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On the Equivalence Conjecture for Proof-Theoretic Harmony

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Abstract The requirement of proof-theoretic harmony has played a pivotal role in a number of debates in the philosophy of logic. Different authors have attempted to precisify the notion in different ways. Among these, three proposals have been prominent in the literature: harmony–as–conservative extension, harmony–as–leveling procedure, and Tennant's harmony–as–deductive equilibrium. In this paper I propose to clarify the logical relationships between these accounts. In particular, I demonstrate that what I call *the equivalence conjecture*—that these three notions essentially come to the same thing—is erroneous.

1 Introduction

The meanings of the logical constants, the logical inferentialist tells us, are determined by the role these expressions play in deduction. And their deductive roles, it is thought, can be profitably summarized in the form of schematic introduction and elimination rules in the framework of a deductive system. At the heart of this logical inferentialist enterprise is the notion of harmony. Only such pairs of inference rules confer coherent meanings on the operators they govern and can be seen to exhibit a certain balance. Pairs of rules that are appropriately matched in this way (and, by extension, the logical constant whose meaning they serve to characterize) are said to be harmonious.[1] The notion of harmony has been deployed for a number of related but distinct ends.[2] Moreover, there has been no consensus as to what form a rigorous formulation of the principle of harmony should take. In the literature three types of proposals are prominent:

1. *harmony–as–conservative extension* (HCE) (see Belnap [1], Dummett [2]),
2. *harmony–as–leveling procedure* (HLP) (see Prawitz [5], Dummett [2]),
3. *harmony–as–deductive equilibrium* (HDE) (see Tennant [12], [13]).

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The purpose of this paper is to investigate the logical relationships between these three approaches. In particular, the thesis, more or less explicitly assumed in parts of the literature, that all three accounts of harmony come to the same thing—call it the *equivalence conjecture*—is disproved.[3] In what follows I briefly introduce the candidate notions in order to then examine the equivalence conjecture by systematically probing the alleged entailments between them, thereby showing the equivalence conjecture to be erroneous.[4]

2 The Three Proposals

2.1 Harmony–as–conservative extension (HCE) The HCE approach was originally proposed by Nuel Belnap [1] in response to Arthur Prior's `tonk`-challenge. In the literature on harmony it is usually presented along the following lines.

> *Conservative extension:* Let S and S' be two deductive systems expressed in the languages $\mathscr{L}$ and $\mathscr{L}'$, respectively, where \$ is a novel logical operator and $\mathscr{L} \cup \{\$\} = \mathscr{L}'$. S' contains all of the inference rules in S plus the pair of rules governing \$.[5] S' is then a conservative extension of S just in case, for any set of formulas Γ and any formula A in $\mathscr{L}$, we have $\Gamma \vdash_{S'} A$ only if we already have $\Gamma \vdash_S A$.

Note that conservativeness (insofar as it is a property attributable to a logical constant at all) is a relational property that a logical constant possesses or lacks *relative to a given base system*. A given constant may conservatively extend one system but not another; everything depends on the features of the rules governing the constant in question and on the particular configuration of the system into which the said constant is introduced. Strictly speaking, conservativeness is thus a property ascribable to pairs (S, (\$-I, \$-E)) where S is a base system and (\$-I, \$-E) is a pair of inference rules governing the newly introduced logical constant \$.

Alternatively, conservativeness could be interpreted as a property of a deductive system alone. A system qualifies as conservative in this sense—call it *full conservativeness*—when, for any one of its logical constants, the system as a whole conservatively extends the remainder system obtained by subtracting that constant. More formally, we have the following.

> *Full conservativeness:* A system S is fully conservative if, for every logical operator \$ in S, S is conservative over $S - \{\$\}$.[6]

Note that both conservativeness demands are *global* constraints. That is, whether or not the requirement is met depends (at least in part) on systemic features. We now turn to a *local* constraint on pairs of inference rules, HLP.

2.2 Harmony–as–leveling procedure (HLP) Like HCE, HLP is a type of noncreativity constraint; its aim is to rule out disharmony caused by inadmissibly strong elimination rules (relatively to the corresponding introduction rules). While the idea is already implicit in Gentzen's work (see [3]), it is elaborated by Dag Prawitz in his *inversion principle*. In Prawitz's words,

> an elimination rule is, in a sense, the inverse of the corresponding introduction rule: by an elimination rule one essentially only restores what had already been established by the major premise of the application of an introduction rule [7, p. 33].

If an elimination rule is really just a device for "undoing" a primitive inferential move effected by an application of an introduction rule, it should not be possible, simply by introducing and subsequently eliminating a logical constant, to arrive at new conclusions not themselves containing the constant in question.

Let us be more precise. Let \$ be a logical operator. Where the introduction rule for \$ is *immediately* followed by a \$-elimination rule, we speak of a *local peak* (with respect to \$). The formula containing \$ as its main connective, which serves simultaneously as the conclusion of the \$-introduction rule and the major premise of the corresponding elimination rule, we shall call a (\$-)*maximum*; "maximum" because such formulas are logically more complex than the formulas in the immediate vicinity on the same deductive path.

Following Dummett, we may refer to procedures that ensure the dispensability of maxima as *leveling procedures*.[7]

Let us briefly illustrate HLP with the aid of a standard example. Consider the case of disjunction with its familiar introduction and elimination rules.[8] Suppose we have a local peak featuring $\vee$ of the following form:

$$\vee\text{-E}, i\;\dfrac{\vee\text{-I}\;\dfrac{\begin{matrix}\Gamma_0\\ \Pi_0\\ A\end{matrix}}{A \vee B} \qquad \begin{matrix}\Gamma_1, [A]^i\\ \Pi_1\\ C\end{matrix} \qquad \begin{matrix}\Gamma_2, [B]^i\\ \Pi_2\\ C\end{matrix}}{C}$$

Our proof is then straightforwardly transformed into one that avoids the detour through the introduction of $A \vee B$; we do this by concatenating the proof Π_0 of A with the proof Π_1 of C from A (and similarly for the other case):

$$\begin{matrix}\Gamma_0\\ \Pi_0\\ \underbrace{\Gamma_1, A}\\ \Pi_1\\ C\end{matrix}$$

The rules for $\vee$ thus satisfy HLP.

2.3 Harmony–as–deductive equilibrium (HDE) So much for HLP. We turn now to our final contender, Tennant's HDE. The idea of a balance or of an equilibrium between introduction and elimination rules that characterized our intuitive notion of harmony is also central to Tennant's account. In order to capture this feature of our pretheoretic understanding of harmony, Tennant introduces the notions of the logically *strongest* and *weakest* propositions with a certain property.

- A is the *strongest* proposition with property P if, for any proposition B with the same property, A entails B.
- A is the *weakest* proposition with property P if A is entailed by any proposition B with the same property.

"Proposition" here takes a nonstandard sense: the proposition A should be understood as the logical equivalence class of which the sentence A is a member, that is, the class of sentences logically equivalent to A.

With these definitions in place, the principle of $\underline{\text{h}}$armony (the lowercase "h" is crucial here) can be defined as follows.

> A set of rules governing a binary operator \$ are $\underline{\text{h}}$armonious if

(i) \$(A, B) is the strongest proposition that can be deduced by means of \$-I, and if
(ii) \$(A, B) is the weakest proposition that may be deduced by means of \$-E.[9]

When h̲armony obtains between \$-I and \$-E we may write h(\$-I, \$-E).

However, h̲armony on its own is not enough. Given a permissible introduction rule it fails to pick out a uniquely matching h̲armonious elimination rule (and vice versa). What is needed is a further requirement of H̲armony (the capital "H" is again significant). Tennant states it as follows:

> Given \$-E we determine \$-I as the *strongest* introduction rule \$-i such that h(\$-i, \$-E). Given \$-I we determine \$-E as the *strongest* elimination rule \$-e such that h(\$-I, \$-e) (see [14, p. 22], with notational adjustments).

Thus, the idea, roughly, is this. Given an introduction rule \$-I we determine the set of h̲armonious elimination rules. Among them we choose the strongest one, \$-E. We then determine the set of h̲armonious introduction rules for \$-E and pick the strongest one, \$-I*. If \$-I = \$-I*, we have a H̲armonious pair, if not we iterate the process until we reach an equilibrium. When \$-I and \$-E are in H̲armony, we write H(\$-I, \$-E).[10] So much for three proposed accounts of harmony. We now turn to our examination of the equivalence conjecture.

3 Putting the Equivalence Conjecture to the Test

3.1 Are HCE and HLP equivalent? So much for the three formal notions of harmony; let us now turn to their logical interrelations. Begin with the left-to-right implication: Does conservativeness entail the existence of a leveling procedure? As it stands, it is not clear whether the question is well posed. Conservativeness, we had said, is a global relational property that is—at least as it is standardly understood—instantiated whenever the rules governing a given constant and the base system to which the constant is adjoined stand in a certain relation. HLP, by contrast, is a local property attributable to a pair of inference rules governing a certain constant.

How, then, are we to understand the alleged entailment? If the question is whether any constant that can be conservatively added to a given system has a reduction procedure, the answer is—trivially—negative. Take any inconsistent system, and add `tonk` to it. The extension will be conservative, yet the rules for `tonk` obviously do not admit of a leveling procedure. A more charitable reading assumes the base system being conservatively extended to be in good working order. But what does it mean for a system to be in good working order? We might be inclined to appeal to our notion of full conservativeness. A system is in good working order just in the case when any one of its constants conservatively extends the system that remains if we delete the constant in question from the initial system. With this constraint in place, we can now restate the left-to-right implication as follows: Let S be any fully conservative system, and let \$ be an arbitrary logical operator. \$ conservatively extends S only if the rules governing \$ admit a leveling procedure. Does the implication hold when reformulated in this way?

The answer, it turns out, is again negative. Fragments of classical logic—fragments not involving $\supset$—provide counterexamples. Consider, for instance, the fragment $S = \{\wedge, \vee\}$. Clearly, S is fully conservative. Now add $\neg$ to the language governed by the rules $\neg$-I, $\neg$-E, and the classical *reductio* rule. The extension is conservative although there appears to be no leveling procedure for the classical negation operator.[11]

Let us turn to the converse implication. The question here is: Is it the case that any logical constant that admits a leveling procedure yields a conservative extension when added to a well-behaved (i.e., fully conservative) system? I think not. Dummett's example involving a weakened quantum logical disjunction operator (see [2, p. 290]) is a case in point, as can be shown. "Quantum-or" is governed by the standard introduction rules for disjunction and the following restricted elimination rule $Q\vee$-E:

$${\scriptstyle Q\vee\text{-E}}\;\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A \sqcup B\end{array}\qquad \begin{array}{c}[A]\\ \vdots\\ C\end{array}\qquad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C}$$

$Q\vee$-E disallows collateral hypotheses alongside A and B in the subproofs. Let us denote quantum-or by $\sqcup$. A moment's reflection reveals that the standard leveling procedure for $\vee$ (see Section 2.2) is equally applicable in the present case—the restrictions on side assumptions imposed by $Q\vee$-E have no bearing on the leveling procedure. It follows that quantum-or satisfies HLP.

Now consider the system containing only, say, $\wedge$ and $\sqcup$. Obviously, our system is fully conservative. Yet, if we now add $\vee$ with its standard rules of inference, we find that $\sqcup$ collapses into $\vee$:

$${\scriptstyle \sqcup\text{-E},\,1}\;\dfrac{A \sqcup B \qquad {\scriptstyle \vee\text{-I}}\,\dfrac{[A]^1}{A \vee B} \qquad {\scriptstyle \vee\text{-I}}\,\dfrac{[B]^1}{A \vee B}}{A \vee B}$$

And this, in turn, allows us to prove the quantum-logically inadmissible law of distributivity $A \wedge (B \sqcup C) \vdash (A \wedge B) \sqcup (A \wedge C)$, thereby showing the extension to be nonconservative:

$${\scriptstyle \vee\text{-E},\,2}\;\dfrac{{\scriptstyle \sqcup\text{-E},\,1}\,\dfrac{{\scriptstyle \wedge\text{-E}}\,\dfrac{A \wedge (B \sqcup C)}{B \sqcup C} \quad {\scriptstyle \vee\text{-I}}\,\dfrac{[B]^1}{B \vee C} \quad {\scriptstyle \vee\text{-I}}\,\dfrac{[C]^1}{B \vee C}}{B \vee C} \qquad {\scriptstyle \sqcup\text{-I}}\,\dfrac{{\scriptstyle \wedge\text{-I}}\,\dfrac{{\scriptstyle \wedge\text{-E}}\,\dfrac{A \wedge (B \sqcup C)}{A} \quad [B]^2}{A \wedge B}}{(A \wedge B) \sqcup (A \wedge C)} \qquad {\scriptstyle \sqcup\text{-I}}\,\dfrac{{\scriptstyle \wedge\text{-I}}\,\dfrac{{\scriptstyle \wedge\text{-E}}\,\dfrac{A \wedge (B \sqcup C)}{A} \quad [C]^2}{A \wedge C}}{(A \wedge B) \sqcup (A \wedge C)}}{(A \wedge B) \sqcup (A \wedge C)}$$

We therefore have an example of a connective, to wit, the ordinary disjunction operator, which admits a leveling procedure, but which may bring about nonconservativeness even when the system into which it is introduced is itself fully conservative.[12] Therefore the converse implication—from HLP to HCE—fails also.

3.2 Are HLP and HDE equivalent? Start again with the left-to-right implication: Is any pair of rules governing a given logical constant that admits of a leveling procedure also in deductive equilibrium in Tennant's sense? The answer is straightforwardly "No." As we have seen in the previous section, quantum-or maxima are readily eliminated, yet, while the rules for $\sqcup$ satisfy <u>h</u>armony, they do not qualify as <u>H</u>armonious on Tennant's account since the restricted $\sqcup$-elimination rule is trumped by the stronger standard elimination rule.[13]

Does the converse implication hold? I argued (see Steinberger [9]) that HDE is vulnerable to counterexamples in the form of patently disharmonious unrestricted quantifier rules.[14] In particular, I show that Tennant's definition of <u>H</u>armony is satisfied by a rogue quantifier that obeys the rules for the existential quantifier with the modification that its elimination rule is devoid of any restrictions on the parameter.

Denote the quantifier whose use is regulated by these excessively permissive rules by $\reflectbox{E}$. We can then construct the following proof:[15]

$$\dfrac{\dfrac{F(a)}{\reflectbox{E}xF(x)}\,\reflectbox{E}\text{-I} \qquad [F(b)]^1}{F(b)}\,\reflectbox{E}\text{-E, }1$$

Clearly, there is no universal procedure to eliminate maxima of this sort; in general there is no direct deductive route from the fact that $F(a)$ to the fact that any arbitrary object b is also F. Consequently, the right-to-left implication from HDE to HLP fails also.

3.3 Are HDE and HCE equivalent? This brings us to the last equivalence between HDE and HCE. The examples introduced so far will suffice to show that both implications fail yet again. Begin once again with the left-to-right implication. The deviant quantifier considered in the previous section is, as we have seen, sanctioned by Tennant's account. Nevertheless, it induces nonconservativeness when adjoined to any reasonable system. This is enough to disprove the HDE-to-HCE entailment. For the right-to-left implication, take again the example involving quantum-or in Section 3.1. While the introduction of quantum-or into any reasonable system (intuitionistic logic, say) leads to a conservative extension, quantum-or violates Tennant's demand for $\underline{\text{H}}$armony.

4 Conclusion

In this paper I investigated the logical relations between three prominent accounts of harmony. We found that the equivalence conjecture fails. Indeed, it turns out that none of the alleged equivalences hold. It follows that there is no core notion of harmony toward which all three accounts of harmony converge. Participants in the debate will henceforth have to justify their choice of one or another of these accounts.

Notes

1. For a more detailed examination of the notion of harmony, see [11].

2. It has been advanced as a response to Arthur Prior's famous '`tonk` challenge' [8]; a criterion of logicality, namely, as the idea that all (and perhaps only) those expressions whose meaning can be exhaustively specified in terms of harmonious inference rules count as properly logical (see [13], [4]); a component of a proof-theoretic justification of the laws of logic (see [6], [2]); finally it is treated by some as a *desideratum* in the dispute between revisionary antirealists and realists defending classical logic (see [5], [2], [13]).

3. The equivalence conjecture remains implicit in Michael Dummett's *The logical basis of metaphysics*, where he appeals to the first two notions and gestures toward something like the third [2, p. 250; p. 291]. The conjecture is perhaps most explicitly endorsed by Neil Tennant who writes:

   > it is in the present author's view, interesting and important unfinished business to show that these three proposals are equivalent [14, p. 15].

4. I note for the record that none of the three proposals is up to the task of furnishing a precise explication of the notion of harmony in my opinion. I will make no attempt

to argue for this contention here, although some of the proposals' shortcomings will become evident from the examples considered below.

5. A logical operator may of course be governed by more than one introduction and/or elimination rule. I will nevertheless speak of 'pairs of rules', where strictly speaking the clunkier 'pairs of sets of rules' would be called for.

6. Dummett mentions this form of conservativeness in [2, p. 250].

7. Dummett assimilates leveling procedures with what he calls *intrinsic* (or *local*) harmony (see [2, p. 250]).

8. The example will prove useful in Section 3.1.

9. Nothing hinges on the fact that \$ is taken to be binary; Tennant's account readily generalizes to operators of different arities. Also, I am skipping certain details that are inessential for our purposes. See [14] for the latest version of the account.

10. See [14] for an illustration of the account's workings.

11. This, of course, hardly constitutes a proof that no such procedure is to be had. And this is no isolated problem: given a pair of inference rules, there is in general no strategy for demonstrating the nonexistence of a leveling procedure. All the same, it seems plausible that the rules governing classical negation are not amenable to leveling.

12. Moreover, all the operators in the system are themselves amenable to leveling.

13. In Tennant's terminology we thus have $h(\sqcup\text{-I}, \sqcup\text{-E})$, but not $H(\sqcup\text{-I}, \sqcup\text{-E})$.

14. See [15] for Tennant's reply to my objection. I respond in [10].

15. The following proof violates the standard restriction imposed upon the $\exists$-E rule that the parameter a ought not to occur in the conclusion of the minor premise.

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