

## A New Conditional for Naive Truth Theory

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**Abstract** In this paper a logic suitable for reasoning disquotationally about truth,  $\text{TJK}^+$ , is presented and shown to have a standard model. This work improves on Hartry Field's recent results establishing consistency and  $\omega$ -consistency of truth theories with strong conditional logics. A novel method utilizing the Banach fixed point theorem for contracting functions on complete metric spaces is invoked, and the resulting logic is shown to validate a number of principles which existing revision theoretic methods have so far failed to provide.

### 1 Introduction

An increasingly popular thesis has it that the claim that  $p$  and the claim that “ $p$ ” is true are equivalent and fully intersubstitutable with one another in contexts which do not contain any intensional or hyperintensional connectives. Formally this is represented by the following intersubstitutivity rule for extensional contexts:

From  $\varphi$  infer  $\varphi'$  and vice versa, (1)

where  $\varphi'$  is any sentence obtained from  $\varphi$  by substituting some occurrences of  $\psi$  for  $\text{Tr}(\ulcorner \psi \urcorner)$ . Here  $\text{Tr}$  is a formal truth predicate, and  $\ulcorner \psi \urcorner$  represents the numeral for the Gödel number of  $\psi$  relative to some suitably chosen Gödel numbering. However, due to the liar paradox, Curry's paradox, and related antinomies, one cannot have this rule without relinquishing some of the principles of classical logic. The remaining question is then: which subclassical logics can consistently support (1) in its full generality?

A now standard example of a logic in which the above rule can be consistently maintained is the 3-valued logic based on the strong Kleene valuation,  $\text{K3}$ , and its paraconsistent dual  $\text{LP}$  (see Kripke [11]). However, among recent defenders of the

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intersubstitutivity principle it is agreed that these logics are just too weak to sustain any kind of substantial reasoning about truth. The former logic has no theorems, although it has many rules, and the latter lacks important rules such as modus ponens.

For recent proponents of the intersubstitutivity principle, Kripke's construction (in its paracomplete or paraconsistent form) does not provide a sufficiently strong logic to support an adequate naive truth theory based on the intersubstitutivity principle. The subsequent proposals have employed instead the revision theoretic techniques first outlined by Brady in [5]—and indeed there is a rich literature tied to this approach (see, e.g., Brady [5], [4], [6], Priest [12], Yablo [15], Field [7], and Beall [2]).<sup>1</sup> However, even the logics generated by the revision theoretic techniques lack a number of natural logical principles. In this paper I present a different method for generating logics supporting the intersubstitutivity principle which has an intuitive geometrical interpretation in terms of the Banach fixed point theorem.

The particular conditional I study here strengthens, but is generally in the same spirit, as the conditional proposed by Field in [7, pp. 272–74]. In particular, Field's conditional lacks a number of very natural principles:

1.  $\varphi \rightarrow (\psi \rightarrow \varphi)$ ,
2.  $(\varphi \rightarrow \psi) \rightarrow ((\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \psi))$ ,
3.  $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$ ,
4.  $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \chi)) \rightarrow (\varphi \rightarrow \chi)$ ,
5.  $(\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi) \rightarrow (\varphi \rightarrow \psi \wedge \chi)$ ,
6.  $(\varphi \rightarrow \chi) \wedge (\psi \rightarrow \chi) \rightarrow (\varphi \vee \psi \rightarrow \chi)$ ,
7.  $\forall x(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall x\psi)$ ,
8.  $\forall x(\varphi \rightarrow \psi) \rightarrow (\exists x\varphi \rightarrow \psi)$ .

However, if one examines the proofs of the liar, Curry, and related paradoxes, these principles look to be innocent. Furthermore, many of these principles seem to be required for an adequate account of quantification. For example, presumably the principle “if all  $F$ 's are  $G$ , then there's a  $G$  if there's an  $F$ ” seems to be an obvious truism, whether or not  $F$  or  $G$  involve the truth predicate, although it is hard to formalize it as a claim which is validated in Field's model.<sup>2</sup> Unlike the less obvious truths of classical logic, such as the law of excluded middle, it is hard to fathom how this principle could fail—if every  $F$  is  $G$ , then how could there be an  $F$  without a  $G$ ?

Failures of the principles “if everything is  $G$  then every  $F$  is  $G$ ” and “if all  $F$ 's are  $G$ , then if all  $G$ 's are  $H$ , then all  $F$ 's are  $H$ ” seem equally unfathomable, but in these cases as well it is hard to formalize these claims in a way that validates them in Field's model. The most obvious way to formalize “every  $F$  is  $G$ ” is as  $\forall x(Fx \rightarrow Gx)$ , and without the principles 1 and 3 above, these principles of quantification fail.

Field also introduces a determinacy operator, which he defines:  $\Delta\varphi := \varphi \wedge (\top \rightarrow \varphi)$ . In order to get a natural logic for this operator, one might want the principle  $\Delta\varphi \wedge \Delta\psi \rightarrow \Delta(\varphi \wedge \psi)$ . However, this principle fails in Field's logic. Since one might want, as Field does, to use this operator to state when something is vague one might think this principle is vital: since the conjunction connective is not vague, then if neither conjunct is vague, how could one be in a situation where the conjunction is not determinate? Where could the vagueness of a complex expression come from if not from one of its components? In a logic containing 5,

however,  $\Delta\varphi \wedge \Delta\psi \rightarrow \Delta(\varphi \wedge \psi)$  becomes provable assuming a fairly uncontroversial background logic.

I therefore think we have some good reasons to study logics with the above axioms if we are to properly assess Field's project. There is a further question of whether these axioms are enough. For example, neither of the following principles are part of the logic I describe:

9.  $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))$ ,
10.  $((\varphi \rightarrow \perp) \rightarrow \perp) \rightarrow \varphi$ .

Principle 9 would allow us the following inference. From  $a$  is  $F$ , infer that if every  $F$  is  $G$ , then  $a$  is  $G$ . One may want to postulate 9 in addition to the principles above. The proposed logic does have another good feature: it validates the axiom

4.  $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \chi)) \rightarrow (\varphi \rightarrow \chi)$ .

We therefore may have the quantificational principle "if every  $F$  is  $G$  and every  $G$  is  $H$ , every  $F$  is  $H$ ." However, it is known that 9 and 4 cannot be consistently combined with principle 1 and the intersubstitutivity of provable equivalents (see Brady [6, Section 6.2]). So it seems that we are forced to make a choice between 9 and 4 on logical grounds. Replacing 4 with 9 in the logic I propose results in the positive fragment of a logic known as **RWK**. It is my view that **RWK** is also worth investigating, although it is currently unknown whether it can  $\omega$ -consistently support a naive truth predicate.

While we may not have as much intuitive reason to adopt 10, it is not obvious that 10 would cause trouble (see Bacon [1]). So it seems at least worth investigating whether 10 can be  $\omega$ -consistently added to the principles 1–8 with or without 9 or 4.<sup>3</sup> Thus what follows is only a step in the right direction—without an explicit proof of the  $\omega$ -inconsistency of the above principles, or, on the other hand, an argument that the resulting truth theory has a standard model, we are still a long way from knowing the truth of the matter.

Finally, there is a worry that no conditional can satisfy an adequate theory of quantification.<sup>4</sup> It is very natural to want a conditional,  $\rightarrow$ , which can be used to formalize "every  $F$  is  $G$ " as  $\forall x(Fx \rightarrow Gx)$ , and a connective,  $*$ , for formalizing relative clauses, so that " $x$  is an  $F$  who is  $G$ " is formalized as  $Fx * Gx$ . In order to ensure the following inference:

Every man is such that everyone he admires is tall.

Therefore every man who admires himself is tall.

it seems that we would need the rule RQ1:  $\varphi \rightarrow (\psi \rightarrow \chi) \vdash (\varphi * \psi) \rightarrow \chi$ . Second, in order to validate:

Every bachelor is a man.

Every bachelor is unmarried.

Therefore every bachelor is a man who is unmarried.

it is natural to want the rule RQ2:  $\varphi \rightarrow \psi, \varphi \rightarrow \chi \vdash \varphi \rightarrow (\psi * \chi)$ . Yet in combination with  $\varphi \rightarrow \varphi$ , transitivity of  $\rightarrow$  and modus ponens RQ1 and RQ2 lead to triviality.<sup>5</sup>

In Section 2 I shall outline the logic, **TJK**<sup>+</sup>, and give a possible world semantics and an equivalent algebraic semantics that it is sound with respect to. In Section 3 I prove a general fixed point theorem for a certain class of functions on the algebraic semantics and show that it is an instance of the Banach fixed point theorem (see

Section 4). In Section 5 a standard model is given for  $\text{TJK}^+$ , and in Section 6 I carry out a brief survey of ways to add an involutive negation operator to the logic.

## 2 Logic and Semantics

Let  $\mathcal{L}$  be the first-order language of Peano arithmetic (consisting of the nonlogical symbols  $0, +, \times, '$  and function symbols,  $f_e$ , for every other primitive recursive function  $e$ ) with a truth predicate,  $\text{Tr}$ , whose primitive connectives are given by the set  $\{\perp, \rightarrow, \vee, \wedge, \forall, \exists\}$ . In what follows I shall use the Greek letters  $\varphi, \psi$ , and  $\chi$  to represent formulas of  $\mathcal{L}$ . I shall assume a bijective Gödel numbering  $\ulcorner \cdot \urcorner$ , and I shall use the shorthand  $\mathbf{n}$  to denote the term of  $\mathcal{L}$  consisting of 0 succeeded by  $n$  “'” symbols. The language  $\mathcal{L}'$  shall be exactly the same, except with two truth predicates,  $\text{Tr}^+$  and  $\text{Tr}^-$ .<sup>6</sup>

The logic  $\text{TJK}^+$  is given by every instance of the following schemata:

1.  $\perp \rightarrow \varphi$ ,
  2.  $\varphi \rightarrow \varphi$ ,
  3.  $\varphi \rightarrow (\psi \rightarrow \varphi)$ ,
  4.  $(\varphi \rightarrow \psi) \rightarrow ((\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \psi))$ ,
  5.  $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$ ,
  6.  $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \chi)) \rightarrow (\varphi \rightarrow \chi)$ ,
  7.  $\varphi \rightarrow \varphi \vee \psi$ ,
  8.  $\psi \rightarrow \varphi \vee \psi$ ,
  9.  $\varphi \wedge \psi \rightarrow \varphi$ ,
  10.  $\varphi \wedge \psi \rightarrow \psi$ ,
  - 11\*.  $(\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi) \rightarrow (\varphi \rightarrow \psi \wedge \chi)$ ,
  - 12\*.  $(\varphi \rightarrow \chi) \wedge (\psi \rightarrow \chi) \rightarrow (\varphi \vee \psi \rightarrow \chi)$ ,
  - 13\*.  $(\varphi \wedge (\psi \vee \chi)) \rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$ ,
- MP**  $\varphi, \varphi \rightarrow \psi \vdash \psi$ ,
- $\wedge$ -intro  $\varphi, \psi \vdash \varphi \wedge \psi$ ,
- M1** If  $\Gamma, \varphi \vdash \chi$ , and  $\Gamma, \psi \vdash \chi$ , then  $\Gamma, \varphi \vee \psi \vdash \chi$ .

I have placed a star next to a principle to indicate that its converse is also to be included (although they are already derivable.) The principles 2–5 are often denoted **I**, **K**, **B**, and **B'**, respectively. Principle 6 appears to be quite distinctive to Routley and Meyer's [14] “dialectical logic” and appears to be related to, albeit weaker in this context than, contraction.<sup>7</sup> I shall refer to the logic as a whole as  $\text{TJK}^+$ , since it is obtained from the negation-free fragment of the relevant logic **TJ** by adding the **K** axiom.<sup>8</sup>

$\text{TJK}^+$  can be extended to a quantificational logic by adding the axioms

1.  $\forall x \varphi \rightarrow \varphi[t/x]$  provided  $t$  is substitutable for  $x$ ,
2.  $\varphi[t/x] \rightarrow \exists x \varphi$  provided  $t$  is substitutable for  $x$ ,
- 3\*.  $\forall x(\varphi \vee \psi) \rightarrow (\varphi \vee \forall x \psi)$  provided  $x$  is not free in  $\varphi$ ,
- 4\*.  $(\varphi \wedge \exists x \psi) \rightarrow \exists x(\varphi \wedge \psi)$  provided  $x$  is not free in  $\varphi$ ,
- 5\*.  $\forall x(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall x \psi)$  provided  $x$  is not free in  $\varphi$ ,
- 6\*.  $\forall x(\varphi \rightarrow \psi) \rightarrow (\exists x \varphi \rightarrow \psi)$  provided  $x$  is not free in  $\varphi$ ,
7.  $t = t$ ,
8.  $s = t \vdash \varphi \rightarrow \varphi[t/s]$ ,

**Gen** If  $\Gamma \vdash \varphi$ , then  $\Gamma \vdash \forall x \varphi$  provided  $x$  does not occur free in  $\Gamma$ .

In the rest of this section I shall introduce a possible worlds semantics for a conditional of this kind. The general form of this semantics has been successfully applied to both intuitionistic and relevance logic. In both cases matters are simplified by including an ordering  $\leq$  on the set of worlds; other than this fact, matters do not differ much from standard modal logic according to which  $n$ -ary connectives are modeled by an  $(n + 1)$ -ary accessibility relation (see Blackburn, de Rijke, and Venema [3]).

**Definition 2.1** A *frame* for a language,  $\mathcal{L}$ , is a quadruple  $\langle W, D, R, \leq \rangle$  where  $W$  is a set,  $D$  is a set,  $R \subseteq W^3$  is a ternary relation on  $W$ , and  $\leq$  is a partial order on  $W$  such that whenever  $Rxyz$ ,  $x \leq x'$ ,  $y \leq y'$ , and  $z \geq z'$ ,  $Rx'y'z'$ .

A model for  $\mathcal{L}$  is a pair  $\langle \mathcal{F}, \|\cdot\| \rangle$  where  $\mathcal{F}$  is a frame. If  $v$  is an assignment, then  $\|\cdot\|^v$  maps constant terms to members of  $D$ , maps the variable  $x$  to  $v(x)$ , maps  $n$ -ary function terms to  $n$ -ary functions on  $D$ , and maps  $n$ -ary relation terms to functions  $D^n \rightarrow V$ , where  $V$  is the set of nonempty downwards closed subsets of  $W$ . A set of worlds,  $p$ , is downwards closed if and only if whenever  $x \leq y$  and  $y \in p$ ,  $x \in p$ . This last condition is called persistence;  $\|\cdot\|^v$  can be extended in the usual way to complex terms.

Note that in the literature on relevance logic and the Kripke semantics for intuitionistic logic the order  $\leq$  goes in the opposite direction from the way I have introduced it here. I shall also assume that  $W$  has a  $\leq$ -least element,  $\perp_W$ . I write  $v[x]u$  to mean that the assignments  $v$  and  $u$  agree everywhere except, possibly, at  $x$ . Throughout the paper I will follow a convention of omitting reference to the variable assignment when it is not playing a role. A model determines a relation,  $\Vdash$ , between formulas of  $\mathcal{L}$ , variable assignments, and worlds as below:

- $w, v \Vdash Pt_1, \dots, Pt_n$  iff  $w \in \|P\|^v(\|t_1\|^v, \dots, \|t_n\|^v)$ ,
- $w \Vdash \perp$  iff  $w = \perp_W$ ,
- $w \Vdash \varphi \wedge \psi$  iff  $w \Vdash \varphi$  and  $w \Vdash \psi$ ,
- $w \Vdash \varphi \vee \psi$  iff  $w \Vdash \varphi$  or  $w \Vdash \psi$ ,
- $w, v \Vdash \forall x\varphi$  iff  $w, u \Vdash \psi$  whenever  $u[x]v$ ,
- $w, v \Vdash \exists x\varphi$  iff  $w, u \Vdash \psi$  for some  $u[x]v$ ,
- $w \Vdash \varphi \rightarrow \psi$  iff whenever  $Rwxy$  and  $x \Vdash \varphi$ ,  $y \Vdash \psi$ .

The clauses for the extensional connectives and quantifiers should be fairly familiar. The clause for the conditional is simply the generalization of the semantics for strict implication, which can be seen as a special case of this semantics in which  $Rwxy$  implies  $x = y$ .

**Definition 2.2** A formula,  $\varphi$ , is *true* in a model  $\langle W, D, R, \leq, \|\cdot\| \rangle$  at an assignment  $v$  if and only if for every  $w \in W$ ,  $w, v \Vdash \varphi$ .

A set of formulas,  $\Gamma$ , *entails* a formula  $\varphi$  if and only if for every model  $\mathcal{M}$  and assignment  $v$  over  $\mathcal{M}$ , if  $\gamma$  is true in  $\mathcal{M}$  at  $v$  for every  $\gamma \in \Gamma$ ,  $\varphi$  is true in  $\mathcal{M}$  at  $v$ .

Various conditional logics can be obtained by placing various restrictions on the kinds of frames we consider (see Restall [13] for a comprehensive survey). From here on out I shall just be concerned with a particular frame which admits a model of the full intersubstitutivity principle and the logic described above.

In what follows we shall be concerned exclusively with *standard* models of  $\mathcal{L}$ .  $\langle \mathcal{F}, \|\cdot\| \rangle$  is a standard model iff  $D := \mathbb{N}$ , and the arithmetical vocabularies receive their standard arithmetical interpretation; that is,  $\|\times\|$  maps  $\langle i, j \rangle$  to  $i \times j$ , and so on for each primitive recursive function.

We can then introduce the underlying set of worlds,  $W$ .

**Definition 2.3** A function  $f : \omega \rightarrow \{0, 1\}$  *flatlines* if and only if for some  $n \in \omega$ ,  $f(m) = 0$  for each  $m > n$ .

We let  $W := \{f : \omega \rightarrow 2 \mid f \text{ flatlines}\}$ , and let  $f \leq g$  iff for each  $n \in \omega$ ,  $f(n) \leq g(n)$ . The lattice theoretic operations of meet and join,  $\sqcap, \sqcup$ , are defined as usual:  $(f \sqcap g)(n) = \min(f(n), g(n))$  and  $(f \sqcup g)(n) = \max(f(n), g(n))$ .

**Definition 2.4** The *rank* of an element  $f \in W$ ,  $r(f)$ , is the smallest  $n$  such that  $f(m) = 0$  for all  $m \geq n$ .

Given this ranking function we define a ternary relation on  $W$  as follows.

**Definition 2.5** The *accessibility relation*,  $R$ , is defined as follows:  $Rxyz$  if and only if  $z \leq x^*$  and  $z \leq y$ .

Here

$$x^*(n) := \begin{cases} x(n) & \text{if } n < r(x) - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that  $r(z) < r(x)$  whenever  $z \leq x^*$  and  $x \neq \perp$ . Note also that whenever  $Rxyz$ ,  $x \leq x'$ ,  $y \leq y'$ , and  $z \geq z'$ ,  $Rx'y'z'$ . Therefore given a domain  $D$ ,  $\langle W, D, R, \leq \rangle$  forms a frame by Definition 2.1.

**Proposition 2.6** *The axioms listed above are validated in any model based on the frame  $\langle W, D, R, \leq \rangle$  described.*

**Proof** Verifying the validities is straightforward but somewhat tedious. We assume the fact, proved in the next section, that if  $x \Vdash \varphi$  and  $y \leq x$ ,  $y \Vdash \varphi$ .

In order to see that  $\varphi \rightarrow (\psi \rightarrow \varphi)$  holds, for example, suppose that  $Rxyz$  and  $y \Vdash \varphi$ . We want to verify that  $z \Vdash (\psi \rightarrow \varphi)$ . Suppose, therefore, that  $Rzuv$  and  $u \Vdash \psi$ . Note that since  $v \leq z^*$ ,  $z^* \leq z$ , and  $z \leq y$  (by the definition of  $R$ ), it follows that  $v \leq y$  and so  $v \Vdash \varphi$ .

The trickiest case is showing that the axiom  $\mathbf{B} (\varphi \rightarrow \psi) \rightarrow ((\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \psi))$  holds. It is sufficient to show that if  $\exists u(Rxyu \wedge Ruzw)$ , then  $\exists u(Ryzu \wedge Rxuw)$  (see Restall [13, Chapter 11.3]).

Suppose that  $Rxyu$  and  $Ruzw$ . So we have

1.  $u \leq x^*$  and  $u \leq y$ ,
2.  $w \leq u^*$  and  $w \leq z$ .

Let

$$u' = u^* \sqcap z = n \mapsto \begin{cases} \min(u(n), z(n)) & \text{if } n < r(u) - 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is then easy to check that

- a.  $u' \leq y^*$  and  $u' \leq z$ ,
- b.  $w \leq x^*$  and  $w \leq u'$ . □

The Kripke semantics for  $\mathcal{L}$  and  $\mathcal{L}'$  can be redescribed algebraically by assigning formulas semantic values from a class of “propositions” (sets of worlds) and by determining the value of a complex formula by certain operations on the values of its parts. This presentation will be more convenient in the following sections.

A proposition is a *nonempty* subset of  $W$  which is downwards closed. In other words,  $p$  is a proposition just in case  $x \in p$  whenever  $y \in p$  and  $x \leq y$ .

Let  $V$  denote the set of propositions over  $W$ . We can define operations on propositions corresponding to logical operations:

$$\begin{aligned} \perp &:= \{\perp_W\}, \\ p \wedge q &:= p \cap q, \\ p \vee q &:= p \cup q, \\ p \rightarrow q &:= \{x \mid z \in q \text{ whenever } y \in p \text{ and } Rxyz\}, \\ \bigwedge X &:= \bigcap_{p \in X} p \quad \text{where } X \subseteq V, \\ \bigvee X &:= \bigcup_{p \in X} p \quad \text{where } X \subseteq V. \end{aligned}$$

**Proposition 2.7** *Propositions are closed under the following operations.*

- The union and the intersection of a set of propositions is a proposition.
- If  $p$  and  $q$  are propositions, then so is  $p \rightarrow q$ .
- $\perp = \{\perp_W\}$  is a proposition.
- For any model, any assignment,  $v$ , and formula  $\varphi \in \mathcal{L}$ , the set  $\{w \mid w, v \Vdash \varphi\}$  is a proposition.

The most important case, that of the conditional, follows from the condition on  $R$  and  $\leq$  stipulated in Definition 2.1. The persistence of  $\| \cdot \|$  ensures that every atomic formula corresponds to a proposition in  $V$ , and Proposition 2.7 ensures that applying the logical operations always results in propositions. Thus for any formula, the set  $\{x \mid x, v \Vdash \varphi\}$  is always a proposition.

### 3 A Fixed Point Theorem

Let us begin with some standard definitions.

**Definition 3.1** A function from  $F : V \rightarrow V$  is *monotonic* iff  $F(p) \sqsubseteq F(q)$  whenever  $p \sqsubseteq q$ .

A function from  $F : V \rightarrow V$  is *antimonotonic* iff  $F(q) \sqsubseteq F(p)$  whenever  $p \sqsubseteq q$ .

Kripke's seminal paper [11] is essentially an application of the Knaster–Tarski theorem.

**Theorem 3.2** *Let  $V$  be a complete lattice. Then every monotonic function  $F : V \rightarrow V$  has a fixed point.*

**Proof** Let  $F^0(x) = F(x)$ ,  $F^{\alpha+1}(x) = F(F^\alpha(x))$ , and  $F^\gamma(x) = \bigsqcup_{\alpha} F^\alpha(x)$  for  $\alpha < \gamma$  whenever  $\gamma$  is a limit ordinal.

Since  $V$  is a set and  $F$  is monotonic,  $F$  cannot grow forever so for some  $\alpha$ ,  $F^{\alpha+1}(\perp) = F^\alpha(\perp)$  (and this is the *least* such fixed point).  $\square$

I shall henceforth use the notation  $\mu_x F(x) := \bigsqcup_{\alpha} F^\alpha(\perp)$  for the least fixed point of the monotonic function  $F$ . Kripke showed that, due to the monotonic nature of the Kleene connectives, there is an interpretation of  $\mathcal{L} \setminus \{\rightarrow\}$  based on the Kleene valuation such that every formula  $\varphi$  formed from the connectives  $\{\neg, \wedge, \vee, \perp\}$  (no conditional),  $\varphi$ , and  $\text{Tr}(\ulcorner \varphi \urcorner)$  get the same value. For the liar sentence,  $\lambda$ , this means

that  $\lambda$  has the same semantic value as  $\neg\lambda$ ; thus the negation operator has a fixed point. Indeed, any definable operation whatsoever,  $\varphi(p)$ , must have a fixed point if we are to accommodate every possible sort of self-reference.

In order to validate the full intersubstitutivity principle in our model, we must also take into account the conditional. However, it is easily seen that the conditional defined in Section 2 is not monotonic, and is in fact antimonotonic, in its left argument. This prevents us from applying the Knaster–Tarski theorem. For example, to find a value for the Curry sentence one must find a fixed point of the function  $F(p) := p \rightarrow \perp$ . However, there is no guarantee that  $F$  will have a fixed point on the basis of the Knaster–Tarski theorem since  $F$  is antimonotonic. In order to guarantee this, we need a more general fixed point theorem.

Let us begin with some definitions. Given a proposition,  $p$ , let  $p \downarrow_n := \{f \in p \mid r(f) \leq n\}$ . Since every element of  $p$  has finite rank we can think of  $p \downarrow_\omega$  as simply  $p$ .

**Definition 3.3** Say that a function  $F : V \rightarrow V$  is accumulating iff  $F(p) \downarrow_{n+1} = F(q) \downarrow_{n+1}$  whenever  $p \downarrow_n = q \downarrow_n$ .

$F$  is weakly accumulating iff  $F(p) \downarrow_n = F(q) \downarrow_n$  whenever  $p \downarrow_n = q \downarrow_n$ .

For monotonic functions  $F$ , we know that we have the following sequence of inclusions  $\perp \subseteq F(\perp) \subseteq FF(\perp) \subseteq FFF(\perp) \subseteq \dots$  the limit of which is a fixed point of  $F$ . This is not the case for accumulating functions. What we instead have is the following sequence of inclusions:  $\perp \downarrow_0 \subseteq F(\perp) \downarrow_1 \subseteq FF(\perp) \downarrow_2 \subseteq FFF(\perp) \downarrow_3 \subseteq \dots$ . It is instead the union of this sequence that provides us with our fixed point.

**Theorem 3.4** *Every accumulating function,  $F$ , on  $V$  has a fixed point.*

**Proof** The fixed point will be the “limit” of a sequence of applications of  $F$  to an initial proposition, say,  $P_0$ . This is somewhat reminiscent of the Knaster–Tarski theorem for monotonic functions:

- $P_{n+1} := F(P_n)$ ,
- $P_\omega := \bigcup_{n < \omega} P_n \downarrow_n$ .

We shall show that for any ordinal  $\omega \geq \beta > n$ ,  $P_\beta \downarrow_n = P_n \downarrow_n$ . Since every proposition is a set of things with rank less than  $\omega$  it follows that  $P_\omega = P_\omega \downarrow_\omega = F(P_\omega) \downarrow_\omega = F(P_\omega)$ , that is,  $P_\omega = F(P_\omega)$ .

**Base case:** Clearly  $P_\beta \downarrow_0 = \{\perp\} = P_0 \downarrow_0$ .

**Inductive step:** Suppose that  $P_n \downarrow_n = P_\beta \downarrow_n$  whenever  $\omega \geq \beta > n$ . Then by the fact that  $F$  is accumulating it follows that  $F(P_n) \downarrow_{n+1} = F(P_\beta) \downarrow_{n+1}$ , that is,  $P_{n+1} \downarrow_{n+1} = P_{\beta+1} \downarrow_{n+1}$  whenever  $\omega > \beta > n$ . So we have the required claim  $P_{n+1} \downarrow_{n+1} = P_\beta \downarrow_{n+1}$  for all the finite ordinals  $\beta > n + 1$ . Note that if  $\beta = \omega$ , then  $P_\omega \downarrow_{n+1} = (\bigcup_{k < \omega} P_k) \downarrow_{n+1} = \bigcup_{k < \omega} (P_k \downarrow_{n+1}) = P_{n+1} \downarrow_{n+1}$  by the above.  $\square$

**Theorem 3.5** *Suppose that  $F(X, Y) : V^2 \rightarrow V$  is accumulating in its first argument and both monotonic and weakly accumulating in its second argument. Then there is some  $Z \in V$  with  $F(Z, Z) = Z$ .*

**Proof** We shall show that the function  $H(X) = \mu Y.F(X, Y)$  is accumulating, where  $\mu Y \cdot F(X, Y)$  is the least fixed point of the monotonic function  $F(X, \cdot)$  determined by the Knaster–Tarski fixed point theorem. Since  $H(X)$  is a fixed point of  $F(X, \cdot)$  it follows that  $F(X, H(X)) = H(X)$  for any  $X$ . Since  $H$  is accumulating, it

will then follow by Theorem 3.4 that  $H$  has a fixed point,  $Z$ , such that  $H(Z) = Z$ . Thus  $Z = H(Z) = F(Z, H(Z)) = F(Z, Z)$  as required.

Note that in general  $H(X) := \bigsqcup_{\alpha} F^{\alpha}(X, \{\perp\})$  where  $F^{\alpha+1}(X, Y) = F(X, F^{\alpha}(X, Y))$  and  $F^{\gamma}(X, Y) = \bigsqcup_{\alpha < \gamma} F^{\alpha}(X, Y)$ . Suppose that  $P \downarrow_n = Q \downarrow_n$ . We shall show by induction that  $F^{\beta}(P, \{\perp\}) \downarrow_{n+1} = F^{\beta}(Q, \{\perp\}) \downarrow_{n+1}$  for all  $\beta$ , and therefore that  $H(P) \downarrow_{n+1} = H(Q) \downarrow_{n+1}$ .

**Base case:** Since  $F$  is accumulating in its first argument,  $F(P, \{\perp\}) \downarrow_{n+1} = F(Q, \{\perp\}) \downarrow_{n+1}$ .

**Successor case:** Suppose  $F^{\beta}(P, \{\perp\}) \downarrow_{n+1} = F^{\beta}(Q, \{\perp\}) \downarrow_{n+1}$ . Then  $F(P, F^{\beta}(P, \{\perp\})) \downarrow_{n+1} = F(Q, F^{\beta}(Q, \{\perp\})) \downarrow_{n+1}$  since  $F$  is weakly accumulating in its right argument.

**Limit case:** Suppose  $F^{\beta}(P, \{\perp\}) \downarrow_{n+1} = F^{\beta}(Q, \{\perp\}) \downarrow_{n+1}$  for  $\beta < \gamma$ . Then  $\bigcup_{\beta < \gamma} F^{\beta}(P, \{\perp\}) \downarrow_{n+1} = \bigcup_{\beta < \gamma} F^{\beta}(Q, \{\perp\}) \downarrow_{n+1}$  and thus  $\bigcup_{\beta < \gamma} F^{\beta}(P, \{\perp\}) \downarrow_{n+1} = \bigcup_{\beta < \gamma} F^{\beta}(Q, \{\perp\}) \downarrow_{n+1}$ .  $\square$

In what follows we shall also want to consider the infinite product of  $V$  with itself,  $V^{\omega}$ , as a space in its own right.

**Definition 3.6** Let  $V^{\omega} := \{\bar{p} \mid \bar{p} : \omega \rightarrow V\}$ . For  $\bar{p}, \bar{q} \in V^{\omega}$ , let  $\bar{p} \leq \bar{q}$  iff  $p_i \leq q_i$  for each  $i \in \omega$ . The rank restriction operation,  $\bar{p} \downarrow_n = \bar{q} \downarrow_n \in V^{\omega}$ , can be extended to  $V^{\omega}$  by letting  $q_i = p_i \downarrow_n$  for each  $i$ .

**Definition 3.7** Let  $\alpha$  be either a finite ordinal or  $\omega$ . We say a function  $F : V^{\alpha} \rightarrow V$  is *accumulating* (or, analogously, *weakly accumulating*) iff  $F(\bar{p}) \downarrow_{n+1} = F(\bar{q}) \downarrow_{n+1}$  whenever  $p_i \downarrow_n = q_i \downarrow_n$  for each  $i \in \alpha$ .

The definitions of monotonicity and antimonotonicity generalize straightforwardly to  $V^{\omega}$  under the above ordering. Similarly we have the following.

**Definition 3.8** We say a function  $F : (V^{\omega})^k \rightarrow V^{\omega}$  is *accumulating* (or, analogously, *weakly accumulating*) iff  $F(\bar{p}_0, \dots, \bar{p}_k) \downarrow_{n+1} = F(\bar{q}_0, \dots, \bar{q}_k) \downarrow_{n+1}$  whenever  $\bar{p}_i \downarrow_n = \bar{q}_i \downarrow_n$  for each  $i \leq k$ .

Theorems 3.4 and 3.5 generalize to  $V^{\omega}$ .

**Theorem 3.9** Every accumulating function  $F : V^{\omega} \rightarrow V^{\omega}$  has a fixed point.

**Theorem 3.10** If  $F(X, Y) : V^{\omega} \times V^{\omega} \rightarrow V^{\omega}$  is accumulating in its first argument and both monotonic and weakly accumulating in its second argument, then there is some  $Z \in V^{\omega}$  with  $F(Z, Z) = Z$ .

The following allows us to construct accumulating functions from  $V^{\omega} \rightarrow V^{\omega}$  given accumulating functions on  $V$ .

**Proposition 3.11** Suppose that  $G_i : V^{\omega} \rightarrow V$  is an accumulating (weakly accumulating) function for each  $i \in \omega$ . Then the function  $F : V^{\omega} \rightarrow V^{\omega}$  given by  $F(\bar{p})(i) := G_i(\bar{p})$  is accumulating (weakly accumulating).

#### 4 A Geometrical Look at the Fixed Point Theorem

In this section we assimilate Theorem 3.4 to a more familiar fixed point theorem: the Banach fixed point theorem for contracting functions on a complete metric space.

**Definition 4.1** Let  $\langle X, d \rangle$  be a metric space. Then a sequence  $(x_n)_{n \in \omega}$  in  $X$  is Cauchy if and only if for every positive  $\epsilon \in \mathbb{R}$ , there is an  $N \in \omega$  such that for every  $m, n > N$ ,  $d(x_n, x_m) < \epsilon$ .

A Cauchy sequence  $(x_n)$  has a limit if and only if there is some  $x \in X$  such that for every  $\epsilon \in \mathbb{R}$  there is some  $N \in \omega$ , with  $d(x, x_n) < \epsilon$  for all  $n > N$ .

Finally, a metric space  $\langle X, d \rangle$  is a complete metric space if and only if every Cauchy sequence has a limit.

Note that  $V$  can be given the structure of a metric space if we set  $d(p, q) = 2^{-n}$  where  $n = \inf\{r(x) \mid x \in (p \setminus q) \cup (q \setminus p)\}$  and  $d(p, q) = 0$  if  $p = q$ . Indeed, under this metric,  $V$  is a complete metric space, where given a Cauchy sequence,  $(p_i)_{i \in \omega}$ ,  $\lim_{i \rightarrow \infty} p_i := \{x \mid x \text{ is in cofinally many } p_i\}$ .

**Definition 4.2** If  $\langle X, d \rangle$  is a complete metric space, a function  $f : X \rightarrow X$  is  $\alpha$ -contracting iff  $d(f(x), f(y)) \leq \alpha \cdot d(x, y)$  for every  $x, y \in X$ .

For an  $n$ -ary function we say that  $f$  is  $\alpha$ -contracting iff  $d(f(x_1, \dots, x_n), f(y_1, \dots, y_n)) \leq \alpha \cdot \max_i d(x_i, y_i)$  for  $x_i, y_i \in X$ .

**Theorem 4.3 (Banach's fixed point theorem)** For each positive  $\alpha < 1$ , every  $\alpha$ -contracting function on a complete metric space has a fixed point.

**Corollary 4.4** Every accumulating function of  $V$  has a fixed point.

**Proof** It is easily verified that every accumulating function,  $F$ , is  $(1/2)$ -contracting. It then follows from the Banach fixed point theorem that  $F$  has a fixed point.  $\square$

## 5 A Standard Model for TJK<sup>+</sup>

The purpose of this section is to obtain a standard model for the proposed naive truth theory in the logic TJK<sup>+</sup>. In Section 2 we defined a standard model to be a model in which the domain is  $\mathbb{N}$  and the interpretation of the arithmetical vocabulary is standard. In order to determine a model for  $\mathcal{L}$  (and  $\mathcal{L}'$ ) all that is left is to provide an interpretation for the truth predicate. We must specify a function,  $\|\text{Tr}\| : \mathbb{N} \rightarrow V$ . Thus our model would be completed once one has provided a function  $\bar{p} : \mathbb{N} \rightarrow V$ , that is,  $\bar{p} \in V^\omega$ .

**Definition 5.1** Fix an assignment,  $v$ . Now given  $\bar{p} \in V^\omega$ , define  $|\varphi|_v(\bar{p})$  (or  $|\varphi|(\bar{p})$  if no ambiguity is present) to be the set  $\{w \mid w, v \Vdash \varphi \text{ relative to the model } \mathcal{M}\}$ , where  $\mathcal{M}$  is the standard model you would get by letting  $\|\text{Tr}\| = \bar{p}$ .

Similarly, for a formula  $\varphi$  of  $\mathcal{L}'$  we can define a binary function  $|\varphi|(\bar{p}, \bar{q})$  whose value is to be the set  $\{w \mid w, v \Vdash \varphi \text{ relative to the model } \mathcal{M}\}$ , where  $\mathcal{M}$  is the model you would get by letting  $\|\text{Tr}^-\| = \bar{p}$  and  $\|\text{Tr}^+\| = \bar{q}$ .

Thus for every formula  $\varphi \in \mathcal{L}$ ,  $|\varphi| : V^\omega \rightarrow V$ , and for  $\varphi \in \mathcal{L}'$ ,  $|\varphi| : V^\omega \times V^\omega \rightarrow V$ .

In order to ensure that every closed formula  $\varphi$  is fully intersubstitutable for  $\text{Tr}(\ulcorner \varphi \urcorner)$  we must pick an interpretation,  $\bar{p}$ , such that for every closed  $\varphi$ ,  $|\varphi|(\bar{p}) = \bar{p}(\ulcorner \varphi \urcorner)$ . It follows that for any world  $x \in W$ ,  $x \Vdash \varphi$  just in case  $x \Vdash \text{Tr}(\ulcorner \varphi \urcorner)$ .

In other words, we want to find a fixed point for the function  $F : V^\omega \rightarrow V^\omega$  given by  $F(\bar{p})(\ulcorner \varphi \urcorner) = |\varphi|_v(\bar{p})$  ( $v$  is some fixed assignment; it does not matter which). We shall see that if the truth predicate only occurs positively in  $\varphi$ , that

is, embedded an even number of times within the left argument of a conditional, then  $|\varphi|$  is monotonic and weakly accumulative on  $V^\omega$ . So if we restricted  $F$  to its arguments where  $\text{Tr}$  occurs positively it would have a fixed point by the Knaster–Tarski theorem. On the other hand, if the truth predicate occurs only negatively in  $\varphi$ , that is, appears in the scope of an odd number of conditionals in the antecedent place, we can show that  $|\varphi|$  is accumulative. Thus restricting  $F$  to these arguments,  $F$  would have a fixed point by Theorem 3.4. However, the truth predicate can appear both negatively and positively simultaneously in a formula, so these two options do not exhaust all the possibilities. In order to obtain a fixed point we instead translate every formula  $\varphi \in \mathcal{L}$  to a formula  $\varphi' \in \mathcal{L}'$  where every negative occurrence of  $\text{Tr}$  in  $\varphi$  is replaced by the predicate  $\text{Tr}^-$ , and every positive occurrence of  $\text{Tr}$  is replaced by the predicate  $\text{Tr}^+$ . We then show instead that the function  $F' : V^\omega \times V^\omega \rightarrow V^\omega$  given by  $F'(\bar{p}, \bar{q})(\ulcorner \varphi' \urcorner) = |\varphi'|(\bar{p}, \bar{q})$  is accumulative in its left argument and weakly accumulative and monotonic in its right argument.  $F'$  thus has a fixed point,  $\bar{p}$ , with  $F'(\bar{p}, \bar{p}) = \bar{p}$  by Theorem 3.5. From this it is easily seen that  $\bar{p}$  is also a fixed point for  $F$ .

**Definition 5.2** The set of atomic formulas involving the truth predicate which occur positively and negatively in  $\varphi$ , denoted  $\text{Pos}(\varphi)$  and  $\text{Neg}(\varphi)$ , are defined as follows:

- $\text{Pos}(\varphi) = \{\varphi\}$  and  $\text{Neg}(\varphi) = \emptyset = \text{Neg}(\perp)$  for each atomic sentence  $\varphi$  of the form  $\text{Tr}(t)$ ;
- $\text{Pos}(\varphi * \psi) = \text{Pos}(\varphi) \cup \text{Pos}(\psi)$ , and  $\text{Neg}(\varphi * \psi) = \text{Neg}(\varphi) \cup \text{Neg}(\psi)$  where  $*$  =  $\wedge, \vee$ ;
- $\text{Pos}(\forall x\varphi) = \text{Pos}(\exists x\varphi) = \text{Pos}(\varphi)$ ,  $\text{Neg}(\forall x\varphi) = \text{Neg}(\exists x\varphi) = \text{Neg}(\varphi)$ ;
- $\text{Pos}(\varphi \rightarrow \psi) = \text{Neg}(\varphi) \cup \text{Pos}(\psi)$ ,  $\text{Neg}(\varphi \rightarrow \psi) = \text{Pos}(\varphi) \cup \text{Neg}(\psi)$ .

**Lemma 5.3** *If  $p \upharpoonright_n = p' \upharpoonright_n$  and  $q \upharpoonright_{n+1} = q' \upharpoonright_{n+1}$ , then  $(p \rightarrow q) \upharpoonright_{n+1} = (p' \rightarrow q') \upharpoonright_{n+1}$ .*

*Thus the function  $F : V^2 \rightarrow V$  mapping  $p$  and  $q$  to  $p \rightarrow q$  is weakly accumulating, is weakly accumulating in its right argument for a fixed  $p$ , and accumulating in its left argument for a fixed  $q$ .*

**Proof** Suppose, for contradiction, that  $p \upharpoonright_n = p' \upharpoonright_n$ ,  $q \upharpoonright_{n+1} = q' \upharpoonright_{n+1}$  but  $p \rightarrow q \upharpoonright_{n+1} \neq p' \rightarrow q' \upharpoonright_{n+1}$ . Without loss of generality, let  $x$  be an element in one of  $p \rightarrow q \upharpoonright_{n+1} \setminus p' \rightarrow q' \upharpoonright_{n+1}$ .  $r(x) \leq n + 1$ . Thus we have the following.

1. For any  $y \in p$ , if  $Rxyz$ ,  $z \in q$ .
2. There is some  $y' \in p'$  and some  $z'$  with  $Rxy'z'$  such that  $z' \notin q'$ .

Let

$$u(n) = \begin{cases} y'(n) & \text{if } n < r(x) - 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that  $Rxy'z$  if and only if  $Rxuz$  for any  $z$ , so in particular,  $Rxuz'$ . By construction  $u \leq y'$ , so  $u \in p'$ . Also  $r(u) < r(x) \leq n + 1$ , so  $u \in p' \upharpoonright_n$  and thus  $u \in p \upharpoonright_n$  since  $p \upharpoonright_n = p' \upharpoonright_n$ .

Now, since  $Rxuz'$  and  $u \in p$ , it follows by (1) that  $z' \in q$ . However since  $Rxuz'$ ,  $r(z') < r(x) \leq n + 1$  and thus  $z' \in q'$  since  $q \upharpoonright_{n+1} = q' \upharpoonright_{n+1}$ . This is a contradiction.  $\square$

**Lemma 5.4** *If  $\text{Pos}(\varphi) = \emptyset$ ,  $|\varphi|(\bar{p})$  defines an accumulating antimonotonic function, and if  $\text{Neg}(\varphi) = \emptyset$ ,  $|\varphi|(\bar{p})$  defines a weakly accumulative monotonic function.*

**Proof** Both claims are proved simultaneously by induction. The tricky case is in showing the claim for conditional formulas. Suppose that the claim holds for formulas  $\varphi$  and  $\psi$  of complexity at most  $n$ .

Suppose  $\text{Pos}(\varphi \rightarrow \psi) = \emptyset$ . Thus it follows that  $\text{Neg}(\varphi) = \text{Pos}(\psi) = \emptyset$ . We need to show that  $|\varphi \rightarrow \psi|(p)$  is antimonotonic and accumulating.

$|\varphi \rightarrow \psi|(p)$  is antimonotonic. Suppose  $\bar{p} \leq \bar{q}$ . Since  $\text{Neg}(\varphi) = \emptyset$ , then  $|\varphi|(\cdot)$  is monotonic by inductive hypothesis, that is,  $|\varphi|(\bar{p}) \subseteq |\varphi|(\bar{q})$ . Since  $\text{Pos}(\psi) = \emptyset$ , then  $|\psi|(\cdot)$  is antimonotonic by inductive hypothesis, that is,  $|\psi|(\bar{q}) \subseteq |\psi|(\bar{p})$ . By the properties of  $\rightarrow$ ,  $|\varphi|(\bar{q}) \rightarrow |\psi|(\bar{q}) \subseteq |\varphi|(\bar{p}) \rightarrow |\psi|(\bar{p})$ .

$|\varphi \rightarrow \psi|(\bar{p})$  is accumulating. Suppose that  $\bar{p} \downarrow_n = \bar{q} \downarrow_n$ . Since  $\text{Neg}(\varphi) = \emptyset$ , then  $|\varphi|(\cdot)$  is weakly accumulative by inductive hypothesis, that is,  $|\varphi|(\bar{p}) \downarrow_n = |\varphi|(\bar{q}) \downarrow_n$ . Since  $\text{Pos}(\psi) = \emptyset$ , then  $|\psi|(\cdot)$  is accumulative by the inductive hypothesis, so  $|\psi|(\bar{p}) \downarrow_{n+1} = |\psi|(\bar{q}) \downarrow_{n+1}$ . Thus by Lemma 5.3,  $|\varphi \rightarrow \psi|(\bar{p}) \downarrow_{n+1} = |\varphi \rightarrow \psi|(\bar{q}) \downarrow_{n+1}$ .

Now suppose that  $\text{Neg}(\varphi \rightarrow \psi) = \emptyset$ . Thus it follows that  $\text{Pos}(\varphi) = \text{Neg}(\psi) = \emptyset$ . We need to show that  $|\varphi \rightarrow \psi|(\bar{p})$  is monotonic and weakly accumulating.

$|\varphi \rightarrow \psi|(\bar{p})$  is monotonic. Suppose  $\bar{p} \leq \bar{q}$ . By the inductive hypothesis  $|\varphi|$  is antimonotonic, so  $|\varphi|(\bar{q}) \subseteq |\varphi|(\bar{p})$ . Similarly  $|\psi|$  is monotonic, so  $|\psi|(\bar{p}) \subseteq |\psi|(\bar{q})$ . So  $|\varphi \rightarrow \psi|(\bar{p}) \subseteq |\varphi \rightarrow \psi|(\bar{q})$  as required.

$|\varphi \rightarrow \psi|(\bar{p})$  is weakly accumulating. Suppose  $\bar{p} \downarrow_n = \bar{q} \downarrow_n$ . Since  $\text{Pos}(\varphi) = \emptyset$ ,  $|\varphi|(\bar{p}) \downarrow_n = |\varphi|(\bar{q}) \downarrow_n$  since  $|\varphi|$  is accumulating and thus weakly accumulating. Similarly since  $\text{Neg}(\psi) = \emptyset$ ,  $|\psi|(\bar{p}) \downarrow_n = |\psi|(\bar{q}) \downarrow_n$  since  $|\psi|$  is weakly accumulating by the inductive hypothesis. In any case,  $|\varphi \rightarrow \psi|(\bar{p}) \downarrow_n = |\varphi \rightarrow \psi|(\bar{q}) \downarrow_n$  since, by Lemma 5.3,  $\rightarrow$  as a binary function is weakly accumulating.  $\square$

**Theorem 5.5**  $\text{TJK}^+$  + the intersubstitutivity rule has a standard model.

**Proof** Let  $v$  be some arbitrary assignment, and let  $F : V^\omega \times V^\omega \rightarrow V^\omega$  be given by  $F(\bar{p}, \bar{q})(\ulcorner \varphi \urcorner) = |\varphi|_v(\bar{p}, \bar{q})$ . By Proposition 3.11 and Lemma 5.4, it follows that  $F(\cdot, \bar{q})$  is accumulating for each  $\bar{q} \in V^\omega$  and  $F(\bar{p}, \cdot)$  is monotonic and weakly accumulating for each  $\bar{p} \in V^\omega$ . By Theorem 3.10  $F(\bar{p}, \bar{p}) = \bar{p}$  for some  $\bar{p} \in V^\omega$ .

Our Kripke model is thus  $\langle \langle W, R, D, \leq \rangle, \|\cdot\| \rangle$  where  $\|\text{Tr}\| = \bar{p}$ , which ensures that for closed formulas  $w \Vdash \varphi$  iff  $w \in \bar{p}(\ulcorner \varphi \urcorner)$  iff  $w \Vdash \text{Tr}(\ulcorner \varphi \urcorner)$ .  $\square$

So  $\text{TJK}^+$  supports a standard model. In particular, we know that all true arithmetical identity statements hold in this model, so we know that true identities of the form  $t = \ulcorner \varphi(t) \urcorner$  hold in this model and that the theory consisting of  $\text{TJK}^+$  and all such identities is closable under the  $\omega$ -rule. It should, of course, be stressed that not all of true arithmetic need hold in such models—for example, due to the contracting nature of the conditional, sentences like  $((0 = 0 \rightarrow \perp) \rightarrow \perp)$  are not validated.

## 6 Adding an Involutive Negation Operator

In the preceding discussion we have ignored the issue of negation. The structure of the consistency argument is clearer when negation is omitted, and a consideration of negation is not needed for the analysis of Curry's paradox and its relatives. However, one might justifiably wonder what happens when a negation operator is added,

and doing so is crucial if we want to compare the current approach to other similar approaches, such as Field's logic (see [7]). Since there are a number of choices one could make about how to implement a negation operator I shall take a more streamlined approach in this section where I concentrate more on possible avenues for adding negation than fully fleshing out the details.

In the language discussed there is a natural candidate for a negation operation, which can be defined as  $\neg\varphi := \varphi \rightarrow \perp$ . However, while we can prove certain desirable principles, such as the following de Morgan law  $(\neg\varphi \wedge \neg\psi) \leftrightarrow \neg(\varphi \vee \psi)$  (see the principle 12 from Section 2) and  $\varphi \rightarrow \psi \vdash \neg\psi \rightarrow \neg\varphi$ , principles like  $\neg\neg p \rightarrow p$  and  $p \rightarrow \neg\neg p$  will not in general be valid.<sup>9</sup>

The situation is analogous in Field's framework with negation defined from the conditional, so Field introduces instead a primitive negation operator,  $\neg$ , which is not defined in terms of the conditional. There are some natural-looking ways to extend the above construction to deal with negation. In what follows I shall discuss one of these.

To get the feel of the idea imagine that instead of adding a primitive negation operator to  $\mathcal{L}$ , we added a primitive falsity *predicate*,  $Fa(x)$  (and, analogously, add two predicates  $Fa^+$  and  $Fa^-$  to  $\mathcal{L}'$ ), and then *introduced* the negation operator by the following recursive definition:

$$\begin{aligned} \neg\varphi &\mapsto \perp && \text{if } \varphi \text{ is a true atomic arithmetical sentence,} \\ \neg\varphi &\mapsto \top && \text{if } \varphi \text{ is a false atomic arithmetical sentence,} \\ \neg\text{Tr}(\mathbf{n}) &\mapsto Fa(\mathbf{n}), \\ \neg Fa(\mathbf{n}) &\mapsto \text{Tr}(\mathbf{n}), \\ \neg(\varphi \wedge \psi) &\mapsto (\neg\varphi \vee \neg\psi), \\ \neg(\varphi \vee \psi) &\mapsto (\neg\varphi \wedge \neg\psi), \\ \neg\forall x\varphi &\mapsto \exists x\neg\varphi, \\ \neg\exists x\varphi &\mapsto \forall x\neg\varphi, \\ \neg(\varphi \rightarrow \psi) &\mapsto \varphi \circ \neg\psi, \\ \neg(\varphi \circ \psi) &\mapsto (\varphi \rightarrow \neg\psi). \end{aligned}$$

Here I have mentioned a new connective,  $\circ$ , which is intended to stand to the conditional as conjunction does in the classical case. It can be added to our frame by adding the following satisfaction clause:<sup>10</sup>

$$x \Vdash \varphi \circ \psi \quad \text{iff} \quad \text{for some } y \text{ and } z \text{ with } x^* \leq y \text{ and } x \leq z, y \Vdash \varphi \text{ and } z \Vdash \psi.$$

It is thus clear that the defined negation operation is both involutive and satisfies the de Morgan laws. In order to find a model for both a truth predicate and a falsity predicate one needs a *pair* of interpretations  $\langle \bar{p}, \bar{q} \rangle \in V \times V$  which satisfy certain conditions. (The condition in question is that  $p_n, q_n$  is a fixed point of a certain *binary* function  $|\varphi|(\cdot, \cdot)$ , whenever  $n = \ulcorner \varphi \urcorner$ .)

A more semantic way to view this idea would be to take our space of semantic values to be *pairs* of propositions (i.e., pairs of downward closed sets) instead of propositions as one's basic semantic value. Given a set of pairs,  $U$ , serving as a space of semantic values there are *two* orderings one could define on  $U$ , which I shall call the *information ordering* and the *logical ordering*. Here I use the notation

$p = \langle p^+, p^- \rangle$  for writing an element of  $U$ :

$$\begin{aligned} p \leq_i q & \text{ iff } p^+ \subseteq q^+ \text{ and } p^- \subseteq q^-, \\ p \leq_l q & \text{ iff } p^+ \subseteq q^+ \text{ and } q^- \subseteq p^-. \end{aligned}$$

The top element in the logical ordering is thus  $\langle \top, \perp \rangle$ , whereas the most informative element is  $\langle \top, \top \rangle$  (assuming  $U$  contains these elements). Similarly, the least informative element is  $\langle \perp, \perp \rangle$ , while the least element in the logical ordering is  $\langle \perp, \top \rangle$  (again assuming that they are in  $U$ ). The algebra  $\langle U, \leq_i, \leq_l \rangle$  is a *bilattice* in the sense of Fitting [8].<sup>11</sup>

In order to define a logic one needs to decide which set of pairs,  $U$ , is to play the role of semantic values, and one must decide on a set of designated values from which a consequence relation can be defined.<sup>12</sup> Here there are a number of options depending on what you want to do. For now I shall merely list three:

- i. Let  $U = \{p \in V \times V \mid p^+ \cap p^- = \perp\}$ , and let the designated value be  $\langle \top, \perp \rangle$ .
- ii. Let  $U = V \times V$ , and let the designated values be  $\{p \mid p \geq_l \langle \top, \top \rangle\}$ .
- iii. Let  $U = \{p \in V \times V \mid p^+ \cap p^- \neq \top\}$ , and let the designated values be  $\{p \mid p >_l \langle \top, \top \rangle\}$ .

I shall restrict attention to proposal (iii), and I shall henceforth use  $U$  to denote  $\{p \in V \times V \mid p^+ \cap p^- < \top\}$ .

We may then extend the nonconditional operations to pairs as follows:

$$\begin{aligned} p \wedge q &= \langle p^+ \cap q^+, p^- \cup q^- \rangle, \\ p \vee q &= \langle p^+ \cup q^+, p^- \cap q^- \rangle, \\ \bigwedge_{i \in I} p_i &= \left\langle \bigcap_{i \in I} p_i^+, \bigcup_{i \in I} p_i^- \right\rangle, \\ \bigvee_{i \in I} p_i &= \left\langle \bigcup_{i \in I} p_i^+, \bigcap_{i \in I} p_i^- \right\rangle, \\ \neg p &= \langle p^-, p^+ \rangle. \end{aligned}$$

If one defines  $p \downarrow_n$  as  $\langle p^+ \downarrow_n, p^- \downarrow_n \rangle$ , one can see that the nonconditional operations are all weakly accumulating and monotonic in the information ordering. There is a certain amount of choice as to how one defines the conditional. Here is a fairly natural definition:

$$\begin{aligned} (p \rightarrow q)^+ &= \{x \mid z \in q^+ \text{ whenever } y \in p^+ \text{ and } Rxyz\}, \\ (p \rightarrow q)^- &= \{x \mid \text{for some } y \in p^+ \text{ with } z \in q^-, x^* \leq y \text{ and } x \leq z\} = p^+ \circ q^-. \end{aligned}$$

Note that this operation takes elements of  $U$  to elements of  $U$ .<sup>13</sup> This operation can be seen to be accumulating in its first argument, and weakly accumulating and monotonic (in both orderings) in its second. Thus by an argument analogous to that in Section 5 we can generate a standard model for the logic augmented by this negation operation.<sup>14</sup>

Given the definition of validity according to (iii) one can ask: what additional logical principles would we get? It is a routine matter to check that the principles listed in Section 2 are validated. For principles specifically about negation we get most of the axioms and rules that Field gets for his negation operation:

1.  $\varphi \rightarrow \neg\neg\varphi$ ,
2.  $\neg\neg\varphi \rightarrow \varphi$ ,
3.  $\neg(\varphi \vee \psi) \rightarrow \neg\varphi \wedge \neg\psi$ ,
4.  $\neg(\varphi \wedge \psi) \rightarrow \neg\varphi \vee \neg\psi$ ,
5.  $\neg\varphi \vee \neg\psi \rightarrow \neg(\varphi \wedge \psi)$ ,
6.  $\neg\varphi \wedge \neg\psi \rightarrow \neg(\varphi \vee \psi)$ ,
7.  $\exists x\neg\varphi \rightarrow \neg\forall x\varphi$ ,
8.  $\neg\forall x\varphi \rightarrow \exists x\neg\varphi$ ,
9.  $\forall x\neg\varphi \rightarrow \neg\exists x\varphi$ ,
10.  $\neg\exists x\varphi \rightarrow \forall x\neg\varphi$ ,
11.  $\varphi, \neg\psi \vdash \neg(\varphi \rightarrow \psi)$ ,
12.  $\varphi, \neg\varphi \vdash \psi$ ,
13.  $\varphi \leftrightarrow \psi, \neg\varphi \leftrightarrow \neg\psi \vdash \chi \leftrightarrow \chi[\varphi/\psi]$ .

One notable absence is any form of contraposition. For example, we do not have the principle

$$\varphi \rightarrow \psi \vdash \neg\psi \rightarrow \neg\varphi.$$

Whether one can tinker with the definition of  $\rightarrow$  to get contraposition without giving up the other principles of  $\text{TJK}^+$  bears further investigation. I shall only note that no matter how one introduces a negation operator, we can always *define* a contraposable conditional as

$$\varphi \Rightarrow \psi := (\varphi \rightarrow \psi) \wedge (\neg\psi \rightarrow \neg\varphi).$$

This conditional continues to satisfy the principles just listed above<sup>15</sup> and satisfies not just the contraposition rules, like the one listed above, but the contraposition *axioms*.<sup>16</sup>

Although this is far from the final word on the matter, I think this is enough to show that the prospects of adding an involutive negation operator to  $\text{TJK}^+$  are promising.

### Notes

1. I refer the reader to the cited texts for further details on these techniques. These techniques are also closely related to the construction behind the revision theory of truth (see Gupta and Belnap [9], Herzberger [10]). The revision theory, however, does not validate the intersubstitutivity rule. Furthermore, revision theorists take their model constructions to not only provide a consistency proof of a particular theory of truth but to offer insight into the diagnosis of the liar paradox; this approach should be sharply distinguished from the approach here.
2. To do this one would presumably want to have the principle  $\forall x(\varphi \rightarrow \psi) \rightarrow (\exists x\varphi \rightarrow \exists x\psi)$ , which is not validated in Field's construction, although it is provable from the above axioms given uncontroversial principles about quantification. In Field's model, if  $\gamma = \text{Tr}(\ulcorner\gamma\urcorner) \rightarrow \perp$  is the Curry sentence, then  $((\gamma \rightarrow \perp) \wedge (\neg\gamma \rightarrow \perp)) \rightarrow (\gamma \vee \neg\gamma \rightarrow \perp)$  is an untrue instance of 6. Since universal and existential quantification are treated analogously in Field's model one can also construct failures of 8. (Similarly 5 and 7 fail—for example,  $((\top \rightarrow \gamma) \wedge (\top \rightarrow \neg\gamma)) \rightarrow (\top \rightarrow (\gamma \wedge \neg\gamma))$  is an untrue instance of 5.)
3. Tore Fjetland Øgaard has since shown me that it is not possible to combine 10 with principles 1, 4, and  $\perp \rightarrow \varphi$ .

4. I am indebted to Cian Dorr here for a useful discussion of this fact.
5. Note that  $\varphi \rightarrow (\varphi * \varphi)$  by RQ2 and  $\varphi \rightarrow \varphi$ . Now consider the Curry sentence,  $\gamma \leftrightarrow (\gamma \rightarrow \perp)$ . We have  $(\gamma * \gamma) \rightarrow \perp$  by RQ1 and  $\gamma \rightarrow (\gamma * \gamma)$ , so by transitivity we have  $\gamma \rightarrow \perp$ , and thus  $\gamma$  and finally  $\perp$  by modus ponens.
6. The predicates  $\text{Tr}^+$  and  $\text{Tr}^-$  are needed for a technical reason that will become apparent later in the proof.
7. In [6] Brady mentions a result of Meyer and Slaney to the effect that in the presence of a fusion connective  $\circ$ , satisfying  $(\varphi \circ \psi \rightarrow \chi) \dashv\vdash (\varphi \rightarrow (\psi \rightarrow \chi))$ , principle 6 entails the fusion form of contraction:  $\varphi \circ \psi \rightarrow \varphi \circ (\varphi \circ \psi)$ . It follows that a fusion connective of this sort cannot be consistently added to a naive truth theory in  $\text{TJK}^+$  (see [6, Section 6.2] for the details).
8. The closest logic to  $\text{TJK}^+$  which supports a naive true predicate is Brady's  $\text{TJ}^d\text{Q}$  (see [6]). However his consistency proof does not validate the principle  $\text{K}$ , and as far as I can see, it is not possible to prove that  $\text{K}$  can be added to anything as strong as  $\text{TJ}^d\text{Q}$  using Brady's methods.
9. It should be noted that in both Field's logic and mine, negation defined from the conditional in this way would not even satisfy the *rule* version of negation elimination:  $\neg\neg\varphi \vdash \varphi$ . However in Field's logic the rule of double negation introduction holds:  $\varphi \vdash \neg\neg\varphi$ . (Thanks to Tore Fjetland Øgaard for pointing this out to me.)
10. It can be verified that this connective is accumulating and monotonic.
11. Bilattices are the natural general setting for transparent logics of truth generated using Kripke-style least fixed point constructions.
12. One might also want to posit a set of *antidesignated* values, such that the argument must not only preserve designated values from premise to conclusion but must also preserve antidesignated values from conclusion to premise. This is a natural requirement if one wants to ensure that the entailment relation is always contraposable.
13. Suppose, for contradiction, that  $(p \rightarrow q)^+ \cap (p \rightarrow q)^- = \top$ . So  $p^+ \circ q^- = \top$ , so  $p^+ = \top$ , and thus  $p^+ = q^- = \top$ . But since  $p^+$ ,  $(p \rightarrow q)^+ = \top$  it follows that  $q^+ = \top$ . This contradicts the fact that  $q \in U$ , since  $q^+ \cap q^- = \top$ .
14. In this variant of the argument we separate occurrences of  $\text{Tr}$  in  $\varphi$  by a slightly different rule: if whenever  $\text{Tr}$  occurs in  $\varphi$  it is in the scope of at least one antecedent argument of a conditional, then  $|\varphi|$  is accumulative. If  $\text{Tr}$  never occurs in antecedent place, then  $|\varphi|$  is monotonic in the information ordering. We can then translate each formula to an appropriate  $\mathcal{L}'$  formula as before and apply the analogue of Theorem 3.5.
15. As well as many of the principles in Section 2.
16. Since writing this paper Tore Fjetland Øgaard has demonstrated to me in a personal communication that one cannot add contraposition and  $\neg(\top \rightarrow \perp)$  (which follows from rule 11 above) to  $\text{TJK}^+$  with a Curry sentence  $\gamma \leftrightarrow (\gamma \rightarrow \perp)$ .

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