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Reducts of the Random Bipartite Graph

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Abstract Let Γ be the random bipartite graph, a countable graph with two infinite sides, edges randomly distributed between the sides, but no edges within a side. In this paper, we investigate the reducts of Γ that preserve sides. We classify the closed permutation subgroups containing the group $\mathrm{Aut}(\Gamma)^*$, where $\mathrm{Aut}(\Gamma)^*$ is the group of all isomorphisms and anti-isomorphisms of Γ preserving the two sides. Our results rely on a combinatorial theorem of Nešetřil and Rödl and a strong finite submodel property for Γ.

1 Introduction

As in Thomas [10], a reduct of a structure Γ is a structure with the same underlying set as Γ, for some relational language, each of whose relations is $\emptyset$-definable in the original structure. If Γ is ω-categorical, then a reduct of Γ corresponds to a closed permutation subgroup in $\mathrm{Sym}(\Gamma)$ (the full symmetric group on the underlying set of Γ) that contains $\mathrm{Aut}(\Gamma)$ (the automorphism group of Γ). Two interdefinable reducts are considered to be equivalent. That is, two reducts of a structure Γ are equivalent if they have the same $\emptyset$-definable sets or, equivalently, if they have the same automorphism groups. There is a one-to-one correspondence between equivalence classes of reducts N and closed subgroups of $\mathrm{Sym}(\Gamma)$ containing $\mathrm{Aut}(\Gamma)$ via $N \mapsto \mathrm{Aut}(N)$ (see [10]).

There are currently a few ω-categorical structures whose reducts have been explicitly classified. In 1977, Higman [5] classified the reducts of the structure $(\mathbb{Q}, <)$. In 2008, Markus Junker and Martin Ziegler [7] classified the reducts of expansions of $(\mathbb{Q}, <)$ by constants and unary predicates. In 2010, Manuel Bodirsky, Hubie Chen, and Michael Pinsker [4] provided a classification of the reducts of the logic of equality. Simon Thomas [9] showed that there are finitely many reducts of the random graph in 1991, and of the random hypergraphs (see [10]) in 1996. In 1996, James Bennett [2] proved similar results for the random tournament and for the random

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k-edge coloring graphs. In this paper, we investigate the reducts of the random bipartite graph that preserve sides. We find it convenient to consider a bipartite graph in a language with two unary predicates (one side R_l, the other side R_r) and two binary predicates (edge P_1, not edge P_2). Equivalently, we analyze the closed subgroups of $\mathrm{Sym}(R_l) \times \mathrm{Sym}(R_r)$ containing $\mathrm{Aut}(\Gamma)$, where R_l, R_r denote the two sides of the random bipartite graph. Let $\mathrm{Aut}(\Gamma)^*$ be a group of all isomorphisms and anti-isomorphisms preserving the two sides. We classified all the closed subgroup of $\mathrm{Sym}(R_l) \times \mathrm{Sym}(R_r)$ containing $\mathrm{Aut}(\Gamma)^*$. We have analyzed some closed groups between $\mathrm{Aut}(\Gamma)$ and $\mathrm{Sym}(\Gamma)$ but do not describe the results here since we do not have a classification of all such groups.

Definition 1.1 A structure $G = (V^G, R_l^G, R_r^G, P_1^G, P_2^G)$, where $R_l^G, R_r^G \subseteq V^G$ and $P_1^G, P_2^G \subseteq R_l^G \times R_r^G$, is a *bipartite graph* if it satisfies the following set of axioms:

$$\exists x R_l(x) \wedge \exists x R_r(x),$$
$$\forall x (R_l(x) \vee R_r(x)),$$
$$\forall x \big((R_l(x) \longrightarrow \neg R_r(x)) \wedge (R_r(x) \longrightarrow \neg R_l(x))\big),$$
$$\forall x \forall y \big((R_l(x) \wedge R_r(y)) \longrightarrow (P_1(x,y) \vee P_2(x,y))\big),$$
$$\forall x \forall y \big((P_1(x,y) \longrightarrow (R_l(x) \wedge R_r(y))) \wedge (P_2(x,y) \longrightarrow (R_l(x) \wedge R_r(y)))\big),$$
$$\forall x \forall y \big((R_l(x) \wedge R_r(y)) \longrightarrow ((P_1(x,y) \longrightarrow \neg P_2(x,y)) \wedge (P_2(x,y) \longrightarrow \neg P_1(x,y)))\big).$$

In the rest of the paper, we will use the following notation: if $E = (a,b) \in R_l \times R_r$, then we call (a,b) a *cross-edge*, and we say that E has *cross-type* P_i if P_i holds for the pair (a,b) for $i = 1,2$. Furthermore, if $g \in \mathrm{Sym}(\Gamma)$ and $E = (a,b) \in R_l \times R_r$, then we denote $(g(a), g(b))$ by $g[E]$. An $(m \times n)$-subgraph is a bipartite graph with m vertices in R_l and n vertices in R_r. $\mathrm{Sym}_{\{l,r\}}(\Gamma)$ denotes the group $\mathrm{Sym}(R_l) \times \mathrm{Sym}(R_r)$.

Definition 1.2 Let $n \in \mathbb{N}$. A bipartite graph satisfies the *extension property* Θ_n if for any two disjoint subsets $X_{l1}, X_{l2} \in [R_l]^{\leq n}$, and any two disjoint subsets X_{r1}, $X_{r2} \in [R_r]^{\leq n}$,

(a) there exists a vertex $v \in R_r$ such that $P_i(x,v)$ for every $x \in X_{li}$ for $i = 1,2$; and
(b) there exists a vertex $w \in R_l$ such that $P_i(w,x)$ for every $x \in X_{ri}$ for $i = 1,2$.

Definition 1.3 A countable bipartite graph, denoted by Γ, is *random* if it satisfies the extension property Θ_n for every $n \in \mathbb{N}$.

The Θ_n's are first-order sentences, and the axioms in Definition 1.1 together with the $\{\Theta_n\}_{n\in\mathbb{N}}$ form a complete and ω-categorical theory. A random bipartite graph can be built by Fraïssé construction for bipartite graphs (see Hodges [6]). It is countable and unique up to isomorphism. It is also easy to show that the random bipartite graph is homogeneous by a back-and-forth argument. In the rest of paper, we denote by Γ the random bipartite graph.

Definition 1.4 Let Γ be the random bipartite graph, and let A be a subset of Γ. A bijection $\sigma : \Gamma \longrightarrow \Gamma$ is a *switch* with respect to A if the following conditions are satisfied: for all $(a,b) \in R_l \times R_r$ and $i = 1,2$, $P_i(a,b) \longleftrightarrow P_i(\sigma(a), \sigma(b))$ if and only if $|\{a,b\} \cap A| \neq 1$.

Note that a switch on any finite set of vertices can be obtained by composing single-vertex switches.

Definition 1.5 Let $X \subseteq \{l, r\}$. The switch group $S_X(\Gamma)$ is the closed subgroup of $\mathrm{Sym}_{\{l,r\}}(\Gamma)$ generated as a topological group by

(1) $\mathrm{Aut}(\Gamma)$; and
(2) the set of all $\sigma \in \mathrm{Sym}_{\{l,r\}}(\Gamma)$ such that σ is a switch with respect to some $v \in R_i$, where $i \in X$.

Since Γ satisfies the extension property Θ_n for $n \in \mathbb{N}$ and $S_{\{l,r\}}(\Gamma)$ is closed, we can construct $\rho \in S_{\{l,r\}}(\Gamma)$ which is a switch w.r.t. R_l. Observe that $\rho \in S_{\{l\}}(\Gamma) \cap S_{\{r\}}(\Gamma)$. Let G^* be the closed group generated by G and ρ. Then the group $S_X(\Gamma)^*$ is the same as the group $S_X(\Gamma)$ except when $X = \emptyset$. Notice $\mathrm{Aut}(\Gamma)^* = S_\emptyset(\Gamma)^*$, which is a group of permutations that either preserve all cross-types on $R_l \times R_r$, or exchange all cross-types on $R_l \times R_r$. Also notice that $\mathrm{Aut}(\Gamma)^* = S_l(\Gamma) \cap S_r(\Gamma)$.

We now state the main result of this paper.

Theorem 1.6 *If G is a closed subgroup with* $\mathrm{Aut}(\Gamma)^* \leq G < \mathrm{Sym}_{\{l,r\}}(\Gamma)$*, then there exists a subset $X \subseteq \{l, r\}$ such that $G = S_X(\Gamma)^*$.*

That is, there are only finitely many closed subgroups of $\mathrm{Sym}_{\{l,r\}}(\Gamma)$ containing $\mathrm{Aut}(\Gamma)^*$: $\mathrm{Aut}(\Gamma)^*$, $S_{\{l\}}(\Gamma)$, $S_{\{r\}}(\Gamma)$, $S_{\{l,r\}}(\Gamma)$, and $\mathrm{Sym}_{\{l,r\}}(\Gamma)$. This theorem relies on a combinatorial theorem of Nešetřil and Rödl [8] and the strong finite submodel property of the random bipartite graph. It is still an open question whether there are finitely many closed subgroups between $\mathrm{Aut}(\Gamma)$ and $\mathrm{Sym}(\Gamma)$.

Here is how the rest of the paper is organized. In Section 2, we study the relations preserved by the groups $S_X(\Gamma)$, where $X \subseteq \{l, r\}$. In Section 3, we show that the random bipartite graph has the strong finite bipartite submodel property. In Section 4, we employ a technique called $(m \times n)$-analysis for the random bipartite graph. These prepare us to give an explicit classification of the closed subgroups of $\mathrm{Sym}_{\{,r\}}(\Gamma)$ containing $\mathrm{Aut}(\Gamma)^*$ in the rest of the paper. In Section 5, we prove the first part of Theorem 1.6, which says that the closed subgroups of $S_{\{l,r\}}(\Gamma)$ containing $\mathrm{Aut}(\Gamma)^*$ are $\mathrm{Aut}(\Gamma)^*$, $S_{\{l\}}(\Gamma)$, $S_{\{r\}}(\Gamma)$, and $S_{\{l,r\}}(\Gamma)$. In Section 6, we prove the existence of some special finite subgraphs of Γ, which will be used in Section 7. Then in Section 7 we show that there is no other proper closed subgroup between $S_{\{l,r\}}(\Gamma)$ and $\mathrm{Sym}_{\{l,r\}}(\Gamma)$, which completes the proof of Theorem 1.6.

2 Relations Preserved by Switch Groups

In this section, we identify the relations preserved by the switch groups $S_{\{l\}}(\Gamma)$, $S_{\{r\}}(\Gamma)$, and $S_{\{l,r\}}(\Gamma)$. For convenience in discussing closures of $G \leq \mathrm{Sym}_{\{l,r\}}(\Gamma)$, we let $\mathfrak{F}(G) = \{g \restriction X \mid g \in G, X \in [\Gamma]^{<\omega}\}$.

Definition 2.1 Let $f \in \mathrm{Sym}_{\{l,r\}}(\Gamma)$, and let S be a finite bipartite subgraph of Γ. We say f *preserves* the parity of cross-types on S if the number of P_1 cross-types in S is even if and only if the number of cross-types in $f[S]$ is even.

Lemma 2.2 *We have $S_{\{l,r\}}(\Gamma) = \{\sigma \in \mathrm{Sym}_{\{l,r\}}(\Gamma) \mid \sigma$ preserves the parity of cross-types in every (2×2)-subgraph of $\Gamma\}$.*

Proof It is easy to show that any $\sigma \in S_{\{l,r\}}(\Gamma)$ preserves the parity of cross-types in every (2×2)-subgraph of Γ. The other direction is proved as follows.

Suppose $\sigma \in \mathrm{Sym}_{\{l,r\}}(\Gamma)$ preserves the parity of cross-types in every (2×2)-subgraph of Γ. Let B be an arbitrary (2×2)-subgraph of Γ. Since σ preserves the parity of the P_i's for $i = l$ and r, only an even number of the cross-types can be changed. That is, 0, 2, or 4 of the cross-types can be changed. We shall prove that in each case, there exists $\theta \in S_{\{l,r\}}(\Gamma)$ such that $\theta \upharpoonright B = \sigma \upharpoonright B$.

Case 1. If none of the cross-types are changed, then there exists $\theta \in \mathrm{Aut}(\Gamma)$ such that $\theta \upharpoonright B = \sigma \upharpoonright B$.

Case 2. If two of the cross-types are changed, then there exists θ which is either a switch with respect to one vertex or a switch with respect to two vertices of B such that $\theta \upharpoonright B = \sigma \upharpoonright B$.

Case 3. If four of the cross-types are changed, then there exists θ which is a switch with respect to R_l of Γ (i.e., $\theta \in \mathrm{Aut}(\Gamma)^*$) such that $\theta \upharpoonright B = \sigma \upharpoonright B$.

We then choose a vertex $v \in \Gamma \backslash B$ and let $\varphi = \theta^{-1} \circ \sigma \upharpoonright B \cup \{v\}$. We may assume $v \in R_l$. Note that if E is a cross-edge in $B \cup \{v\}$ and φ does not preserve the cross-type on E, then $E = (v, u)$ for some $u \in R_r$. Also notice that θ and σ both preserve the parity of cross-types in (2×2)-subgraphs of Γ; hence so does φ. Then it is easy to check that either for every $w \in B \cap R_r$, $P_i(v, w) \longrightarrow P_i(\varphi(v), \varphi(w))$, or for every $w \in B \cap R_r$, $P_i(v, w) \longrightarrow \neg P_i(\varphi(v), \varphi(w))$, where $i = 1$ and 2. Therefore $\varphi \in \mathfrak{F}(S_{\{l,r\}}(\Gamma))$, and so $\sigma \upharpoonright B \cup \{v\} \in \mathfrak{F}(S_{\{l,r\}}(\Gamma))$. Continuing in this manner for the vertices in $\Gamma \backslash B \cup \{v\}$, we see that for any finite bipartite graph $S \subset \Gamma$, there exists an element $\theta_S \in \mathfrak{F}(S_{\{l,r\}}(\Gamma))$ such that $\sigma \upharpoonright S = \theta_S$. Thus $\sigma \in S_{\{l,r\}}(\Gamma)$, since $S_{\{l,r\}}(\Gamma)$ is closed. This complete the proof of Lemma 2.2. □

Similarly, we can prove the following results.

Lemma 2.3 *We have* $S_{\{l\}}(\Gamma) = \{\sigma \in \mathrm{Sym}_{\{l,r\}}(\Gamma) \mid \sigma$ *preserves the parity of cross-types in every* (1×2)*-subgraph of* $\Gamma\}$.

Lemma 2.4 *We have* $S_{\{r\}}(\Gamma) = \{\sigma \in \mathrm{Sym}_{\{l,r\}}(\Gamma) \mid \sigma$ *preserves the parity of cross-types in every* (2×1)*-subgraph of* $\Gamma\}$.

3 The Strong Finite Bipartite Submodel Property

In this section, we define the strong finite bipartite submodel property (SFBSP), inspired by the strong finite submodel property introduced by Thomas in [10], and we prove that the random bipartite graph has the SFBSP. This result will be used in the proof of Lemma 5.4 in Section 5.

Definition 3.1 A countable infinite bipartite graph Γ has the *strong finite bipartite submodel property (SFBSP)* if $\Gamma = \bigcup_{i \in \mathbb{N}} \Gamma_i$ is a union of an increasing chain of substructures Γ_i such that

(1) $\Gamma_i \subset \Gamma_{i+1}$ and $|\Gamma_i| = i$ for each $i \in \mathbb{N}$; in particular,
  - if i is even, then $|\Gamma_i \cap R_l| = |\Gamma_i \cap R_r|$;
  - otherwise, $|\Gamma_i \cap R_l| = |\Gamma_i \cap R_r| + 1$;

(2) for any sentence φ with $\Gamma \models \varphi$, there exists $N \in \mathbb{N}$ such that $\Gamma_i \models \varphi$ for all $i \geq N$.

Theorem 3.2 *The countable random bipartite graph* Γ *has the SFBSP.*

Theorem 3.2 is a consequence of the Borel–Cantelli lemma, as below.

Definition 3.3 (see [10]) If $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of events in a probability space, then $\bigcap_{n\in\mathbb{N}}\left[\bigcup_{n\le k\in\mathbb{N}} A_k\right]$ is the event that consists of realization of infinitely many of A_n, denoted by $\overline{\lim} A_n$.

Lemma 3.4 (Borel and Cantelli; see Billingsley [3]) *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of events in a probability space. If $\sum_{n=0}^{\infty} P(A_n) < \infty$, then $P(\overline{\lim} A_n) = 0$.*

Proof of Theorem 3.2 Since the extension properties Θ_n's axiomatize the random bipartite graph Γ and Θ_i implies Θ_{i-1} for all $i \in \mathbb{N}$, for every sentence φ true in Γ, there exists some $k \in \mathbb{N}$ such that Θ_k holds if and only if φ holds. Let Ω be the probability space of all countable bipartite graphs (S, R_l, R_r, P_1, P_2), where $|R_l| = |R_r| = \omega$ and every cross-edge $E \in R_l \times R_r$ has cross-type P_1 with probability $\frac{1}{2}$. For each $n \in \mathbb{N}$ with $n \ge k$, let $S_n \in [S]^n$ such that if n is even, then $|S_n \cap R_l| = \frac{n}{2}$; otherwise $|S_n \cap R_l| = |S_n \cap R_r| + 1$. Let A_n be the event for which the induced graph on S_n does not satisfy the extension property Θ_k. Then by simple computation,

$$\sum_{n=0}^{\infty} P(A_n) = \sum_{m=0}^{\infty} P(A_{2m}) + \sum_{m=0}^{\infty} P(A_{2m+1}) \\ \le 4\sum_{m=0}^{\infty} \binom{m+1}{k}\binom{m+1-k}{k}\left(1-\left(\frac{1}{4}\right)^k\right)^{m-2k}, \qquad (1)$$

where $\binom{n}{i}$ is the number of combinations of n objects taken i at a time. Let $C_m = \binom{m+1}{k}\binom{m+1-k}{k}\left(1-\left(\frac{1}{4}\right)^k\right)^{m-2k}$. Then $\lim_{m\to+\infty} \frac{C_{m+1}}{C_m} = 1-\left(\frac{1}{4}\right)^k < 1$. By the ratio test for infinite series, we have that $\sum_{m=0}^{\infty} C_m$ converges, and so does $\sum_{n=0}^{\infty} P(A_n)$. Thus by Lemma 3.4, $P(\overline{\lim} A_n) = 0$. So there exists a bipartite graph $S \in \Omega$ and an integer N such that for all $n \ge N$, the subgraph on $S_n \in [S]^n$ satisfies the extension property Θ_k, and so φ. Notice that the choice of S ensures that S is countable and satisfies all the axioms for the random bipartite graph. Hence S is isomorphic to Γ. Then Γ has the SFBSP, which completes the proof of Theorem 3.2. □

4 $(m \times n)$-Analysis

In [10], Thomas used a helpful tool called "m-analysis" to classify the reducts of the random hypergraphs. Using a similar approach, we give the definition of $(m \times n)$-analysis in this section, and we prove that if $f \in \mathfrak{F}(S_{\{l,r\}}(\Gamma))$ and if $|\operatorname{dom} f|$ is sufficiently large, then f has an $(m \times n)$-analysis. This rather technical concept will be used in the proof of Theorem 1.6.

Definition 4.1 Let $m, n > 2$. Suppose $f \in \mathfrak{F}(S_{\{l,r\}}(\Gamma))$ and $Z = \operatorname{dom} f$ satisfies $|Z \cap R_l| \ge m$ and $|Z \cap R_r| \ge n$. An $(m \times n)$-*analysis* of f consists of a finite sequence of elements $f_0, f_1, \ldots, f_s \in \mathfrak{F}(S_{\{l,r\}}(\Gamma))$ satisfying the following conditions:

(1) $f_0 = \theta \circ f$ where $\theta \in \mathfrak{F}(\operatorname{Aut}(\Gamma)^*)$.

(2) For each $0 \le j \le s-1$, there exist a finite $(m \times n)$-subgraph Y_j in Z, and an element $\theta_j \in S_{\{l,r\}}(\Gamma)$ such that

  (a) θ_j is either an automorphism or a switch with respect to some vertex $v_j \in Y_j \cap R_{i_j}$ where $i_j \in \{l, r\}$;

(b) $\theta_j \restriction Y_j = (f_j \circ f_{j-1} \circ \cdots \circ f_0) \restriction Y_j$;
(c) $f_{j+1} = \theta_j^{-1} \restriction \operatorname{ran}(f_j \circ \cdots \circ f_0)$.

(3) $f_s \circ \cdots \circ f_0 : Z \longrightarrow \Gamma$ is an isomorphic embedding.

We now prove the existence of an $(m \times n)$-analysis for a given f.

Theorem 4.2 *Let $m, n \in \mathbb{N}$ and $m, n > 2$. For every $f \in \mathfrak{F}(S_{\{l,r\}}(\Gamma))$, there exists an integer $s(m,n)$ such that if $|\operatorname{dom} f \cap R_i| \geq s(m,n)$ for $i = l$ and r, then there exists an $(m \times n)$-analysis of f.*

Proof Let $f \in \mathfrak{F}(S_{\{l,r\}}(\Gamma))$ be such that $Z = \operatorname{dom} f$ is a very large subset of Γ. By Ramsey's theorem, there exists a large subset S of Z such that S satisfies one of the following two conditions for every cross-edge E in S, where $i = 1, 2$:

(a) $P_i(E)$ implies $P_i(f[E])$;
(b) $P_i(E)$ implies $\neg P_i(f[E])$.

We will construct a sequence of f_i's as follows.

If (a) holds, then we let $f_0 = \theta \circ f$ where $\theta \in \mathfrak{F}(\operatorname{Aut}(\Gamma)^*)$ is the identity map on $\operatorname{dom} f$. Let Y_0 be an arbitrary $(m \times n)$-subgraph in S, and choose $\theta_0 \in \operatorname{Aut}(\Gamma)$ such that $\theta_0 \restriction S = f_0 \restriction S$. Define $f_1 = \theta_0^{-1} \restriction \operatorname{ran}(f_0)$.

Next we choose $w_1 \in Z \setminus S$ if it exists and consider $f_1 \circ f_0 \restriction S \cup \{w_1\}$. Since $f_1 \circ f_0 \in \mathfrak{F}(S_{\{l,r\}}(\Gamma))$ and $f_1 \circ f_0 \restriction S$ is the identity map, $f_1 \circ f_0 \restriction S \cup \{w_1\}$ is either an isomorphism or a switch with respect to w_1 by Lemma 2.2. Let Y_1 be an arbitrary $(m \times n)$-subgraph of $S \cup \{w_1\}$ containing w_1. Then there exists $\theta_1 \in \mathfrak{F}(S_{\{l,r\}}(\Gamma))$ which is either an isomorphism or a switch with respect to w_1 and $\theta_1 \restriction S \cup \{w_1\} = f_1 \circ f_0 \restriction S \cup \{w_1\}$. Define $f_2 = \theta_1^{-1} \restriction \operatorname{ran}(f_1 \circ f_0)$.

Continuing in this manner, for $0 \leq j < s = |Z/S|$, we can find an $(m \times n)$-subgraph Y_j of Z and $\theta_j \in S_{\{l,r\}}(\Gamma)$ such that

(1) θ_j is either an isomorphism or a switch with respect to some vertex $w_j \in Y_j \cap R_{i_j}$ where $i_j \in \{l, r\}$;
(2) $\theta_j \restriction Y_j = (f_j \circ f_{j-1} \circ \cdots \circ f_0) \restriction Y_j$;
(3) $f_{j+1} = \theta_j^{-1} \restriction \operatorname{ran}(f_j \circ \cdots \circ f_0)$.

Also $f_s \circ \cdots \circ f_0 : Z \longrightarrow \Gamma$ is an isomorphic embedding.

If (b) holds, then there exists $\theta \in \mathfrak{F}(\operatorname{Aut}(\Gamma)^*)$ with $\operatorname{dom}(\theta) = \operatorname{ran}(f)$, which exchanges all the cross-types on Γ. Let $f_0 = \theta \circ f$. Hence $f_0 \restriction S$ is an isomorphism. The rest of the proof will be the same as in (a).

Hence $f_0, f_1, \ldots, f_s$ is an $(m \times n)$-analysis of f. This completes the proof of Theorem 4.2. □

5 Closed Subgroups of $S_{\{l,r\}}(\Gamma)$ Containing $\operatorname{Aut}(\Gamma)^*$

In this section, we prove the first part of Theorem 1.6, which says that the closed subgroups of $S_{\{l,r\}}(\Gamma)$ containing $\operatorname{Aut}(\Gamma)^*$ are $\operatorname{Aut}(\Gamma)^*$, $S_{\{l\}}(\Gamma)$, $S_{\{r\}}(\Gamma)$, and $S_{\{l,r\}}(\Gamma)$. Notice that in the rest of the paper, we only consider maps in $\operatorname{Sym}_{\{l,r\}}(\Gamma)$. Hence from now on, we call $f \restriction E$ an *isomorphism* if $E = (a, b)$ is a cross-edge and $P_i(a, b)$ implies $P_i(f(a), f(b))$ for $i = 1, 2$. We call $f \restriction E$ an *anti-isomorphism* if $E = (a, b)$ is a cross-edge and $P_i(a, b)$ implies $\neg P_i(f(a), f(b))$ for $i = 1, 2$.

Theorem 5.1 *Suppose that G is a closed subgroup with* $\operatorname{Aut}(\Gamma)^* \leq G \leq S_{\{l,r\}}(\Gamma)$. *Let X be the largest subset of $\{l, r\}$ such that $S_X(\Gamma)^* \subseteq G$. Then $G \subseteq S_X(\Gamma)^*$, and so $G = S_X(\Gamma)^*$.*

In the rest of this section, we let G be a closed subgroup with $\mathrm{Aut}(\Gamma)^* \leq G \leq S_{\{l,r\}}(\Gamma)$ and let X be the largest subset of $\{l, r\}$ such that $S_X(\Gamma)^* \subseteq G$.

Lemma 5.2 *Suppose that $g \in G$ is a bijection such that for every finite $T \subseteq \Gamma$ with $|T \cap R_i| \geq 2$ for $i = l$ and r, we have $g \upharpoonright T \in \mathfrak{F}(S_X(\Gamma)^*)$. Then $g \in S_X(\Gamma)^*$.*

Proof If $X \neq \emptyset$, from Lemmas 2.2, 2.3, and 2.4, we know that $g \upharpoonright T \in \mathfrak{F}(S_X(\Gamma))$ implies $g \in S_X(\Gamma)$. Then we are done. If $X = \emptyset$, then $S_\emptyset(\Gamma)^* = \mathrm{Aut}(\Gamma)^*$. If $g \upharpoonright T \in \mathfrak{F}(\mathrm{Aut}(\Gamma)^*)$, then $\mathrm{Aut}(\Gamma)^* = S_{\{l\}}(\Gamma) \cap S_{\{r\}}(\Gamma)$ implies $g \upharpoonright T \in \mathfrak{F}(S_l(\Gamma))$ and $g \upharpoonright T \in \mathfrak{F}(S_r(\Gamma))$. Thus $g \in S_{\{l\}}(\Gamma) \cap S_{\{r\}}(\Gamma)$, and so $g \in \mathrm{Aut}(\Gamma)^*$. This completes the proof of Lemma 5.2. □

Now let $g \in G$. Let $T \subseteq \Gamma$ be an arbitrary finite bipartite graph with $|T \cap R_i| \geq 2$ for $i = l$ and r. Then it will be sufficient to show that $g \upharpoonright T \in \mathfrak{F}(S_X(\Gamma)^*)$. To achieve this, we adjust g repeatedly via composition with elements of $S_X(\Gamma)^*$ until we eventually obtain an element $h \in \mathfrak{F}(S_{\{l,r\}}(\Gamma))$ such that $h \upharpoonright T$ is an isomorphism. Our strategy is based upon the following lemma.

Lemma 5.3 *Suppose that $h \in \mathfrak{F}(S_{\{l,r\}}(\Gamma))$ and that U, $T \subset \mathrm{dom}(h)$ are two disjoint bipartite subgraphs such that for every cross-edge E in $(T \cup U)\backslash T$, $h \upharpoonright E$ is an isomorphism. Then $h \upharpoonright T$ is an isomorphism.*

Proof We prove this by contradiction. Suppose $h \upharpoonright T$ is not an isomorphism; then there exists a cross-edge $A \in [T]^2$ such that $h \upharpoonright A$ is not an isomorphism. Let W be a (2×2)-subgraph of $T \cup U$ such that $W \cap T = A$. By assumption, $h \upharpoonright E$ is an isomorphism for every cross-edge $E \in [W]^2\backslash A$. Thus h does not preserve the parity of the cross-types on the (2×2)-subgraph W, which contradicts Lemma 2.2. This completes the proof of Lemma 5.3. □

We shall make use of the following property of X.

Lemma 5.4 *Let X be the largest subset of $\{l, r\}$ such that $S_X(\Gamma)^* \subseteq G$. There exists a nonempty finite bipartite subgraph H of Γ satisfying the following.*

For any $i \in \{l, r\}$, if there exists some vertex $v_i \in H \cap R_i$ and $g \in G$ such that $g \upharpoonright H$ is a switch w.r.t. v_i, then $i \in X$.

Proof We prove the equivalent statement: there exists a nonempty finite bipartite subgraph H of Γ satisfying the following. If $i \in \{l, r\}$ and $i \notin X$, then for every $v_i \in H \cap R_i$ and every $g \in G$, $g \upharpoonright H$ is not a switch w.r.t. v_i.

Since $i \in \{l, r\}$ and $i \notin X$, there exists a map f which is a switch with respect to some vertex $a_i \in R_i$, but not in G. Otherwise the closed group generated by $\mathrm{Aut}(\Gamma)$ and f is $S_{\{i\}}(\Gamma)$, and so $S_{\{i\}}(\Gamma) = S_{\{i\}}(\Gamma)^*$ is a subgroup of G, a contradiction with the definition of X. Then $f \notin G$ implies that for every $g \in G$, g is not a switch with respect to a_i. So there exists a finite set $A \subseteq \Gamma$ containing a_i such that for every $g \in G$, $g \upharpoonright A$ is not a switch with respect to a_i.

Since Γ has the extension property, the following holds.

For every vertex $v_i \in R_i$, there exists a bipartite graph $A' \subseteq \Gamma$ containing v_i which is isomorphic to A mapping v_i to a_i. This can be expressed by the first-order sentence σ_i. If σ is the sentence $\bigwedge_{i \notin X} \sigma_i$, then $\Gamma \models \sigma$. Hence by Theorem 3.2, there exists a nonempty finite bipartite H of Γ such that $H \models \sigma$. This H satisfies our requirement, which completes the proof of Lemma 5.4. □

We shall also make use of a combinatorial theorem of Nešetřil and Rödl, which is a generalization of Ramsey's theorem. The following formulation, convenient for our use, is due to Abramson and Harrington [1].

Definition 5.5 (see [10]) A *system of colors of length* n, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-sequence of finite nonempty sets. An α*-colored set* consists of a finite ordered set X and a function $\tau : [X]^{\leq n} \longrightarrow \alpha_1 \cup \cdots \cup \alpha_n$ such that $\tau(A) \in \alpha_k$ for each $A \in [X]^k$ where $1 \leq k \leq n$. For each $A \in [X]^{\leq n}$, $\tau(A)$ is called the *color* of A. An α*-pattern* is an α-colored set whose underlying ordered set is an integer.

Theorem 5.6 (see Abramson and Harrington [1]) *Given* n, e, $M \in \mathbb{N}$, *a system* α *of colors of length* n *and an* α*-pattern* P*, there exists an* α*-pattern* Q *with the following property. For any* α*-colored set* (X, τ) *with* α*-pattern* Q *and for any function* $F : [X]^e \longrightarrow M$*, there exists* $Y \subseteq X$ *such that* $(Y, \tau \restriction Y)$ *has an* α*-pattern* P *and such that for any* $A \in [Y]^e$*,* $F(A)$ *depends only on the* α*-pattern of* $(A, \tau \restriction A)$*. (We say that such a* Y *is* F-homogeneous.*)*

Proof of Theorem 5.1 Let X be the largest subset of $\{l, r\}$ such that $S_X(\Gamma)^* \subseteq G$. Suppose $g \in G$, and let $T \subseteq \Gamma$ be finite with $|T \cap R_l| > 2$ and $|T \cap R_r| > 2$. By Lemma 5.2, it is enough to show now that $g \restriction T \in \mathfrak{F}(S_X(\Gamma)^*)$. The proof of Theorem 5.1 proceeds via a sequence of claims.

Fix an ordering $\prec$ of vertices in Γ such that T is an initial segment of this ordering of Γ. For a suitable system of colors α, we define an α-coloring τ of $[\Gamma \backslash T]^{\leq 2}$ by setting $\tau(A) = \tau(B)$ if and only if $|A| = |B|$ and the order-preserving bijection $T \cup A \longrightarrow T \cup B$ is an isomorphism.

Now we define the partition function $F_g : [\Gamma \backslash T]^2 \longrightarrow 2$ such that for $E \in [\Gamma \backslash T]^2$,

- $F_g(E) = 1$ if $E \in [R_i]^2$ for $i = 1, 2$; or if $E \in R_l \times R_r$ with $g \restriction E$ is an isomorphism;
- $F_g(E) = 0$ otherwise.

Let H be the finite bipartite graph given by Lemma 5.4, and let $m = |H \cap R_l|$, $n = |H \cap R_r|$. Since Γ satisfies the extension properties, the following conditions hold:

(a) $|\Gamma \cap R_i| \geq s(m, n) + |T|$ for $i = l$ and r, where $s(m, n)$ is as in Lemma 4.2;
(b) Γ contains all different copies of (2×2)-graphs, each connecting to T in all possible ways;
(c) Γ contains isomorphic copies of an $(m \times n)$-subgraph H connecting to T in all possible ways;
(d) for every $v \in T$, there exists a finite bipartite subgraph $V \subseteq (\Gamma \backslash T) \cup \{v\}$ containing v such that V is isomorphic to the $(m \times n)$-subgraph H.

Since Γ has the extension property, there exists a finite subgraph $U \subset \Gamma \backslash T$ such that the conditions (a)–(d) hold in U. Now let the α-pattern P be the one derived from $(U, \tau \restriction U)$. By Theorem 5.6 there exists $U' \subset \Gamma \backslash T$ such that U' is F_g-homogeneous and has the α-pattern P. Thus $T \cup U'$ is isomorphic to $T \cup U$ sending T to T. Without loss of generality, we assume $U = U'$ in the rest of this section. Now we will use the following claims.

Claim A Suppose that X_1, $X_2 \subseteq U$ and that $|X_1 \cap R_i| = |X_2 \cap R_i|$ for $i = l$ and r. Let $\varphi : T \cup X_1 \longrightarrow T \cup X_2$ be an order-preserving bijection such that $\varphi \restriction E$

is an isomorphism for all $E \in [T \cup X_1]^2 \backslash [X_1]^2$. Then for all $E \in [X_1]^2$, $g \restriction E$ is an isomorphism if and only if $g \restriction \varphi(E)$ is an isomorphism.

Proof We prove this by contradiction. We may assume that there exists some $E \in [X_1]^2$ such that $g \restriction E$ is an isomorphism while $g \restriction \varphi[E]$ is not. Since U satisfies condition (b), there exist (2×2)-subgraphs $V, W \subset U$ and $F \in [V]^2$, $F' \in [W]^2$ with $\tau(E) = \tau(F)$ and $\tau(\varphi[E]) = \tau(F')$ satisfying the following condition.

There exists an order-preserving bijection $\alpha : T \cup V \longrightarrow T \cup W$ mapping F to F' such that for every $A \in [T \cup V]^2 \backslash F$, $\alpha \restriction A$ is an isomorphism.

In particular, $\tau(A) = \tau(\alpha(A))$ for all $A \in [V]^2 \backslash F$. Since U is F_g-homogeneous, it follows that for all $A \in [V]^2 \backslash F$, $g \restriction A$ is an isomorphism if and only if $g \restriction \alpha(A)$ is an isomorphism. Since $\tau(E) = \tau(F)$ and $\tau(\varphi[E]) = \tau(F')$, we have $g \restriction F$ is an isomorphism but $g \restriction F'$ is not an isomorphism. Let $P = |\{A \in [V]^2 \mid g \restriction A$ is not an isomorphism$\}|$, and let $Q = |\{A \in [W]^2 \mid g \restriction A$ is not an isomorphism$\}|$. Then $Q = P + 1$ because of the effect of g on F and F'. But by Lemma 2.2, $g \in S_{\{l,r\}}(\Gamma)$ implies that g preserves the parity of cross-types in V and W. Thus P and Q must be even, which contradicts $Q = P + 1$. This completes the proof of Claim A. □

Claim B We have $g \restriction U \in \mathfrak{F}(S_X(\Gamma)^*)$.

Proof Since U satisfies condition (a), by Theorem 4.2 there exists an $(m \times n)$-analysis of $g \restriction U$: $g_0, g_1, \ldots, g_t \in \mathfrak{F}(S_{\{l,r\}}(\Gamma))$. That is, for each $0 \leq j \leq t-1$, there exists a finite $(m\times n)$-subgraph Y_j in U and an element $\theta_j \in S_{\{l,r\}}(\Gamma)$ such that

(1) $g_0 = \theta \circ g \restriction U$ where $\theta \in \mathfrak{F}(\mathrm{Aut}(\Gamma)^*)$;
(2) θ_j is either an isomorphism or a switch with respect to some vertex $a_j \in Y_j \cap R_{i_j}$ where $i_j \in \{l, r\}$;
(3) $\theta_j \restriction Y_j = (g_j \circ g_{j-1} \circ \cdots \circ g_0) \restriction Y_j$;
(4) $g_{j+1} = \theta_j^{-1} \restriction \mathrm{ran}(g_j \circ \cdots \circ g_0)$;
(5) $(g_t \circ \cdots \circ g_0) : U \longrightarrow \Gamma$ is an isomorphic embedding.

If all $\{i_0, \ldots, i_{t-1}\} \subseteq X$, then $g_0 \restriction U \in \mathfrak{F}(S_X(\Gamma)^*)$, and so $g \restriction U \in \mathfrak{F}(S_X(\Gamma)^*)$. Otherwise, let j be the least integer such that $i_j \notin X$ and the corresponding θ_j is a switch with respect to $a_j \in R_{i_j} \cap Y_j$. Note $\theta_0, \ldots, \theta_{j-1} \in S_X(\Gamma)^*$, which implies $g_1, \ldots, g_j \in \mathfrak{F}(S_X(\Gamma)^*)$. We prove that this situation cannot occur. Note that $(g_j \circ \cdots \circ g_0) \restriction Y_j = \theta_j \restriction Y_j$ is a switch with respect to a vertex $a_j \in R_{i_j} \cap Y_j$.

Since U satisfies condition (c), there exist an $(m \times n)$-subgraph $H' \subseteq U$ which is an isomorphic copy of H, and a map φ satisfying that $\varphi : T \cup Y_j \longrightarrow T \cup H'$ is an order-preserving bijection such that $\varphi \restriction E$ is an isomorphism for all $E \in [T \cup Y_j]^2 \backslash [Y_j]^2$.

By Claim A, for every $E \in [Y_j]^2$, $g \restriction E$ is an isomorphism if and only if $g \restriction \varphi[E]$ is an isomorphism. Next we will show there exist $g_1^*, \ldots, g_j^* \in \mathfrak{F}(S_X(\Gamma)^*)$ such that $g_j^* \circ \cdots \circ g_1^* \circ g_0 \restriction H'$ is a switch with respect to $\varphi(a_j)$ of H' in R_{i_j}. But then Lemma 5.4 implies that $i_j \in X$, contrary to our assumption. We define g_l^* inductively for $1 \leq l \leq j$ such that for all $E \in [Y_j]^2, g_l \circ \cdots \circ g_0 \restriction E$ is an isomorphism if and only if $g_l^* \circ \cdots \circ g_1^* \circ g_0 \restriction \varphi[E]$ is an isomorphism.

Suppose $g_1^*, \ldots, g_{l-1}^*$ have been defined; we now define g_l^* for $1 \leq l \leq j$.

(a) If θ_{l-1} is an isomorphism, or if θ_{l-1} is a switch w.r.t. $a_{l-1} \in R_{i_{l-1}}$ but $a_{l-1} \notin Y_j$, then g_l is an isomorphism on $g_{l-1} \circ \cdots \circ g_0[Y_j]$, which is in $\mathfrak{F}(S_X(\Gamma))$. We define g_l^* as the identity map on $\operatorname{ran}(g_{l-1}^* \circ \cdots \circ g_1^* \circ g_0)$.

(b) Otherwise, θ_{l-1} is a switch w.r.t. $a_{l-1} \in R_{i_{l-1}}$ and $a_{l-1} \in Y_j$; then g_l is a switch with respect to $g_{l-1} \circ \cdots \circ g_0(a_{l-1}) \in R_{i_{l-1}} \cap g_{l-1} \circ \cdots \circ g_0[Y_j]$. Then $g_l \in \mathfrak{F}(S_X(\Gamma))$. Let $\theta^* \in S_X(\Gamma)$ be a switch with respect to $g_{l-1}^* \circ \cdots \circ g_1^* \circ g_0(\varphi(a_{l-1}))$, and define g_l^* as $\theta^* \restriction \operatorname{ran}(g_{l-1}^* \circ \cdots \circ g_1^* \circ g_0)$.

This completes the proof of Claim B. □

Now choose $\psi_0 \in S_X(\Gamma)^*$ such that $\psi_0 \restriction U = g \restriction U$, and let $h_1 = \psi_0^{-1} \circ g \restriction T \cup U$. Then $h_1 \restriction E$ is the identity for every $E \in [U]^2$.

Next, we choose a vertex v_1 in T. Without loss of generality, we let $v_1 \in R_l$ and consider $h_1 \restriction U \cup \{v_1\}$. Notice that if $E \in [U \cup \{v_1\}]^2$ and $h_1 \restriction E$ is not an isomorphism, then $v_1 \in E$.

Claim C We have $h_1 \restriction U \cup \{v_1\} \in \mathfrak{F}(S_X(\Gamma)^*)$.

Proof Since $h_1 \restriction U = \mathrm{id}$ and $h_1 \in \mathfrak{F}(S_{\{l,r\}}(\Gamma))$, by Lemma 2.2, h_1 preserves the parity of cross-types in every (2×2)-subgraph of $U \cup \{v_1\}$. So $h_1 \restriction U \cup \{v_1\}$ is either an isomorphism or a switch with respect to v_1. We may assume $h_1 \restriction U \cup \{v_1\}$ is a switch with respect to v_1. Then there exists a switch $\psi_1 \in S_{\{l\}}(\Gamma)$ such that $h_1 \restriction U \cup \{v_1\} = \psi_1 \restriction U \cup \{v_1\}$, and for all $E \in [T \cup U]^2$ with $v_1 \notin E$, $\psi_1 \restriction E$ is an isomorphism.

If $l \in X$, then $\psi_1 \in S_X(\Gamma)$ and so $\psi_1 \in S_X(\Gamma)^*$; then we are done. Otherwise, we show that there will be contradiction. Since U satisfies condition (d), there exists an $(m \times n)$-subgraph V in $U \cup \{v\}$ such that $v \in V$ and $V \simeq H$. Then $h_1 \restriction V$ is a switch with respect to $v_1 \in R_l$. By Lemma 5.4, we have $l \in X$, a contradiction with our assumption. This completes the proof of Claim C. □

By Claim C, there exists $\psi_1 \in S_X(\Gamma)^*$ that is either an isomorphism or a switch w.r.t. $v_1 \in R_i$ for $i \in X$ such that

(a) $\psi_1 \restriction U \cup \{v_1\} = h_1 \restriction U \cup \{v_1\}$;

(b) for all $E \in [T \cup U]^2$, if $v_1 \notin E$, then $\psi_1 \restriction E$ is an isomorphism.

Let $h_2 = \psi_1^{-1} \circ h_1 \restriction T \cup U$; then for all $E \in [T \cup \{v_1\}]^2$, $h_2 \restriction E$ is an isomorphism.

Now choose a second vertex $v_2 \in T \backslash \{v_1\}$. Arguing similarly as in Claim C, there exists $\psi_2 \in S_X(\Gamma)^*$ which is either an isomorphism or a switch w.r.t. $v_2 \in R_i$ for $i \in X$ such that

(a) $\psi_2 \restriction U \cup \{v_2\} = h_2 \restriction U \cup \{v_2\}$;

(b) for all $E \in [T \cup U]^2$, if $v_2 \notin E$, then $\psi_2 \restriction E$ is an isomorphism.

Note that such ψ_2 is an isomorphism for all the cross-edges E such that $E \subseteq U$ or $E \cap T = \{v_1\}$. Thus when we next adjust h_2 to $h_3 = \psi_2^{-1} \circ h_2 \restriction T \cup U$, we do not spoil the progress which we make with our earlier adjustments. Hence for all $E \in [T \cup \{v_1, v_2\}]^2 \backslash \{v_1, v_2\}$, $h_3 \restriction E$ is an isomorphism.

By continuing in this fashion, we can deal with the other vertices in $T \backslash \{v_1, v_2\}$. After $|T|$-1 steps, we obtain a map $h^* : T \cup U \longrightarrow T \cup U$ such that

(a) there exists $\psi^* \in S_X(\Gamma)^*$ such that $h^* = \psi^* \circ g \restriction T \cup U$;

(b) for all $E \in [T \cup U]^2 \backslash [T]^2$, $h^* \restriction E$ is an isomorphism.

Now Lemma 5.3 implies $h^* \restriction T$ is an isomorphism; hence $g \restriction T = {\psi^*}^{-1} \circ h^* \restriction T \in \mathfrak{F}(S_X(\Gamma)^*)$. This completes the proof of Theorem 5.1. □

6 Some Special Finite Subgraphs of Γ

In the rest of paper, we express Γ as a union of an increasing chain of substructures Γ_i as mentioned in Theorem 3.2. That is, $\Gamma = \bigcup_{i \in \mathbb{N}} \Gamma_i$ where $\Gamma_i \subset \Gamma_{i+1}$ and $|\Gamma_i| = i$ for each $i \in \mathbb{N}$. In particular, if i is even, then $|\Gamma_i \cap R_l| = |\Gamma_i \cap R_r|$; otherwise, $|\Gamma_i \cap R_l| = |\Gamma_i \cap R_r| + 1$. In this section we show the existence of some special finite bipartite subgraphs Γ_{N_G} and Z. We will use the following two lemmas, each of which witnesses the fact that G is a nontrivial reduct.

Lemma 6.1 *Let G be a proper closed subgroup of* $\mathrm{Sym}_{\{l,r\}}(\Gamma)$. *There exists a finite bipartite subgraph B_0 of Γ such that for every $g \in G$, there exist cross-edges E_1, E_2 in B_0 such that $P_1(g[E_1])$ and $P_2(g[E_2])$.*

Proof Suppose no such B_0 exists; then for every finite bipartite subgraph B of Γ, there exists some $g \in G$ such that either $P_1(g[E])$ for every cross-edge E in B, or $P_2(g[E])$ for every cross-edge E in B.

Express $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$ as a union of an increasing chain of finite bipartite subgraphs Γ_n. There exists an infinite subset I of $\mathbb{N}$ such that either for every $n \in I$, there is $g_n \in G$ such that $P_1(g_n[E])$ for every cross-edge E in Γ_n; or for every $n \in I$, there is $g_n \in G$ such that $P_2(g_n[E])$ for every cross-edge E in Γ_n.

We may assume the first situation holds. For any $(m \times n)$-subgraph $C \subset \Gamma$ where $m, n \in \mathbb{N}$, there exists $N \in I$ such that $C \subseteq \Gamma_N$. Hence there exists some $g_c \in G$ such that $P_1(g_c[E])$ for every cross-edge E in C. Then for any two $(m \times n)$-subgraphs A, B of Γ, we can find $\sigma \in \mathrm{Aut}(\Gamma)$ sending $g_A[A]$ to $g_B[B]$. Then the map $f = g_B^{-1} \circ \sigma \circ g_A \in G$, and f takes A to B. But A and B are arbitrary $(m \times n)$-subgraphs of Γ, and so such f's generate all of $\mathrm{Sym}_{\{l,r\}}(\Gamma)$, a contradiction with the fact that G is a proper subgroup of $\mathrm{Sym}_{\{l,r\}}(\Gamma)$. This completes the proof of Lemma 6.1. □

Lemma 6.2 *Let $i \in \{l, r\}$ and $j \in \{1, 2\}$, and let G be as above. There exists a nonempty finite bipartite subgraph B^i_j of Γ satisfying the following property for every $g \in G$:*

(†) *No vertex $v \in B^i_j \cap R_i$ has the property that for every cross-edge E in B^i_j, $\neg P_j(g[E])$ if and only if $P_j(E)$ and $v \in E$.*

Proof Fix i and j. Let $m = |B_0 \cap R_l|$ and $n = |B_0 \cap R_r|$ for B_0 in Lemma 6.1. We prove this by contradiction. Suppose there is no nonempty finite bipartite graph satisfying the property (†) for every $g \in G$. Then B_0 does not satisfy the property (†) for all $g \in G$, and then there exist some $g_0 \in G$ and $v_0 \in B_0$ such that g_0 preserves the cross-types on all the cross-edges in B_0 except those cross-edges E where $P_j(E)$ and $v_0 \in E$. Now compared with B_0, $g_0[B_0]$ has fewer cross-edges with P_j holding on them. Note that $g_0[B_0]$ is finite, so it does not satisfy the property (†) by assumption. Similarly, we can find g_1 and $v_1 \in g_0[B_0]$ witnessing this failure, and such that $g_1 g_0[B_0]$ has even fewer cross-edges with P_j. Thus we can find a sequence of elements of G successively reducing the number of instances of P_j, and finally we get their composite g which, when applied to B_0, has eliminated all instances of P_j. But this contradicts the property of B_0 in Lemma 6.1. Thus some $(m \times n)$-subgraph must satisfy the requirement for B^i_j. □

Note that the following graphs exist in Γ:

(a) the finite bipartite subgraph B_0 as in Lemma 6.1;

(b) the finite bipartite subgraph B_i^j for $i \in \{l, r\}$ and $j \in \{1, 2\}$ as in Lemma 6.2.
Then it follows that there exists $N_G \in \mathbb{N}$ such that Γ_{N_G} contains subgraphs (a) and (b).

In the rest of the section, we will prove the existence of a finite bipartite graph $Z \subset \Gamma$ which contains an isomorphic copy of B_0 and also has the properties that every $f \in G$ either preserves or interchanges cross-types on Z.

Theorem 6.3 *Let G be a proper closed subgroup of $\mathrm{Sym}_{\{l,r\}}(\Gamma)$. There exists a finite bipartite subgraph $Z \subset \Gamma$ containing an isomorphic copy of B_0 such that for every $f \in G$ and every cross-edge E in Z, either $P_i(E)$ implies $P_i(f[E])$, or $P_i(E)$ implies $\neg P_i(f[E])$, where $i = 1$ and 2. That is, f either preserves or interchanges cross-types on Z.*

Proof Fix an ordering of the vertices of Γ. For a suitable system of colors α, define an α-coloring χ of $[\Gamma]^{\leq 2}$ by setting $\chi(A) = \chi(B)$ if and only if $A, B \in [\Gamma]^{\leq 2}$ and the bijection $A \to B$ is an isomorphism.

Let P be the α-pattern such that if U is a finite bipartite U of Γ and $(U, \chi \upharpoonright U)$ has an α-pattern P, then $(U, \chi \upharpoonright U) \cong \Gamma_{N_G}$. By Theorem 5.6 there exists an α-pattern Q such that for any α-colored set $(X, \chi \upharpoonright X)$ with α-pattern Q and for any partition $F : [X]^2 \longrightarrow 2$, there exists Z of X such that Z has the α-pattern P; hence $Z \cong \Gamma_{N_G}$, and $(Z, \chi \upharpoonright Z)$ is F-homogeneous.

We define a particular partition $F : [X]^2 \longrightarrow 2$ such that for every $E \in [X]^2$,

- $F(E) = 1$ if $E \in [R_i]^2$ for $i = l, r$, or if E is a cross-edge and f preserves P_j on E for $j = 1, 2$;
- $F(E) = 0$ otherwise.

Then one of the following conditions must hold in Z for every cross-edge E where $i = 1, 2$:

(1) $P_i(E)$ implies $P_i(f[E])$;
(2) $P_i(E)$ implies $\neg P_i(f[E])$;
(3) $P_1(f[E])$;
(4) $P_2(f[E])$.

Note that $Z \cong \Gamma_{N_G}$, which contains B_0. This guarantees that only (1) or (2) holds for Z, as desired. This completes the proof of Theorem 6.3. □

7 The Closed Groups between $S_{\{l,r\}}(\Gamma)$ and $\mathrm{Sym}_{\{l,r\}}(\Gamma)$

In this section, we will prove the following theorem.

Theorem 7.1 *If G is a closed subgroup such that $\mathrm{Aut}(\Gamma)^* \leq G < \mathrm{Sym}_{\{l,r\}}(\Gamma)$, then $G \leq S_{\{l,r\}}(\Gamma)$.*

For the rest of this section, we fix G as a closed subgroup such that $\mathrm{Aut}(\Gamma)^* \leq G < \mathrm{Sym}_{\{l,r\}}(\Gamma)$. Let X be the largest subset of $\{l, r\}$ such that $S_X(\Gamma)^* \subseteq G$; and so X is also the largest subset of $\{l, r\}$ such that $S_X(\Gamma)^* \subseteq G \cap S_{\{l,r\}}(\Gamma)$. Note that $G \cap S_{\{l,r\}}(\Gamma)$ is a closed subgroup of $S_{\{l,r\}}(\Gamma)$ containing $\mathrm{Aut}(\Gamma)^*$, then by Theorem 5.1, $G \cap S_{\{l,r\}}(\Gamma) = S_X(\Gamma)^*$.

Proof We prove this by contradiction. Assume G is a closed subgroup with $\mathrm{Aut}(\Gamma) \leq G < \mathrm{Sym}_{\{l,r\}}(\Gamma)$ but $G \not\leq S_{\{l,r\}}(\Gamma)$. Then there exist a map $f \in G \backslash S_{\{l,r\}}(\Gamma)$ and a (2×2)-subgraph Y of Γ such that $f \upharpoonright Y$ does not preserve the parity of cross-types in Y. Let $Z \subset \Gamma$ be the finite bipartite subgraph as in Theorem 6.3. Since Γ is homogeneous, there is $\varphi \in \mathrm{Aut}(\Gamma)$ such that $\varphi(Z) = \Gamma_{N_G}$. Then there exists $s \in \mathbb{N}$ such that $\varphi(Y \cup Z) \subseteq \Gamma_s$. Let $M = \varphi^{-1}[\Gamma_s]$. Then $Y \cup Z \subseteq M$, and $\tau = \varphi \upharpoonright M$ is an isomorphism from M onto Γ_s with $\tau[Z] = \Gamma_{N_G}$.

For any m with $N_G \leq m \leq s$, let $Z_m = \tau^{-1}[\Gamma_m]$ (note that $Z_{N_G} = Z$). By Theorem 6.3, $f \upharpoonright Z_{N_G} \in \mathfrak{F}(S_{\{l,r\}}(\Gamma))$. Let a be the greatest integer such that $N_G \leq a \leq s$ and $f \upharpoonright Z_a \in \mathfrak{F}(S_{\{l,r\}}(\Gamma))$. By the definition of a, Theorem 5.1 implies that there exists a map $\theta \in S_X(\Gamma)^*$ such that $f \upharpoonright Z_a = \theta \upharpoonright Z_a$. The existence of $Y \subseteq M$ ensures that $a < s$. Suppose $Z_{a+1} = Z_a \cup \{v\}$. Without loss of generality, let $v \in R_l$. We let $f_1 = (\theta^{-1} \circ f \circ \tau^{-1}) \upharpoonright \Gamma_{a+1}$ and $w = \tau(v)$. By the maximality of a, $f \upharpoonright Z_{a+1} \notin \mathfrak{F}(S_{\{l,r\}}(\Gamma))$. Thus $f_1 \in \mathfrak{F}(G)\backslash\mathfrak{F}(S_{\{l,r\}}(\Gamma))$.

Fix an ordering $\prec$ of Γ_{a+1} such that w is the initial element. For a suitable system of colors α, define an α-coloring η of $[\Gamma\backslash\{w\}]^{\leq 2}$ by setting $\eta(A) = \eta(B)$ if and only if the order-preserving bijection $\{w\} \cup A \longrightarrow \{w\} \cup B$ is an isomorphism.

Let the α-pattern P be such that if $(S, \eta \upharpoonright S)$ has an α-pattern P, then $S \cup \{w\} \simeq \Gamma_{a+1}$. By Theorem 5.6 there exists a finite bipartite graph $Q \subseteq \Gamma\backslash\{w\}$ such that for any partition $F : [Q]^2 \longrightarrow 2$, there exists V of Q such that there exists an isomorphism $\sigma : V \cup \{w\} \longrightarrow \Gamma_{a+1}$ sending w to w. Furthermore, $(V, \eta \upharpoonright V)$ is F-homogeneous. Now we define the partition function $F : Q \longrightarrow 2$ for every $a \in Q$:

- $F(a) = 1$ if $a \in R_r$ and $f_1 \upharpoonright (w, a)$ is an anti-isomorphism;
- $F(a) = 0$ if $a \in R_l$, or $a \in R_r$ with $f_1 \upharpoonright (w, a)$ is an isomorphism.

Let $U = V \cup \{w\}$. Then one of the following conditions must hold on U:

(a) $f_1 \circ \sigma$ is an isomorphism;
(b) $f_1 \circ \sigma$ is a switch with respect to w;
(c) for all $E \in [U]^2$, $f_1 \circ \sigma \upharpoonright E$ is not an isomorphism if and only if $P_2(E)$ and $w \in E$;
(d) for all $E \in [U]^2$, $f_1 \circ \sigma \upharpoonright E$ is not an isomorphism if and only if $P_1(E)$ and $w \in E$.

Note that $U \cong \Gamma_{a+1}$ and $\Gamma_{a+1} \supseteq \Gamma_{N_G}$, and that Γ_{N_G} contains an isomorphic copy of B_1^l, B_2^l, so U contains isomorphic copies of B_1^l and of B_2^l, which fail to obey conditions (3) and (4). Thus only condition (1) or (2) holds in U, which implies that $f_1 \circ \sigma \upharpoonright U \in \mathfrak{F}(S_{\{l,r\}}(\Gamma))$, and so $f_1 \in \mathfrak{F}(S_{\{l,r\}}(\Gamma))$. This contradicts the fact that $f_1 \notin \mathfrak{F}(S_{\{l,r\}}(\Gamma))$. This completes the proof of Theorem 7.1. □

The result of Theorem 7.1, together with Theorem 5.1, completes our proof of the main result.

Proof of Theorem 1.6 Let G be a closed subgroup with $\mathrm{Aut}(\Gamma)^* \leq G < \mathrm{Sym}_{\{l,r\}}(\Gamma)$. Then by Theorem 7.1, $G \leq S_{\{l,r\}}(\Gamma)$. Using the result of Theorem 5.1, we have $G = S_X(\Gamma)^*$ for some subset $X \subseteq \{l, r\}$. This completes the proof of Theorem 1.6. □

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