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Consecutive Singular Cardinals and the Continuum Function

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Abstract We show that from a supercompact cardinal κ, there is a forcing extension $V[G]$ that has a symmetric inner model N in which ZF + $\neg$AC holds, κ and κ^+ are both singular, and the continuum function at κ can be precisely controlled, in the sense that the final model contains a sequence of distinct subsets of κ of length equal to any predetermined ordinal. We also show that the above situation can be collapsed to obtain a model of ZF + $\neg\text{AC}_\omega$ in which either (1) $\aleph_1$ and $\aleph_2$ are both singular and the continuum function at $\aleph_1$ can be precisely controlled, or (2) $\aleph_\omega$ and $\aleph_{\omega+1}$ are both singular and the continuum function at $\aleph_\omega$ can be precisely controlled. Additionally, we discuss a result in which we separate the lengths of sequences of distinct subsets of consecutive singular cardinals κ and κ^+ in a model of ZF. Some open questions concerning the continuum function in models of ZF with consecutive singular cardinals are posed.

1 Introduction

In this paper we will be motivated by the following question: Are there models of Zermelo–Fraenkel (ZF) set theory with consecutive singular cardinals κ and κ^+ such that "the generalized continuum hypothesis (GCH) fails at κ" in the sense that there is a sequence of distinct subsets of κ of length greater than κ^+? Let us start by considering some known models of ZF that have consecutive singular cardinals.

Gitik showed in [8] that from a proper class of strongly compact cardinals, $\langle\kappa_\alpha \mid \alpha \in \text{ORD}\rangle$, there is a model of ZF + $\neg\text{AC}_\omega$ in which all uncountable cardinals are singular. Essentially he uses a certain type of generalized Prikry forcing that simultaneously singularizes and collapses each κ_α, thereby resulting in a model in which the class of uncountable well-ordered cardinals consists of the previously

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strongly compact κ_α's and their limits. In this model, every uncountable cardinal is singular, and for each $\alpha \in$ ORD and for each limit ordinal λ, all cardinals in the open intervals $(\kappa_\alpha, \kappa_{\alpha+1})$ and $(\sup_{\beta<\lambda} \kappa_\beta, \kappa_\lambda)$ have been collapsed to have size κ_α and $\sup_{\beta<\lambda} \kappa_\beta$, respectively. Since each κ_α is a strong limit cardinal in the ground model, it follows that in Gitik's final model there is no cardinal κ that has a sequence of distinct subsets of length greater than—or even equal to—κ^+. (Of course, trivially, in any model of ZF, for any cardinal κ, there is always a κ-sequence of distinct subsets of κ given by the sequence of intervals $\langle [\alpha, \kappa) \mid \alpha < \kappa \rangle$. This also trivially implies that, for any $\beta \in (\kappa, \kappa^+)$, there is a β-sequence of distinct subsets of κ as well.) For similar reasons, the models constructed in Gitik [9] and Apter, Dimitriou, and Koepke [4] also will not have consecutive singular cardinals κ and κ^+ with a sequence of distinct subsets of κ of length even κ^+.

There has been a great deal of work, involving forcing over models of AD, in which models are constructed having consecutive singular cardinals, as exemplified by Apter [3]. However, in any model of AD, no cardinal $\kappa < \Theta$ has a sequence of distinct subsets of length κ^+ let alone of longer length (see Steel [17]). Thus, forcing over a model of AD does not seem to yield, in any obvious way, a model containing consecutive singular cardinals, κ and κ^+, in which there is a sequence of distinct subsets of κ of length κ^+.

In this article, we will show that from a supercompact cardinal, there are models of ZF + $\neg$AC that have consecutive singular cardinals, say, κ and κ^+, such that there is a sequence of distinct subsets of κ of length equal to any predetermined ordinal. Indeed, we will prove the following.

Theorem 1 *Suppose that κ is supercompact,* GCH *holds, and θ is an ordinal. Then there is a forcing extension $V[G]$ that has a symmetric inner model $N \subseteq V[G]$ of* ZF + $\neg$AC *in which the following hold:*

(1) κ and κ^+ are both singular with $\mathrm{cf}(\kappa)^N = \omega$ and $\mathrm{cf}(\kappa^+)^N < \kappa$;
(2) κ is a strong limit cardinal that is a limit of inaccessible cardinals;
(3) there is a sequence of distinct subsets of κ of length θ.

Let us remark here that property (3) in Theorem 1 makes this result interesting, since none of the previously known models with consecutive singular cardinals discussed above satisfies it when $\theta \geq \kappa^+$. Since the definitions of "strong limit cardinal" and "inaccessible cardinal" generally do not make sense in models of $\neg$AC, let us explain why the assertion in Theorem 1 that (2) holds in N makes sense. It will be the case that N and V have the same bounded subsets of κ, and from this it follows that the usual definitions of "κ is a strong limit cardinal" and "$\delta < \kappa$ is an inaccessible cardinal" make sense in N.

Using the methods of Bull [6], Apter [1], and Apter and Henle [5], we also obtain the following two results.

Theorem 2 *Suppose that κ is supercompact,* GCH *holds, and θ is an ordinal. Then there is a model of* ZF + $\neg\mathrm{AC}_\omega$ *in which $\mathrm{cf}(\aleph_1) = \mathrm{cf}(\aleph_2) = \omega$, and there is a sequence of distinct subsets of $\aleph_1$ of length θ.*

Theorem 3 *Suppose that κ is supercompact,* GCH *holds, and θ is an ordinal. Then there is a model of* ZF + $\neg\mathrm{AC}_\omega$ *in which $\aleph_\omega$ and $\aleph_{\omega+1}$ are both singular with $\omega \leq \mathrm{cf}(\aleph_{\omega+1}) < \aleph_\omega$, and there is a sequence of distinct subsets of $\aleph_\omega$ of length θ.*

We note that in Theorem 2, $\aleph_1$ and $\aleph_2$ can be replaced with δ and δ^+, respectively, where δ is the successor of any ground-model regular cardinal less than κ. Also, in Theorem 3, we note that $\aleph_\omega$ and $\aleph_{\omega+1}$ can be replaced by η and η^+, respectively, where $\eta < \kappa$ can be any reasonably defined singular limit cardinal of cofinality ω. We will return to these issues later.

Let us now give a brief outline of the rest of the paper. In Section 2, we include a definition of the basic forcing notion we will use and outline its important properties. In Section 3, we give a detailed proof of Theorem 1. In Section 4, we sketch the proofs of Theorems 2 and 3. In Section 5, we discuss a result in which we separate the lengths of distinct subsets of consecutive singular cardinals, and we also pose some open questions.

2 Preliminaries

In this section, we will briefly discuss the various forcing notions used. If κ is a regular cardinal and λ is an ordinal, $\mathrm{Add}(\kappa, \lambda)$ denotes the standard partial order for adding λ Cohen subsets to κ. If $\lambda > \kappa$ is an inaccessible cardinal, $\mathrm{Coll}(\kappa, {<}\lambda)$ is the standard partial order for collapsing λ to κ^+ and all cardinals in the interval $[\kappa, \lambda)$ to κ. For further details, we refer the reader to Jech [12]. For a given partial order $\mathbb{P}$ and a condition $p \in \mathbb{P}$, we define $\mathbb{P}/p := \{q \in \mathbb{P} \mid q \leq p\}$. If φ is a statement in the forcing language associated with $\mathbb{P}$ and $p \in \mathbb{P}$, we write $p \parallel \varphi$ if and only if p decides φ.

We will now review the definition and important features of supercompact Prikry forcing and refer the reader to Gitik [11] or Apter [2] for details. Suppose that κ is λ-supercompact and that U is a normal fine measure on $P_\kappa\lambda$ satisfying the Menas partition property (see Menas [15] for a definition and a proof of the fact that if κ is 2^λ-supercompact, then $P_\kappa\lambda$ has a normal fine measure with this property). For $P, Q \in P_\kappa\lambda$ we say that P is *strongly included* in Q and write $P \precsim Q$ if $P \subseteq Q$ and $\mathrm{ot}(P) < \mathrm{ot}(Q \cap \kappa)$. We define *supercompact Prikry forcing* $\mathbb{P}$ to be the set of all ordered tuples of the form $\langle P_1, \dots, P_n, A\rangle$ such that

(1) $P_1, \dots, P_n$ is a finite $\precsim$-increasing sequence of elements of $P_\kappa\lambda$,
(2) $A \in U$, and
(3) for every $Q \in A$, $P_n \precsim Q$.

Given $\langle P_1, \dots, P_n, A\rangle, \langle Q_1, \dots, Q_m, B\rangle \in \mathbb{P}$ we say that $\langle P_1, \dots, P_n, A\rangle$ *extends* $\langle Q_1, \dots, Q_m, B\rangle$ and write $\langle P_1, \dots, P_n, A\rangle \leq \langle Q_1, \dots, Q_m, B\rangle$ if and only if

(1) $n \geq m$,
(2) for each $k \leq m$, $P_k = Q_k$,
(3) $A \subseteq B$, and
(4) $\{P_{m+1}, \dots, P_n\} \subseteq B$.

Since any two conditions of the form $\langle P_1, \dots, P_n, A\rangle$ and $\langle P_1, \dots, P_n, B\rangle$ in $\mathbb{P}$ are compatible, one may easily show that $\mathbb{P}$ is $(\lambda^{<\kappa})^+$-c.c. Since U satisfies the Menas partition property, it follows that forcing with $\mathbb{P}$ does not add new bounded subsets to κ. In the forcing extension by $\mathbb{P}$, κ has cofinality ω, and if $\lambda > \kappa$, then certain cardinals will be collapsed according to the following.

Lemma 4 *Every $\gamma \in [\kappa, \lambda]$ of cofinality at least κ (in V) changes its cofinality to ω in $V[G]$. Moreover, in $V[G]$, every cardinal in $(\kappa, \lambda]$ is collapsed to have size κ.*

3 The Proof of Theorem 1

Now we will begin the proof of Theorem 1. We note that our proof amalgamates the methods used in [5] with those of [2].

Proof of Theorem 1 Suppose that κ is supercompact and that θ is an ordinal in some initial model V_0 of ZFC + GCH. We will show that there is a forcing extension of V_0 that has a symmetric inner model N in which κ and κ^+ are both singular with $\mathrm{cf}(\kappa)^N = \omega$ and $\mathrm{cf}(\kappa^+)^N < \kappa$, and there is a θ-sequence of subsets of κ. By first forcing the supercompactness of κ to be Laver indestructible, as in Laver [13], and then forcing with $\mathrm{Add}(\kappa, \theta)$, we may assume without loss of generality that κ is supercompact and $2^\kappa = \theta$ in a forcing extension V of V_0. Let λ be a cardinal such that $\kappa < \lambda$ and $\mathrm{cf}(\lambda)^V < \kappa$. In V, let $\mathbb{P}$ be the supercompact Prikry forcing relative to some normal fine measure U on $P_\kappa\lambda$ satisfying the Menas partition property. Let G be V-generic for $\mathbb{P}$, and let $\langle P_n \mid n < \omega\rangle$ be the supercompact Prikry sequence associated with G; that is, $\langle P_n \mid n < \omega\rangle$ is the sequence of elements of $P_\kappa\lambda$ such that for each $n < \omega$, there is an $A \in U$ with $(P_1, \ldots, P_n, A) \in G$.

By Lemma 4, it follows that in $V[G]$, the cofinality of κ is ω, and every ordinal in the interval $(\kappa, \lambda]$ has size κ. Furthermore, since the supercompact Prikry forcing adds no new bounded subsets to κ, it follows that κ remains a cardinal in $V[G]$. We will now define a symmetric inner model $N \subseteq V[G]$ in which $\kappa^+ = \lambda$, and we will argue that the conclusions of Theorem 1 hold in N.

In order to define N, we need to discuss a way of restricting the forcing conditions in $\mathbb{P}$. First note that, as in [2], for $\delta \in [\kappa, \lambda]$ a regular cardinal, $U \restriction \delta := U \cap P(P_\kappa\delta)$ is a normal fine measure on $P_\kappa\delta$ satisfying the Menas partition property. Let $\mathbb{P}_{U\restriction\delta}$ denote the supercompact Prikry forcing associated with $U \restriction \delta$. If $p = \langle Q_1, \ldots, Q_n, A\rangle \in \mathbb{P}$, we define $p \restriction \delta := \langle Q_1 \cap \delta, \ldots, Q_n \cap \delta, A \cap P_\kappa\delta\rangle$ and note that $p \in \mathbb{P}_{U\restriction\delta}$. If $A \in P_\kappa\lambda$, we define $A \restriction \delta := A \cap P_\kappa\delta$. The Mathias genericity criterion (see Mathias [14]) for supercompact Prikry forcing yields that $r_\delta := \langle P_n \cap \delta \mid n < \omega\rangle$ generates a V-generic filter for $\mathbb{P}_{U\restriction\delta}$. Indeed, $G \restriction \delta := G \cap \mathbb{P}_{U\restriction\delta}$ is the generic filter for $\mathbb{P}_{U\restriction\delta}$ generated by r_δ. N is now defined informally as the smallest model of ZF extending V which contains r_δ for each regular cardinal $\delta \in [\kappa, \lambda)$ but not the full supercompact Prikry sequence $r := \langle P_n \mid n < \omega\rangle$.

We may define N more formally as follows. Let $\mathcal{L}$ be the forcing language associated with $\mathbb{P}$, and let $\mathcal{L}_1 \subseteq \mathcal{L}$ be the ramified sublanguage containing symbols $\check{v}$ for each $v \in V$, a unary predicate $\check{V}$ (interpreted as $\check{V}(\check{v})$ if and only if $v \in V$), and symbols $\dot{r}_\delta$ for each regular cardinal $\delta \in [\kappa, \lambda)$. We define N inductively inside $V[G]$ as follows:

$$\begin{aligned} N_0 &= \emptyset, \\ N_\delta &= \bigcup_{\alpha<\delta} N_\alpha \quad \text{for } \delta \text{ a limit ordinal}, \\ N_{\alpha+1} &= \big\{x \subseteq N_\alpha \;\big|\; x \text{ can be defined over } \langle N_\alpha, \in, c\rangle_{c\in N_\alpha} \\ &\qquad \text{using a forcing term } \tau \in \mathcal{L}_1 \text{ of rank} \le \alpha\big\}, \end{aligned}$$

and

$$N = \bigcup_{\alpha \in \mathrm{ORD}} N_\alpha.$$

Standard arguments show that $N \models$ ZF. As usual, each $\check{v}$ for $v \in V$ may be chosen so as to be invariant under any isomorphism $\Psi : \mathbb{P}/p \to \mathbb{P}/q$ for $p, q \in \mathbb{P}$. Further, terms τ mentioning only $\dot{r}_\delta$ may be chosen so as to be invariant under any isomorphism $\Psi : \mathbb{P}/p \to \mathbb{P}/q$ which preserves the meaning of r_δ.

The following lemma provides the key to showing that N has the desired features.

Lemma 5 *If $x \in N$ is a set of ordinals, then for some regular cardinal $\delta \in [\kappa, \lambda)$, $x \in V[r_\delta]$.*

Proof Let us note that the following proof of Lemma 5 blends ideas found in the proofs of [2, Lemma 1.5] and [5, Lemma 2.1]. Let τ be a term in $\mathcal{L}_1$ for x. Suppose β is an ordinal, $p \Vdash_{\mathbb{P},V} \tau \subseteq \beta$, and $p \in G$. Since $\tau \in \mathcal{L}_1$, it follows that τ mentions finitely many terms of the form $\dot{r}_\delta$. Without loss of generality, we may assume that τ mentions a single $\dot{r}_\delta$. We will show that $x \in V[r_\delta]$. Let

$$y := \left\{\alpha < \beta \;\middle|\; \exists q \leq p(q \restriction \delta \in G \restriction \delta \text{ and } q \Vdash_{\mathbb{P},V} \alpha \in \tau)\right\}.$$

We will show that $x = y$. Since it is clear that $y \in V[r_\delta]$, this will suffice. Suppose $\alpha \in x$, and choose $p' \leq p$ with $p' \in G$ such that $p' \Vdash_{\mathbb{P},V} \alpha \in \tau$. Since $p' \restriction \delta \in G \restriction \delta$, we conclude that $\alpha \in y$. Thus, $x \subseteq y$. Now suppose $\alpha \in y$, and let $q \leq p$ with $q \restriction \delta \in G \restriction \delta$ and $q \Vdash_{\mathbb{P},V} \alpha \in \tau$. There is a $q' \in G$ such that $q' \parallel \alpha \in \tau$. If $q' \Vdash \alpha \in \tau$, then $\alpha \in x$ and we are done; thus we assume that $q' \Vdash \alpha \notin \tau$. Write $q = \langle Q_1, \ldots, Q_l, A\rangle$ and $q' = \langle Q'_1, \ldots, Q'_m, A'\rangle$, where without loss of generality we assume that $l < m$. Since $q' \restriction \delta, q \restriction \delta \in G \restriction \delta$ and $l < m$, we know that $Q_i \cap \delta = Q'_i \cap \delta$ for $1 \leq i \leq l$. Furthermore, there is some $q^* := \langle Q_0 \cap \delta, \ldots, Q_l \cap \delta, R^*_{l+1}, \ldots, R^*_m, A^*\rangle \in G \restriction \delta$ extending $q \restriction \delta$ with $R^*_i = Q'_i \cap \delta$ for $l + 1 \leq i \leq m$. (To find such a condition one could just take a common extension of $q' \restriction \delta$ and $q \restriction \delta$ in $G \restriction \delta$ and then obtain the appropriate stem by throwing unwanted points back into the measure one set.) Now let us argue that there is a $q'' \leq q$ in $\mathbb{P}$ such that $q'' = \langle Q_0, \ldots, Q_l, S_{l+1}, \ldots, S_m, A''\rangle$, and for $l + 1 \leq i \leq m$ we have $S_i \cap \delta = R^*_i = Q'_i \cap \delta$. Since $q^* \leq_{\mathbb{P}\restriction\delta} q \restriction \delta$, it follows by the definition of $\leq_{\mathbb{P}\restriction\delta}$ that for $l + 1 \leq i \leq m$, $R^*_i \in A \restriction \delta = A \cap P_\kappa\delta$, which implies $R^*_i = S_i \cap \delta$ for some $S_i \in A$. Also by the definition of $\leq_{\mathbb{P}\restriction\delta}$, we have $A^* \subseteq A \restriction \delta$, and since $q^* \in \mathbb{P} \restriction \delta$, we have $A^* = B \restriction \delta$ for some $B \in U$. Now let $A'' := A' \cap A \cap B$, and notice that $A'' \restriction \delta \subseteq A^*$. Indeed we have $q'' \restriction \delta \leq_{\mathbb{P}\restriction\delta} q^*$ and $q'' \leq_{\mathbb{P}} q$. We let q''' be the condition extending q' defined by $q''' := \langle Q'_1, \ldots, Q'_m, A''\rangle$.

Now we define an isomorphism from $\mathbb{P}/q''$ to $\mathbb{P}/q'''$ that sends q'' to q''' and fixes τ. Let $\Psi : P_\kappa\lambda \to P_\kappa\lambda$ be the permutation defined by $\Psi(Q_i) = Q'_i$ and $\Psi(Q'_i) = Q_i$ for $1 \leq i \leq l$, by $\Psi(S_i) = Q'_i$ and $\Psi(Q'_i) = S_i$ for $l+1 \leq i \leq m$, and by letting Ψ be equal to the identity function otherwise. This permutation induces a map $\Psi : \mathbb{P}/p'' \to \mathbb{P}/p'''$ defined by $\Psi(\langle P_1, \ldots, P_n, C\rangle) = \langle \Psi(P_1), \ldots, \Psi(P_n), \Psi''C\rangle$. Note that since Ψ fixes all but finitely many elements of $P_\kappa\lambda$, it follows that $\Psi''C \in U$. One may check that Ψ is an isomorphism, and it easily follows that $\Psi(q'') = \langle Q'_1, \ldots, Q'_m, \Psi''A''\rangle = \langle Q'_1, \ldots, Q'_m, A''\rangle = q'''$. Furthermore, since τ

mentions only $\dot{r}_\delta$, since

$$\langle Q_1 \cap \delta, \ldots, Q_l \cap \delta, S_{l+1} \cap \delta, \ldots, S_m \cap \delta\rangle = \langle Q'_1 \cap \delta, \ldots, Q'_m \cap \delta\rangle,$$

and since any condition $\langle Q_1, \ldots, Q_l, S_{l+1}, \ldots, S_m, S_{m+1}, \ldots, S_k, D\rangle$ extending q'' must have $S_i \notin \{Q_1, \ldots, Q_l, S_{l+1}, \ldots, S_m, Q'_1, \ldots, Q'_m\}$ for $m+1 \leq i \leq k$, it follows that Ψ does not affect the meaning of τ. By extending Ψ to the relevant $\mathbb{P}$-terms, since $q'' \Vdash \alpha \in \tau$, we have $\Psi(q'') \Vdash \Psi(\alpha) \in \Psi(\tau)$. This implies $\Psi(q'') = q''' \Vdash \alpha \in \tau$. This contradicts the fact that $q''' \leq q' \Vdash \alpha \notin \tau$. □

Since $V \subseteq N \subseteq V[G]$ and $\mathbb{P}$ does not add bounded subsets to κ, it follows that N and V have the same bounded subsets of κ. Thus, in N, κ is a limit of inaccessible cardinals and hence is also a strong limit cardinal.

We will now use Lemma 5 to show that λ, which was collapsed to have size κ in $V[G]$, is a cardinal in N and, furthermore, $(\kappa^+)^N = \lambda$ and $\mathrm{cf}(\lambda)^N = \mathrm{cf}(\lambda)^V$.

Let us argue that if $\gamma \geq \lambda$ is a cardinal in V, then γ remains a cardinal in N. Suppose, for a contradiction, that γ is not a cardinal in N. Then there is a bijection from some $\alpha < \gamma$ to γ which is coded by a set of ordinals in N. By Lemma 5, there is a regular cardinal $\delta \in (\kappa, \lambda)$ such that the code and hence the bijection are in $V[G \restriction \delta]$. This implies that γ is not a cardinal in $V[G \restriction \delta]$. We will obtain a contradiction by using the chain condition of $\mathbb{P}_{U\restriction\delta}$ to show that γ is a cardinal in $V[G \restriction \delta]$. Indeed, we will show that even though GCH may fail at κ in V, the supercompact Prikry forcing $\mathbb{P}_{U\restriction\delta}$ is δ^+-c.c. in V. It follows from our remarks in Section 2 that $\mathbb{P}_{U\restriction\delta}$ is $(\delta^{<\kappa})^+$-c.c. in V. Since GCH holds in V_0 we have $(\delta^{<\kappa})^{V_0} = \delta$, and since $\mathrm{Add}(\kappa, \theta)$ preserves cardinals and adds no sequences of ordinals of length less than κ, we conclude that $(\delta^{<\kappa})^V = (\delta^{<\kappa})^{V_0} = \delta$. This shows that $\mathbb{P}_{U\restriction\delta}$ is δ^+-c.c. in V, and thus γ is a cardinal in $V[G \restriction \delta]$, a contradiction.

For each regular cardinal $\delta \in (\kappa, \lambda)$, we have $V[G \restriction \delta] \subseteq N$, and this implies that $\mathrm{cf}^N(\kappa) = \omega$ and that every ordinal in (κ, λ) which is a cardinal in V is collapsed to have size κ in N. Thus, we have $(\kappa^+)^N = \lambda$. Furthermore, since N and V agree on bounded subsets of κ, we see that $\mathrm{cf}^N(\lambda) = \mathrm{cf}^V(\lambda) < \kappa$. This shows that $\mathrm{cf}^N((\kappa^+)^N) = \mathrm{cf}^V(\lambda) < \kappa$, and this implies that N satisfies $\neg$AC. Since $V \subseteq N$, and since $(2^\kappa = \theta)^V$, it follows that there is a θ-sequence of distinct subsets of κ in N.

This completes the proof of Theorem 1. □

Let us emphasize: The fact that GCH can potentially fail at κ in V, depending on the size of θ, together with the cardinal preservation to N, is the feature of our construction that sets the results of this paper apart from those previously discussed in the literature.

4 The Proofs of Theorems 2 and 3

In this section, we sketch the proofs of Theorems 2 and 3. We begin with Theorem 2.

Proof of Theorem 2 Suppose that the model N is such that $\mathrm{cf}(\kappa)^N = \omega$, $\mathrm{cf}(\kappa^+)^N < \kappa$, and that there is, in N, a sequence of distinct subsets of κ of length θ. We will now argue that in a symmetric inner model M of a forcing extension of N, we have $\mathrm{cf}(\aleph_1) = \mathrm{cf}(\aleph_2) = \omega$, and there is a sequence of distinct subsets of $\aleph_1$ of length θ.

Working in N, let $\langle \kappa_n \mid n < \omega \rangle$ be a sequence of inaccessible cardinals less than κ which is cofinal in κ. Let $\mathbb{P} := \mathrm{Coll}(\omega, {<}\kappa)$, and let G be N-generic for $\mathbb{P}$. Let $\mathbb{P}_n := \mathrm{Coll}(\omega, {<}\kappa_n)$. Standard arguments show that $G_n := G \cap \mathbb{P}_n$ is N-generic for $\mathbb{P}_n$ (see [2, proof of Theorem 2]).

As in the proof of Theorem 1, we let M be the least model of ZF extending N containing each G_n but not G. More formally, let $\mathcal{L}_2$ be the ramified sublanguage of the forcing language associated with $\mathbb{P}$ containing terms $\check{x}$ for each $x \in N$, a unary predicate $\check{N}$ for N, and canonical terms $\dot{G}_n$ for each G_n. We now define M inductively inside $N[G]$ as follows:

$$\begin{aligned} M_0 &= \emptyset, \\ M_\delta &= \bigcup_{\alpha<\delta} M_\alpha \quad \text{for } \delta \text{ a limit ordinal}, \\ M_{\alpha+1} &= \big\{x \subseteq M_\alpha \;\big|\; x \text{ can be defined over } \langle M_\alpha, \in, c\rangle_{c\in M_\alpha} \\ &\qquad \text{using a forcing term } \tau \in \mathcal{L}_2 \text{ of rank} \le \alpha\big\}, \end{aligned}$$

and

$$M = \bigcup_{\alpha \in \mathrm{ORD}} M_\alpha.$$

As before, standard arguments show that $M \models \mathrm{ZF}$. Since M contains G_n for each n, it follows that cardinals in $[\omega, \kappa)$ are collapsed to have size ω, and hence $\aleph_1^M \ge \kappa$. However, standard arguments (see [6, Lemmas 6.2 and 5.3]) also show that if $x \in M$ is a set of ordinals, then $x \in N[G_n]$ for some $n < \omega$. Since $\mathrm{Coll}(\omega, {<}\kappa_n)$ is canonically well orderable in N with order type κ_n, the usual proofs show that cardinals and cofinalities greater than or equal to κ are preserved to $N[G_n]$. Since $\kappa = \aleph_1^M$, $\mathrm{cf}(\aleph_1)^M = \mathrm{cf}(\aleph_2)^M = \omega$. It therefore follows that $M \models \neg\mathrm{AC}_\omega$. Thus, M is the desired model. □

We remark here that the above proof may be easily adapted to collapse κ and κ^+ to δ and δ^+, respectively, where δ is the successor of a regular cardinal, say, $\delta = \mu^+$. The main difference between the above proof of Theorem 2 and the proof in this more general setting is that the restricted version of the collapse forcing, call it $\mathbb{P}'_n := \mathrm{Coll}(\mu, {<}\kappa_n)$, is no longer canonically well orderable. However, since N and V have the same bounded subsets of κ, and $V \subseteq N$, it follows that $\mathbb{P}'_n$ can be well ordered in both V and N with order type less than κ. In this way, we obtain a model M of ZF + $\neg$AC in which $\mathrm{cf}(\delta) = \mathrm{cf}(\delta^+) = \omega$ and in which there is a sequence of distinct subsets of δ of length θ.

Below we present a sketch of our proof of Theorem 3. As in the above proof sketch of Theorem 2, we will argue that in a symmetric inner model M of a forcing extension of N, we have $\omega \le \mathrm{cf}(\aleph_{\omega+1}) < \aleph_\omega$, and there is a sequence of distinct subsets of $\aleph_\omega$ of length θ.

Proof of Theorem 3 Let N be constructed so that $\mathrm{cf}(\kappa)^N = \omega$, $\mathrm{cf}(\kappa^+)^N < \kappa$, and there is a sequence of distinct subsets of κ of length θ in N. Let $\langle \kappa_i \mid i < \omega \rangle$ be a sequence of inaccessible cardinals cofinal in κ. Let $\mathbb{P}_0 := \mathrm{Coll}(\omega, {<}\kappa_0)$ and $\mathbb{P}_i := \mathrm{Coll}(\kappa_{i-1}, {<}\kappa_i)$ for $i \in [1, \omega)$. Let $\mathbb{P} := \prod_{i<\omega} \mathbb{P}_i$, where the product has finite support. For each $n < \omega$, we may factor $\mathbb{P}$ as $\mathbb{P} \cong \mathbb{P}^*_n \times \mathbb{P}^n$, where $\mathbb{P}^*_n := \prod_{i\in[0,n]} \mathbb{P}_i$ and $\mathbb{P}^n := \prod_{i\in[n+1,\omega)} \mathbb{P}_i$. Let $G \cong G^*_n \times G^n$ be N-generic for $\mathbb{P}$. As in [2, proof of

Theorem 2], each G_n^* is N-generic for $\mathbb{P}_n^*$. As before, we let M be the least model of ZF extending N containing each G_n^* but not $\langle G_n^* \mid n < \omega\rangle$. More formally, let $\mathcal{L}_3$ be the ramified sublanguage of the forcing language associated with $\mathbb{P}$ containing terms $\check{x}$ for each $x \in N$, a unary predicate $\check{N}$ for N, and canonical terms $\dot{G}_n^*$ for each G_n^*. We now define M inductively inside $N[G]$ as follows:

$$\begin{aligned} M_0 &= \emptyset, \\ M_\delta &= \bigcup_{\alpha<\delta} M_\alpha \quad \text{for } \delta \text{ a limit ordinal}, \\ M_{\alpha+1} &= \big\{x \subseteq M_\alpha \;\big|\; x \text{ can be defined over } \langle M_\alpha, \in, c\rangle_{c\in M_\alpha} \\ &\qquad \text{using a forcing term } \tau \in \mathcal{L}_3 \text{ of rank} \leq \alpha\big\}, \end{aligned}$$

and

$$M = \bigcup_{\alpha\in \mathrm{ORD}} M_\alpha.$$

Since $G_n^* \in M$ for each $n < \omega$, it follows that in M, $\aleph_\omega \geq \kappa$, and hence $\aleph_{\omega+1} \geq (\kappa^+)^N$. To show that $\kappa = \aleph_\omega$ and $(\kappa^+)^N = \aleph_{\omega+1}$ in M, we will use the following lemma.

Lemma 6 *If x is a set of ordinals in M, then $x \in N[G_n^*]$ for some $n < \omega$.*

For a proof of Lemma 6, one may consult [2, Lemma 2.1].

We now argue as in our sketch of the proof of Theorem 2. Since N and V contain the same bounded subsets of κ, and $V \subseteq N$, $\mathbb{P}_n^*$ can be well ordered in both V and N with order type less than κ. Therefore, as before, the usual proofs show that cardinals and cofinalities greater than or equal to κ are preserved. Furthermore, $M \models \neg\mathrm{AC}_\omega$ since $\langle G_n^* \mid n < \omega\rangle \notin M$. It follows that M is thus once again the desired model. □

We remark that, as in [2, Theorem 2], in the model M constructed in the above proof of Theorem 3, $\aleph_\omega$ is a strong limit cardinal. Also, as we mentioned earlier, by changing the cardinals to which each κ_i is collapsed, it is possible to collapse κ to $\aleph_{\omega+\omega}$, $\aleph_{\omega^2}$, and so on.

5 An Additional Result and Some Open Questions

In the above results, from GCH and a supercompact cardinal κ we obtain models of ZF with consecutive singular cardinals, κ and κ^+, in which there is a sequence of distinct subsets of κ with any predetermined length, and hence there is a sequence of distinct subsets of κ^+ with this same length. This suggests the following question.

Question 1 Suppose that θ_1 and θ_2 are arbitrary ordinals. Are there models of ZF with consecutive singular cardinals, κ and κ^+, in which there are sequences of distinct subsets of κ and κ^+ having lengths θ_1 and θ_2, respectively?

To avoid trivialities, we also require in Question 1 that there be no sequence of subsets of κ of length θ_2.

Let us remark that in Gitik's model in which all uncountable cardinals are singular (see [8]), for every pair of cardinals κ and κ^+, there is a sequence of distinct subsets of κ of length θ_1 and a sequence of distinct subsets of κ^+ of length θ_2, where θ_1 and θ_2 are ordinals satisfying $\kappa < \theta_1 < \kappa^+ < \theta_2 < \kappa^{++}$. In this sense, Question 1 is partially answered by Gitik's model, for some particular θ_1 and θ_2. However, neither Gitik's model nor our previous theorems address Question 1 if we require, for example, that $\theta_1 = \kappa^+$ and $\theta_2 \geq \kappa^{++}$. The following theorem provides more information toward an answer to Question 1, for the case in which $\kappa < \theta_1 < \kappa^+$ and $\theta_2 \geq \kappa^+$.

Theorem 7 *Suppose that* GCH *holds, $\kappa < \lambda$ are such that κ is 2^λ-supercompact, and λ has cofinality ω with $\{\alpha < \lambda \mid o(\alpha) \geq \alpha^{+n}\}$ cofinal in λ for every $n < \omega$. Then there is a forcing extension $V[G]$ with a symmetric inner model $N \subseteq V[G]$ of* ZF *in which*

(1) $\mathrm{cf}(\kappa) = \mathrm{cf}(\kappa^+) = \omega$,
(2) there is no κ^+-sequence of distinct subsets of κ, and
(3) there is a sequence of distinct subsets of κ^+ of length κ^{+17}.

Let us remark that the hypotheses of Theorem 7 follow from GCH and the existence of $\kappa < \delta$ such that κ is δ-supercompact and δ is δ^+-supercompact. We also note that by Gitik [10], in Theorem 7(3) above, one can replace 17 with $\delta + 1$ for any $\delta < \aleph_1$. In addition, note that the hypotheses of Theorem 7 imply that λ is a strong limit cardinal, since it is a limit of inaccessible cardinals.

Proof of Theorem 7 In [10], Gitik shows that under these hypotheses on λ, if $\delta < \aleph_1$, then there is a forcing notion, call it $\mathbb{P}$, that preserves cardinals, adds no new bounded subsets to λ, and forces $2^\lambda = \lambda^{+\delta+1}$. It will suffice for us to take $\delta = 16$ so that we achieve (3).

Let V_0 satisfy the hypotheses of Theorem 7. Let $G \subseteq \mathbb{P}$ be V_0-generic, and let $V := V_0[G]$. Then it follows by Gitik's result that there is an injection $f : \lambda^{+17} \to P(\lambda)$ in V. Since κ is 2^λ-supercompact in V_0, we may let $U \in V_0$ denote a normal fine measure on $(P_\kappa\lambda)^{V_0}$ satisfying the Menas partition property. Since $\mathbb{P}$ does not add bounded subsets to λ, it follows that λ remains a strong limit cardinal in $V = V_0[G]$, and κ remains γ-supercompact in V for each cardinal $\gamma < \lambda$. Indeed, if we let $U \restriction \gamma := U \cap P(P_\kappa\gamma)$ for each regular cardinal $\gamma < \lambda$, then $U \restriction \gamma$ is a normal fine measure on $P_\kappa\gamma$ in V satisfying the Menas partition property.

In V, let $\langle \gamma_n \mid n < \omega\rangle$ be a sequence of regular cardinals cofinal in λ, and let $\mathbb{Q}_{U\restriction\gamma_n}$ denote the supercompact Prikry forcing over $P_\kappa(\gamma_n)$ defined using $U \restriction \gamma_n$. Even though U will not be a normal measure on $P_\kappa\lambda$ in V, we can use it in the definition of supercompact Prikry forcing over $P_\kappa\lambda$. Call this forcing $\mathbb{Q}$. Let H be V-generic for $\mathbb{Q}$, and let r_{γ_n} be the supercompact Prikry sequence for $\mathbb{Q}_{U\restriction\gamma_n}$ obtained from H as in the proof of Theorem 1. Let N be the smallest inner model of $V[H]$ that contains r_{γ_n} for each $n < \omega$ but does not contain H. More formally, let $\mathcal{L}_4$ be the ramified sublanguage of the forcing language associated with $\mathbb{Q}$ containing terms $\check{v}$ for each $v \in V$, a unary predicate $\check{V}$ for V, and canonical terms $\dot{r}_{\gamma_n}$ for each r_{γ_n}. We now define N inductively inside $V[H]$ as follows:

$$N_0 = \emptyset,$$
$$N_\delta = \bigcup_{\alpha<\delta} N_\alpha \quad \text{for } \delta \text{ a limit ordinal},$$
$$N_{\alpha+1} = \Big\{x \subseteq N_\alpha \;\Big|\; x \text{ can be defined over } \langle N_\alpha, \in, c\rangle_{c\in N_\alpha} \text{ using a forcing term } \tau \in \mathcal{L}_4 \text{ of rank} \leq \alpha\Big\},$$

and

$$N = \bigcup_{\alpha \in \mathrm{ORD}} N_\alpha.$$

Lemma 8 *If $x \in N$ is a set of ordinals, then there is an $n < \omega$ such that $x \in V[r_{\gamma_n}] = V_0[G][r_{\gamma_n}]$.*

The proof of Lemma 8 is the same as that of Lemma 5 above. Using Lemma 8, it is straightforward to verify that the conclusions of Theorem 7 hold in N. Just as in the above proof of Theorem 1, it follows from Lemma 8 that $(\kappa^+)^N = \lambda$ and $\mathrm{cf}(\kappa)^N = \mathrm{cf}(\kappa^+)^N = \omega$, which implies that (1) holds in N. Furthermore, since the injection f is in $V_0[G] = V \subseteq N$, we conclude that (3) holds in N. It remains to show that (2) holds in N. Working in N, suppose that $\vec{x} = \langle x_\alpha \mid \alpha < \kappa^+ \rangle$ is a sequence of distinct subsets of κ. Then by Lemma 8, $\vec{x} \in V[r_{\gamma_n}]$ for some $n < \omega$. This is impossible, since $\lambda = (\kappa^+)^N$ remains a strong limit cardinal in $V[r_{\gamma_n}]$ because $|\mathbb{Q}_{U \restriction \gamma_n}| < \lambda$. □

The results in this paper suggest the question as to whether one can prove an Easton theorem-like result, but for models of ZF with consecutive singular cardinals. Let us state two seemingly very difficult related open questions.

Question 2 From large cardinals, is there a model of ZF in which every cardinal is singular and in which for every cardinal κ, there is a sequence of κ^+ distinct subsets of κ?

Question 3 From large cardinals, is there a model of ZF in which every cardinal is singular and in which GCH fails everywhere in the sense that for every cardinal κ, there is a sequence of κ^{++} distinct subsets of κ?

Addressing Question 1, one would also like to obtain models of ZF with consecutive singular cardinals, say, κ and κ^+, where κ^+ has uncountable cofinality, $\theta_1 < \theta_2$ are cardinals, and $\theta_2 \geq \kappa^{+3}$. Notice that Gitik's methods for violating GCH at ground-model singular cardinals do not seem to work for singular cardinals of uncountable cofinality. This suggests the following alternative strategy. Let $\kappa < \lambda$ be the appropriate large cardinals. Using standard techniques, blow up the size of the power set of λ while preserving "sufficiently many" of the large cardinal properties of κ and λ. This will allow us to change the cofinality of λ to some uncountable cardinal and to change the cofinality of κ, while simultaneously collapsing all cardinals in the interval (κ, λ) to κ. However, the standard forcings for changing to uncountable cofinality at λ, for example, Radin or Magidor forcing, will introduce Prikry sequences to unboundedly many cardinals in the interval (κ, λ) (see [11]). By Cummings, Foreman, and Magidor [7, Theorem 11.1(1)], this will introduce nonreflecting stationary subsets of ordinals of cofinality ω to unboundedly many regular cardinals δ in the interval (κ, λ). By Solovay, Reinhardt, and Kanamori [16, Theorem 4.8] and the succeeding remarks, no cardinal below λ is strongly compact up to λ. Thus one cannot use the standard forcings for changing the cofinality of κ while simultaneously collapsing cardinals in the interval (κ, λ) to κ. This suggests that one would like some forcing notion that changes the cofinality of $\lambda > \kappa$ to an uncountable cardinal, and also preserves enough of the original large cardinal properties of κ to allow these collapses to occur. As pointed out by the referee of this paper, by the work of

Woodin [18] on inner models for supercompact cardinals, it appears as though this is impossible.

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