c_{m-1}, c_m)$	[at degree 1]
and	$P(c_{m-1})$	[at degree ε]
we can state		
	$E(c_1, c_{m-1}) \wedge E(c_{m-1}, c_m) \wedge P(c_{m-1})$.	[at degree $\varepsilon^{(m-2)}$]
Therefore, since		
	$E(c_1, c_m) \wedge E(c_m, c_{m+1}) \wedge P(c_m) \Rightarrow E(c_1, c_{m+1})$	[at degree 1]
we can state		
	$E(c_1, c_m)$.	[at degree $\varepsilon^{(m-2)}$]

Thus, such proof entails that the conclusion $E(c_1, c_m)$ is true at least at degree $\varepsilon^{(m-2)} = 0$. As such, this conclusion is not contradictory with the axiom $(\neg E(c_1, c_m), 1)$. □

We emphasize that the fact that a proof π of $E(c_1, c_m)$ is evaluated to 0 does not mean that such a formula is false but rather that π gives no information on its truth value. As an example imagine that we add the axiom $E(c_1, c_3)$ to our theory. Then there is a proof to prove $E(c_1, c_m)$ at degree $\varepsilon^{(m-3)}$ and it is possible that such a value is different from 0.

8 Some Connections with the Sorites Paradox

We will compare the solution we have just proposed for the Poincaré paradox with the solutions proposed by fuzzy logic to another famous paradox: the sorites paradox. In particular, we refer to [12] by Hájek and Novák. This paradox runs as follows. Consider a predicate Small and an infinite sequence $c_1, c_2, \ldots$ of constants. The intended meaning is that c_n denotes a heap d_n with n grains and $\text{Small}(c_n)$ means that such a heap is small. In accordance with such an interpretation, we assume that the formulas

$$\text{Small}(c_1), \text{Small}(c_1) \Rightarrow \text{Small}(c_2), \ldots, \text{Small}(c_n) \Rightarrow \text{Small}(c_{n+1}), \ldots$$

hold true. On the other hand, it is evident that, given any $n \in N$, from these formulas we can prove $\text{Small}(c_n)$ by a suitable number of applications of modus ponens. This contradicts our intuition suggesting that there is $m \in N$ such that $\text{Small}(c_m)$ is false.

A first analysis of such a paradox in fuzzy logic was proposed by Goguen in [8]. Successively, Hájek and Novák in [12] rendered his considerations more precise and introduced new interesting ideas. In particular, two approaches are considered. The first one is based on the idea for which if "d_n is small", then it is *almost true* that "d_{n+1} is small". The second idea is that the implication "if d_n is small then d_{n+1} is small" is almost true.

In the first case the logical connective "almost true" At is considered and it is interpreted in such a way that the two axiom schemata $\alpha \Rightarrow At(\alpha)$ and $(\alpha \Rightarrow \beta) \Rightarrow (At(\alpha) \Rightarrow At(\beta))$ are satisfied. A simple example is obtained by considering a value ε different from 0 and 1 and by interpreting At by the function $at(x) = \varepsilon \rightarrow x$. In particular, by assuming that $\rightarrow$ is the Łukasiewicz implication, we obtain that $at(x) = 1 \wedge (x + 1 - \varepsilon)$. In such a case, since $at^n(x) = 1 \wedge (x + n(1 - \varepsilon))$, we have that $at^n(0) = 1 \wedge n \cdot (1 - \varepsilon)$ and, therefore, if $\varepsilon \neq 1, at^n(0) = 1$ for every n such that $n \geq 1/(1 - \varepsilon)$. This means that given a false formula α, there is $m \in N$ such that $At^m(\alpha)$ is true. Now, in this enriched fuzzy logic we can formulate the heap's axioms as follows:

$$\text{Small}(c_1), \text{Small}(c_1) \Rightarrow At(\text{Small}(c_2)), \ldots, \text{Small}(c_n) \Rightarrow At(\text{Small}(c_{n+1})), \ldots.$$

From these axioms we can derive $At^m(\text{Small}(c_m))$ (and not $\text{Small}(c_m)$). If $m \geq 1/(1 - \varepsilon)$, since $at^m(0) = 1$, this conclusion is not in contradiction with the falsity of $\text{Small}(c_m)$. Thus such a reformulation of the paradoxical argument gives no paradox. Observe that the price to pay for such a solution is to admit that $At^m(\alpha)$ is true in spite of the falsity of α. Now, in spite of the justification given in the paper for which "we may not be 100% sure that something is false," there is something unsatisfactory in such an acceptance. Indeed, the interpretation of a logic connective is independent of the meaning of the formulas since it depends only on the considered truth values. Then it is sufficient that there is only a formula which is surely false (as an example $\underline{0}$, or $\beta \wedge \neg\beta$) to impose that $at(0) = 0$ and therefore that $at^n(0) = 0$ for every natural n. Then the property admitted for the logical connective At is no less "paradoxical" than the heap paradox.

The second idea exposed in [12] leads us to assume the axioms $\text{Small}(c_n) \Rightarrow \text{Small}(c_{n+1})$ at a degree $\varepsilon \neq 1$. Indeed, it is easy to see that in the deduction apparatus of fuzzy logic the sorites argument enables us to prove $\text{Small}(c_m)$ only at a degree $\varepsilon^{(m)} = 0$. Such an approach is the same proposed first by Goguen in [8] (see also [4]).

Now the first approach suggests the possibility to solve the Poincaré paradox by considering the following class of fuzzy relations defined by using the logical operator At.

Definition 8.1 A fuzzy relation eq in a set S is called an *almost-⊗-similarity* provided that (S, eq) satisfies the following axioms:

$$\begin{array}{ll} \forall x E(x,x) & \text{(reflexive)}, \\ \forall x \forall y (E(x,y) \Rightarrow E(y,x)) & \text{(symmetric)}, \\ \forall z \forall x \forall y (E(x,z) \wedge E(y,z) \Rightarrow At(E(x,y)) & \text{(almost-transitive)}. \end{array}$$

Then in an almost-⊗-similarity e the transitivity is expressed by the inequality

$$e(x,z) \otimes e(y,z) \leq at(e(x,y)).$$

Now, in such a case the Poincaré argument gives as a theorem the formula $At^m(E(c_1, c_m))$ and not $E(c_1, c_m)$. Such a theorem is not in contradiction with the falsity of $E(c_1, c_m)$.

As a matter of fact if we consider as an interpretation of At the fuzzy function $\varepsilon \rightarrow x$, the notion of almost-⊗-similarity coincides with the one of fuzzy ⊗-similarity with respect to the fuzzy subset p constantly equal to ε. In fact, we have

$$\begin{aligned} e(x,z) \otimes e(y,z) \leq at(e(x,y)) &\Leftrightarrow e(x,z) \otimes e(y,z) \leq (\varepsilon \rightarrow e(x,y)) \\ &\Leftrightarrow e(x,z) \otimes e(y,z) \otimes \varepsilon \leq e(x,y) \\ &\Leftrightarrow e(x,z) \otimes e(y,z) \otimes p(z) \leq e(x,y). \end{aligned}$$

Then it is not surprising that the notion of almost-⊗-similarity enables us to prove one analogous to Theorem 7.1 and therefore to solve the paradox.

The second approach suggests that we consider the transitivity property as an axiom at degree ε with $\varepsilon \neq 1$, that is, to represent such a property by the signed formula $(\forall z \forall x \forall y (E(x,z) \wedge E(y,z) \Rightarrow E(x,y)), \varepsilon)$. A fuzzy relation satisfies such a property provided that

$$e(x,z) \otimes e(y,z) \rightarrow e(x,y) \geq \varepsilon.$$

Again, the presence of such a weak formulation of the transitivity enables us to solve the paradox. Again, this is not surprising since, in accordance with Proposition 5.6, assuming the transitivity at degree ε is equivalent to referring to the ⊗-similarities in which p is constantly equal to ε.

Finally, we observe that the sorites paradox can be expressed by involving the distinguishability relation E, too. As an example, a *phenomenal* reformulation of such a paradox is obtained by assuming that

(i) the heaps $d_1, \ldots, d_{1000}$ look to be small for me,
(ii) for $n \geq 1000$ I am not able to distinguish d_n from d_{n+1},
(iii) if a heap x is small and we are not able to distinguish x from a heap y, then y is small,
(iv) d_m is not small,

where m is a sufficiently big number. In classical logic we can express this by the following system of axioms

$$
\begin{array}{ll}
\mathrm{Small}(c_n) & \text{for } n \leq 1000 \\
E(c_n, c_{n+1}) & \text{for every } n \geq 1000 \\
\neg\mathrm{Small}(c_m) & \\
\forall x \forall y(\mathrm{Small}(x) \wedge E(x, y) \Rightarrow \mathrm{Small}(y)). &
\end{array}
$$

Trivially, for every $n \in N$, the formula $\mathrm{Small}(c_n)$ is a theorem of such a system and this contradicts the axiom $\neg\mathrm{Small}(c_m)$. In fact, $\mathrm{Small}(c_1), \ldots, \mathrm{Small}(c_{1000})$ are axioms (and therefore theorems). Assume that $\mathrm{Small}(c_n)$ is a theorem with $n \geq 1000$; then since $E(c_n, c_{n+1})$ and $\mathrm{Small}(c_{n-1}) \wedge E(c_{n-1}, c_n) \Rightarrow \mathrm{Small}(c_n)$ are axioms, $\mathrm{Small}(c_n)$ is a theorem, too. It is interesting to observe that in such a version of sorites paradox the transitivity property of E plays no role and the only hypothesis is that the formulas $E(c_n, c_{n+1})$ are assumed at degree 1 for every $n \geq 1000$. Then, independently from the fact that E is interpreted by a fuzzy equivalence relation, by a resemblance relation, or by an approximate $\otimes$-similarity, the paradoxical argument remains valid. Obviously, fuzzy logic is able to give a solution in the case the formulas $E(c_n, c_{n+1})$ are substituted with the signed formulas $(E(c_n, c_{n+1}), \lambda)$ where λ is a suitable number different from 1.

Theorem 8.2 *Given $q \in N$ and $\lambda \neq 1$, consider in Łukasiewicz fuzzy logic the fuzzy theory obtained by adding to the axioms for the fuzzy $\otimes$-equivalences the axioms*

$$
\begin{array}{ll}
\mathrm{Small}(c_n) & \textit{for } n \leq q \\
(E(c_n, c_{n+1}), \lambda) & \textit{for every } n > q \\
\forall x \forall y(\mathrm{Small}(x) \wedge E(x, y) \Rightarrow \mathrm{Small}(y)). &
\end{array}
$$

Then there is a fuzzy model of such a theory satisfying the formula $\neg\mathrm{Small}(c_m)$ for a suitable $m \in N$.

Proof Let S be the set of heaps and let $(a_n)_{n \in N}$ be a sequence of real numbers in $[0, 1]$. Define eq by setting

$$
\begin{array}{ll}
\mathrm{eq}(d_h, d_k) = 1 & \text{if } h = k \\
\mathrm{eq}(d_h, d_k) = a_{h-1} \otimes a_{h-2} \otimes \cdots \otimes a_k & \text{if } h > k \\
\mathrm{eq}(d_h, d_k) = \mathrm{eq}(d_k, d_h) & \text{otherwise.}
\end{array}
$$

Then e is a fuzzy $\otimes$-equivalence. In fact, while it is evident that eq is reflexive and symmetric, to prove that, for every d_h and d_k in S,

$$\mathrm{eq}(d_h, d_k) \geq \mathrm{eq}(d_h, d_i) \otimes \mathrm{eq}(d_i, d_k),$$

it is not restrictive to assume that $h > k$. Then, in the case $i \geq h$,

$$\mathrm{eq}(d_h, d_k) \geq \mathrm{eq}(d_i, d_k) \geq \mathrm{eq}(d_h, d_i) \otimes \mathrm{eq}(d_i, d_k);$$

in the case $i \leq k$,

$$\mathrm{eq}(d_h, d_k) \geq \mathrm{eq}(d_h, d_i) \geq \mathrm{eq}(d_h, d_i) \otimes \mathrm{eq}(d_i, d_k).$$

Finally, in the case $h > i > k$,

$$
\begin{aligned}
\mathrm{eq}(d_h, d_k) = a_{h-1} \otimes a_{h-2} \otimes a_i \otimes a_{i-1} \otimes \cdots \otimes a_k = \\
(a_{h-1} \otimes a_{h-2} \otimes a_i) \otimes (a_{i-1} \otimes \cdots \otimes a_k) = \mathrm{eq}(d_h, d_i) \otimes \mathrm{eq}(d_i, d_k).
\end{aligned}
$$

Define the fuzzy relation small by setting $\text{small}(d_1) = 1$ and, for $n > 1$,

$$\text{small}(d_n) = a_{n-1} \otimes \cdots \otimes a_1.$$

Obviously, $(\text{small}(d_n))_{n\in N}$ is an order-reversing sequence. We claim that

$$\text{small}(d_h) \geq \text{small}(d_k) \otimes \text{eq}(d_h, d_k)$$

for every pair d_h, d_k in S. In fact, in the case $h > k$,

$$\text{small}(d_h) = (a_{h-1} \otimes a_{h-2} \otimes \cdots \otimes a_k) \otimes (a_{k-1} \otimes a_{k-2} \otimes \cdots \otimes a_1) = \text{eq}(d_h, d_k) \otimes \text{small}(d_k).$$

In the case $h \leq k$,

$$\text{small}(d_h) \geq \text{small}(d_k) \geq \text{small}(d_k) \otimes \text{eq}(d_h, d_k).$$

In order to satisfy the remaining axioms, we have to consider a sequence $(a_n)_{n\in N}$ satisfying suitable properties. Now, consider a sequence $(c_n)_{n\in N}$ of elements in $[0, 1]$ such that $c_n \leq 1 - \lambda$ and $\sum_{n=1}^{n=\infty} c_n > 1$ and define $(a_n)_{n\in N}$ by setting $a_n = 1$ if $n \leq q$ and $a_n = 1 - c_{n-q}$ otherwise. Then, $\text{small}(d_1) = \text{small}(d_2) = \cdots = \text{small}(d_q) = 1$ and $\text{eq}(d_n, d_{n+1}) = a_n = 1 - c_{n-q} \geq \lambda$ for every $n > p$. Let $k \in N$ such that $\sum_{n=l}^{n=k} c_n \geq 1$. Then, since $1 - a_{q+1} + \cdots + 1 - a_{q+k} = c_1 + \cdots + c_k \geq 1$, we have that $a_{q+1} + \cdots + a_{q+k} - k + 1 \leq 0$. Thus, if we set $m = k + 1 + q$, we obtain that $a_{q+1} + \cdots + a_{m-1} + q - m + 2 \leq 0$ and, therefore,

$$\text{small}(d_m) = a_1 \otimes \cdots \otimes a_{m-1} = a_{q+1} \otimes \cdots \otimes a_{m-1} = (a_{q+1} + \cdots + a_{m-1} - m + q + 2) \vee 0 = 0$$

and this proves that $(S, \text{eq}, \text{small})$ is a fuzzy model of our fuzzy theory satisfying $\text{Small}(c_m)$.

We conclude the proof by observing that there is no difficulty in exhibiting a sequence $(c_n)_{n\in N}$ satisfying the required property. As an example, we can set $c_i = 1 - \lambda$ and, therefore, $a_{q+i} = \lambda$ for every $n \in N$. In such a case, it is sufficient to assume that $m \geq 1 + q + 1/(1 - \lambda)$ to obtain that $\text{small}(d_m) = \lambda^{(m-1-q)} = 0$. A more interesting example is obtained, under the hypothesis $\lambda < 1/2$, by considering a real number h such that $1/2 < h \leq 1 - \lambda$ and set $c_n = h^n$. Then $\sum_{n=1}^{n=\infty} c_n = \frac{1}{1-h} - 1 = \frac{h}{1-h} \geq 1$ and it is evident that $c_n \leq 1 - \lambda$. In such a case, since

$$c_1 + \cdots + c_k = (1 - h^{k+1})/(1 - h) - 1 \text{ and}$$
$$c_1 + \cdots + c_k \geq 1 \Leftrightarrow 1 - h^{k+1} \geq 2(1 - h) \Leftrightarrow k \geq \log_h(2h - 1),$$

we can set m equal to any natural number such that $m \geq \log_h(2h - 1) - 1 - q$. □

It is an open question to find a similar solution of the sorites paradox in which the axioms $E(c_n, c_{n+1})$ are assumed at degree 1 and E is interpreted by an approximate $\otimes$-similarity.

9 Distances and Diameters in Point-free Geometry

Theorem 4.5 points to a bridge between a notion logical in nature and a notion metrical in nature. We can extend such a connection to the approximate $\otimes$-similarities provided we refer to the notion of pointless metric space. Such a notion was proposed in a series of papers as a basis for a metrical approach to point-free geometry (see [5] and [6]) in which the notion of region, distance, diameter, and inclusion are assumed as primitive and the points are defined in a suitable way. This is in accordance with the ideas of Whitehead (see [21]).

Definition 9.1 A *pointless pseudo-metric space*, in short a ppm-*space*, is a structure $(S, \leq, \delta, |\,|)$, where $(S, \leq)$ is an ordered set, $\delta : S \times S \to [0, \infty)$ is order-reversing, $|\,| : S \to [0, \infty]$ is order-preserving, and for every $x, y, z \in S$,

(a1) $\delta(x, x) = 0$,
(a2) $\delta(x, y) = \delta(y, x)$,
(a3) $\delta(x, y) \leq \delta(x, z) + \delta(z, y) + |z|$.

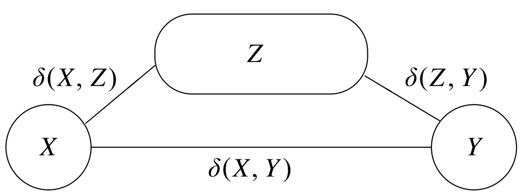
The elements in S are called *region*s, the order $\leq$ is called *inclusion relation*, $\delta(x, y)$ *distance between* x and y, $|x|$ the *diameter of* x. Observe that (a3) is a weak form of the triangular inequality taking in account the diameters of the regions. In fact, if all the diameters are equal to zero, then (a3) coincides with the triangular inequality and the ppm-space is a pseudo-metric space. Then the notion of ppm-space extends the one of pseudo-metric space (and therefore of metric space). More precisely, we can identify the pseudo-metric spaces as the ppm-spaces in which $\leq$ is the identity and all the diameters are equal to zero. The prototypical examples of ppm-space are given in the following proposition (see [6]).

Proposition 9.2 *Let (M, d) be a pseudo-metric space and let C be a nonempty class of bounded and nonempty subsets of M. Define δ and $|\,|$ by setting*

$$\delta(X, Y) = \inf\{d(x, y) : x \in X, y \in Y\} \tag{9.1}$$
$$|X| = \sup\{d(x, y) : x, y \in X\}, \tag{9.2}$$

respectively. Then $(C, \subseteq, \delta, |\,|)$ is a ppm-*space.*

We call the so-defined spaces *canonical*. Such an interpretation enables us to illustrate the meaning of (a3). Indeed, by referring to the Euclidean plane, in the following picture



it is evident that $\delta(X, Y) > \delta(X, Z) + \delta(Z, Y)$ and therefore that the usual triangular inequality cannot be assumed. Instead, it is routine matter to prove that $\delta(X, Y) \leq \delta(X, Z) + \delta(Z, Y) + |Z|$.

For instance, let (E, d) be a Euclidean metric space, and $s : E \to [0, \infty)$ be a function. Also, denote by $B(P, s(P))$ the closed ball centered in P and whose diameter is $s(P)$ and set $S = \{B(P, s(P)) : P \in E\}$. Then we can consider the canonical space defined by this class of closed balls. In such a space the order is the identity relation, the diameter $|\,|$ coincides with the function s, and

$$\delta(B(P, s(P)), B(Q, s(Q))) = 0 \qquad \text{if } d(P, Q) \leq (s(P) + s(Q))/2$$
$$\delta(B(P, s(P)), B(Q, s(Q))) = d(P, Q) - (s(P) + s(Q))/2 \qquad \text{otherwise.}$$

If we denote by P the ball $B(P, s(P))$, we can define directly in E a ppm-space by setting

$$\begin{array}{ll}\delta(P, Q) = 0 & \text{if } d(P, Q) \leq (s(P) + s(Q))/2 \\ \delta(P, Q) = d(P, Q) - (s(P) + s(Q))/2 & \text{otherwise.}\end{array}$$

This suggests the definition of a simple class of ppm-spaces we call *formal balls space*.

Proposition 9.3 *Given a pseudo-metric space (S, d) and a function $s : S \to [0, \infty)$, define δ_s by setting*

$$\delta_s(x, y) = (d(x, y) - (s(x) + s(y))/2) \vee 0.$$

Then, the structure (S, δ_s, s) is a ppm-*space.*

Proof In the case $d(x, y) < (s(x) + s(y))/2$ the inequality

$$\delta_s(x, z) + \delta_s(z, y) + s(z) \geq \delta_s(x, y)$$

is trivial. So, we assume that $d(x, y) \geq (s(x)+s(y))/2$. In the case $d(x, z) \geq (s(x)+s(z))/2$ and $d(z, y) \geq (s(z) + s(y))/2$,

$$\begin{aligned}&\delta_s(x, z) + \delta_s(z, y) + s(z) = \\ &\qquad d(x, z) - (s(x) + s(z))/2 + d(z, y) - (s(y) + s(z))/2 + s(z) \geq \\ &\qquad\qquad d(x, y) - (s(x) + s(y))/2 = \delta_s(x, y).\end{aligned}$$

Assume that $d(x, z) < (s(x) + s(z))/2$ and therefore that

$$s(z)/2 - s(y)/2 = (s(x) + s(z))/2 - (s(x) + s(y))/2 \geq d(x, z) - (s(x) + s(y))/2.$$

Then, in the case $d(z, y) \geq (s(z) + s(y))/2$,

$$\begin{aligned}&\delta_s(x, z) + \delta_s(z, y) + s(z) = d(z, y) - (s(z) + s(y))/2 + s(z) = \\ &\qquad d(z, y) + s(z)/2 - s(y))/2 \geq d(z, y) + d(x, z) - (s(x) + s(y))/2 \geq \\ &\qquad\qquad d(x, y) - (s(x) + s(y))/2 = \delta_s(x, y).\end{aligned}$$

In the case $d(z, y) \leq (s(z) + s(y))/2$, we have that $d(x, z) + d(z, y) \leq (s(x) + s(z))/2 + (s(z) + s(y))/2$ and, therefore,

$$s(z) \geq d(x, z) + d(z, y) - (s(x) + s(y))/2.$$

So,

$$\begin{aligned}&\delta_s(x, z) + \delta_s(z, y) + s(z) = \\ &\qquad s(z) \geq d(x, z) + d(z, y) - (s(x) + s(y))/2 \geq \\ &\qquad\qquad d(x, y) - (s(x) + s(y))/2 = \delta_s(x, y).\end{aligned}$$

In a similar way we go on in the remaining cases. □

In a canonical space a region can be interpreted as an incomplete information (that is, a constraint) on a point. This means that the approximation originates from the imprecision of the objects whose distance we have to calculate. We can consider also the case in which the objects are given in a precise way but the approximation originates from the instrument used to measure distances. As an example, given a natural number n, denote by $\text{trunc}_n(x)$ the n-decimal truncation of a real number x. Then the proof of the following proposition is routine matter.

Proposition 9.4 *Let* (M, d) *be a pseudo-metric space and* n *be a fixed natural number. Also, set* $\delta(x, y) = \mathrm{trunc}_n(d(x, y))$ *and* $|x| = 2 \cdot 10^{-n}$. *Then* $(M, =, \delta, |\,|)$ *is a* ppm-*space.*

10 Geometrical Interpretations of the Approximate ⊗-Similarities

This section is devoted to show that the logical notion of approximate ⊗-similarity is strictly connected with the metrical notion of ppm-space. We are not interested in the order relation $\leq$ and therefore we confine ourselves to the cases in which $\leq$ is the identity. For these structures we write $(S, \delta, |\,|)$ instead of $(S, =, \delta, |\,|)$. Notice that if $(S, \leq, \delta, |\,|)$ is a ppm-space, then $(S, =, \delta, |\,|)$ is a ppm-space too.

Theorem 10.1 *Let* h *be an additive generator and* $\otimes$ *be the related* t*-norm. Then we can associate any* ppm-*space*$(S, \delta, |\,|)$ *with an approximate* ⊗*-similarity space* (S, e, p) *such that*

$$e(x, y) = h^{[-1]}(\delta(x, y));\ p(x) = h^{[-1]}(|x|).$$

Conversely, we can associate any approximate ⊗*-similarity space* (S, e, p) *with a* ppm-*space* $(S, \delta, |\,|)$ *by setting*

$$\delta(x, y) = h(e(x, y));\ |x| = h(p(x)).$$

Proof Let $(S, \delta, |\,|)$ be a ppm-space. Then, since

$$\delta(x, y) \wedge h(0) \leq \delta(x, z) \wedge h(0) + \delta(z, y) \wedge h(0) + |z| \wedge h(0),$$

we have

$$\begin{aligned}
e(x, y) = h^{-1}(\delta(x, y) \wedge h(0)) \geq \\
h^{-1}((\delta(x, z) \wedge h(0)) + (\delta(z, y) \wedge h(0)) + (|z| \wedge h(0))) = \\
h^{-1}(\delta(x, z) \wedge h(0)) \otimes h^{-1}(\delta(z, y) \wedge h(0)) \otimes h^{-1}(|z| \wedge h(0))) = \\
e(x, z) \otimes e(y, z) \otimes p(z).
\end{aligned}$$

This proves that (S, e, p) is an approximate similarity space.

Conversely, let (S, e, p) be an approximate similarity space and set $\lambda +^t \mu = (\lambda + \mu) \wedge h(0)$. Then, since h is an order-reversing isomorphism from $([0, 1], \otimes, 1)$ to $([0, h(0)], +^t, 0)$,

$$\begin{aligned}
\delta(x, y) = h(e(x, y)) \leq h(e(x, z) \otimes e(y, z) \otimes p(z)) = \\
h(e(x, z)) +^t h(e(y, z)) +^t h(p(z)) \leq \delta(x, z) + \delta(z, y) + |z|.
\end{aligned}$$

This proves that $(S, \delta, |\,|)$ is a ppm-space. □

In accordance with such a theorem, every example of ppm-space gives an example of approximate ⊗-similarity. As a first example, we consider the ppm-spaces of the formal balls.

Proposition 10.2 *The Łukasiewicz generator* l *defines a connection between the* ppm-*spaces of the formal-balls and the class of the approximate* ⊗*-similarities defined in Proposition 6.1.*

Proof Consider a pseudo-metric (S, d) and let $s : S \to [0, 1]$ be a function. Define d_1 by setting $d_1(x, y) = d(x, y) \wedge 1$ and let (S, δ_s, s) be the ppm-space associated with (S, d_1) and s by Proposition 9.3. Then by the Łukasiewicz generator l we can obtain an approximate $\otimes$-similarity (S, e, p) where $e(x, y) = 1 - \delta_s(x, y)$ and $p(x) = 1 - s(x)$. Moreover, if we set $\text{eq}(x, y) = 1 - d_1(x, y)$, we have that eq is a fuzzy $\otimes$-equivalence and that $e(x, y) = 1$ in the case $d_1(x, y) \leq (s(x) + s(y))/2$; that is, in the case

$$\text{eq}(x, y) = 1 - d_1(x, y) \geq 1 - (s(x) + s(y))/2 = (1 - s(x) + 1 - s(y))/2 = m(x, y).$$

Otherwise, we have

$$e(x, y) = 1 - (d_1(x, y) - (s(x) + s(y))/2) \vee 0 = \text{eq}(x, y) + 1 - (p(x) + p(y))/2 = \text{eq}(x, y) + m(x, y).$$

This proves that $e(x, y) = \text{eq}(x, y) \oplus m(x, y)$.

Conversely let (S, e, p) be the approximate $\otimes$-similarity defined in Proposition 6.1 from the fuzzy $\otimes$-equivalence eq and the function p. Also, set $d(x, y) = 1 - \text{eq}(x, y)$ and $s(x) = 1 - p(x)$; then d is a pseudo-metric and, in accordance with Proposition 9.3, the pair d, s define a ppm-space (S, δ_s, s). Trivially, the approximate $\otimes$-similarity associated with such a space coincides with (S, e, p). □

In accordance with Proposition 6.2, we obtain a geometrical interpretation for the resemblance relations defined in Proposition 2.2. In fact, these relations are the dual ones of the *ppm*-spaces of the formal balls with a fixed diameter.

A second class of examples is furnished by the ppm-spaces defined in Proposition 9.4.

Proposition 10.3 *Consider a pseudo-metric space (M, d) and a fixed natural number n. Then we obtain an approximate $\otimes$-similarity by setting*

$$e(x, y) = l^{[-1]}(\text{trunc}_n(d(x, y))) = 1 - \text{trunc}_n(d(x, y)) \wedge 1;\ p(x) = 1 - 2 \cdot 10^{-n}.$$

Such a $\otimes$-similarity is a resemblance relation with respect to d.

Notice that, since $e(x, y) = 1$ for every x, y such that $d(x, y) \leq 10^{-n}$, the similarity so defined is not able to detect small differences.

Finally, we show that the duality defined in Theorem 10.1 gives geometrical examples of $[0, 1]$-valued equalities.

Proposition 10.4 *Let (S, d) be a pseudo-metric, $\varepsilon \in [0, 1]$, and consider the function δ^ε defined by setting*

$$\delta^\varepsilon(x, y) = d(x, y) + \varepsilon.$$

Then the fuzzy relation e defined by setting

$$\text{eq}(x, y) = l^{[-1]}(\delta(x, y)),$$

is a $[0, 1]$-valued equality.

Proof Properties (e1) and (e2) are evident. To prove (e3) at first observe that

$$\text{eq}(x, y) = (1 - \delta^\varepsilon(x, y)) \vee 0 = (1 - d(x, y) - \varepsilon) \vee 0.$$

Since it is not restrictive to assume that $\mathrm{eq}(x,z) \otimes (\mathrm{eq}(z,z) \to \mathrm{eq}(y,z))$ is different from 0, we have that $\mathrm{eq}(x,z) \neq 0$ and therefore that $\mathrm{eq}(x,z) = 1 - \delta^{\varepsilon}(x,z) = 1 - d(x,z) - \varepsilon > 0$. Also, since

$$\begin{aligned}
&\mathrm{eq}(x,z) \otimes (\mathrm{eq}(z,z) \to \mathrm{eq}(y,z)) = \\
&\qquad 1 - d(x,z) - \varepsilon + (\mathrm{eq}(y,z) + \varepsilon) \wedge 1 - 1 \leq \\
&\qquad 1 - d(x,z) - \varepsilon + (\mathrm{eq}(y,z) + \varepsilon) - 1 = \\
&\qquad\qquad - d(x,z) + \mathrm{eq}(y,z),
\end{aligned}$$

we have, $-d(x,z) + \mathrm{eq}(y,z) > 0$ and therefore $\mathrm{eq}(y,z) > 0$. Then,

$$\begin{aligned}
&\mathrm{eq}(x,z) \otimes (\mathrm{eq}(z,z) \to \mathrm{eq}(y,z)) \leq \\
&\qquad - d(x,z) + 1 - d(y,z) - \varepsilon \leq -d(x,y) - \varepsilon + 1 \leq \\
&\qquad\qquad (-d(x,y) - \varepsilon + 1) \vee 0 = \mathrm{eq}(x,y).
\end{aligned}$$

$\square$

11 Conclusions and Future Works

This paper is addressed mainly to face the “paradoxes” arising from the indistinguishability relation and this was done by proposing a weakened form of the transitivity property in the framework of fuzzy logic. Patently, I do not affirm that the solution I propose is the definitive one. Indeed, any genuine paradox admits several different solutions, in general. Moreover, all these solutions are interesting from some point of view and no solution is definitive. For example, in set theory the paradoxes where faced by proposing totally different systems of axioms or mathematical philosophies and we cannot exclude that further answers will be given in the future. Then the main role of a paradox is to stimulate analyses and discussions and to suggest new mathematical formalisms.

From a theoretical point of view there is a lot of work to do. As an example, an important task is to give a suitable notion of morphism and to investigate the properties of the resulting category. This in analogy with the papers of Höhle. Also, in order to make the duality established in Theorem 10.1 complete, it should be opportune to extend the notion of approximate similarity structure by introducing an order relation over the set S of elements under consideration. Once we interpret the elements in S as pieces of information, the interpretation of $x \leq y$ should be that x is obtained from y by adding further information.

Finally, another interesting task is to investigate the potentialities of the notion of approximate similarity for applications. Now, assume that the elements in S are pieces of information on the elements we are interested in and that p is a measure of the completeness of the information. Then perhaps applications are possible in all the frameworks in which

(i) the notion of similarity (or distance) plays a basic role,
(ii) there is not complete information on the objects under consideration.

Nevertheless, due to the initial state of my research on this subject, I have no concrete example to support this claim.

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