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New Consecution Calculi for $R^{t}_{\to}$

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Abstract The implicational fragment of the logic of relevant implication, $R_{\to}$ is one of the oldest relevance logics and in 1959 was shown by Kripke to be decidable. The proof is based on $LR_{\to}$, a Gentzen-style calculus. In this paper, we add the truth constant $\boldsymbol{t}$ to $LR_{\to}$, but more importantly we show how to reshape the sequent calculus as a *consecution calculus* containing a binary structural connective, in which permutation is replaced by two structural rules that involve $\boldsymbol{t}$. This calculus, $LT^{\textcircled{t}}_{\to}$, extends the consecution calculus $LT^{\boldsymbol{t}}_{\to}$ formalizing the *implicational fragment of ticket entailment*. We introduce two other new calculi as alternative formulations of $R^{\boldsymbol{t}}_{\to}$. For each new calculus, we prove the *cut theorem* as well as the equivalence to the original Hilbert-style axiomatization of $R^{\boldsymbol{t}}_{\to}$. These results serve as a basis for our positive solution to the long open problem of the decidability of $T_{\to}$, which we present in another paper.

1 Introduction

In this paper we present some new consecution calculi including $LR^{\boldsymbol{t}}_{\to;}$. What part of $LR^{\boldsymbol{t}}_{\to;}$ don't you understand? In this introduction we will "break down" $LR^{\boldsymbol{t}}_{\to;}$ into its component parts, and by this, we give some relevant historical and conceptual background for our paper (which is otherwise somewhat technical).

1.1 The system R of relevant implication Anderson and Belnap [3] is the seminal source for various relevance logics formalized by themselves and others. It focuses on their system E of *entailment*, which according to them was initiated in large part by Ackermann [1]. But on [3, p. xxiii], they do say good things about the system R of *relevant implication*, and they also present and motivate the system T of *ticket entailment* that was introduced in Anderson [2]:

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> The book is intended to be "encyclopedic," in the modest sense that we have tried to tell the reader everything that is known (at present writing) about the family of systems of logic that grew out of Ackermann's 1956 paper. But there are still many entertaining open questions, chief of which are the decision problems for **R** and **E** (and perhaps **T**), which have proved to be especially recalcitrant.

Urquhart [39] proved that all of the above systems are undecidable, which is somewhat surprising given that early work by Kripke [27] had shown that the implicational fragment of E and R are decidable. To our knowledge, the same question for the implicational fragment of T, the last important open problem in relevance logic, has remained unanswered until recently. We have solved the decidability problem of $T_{\to}$, relying on the consecution calculus $LT^{\textcircled{t}}_{\to}$, which we introduce in this paper.[1]

1.2 The pure implicational relevance logic $R_{\to}$ This system was developed independently by Moh [35] and Church [15], both of whom motivated it by a version of the *deduction theorem* that says in effect if $\mathcal{B}$ is deducible from the premises $\mathcal{A}_1, \ldots, \mathcal{A}_{n-1}, \mathcal{A}_n$, then $\mathcal{A}_n \to \mathcal{B}$ is deducible from $\mathcal{A}_1, \ldots, \mathcal{A}_{n-1}$ unless $\mathcal{B}$ is deducible from $\mathcal{A}_1, \ldots, \mathcal{A}_{n-1}$, that is, in effect the presence of $\mathcal{A}_n$ was irrelevant to the deducibility of $\mathcal{B}$.[2]

The (slightly different) axiomatizations of Moh and Church were later discovered by Anderson and Belnap to be equivalent to each other and to determine the putative pure implicational fragment of their relevance logic R (see [3, Section 3]; see also [3, Section 28.3] for R. K. Meyer's proof that these axiomatizations actually give the pure implicational fragment). As Anderson and Belnap say on [3, p. 349]:

> The pure implication fragment $\mathbf{R}_{\to}$ of **R** is the oldest of the relevance logics, having been formulated independently in Moh 1950 and Church 1951; both papers contain appropriate deduction theorems, as remarked in §3. Neither Moh nor Church considered the possibility of obtaining **R** by adding axioms for truth functions to $\mathbf{R}_{\to}$ in the straightforward way suggested by Ackermann's addition of truth functional axioms to $\mathbf{E}_{\to}$. However, the heart of relevance in **R** lies in the aged Moh–Church implicational fragment, which is one reason why **R** deserves at least respect if not outright veneration.[3]

1.3 Gentzen's prefix L Gentzen [25] developed new formal calculi for the classical and intuitionistic logics, which he denoted, respectively, by LK and LJ. As it has customarily been presented, a *sequent* in LK (or as [3] calls it, using a word from Tryg Ager, a *consecution*) is an ordered pair consisting of two finite sequences of formulas $\mathcal{A}_1, \ldots, \mathcal{A}_m$ (the antecedent) and $\mathcal{B}_1, \ldots, \mathcal{B}_n$ (the succedent), which we write as $\mathcal{A}_1, \ldots, \mathcal{A}_m \vdash \mathcal{B}_1, \ldots, \mathcal{B}_n$. Gentzen had *operational rules* for each connective. These rules introduce a formula into the antecedent or succedent with the main connective of the formula being prescribed by the rule. Gentzen also had *structural rules* for permuting formulas in the antecedent, contracting multiple occurrences of a formula, and inserting ("thinning in") an occurrence of a formula (which has as a special case duplicating an occurrence of a formula). Gentzen's system LJ is essentially the same except for one crucial difference; the succedent must consist of a single formula or else be empty. It is well known that if one is interested in just the pure implicational fragment $LJ_{\to}$ one can drop the option that the succedent might be empty, as we shall do with the systems we develop here.

1.4 The semicolon and grouping In this paper we develop a number of Gentzen-style calculi for the implicational fragment of $R_{\rightarrow}$; one of them is $LR^{\boldsymbol{t}}_{\rightarrow;}$. We now say a few words about ;. We take ; as a *structural connective*, and moreover, we take it as forming ordered pairs, which we call *structures*. It is much more standard to use a structural connective—as did Gentzen—to form *sequences*. We use the term "consecution calculus" when ; is a structural connective and reserve "sequent calculus" for those systems in which , is used. This paper can be viewed as being about alternative ways to view sequents as constructed from finite multiplicities. Because of his structural rules, it is easy to see that Gentzen did not really need to have a sequence of formulas but could have instead just had a set of formulas (cf. Curry [17]).

Can you see any difference between the following?

$$\mathcal{A}, \mathcal{B} \qquad\qquad \mathcal{B}, \mathcal{A} \qquad\qquad \mathcal{B}, \mathcal{B}, \mathcal{A}$$

If you cannot see any difference between the three, it must be because you are thinking of them as *sets*, and all that matters is that they are made up out of the elements $\mathcal{A}$ and $\mathcal{B}$; the order in which they are displayed and their number make no difference. If you saw no difference between the first two but saw that they differed from the third, you must have been thinking of them as *multisets*: the order does not matter but the number of occurrences does.[4] Lastly, if you saw a difference between all three it is because you were thinking of them as *finite sequences* (alternatively, ordered n-tuples), where the order and the multiplicity both make a difference.[5] It is customary to denote sets using braces, multisets using square brackets, and sequences using corner brackets:[6]

$$\{\mathcal{A}, \mathcal{B}\} = \{\mathcal{B}, \mathcal{A}\} = \{\mathcal{B}, \mathcal{B}, \mathcal{A}\},$$
$$[\mathcal{A}, \mathcal{B}] = [\mathcal{B}, \mathcal{A}] \neq [\mathcal{B}, \mathcal{B}, \mathcal{A}],$$
$$\langle \mathcal{A}, \mathcal{B}\rangle \neq \langle \mathcal{B}, \mathcal{A}\rangle \neq \langle \mathcal{B}, \mathcal{B}, \mathcal{A}\rangle.$$

To complicate matters further we can imagine the items grouped as pairs, according, say, to the order in which they were placed. Thus $\mathcal{A}, \mathcal{B}, \mathcal{C}$ might be viewed as either $(\mathcal{A}, (\mathcal{B}, \mathcal{C}))$ or $((\mathcal{A}, \mathcal{B}), \mathcal{C})$. We will use ; in place of , to indicate this grouping, with pairing done to the left unless otherwise indicated (e.g., we might write $\mathcal{A}; \mathcal{B}; \mathcal{C}$ for $((\mathcal{A}, \mathcal{B}), \mathcal{C})$, but we will write $\mathcal{A}; (\mathcal{B}; \mathcal{C})$ for $(\mathcal{A}, (\mathcal{B}, \mathcal{C}))$).

1.5 The sentential constant $\boldsymbol{t}$ Ackermann [1] had already introduced the related sentential constant $\curlywedge$ for "das Absurde," with the idea of then defining "necessary $\mathcal{A}$" as $\sim \mathcal{A} \rightarrow \curlywedge$, which is equivalent to $\sim \curlywedge \rightarrow \mathcal{A}$. Think of $\boldsymbol{t}$ as $\sim \curlywedge$. Anderson and Belnap show that the addition of $\curlywedge$ is conservative (i.e., no new theorems become provable in the original language without $\curlywedge$), because in any given proof $\curlywedge$ can be defined as the negation of the conjunction of all of the formulas $p \rightarrow p$, where p is an atomic sentence occurring in the proof. Dunn [20] made the obvious modifications to show that $\boldsymbol{t}$ can be conservatively added to R, the point being that in the Lindenbaum algebra of $R^{\boldsymbol{t}}$ the equivalence class $[\boldsymbol{t}]$ can be seen as an identity, which means that the Lindenbaum algebra of $R^{\boldsymbol{t}}$ is a *De Morgan monoid*. It turns out that $\boldsymbol{t}$ can serve R in many other useful ways. Meyer [29] showed how it can be used to give a translation of intuitionistic implication into R. An example relevant to this paper is that Dunn [21] introduced $\boldsymbol{t}$ into a sequent system LR^{+} for negation-free R, thus allowing one to avoid a sequent $\vdash \mathcal{A}$ with an empty left-hand side, replacing it instead with $\boldsymbol{t} \vdash \mathcal{A}$ (see [3, Section 28.5]). Meyer [30] provides a deep investigation of $\boldsymbol{t}$, and other sentential constants, in R, and other relevance logics.

1.6 How we put the pieces together In our paper, we develop a number of Gentzen-style calculi for the system $R^{\boldsymbol{t}}_{\rightarrow}$. But first, we recall $LR_{\rightarrow}$ from Dunn [22], which differs from the usual formulation only in using multisets in place of finite sequences, thus dispensing with the need for a rule of permutation. Then we mention $[LR_{\rightarrow}]$, which builds upon $LR_{\rightarrow}$ but follows Kripke by building contraction into an operational rule and thus bounding the number of contractions. We then consider a sequent system $LR^{\boldsymbol{t}}_{\rightarrow}$ obtained by adding the sentential constant $\boldsymbol{t}$ to $LR_{\rightarrow}$, and then again a variant, denoted by $[LR^{\boldsymbol{t}}_{\rightarrow}]$, which limits contractions.

To obtain a consecution system $LT^{\boldsymbol{t}}_{\rightarrow}$ for $T^{\boldsymbol{t}}_{\rightarrow}$ one has to replace the flat notion of a multiset with the notion of a *structure*, using ; as a binary structural connective, because $T^{\boldsymbol{t}}_{\rightarrow}$ does not permit unlimited permutation or a multiset view of structures. Then appropriate structural rules, which correspond to various combinators, have to be added together with some special rules involving $\boldsymbol{t}$ (see Bimbó [10]). We show how one can obtain another consecution calculus for $R^{\boldsymbol{t}}_{\rightarrow}$ by adding to $LT^{\boldsymbol{t}}_{\rightarrow}$ a rule corresponding to the combinator C. To emphasize that this is built on binary grouping, we use the notation $LR^{\boldsymbol{t}}_{\rightarrow;}$.

All the systems mentioned so far are built by incremental steps. But finally, we develop yet another consecution calculus, which we denote by $LT^{\textcircled{t}}_{\rightarrow}$. We do this following the insights that were implemented in Bimbó [9] in the definition of a consecution calculus for the implicational fragment of the system E. $LT^{\textcircled{t}}_{\rightarrow}$ has no explicit rule of permutation for arbitrary formulas or structures but instead has a special permutation rule that involves the constant $\boldsymbol{t}$. $LT^{\textcircled{t}}_{\rightarrow}$ also has a special instance of the rule of thinning, namely, thinning the constant $\boldsymbol{t}$ into the right-hand side of a structure. (Thinning $\boldsymbol{t}$ into the left-hand side of a structure is already a rule in $LT^{\boldsymbol{t}}_{\rightarrow}$.) It turns out that these rules together with the rules in $LT^{\boldsymbol{t}}_{\rightarrow}$ give the *effect* of permutation.[7]

2 A Sequent Calculus for the Implicational Fragment of R with t

The logic of relevant implication was created with the intention of preserving as much from classical logic as reasonably possible. One of the features of combining premises, which is rarely questioned, is the indifference toward the order of the presentation of the premises. In other words, the premise-combining operation—when thought to be a binary operation—is often taken to be *commutative*. In Gentzen's sequent calculi LK and LJ, the premise-building operation is , (on the left of the turnstile), which corresponds to $\wedge$ (conjunction). Both classical and intuitionistic logic contain a commutative $\wedge$. Our main concern in connection to $R_{\rightarrow}$ in this paper is *permutation*, and how the commutativity of the structural connective in consecution calculi for $R^{\boldsymbol{t}}_{\rightarrow}$ arises. Since $R_{\rightarrow}$ has contraction, in the first couple of sequent calculi we will view formulas as combined into multisets rather than forming sequences, sets, or ordered pairs.

There are calculi of various types for the implicational fragment of the logic of relevant implication, and we briefly overview some of them (see, e.g., [3], [22], or Dunn and Restall [24] for a natural deduction, for a merge, and for two sequent calculi).

The only connective in $R_{\to}$ is $\to$, which is binary and stands for *relevant implication*. There is a denumerable set of propositional variables in the language of $R_{\to}$, which we denote by $p_0, p_1, \dots$. The set of formulas is generated from that base set by finitely many applications of $\to$ to a pair of formulas. Our notation for arbitrary formulas is $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$. Parentheses are omitted from formulas according to the usual convention for simple types, that is, by association to the right. All the calculi we consider in detail here are *right singular*; that is, there is exactly one formula on the right-hand side of the $\vdash$ in a sequent or consecution.

First, we consider $LR_{\to}$, which is derived from [27] and is, perhaps, the simplest sequent calculus for $R_{\to}$ (see [22, Section 3.6]).

Definition 2.1 (Sequents in $LR_{\to}$) An *antecedent* is a finite, possibly empty, multiset of formulas. We denote antecedents by $\alpha, \beta, \gamma, \dots$, and if the antecedent is known to be empty, then it may be denoted by a small space. A *sequent* comprises an antecedent followed by $\vdash$ and then by a formula.

Both $\alpha, \mathcal{A}$ and $\mathcal{A}, \alpha$ denote the multiset that is identical to α except that the wff $\mathcal{A}$ occurs one more time than in α. Similarly, α, β denotes the union of the multisets α and β. In other words, $\mathcal{A}$ is an element of α, β iff it is an element of either, and the multiplicity of $\mathcal{A}$ is the sum of multiplicities of $\mathcal{A}$ in α and in β.

The *sequent calculus* $LR_{\to}$ contains an axiom, two connective rules, and a structural rule:

$$\mathcal{A} \vdash \mathcal{A} \ \text{id},$$

$$\frac{\alpha \vdash \mathcal{A} \qquad \mathcal{B}, \beta \vdash \mathcal{C}}{\alpha, \mathcal{A} \to \mathcal{B}, \beta \vdash \mathcal{C}} \to\vdash, \qquad \frac{\alpha, \mathcal{A} \vdash \mathcal{B}}{\alpha \vdash \mathcal{A} \to \mathcal{B}} \vdash\to, \qquad \frac{\alpha, \mathcal{A}, \mathcal{A} \vdash \mathcal{B}}{\alpha, \mathcal{A} \vdash \mathcal{B}} W\vdash.$$

The notion of a proof is standard, that is, a *proof* is a tree comprising occurrences of sequents such that all the leaves are instances of the axiom; all other nodes are obtained by applications of the rules in $LR_{\to}$ from top to bottom. The *root* of the tree is the sequent that is proved. A sequent $\alpha \vdash \mathcal{A}$ is *provable* if there is a proof in which this sequent is the root. Further, if α is empty, then $\mathcal{A}$ is a *theorem* of $LR_{\to}$.

The *single cut* rule, which is of the following form, is *admissible* in this calculus:

$$\frac{\alpha \vdash \mathcal{A} \qquad \mathcal{A}, \beta \vdash \mathcal{B}}{\alpha, \beta \vdash \mathcal{B}} \ \text{cut}.$$

Given the admissibility of cut, it may be proved that $LR_{\to}$ is equivalent to $R_{\to}$, which is defined by the principal type schemas (pt's, for short) of the combinators I, B, C, and W.[8] In other words, $R_{\to}$ is defined by axioms (A1)–(A4) and rule (R1) (given below). $\boldsymbol{t}$ may be added conservatively to $R_{\to}$ by the two rules compressed into one line in (R2); the resulting axiom system is denoted by $R^{\boldsymbol{t}}_{\to}$:

(A1)	$\mathcal{A} \to \mathcal{A}$	[self-implication, pt(I)],
(A2)	$(\mathcal{A} \to \mathcal{B}) \to (\mathcal{C} \to \mathcal{A}) \to \mathcal{C} \to \mathcal{B}$	[prefixing, pt(B)],
(A3)	$(\mathcal{A} \to \mathcal{B} \to \mathcal{C}) \to \mathcal{B} \to \mathcal{A} \to \mathcal{C}$	[permutation, pt(C)],
(A4)	$(\mathcal{A} \to \mathcal{A} \to \mathcal{B}) \to \mathcal{A} \to \mathcal{B}$	[contraction, pt(W)],
(R1)	$\mathcal{A} \to \mathcal{B}, \mathcal{A} \Rightarrow \mathcal{B}$	[detachment],
(R2)	$\vdash \mathcal{A} \Leftrightarrow \vdash \boldsymbol{t} \to \mathcal{A}$	[$\boldsymbol{t}$ introduction and elimination].

The addition of $\boldsymbol{t}$ to $LR_{\to}$ is not difficult. The notion of a multiset guarantees—together with the cut theorem—that the structural connective performs as an Abelian semigroup operation. We have to ensure that $\boldsymbol{t}$ can be proved to be the identity for the structural connective. Obviously, the inclusion of a second axiom and of a rule suffice for this. We denote the extended sequent calculus by $LR^{\boldsymbol{t}}_{\to}$:

$$\frac{\alpha \vdash \mathcal{A}}{\boldsymbol{t}, \alpha \vdash \mathcal{A}}\ \boldsymbol{t}\vdash, \qquad \vdash \boldsymbol{t} \quad \vdash \boldsymbol{t}.$$

Theorem 2.2 *The single cut rule is* admissible *in* $LR^{\boldsymbol{t}}_{\to}$.

Proof The proof comprises many inductions.[9] We include only some essential features of the proof without giving all the steps and the transformation. The parameters for a triple induction are δ, ϱ, and μ, where δ is the *degree* of the cut formula, ϱ is the *rank* of an application of the cut rule in a proof, and μ is the *contraction measure* of an application of the cut rule in a proof.

A separate induction on χ, the height of the derivation of $\alpha \vdash \boldsymbol{t}$, may be used to prove that the cut is eliminable when the cut formula is $\boldsymbol{t}$.

Lastly, an induction on τ, the height of the proof of $\alpha, \boldsymbol{t} \vdash \mathcal{A}$ allows us to prove that the provability of this sequent implies the provability of $\alpha \vdash \mathcal{A}$. We consider the base case (for $\tau = 1$) and one of the inductive steps.

1. If $\tau = 1$, then $\alpha, \boldsymbol{t} \vdash \mathcal{A}$ is an axiom. An instance of id is $\boldsymbol{t} \vdash \boldsymbol{t}$, which can be replaced by the axiom for $\boldsymbol{t}$.

2. If $\tau > 1$, then there are three possibilities—according to what is the last rule resulting in $\alpha, \boldsymbol{t} \vdash \mathcal{A}$. We consider when the last rule is $W\vdash$ (and leave the two other cases to the reader). α is $\beta, \mathcal{B}, \mathcal{B}$, and the given segment of the proof (on the left) is transformed into the segment on the right:

$$\frac{\begin{array}{c}\vdots\\ \boldsymbol{t}, \beta, \mathcal{B}, \mathcal{B} \vdash \mathcal{A}\end{array}}{\boldsymbol{t}, \beta, \mathcal{B} \vdash \mathcal{A}} \qquad \rightsquigarrow \qquad \frac{\begin{array}{c}\vdots\\ \beta, \mathcal{B}, \mathcal{B} \vdash \mathcal{A}\end{array}}{\beta, \mathcal{B} \vdash \mathcal{A}}.$$

The upper sequent on the right-hand side is provable, by inductive hypothesis, and the rule $W\vdash$ is applicable as it was in the original proof. □

Lemma 2.3 *A formula* $\mathcal{A}$ *is* provable *in* $R^{\boldsymbol{t}}_{\to}$ *if and only if* $\mathcal{A}$ *is a* theorem *of* $LR^{\boldsymbol{t}}_{\to}$.

Given Theorem 2.2, the proof of this lemma is not difficult, and we do not include the details here.

A well-known metatheorem about $R_{\to}$, due to [27], is that this logic is *decidable* (see also Meyer [31]). In other words, given any formula $\mathcal{A}$ in the signature of $R_{\to}$, there is an algorithm that (in finitely many steps) yields a provably correct answer to the question whether $\mathcal{A}$ is a theorem of $R_{\to}$. A transparent presentation of the proof of the decidability of $R_{\to}$ can be found in [22]. A core component in the decision procedure is a modified version of $LR_{\to}$, which we will describe in what follows.

Another well-known metatheorem is that the classical and intuitionistic propositional logics are decidable. The second of these decidability results was first proved in [25] (which is available in English as Gentzen [26]). Gentzen's idea is that, if a sequent is provable, then the same sequent has a proof that is *cut-free* and also *clutter-free* (so to speak). The former means that no applications of the cut rule are necessary for the proof of the sequent in question. The latter means that a proof can be constructed from *reduced sequents* which do not accumulate too many copies of a formula. Given the possibility of lax handling of the number and order of occurrences of formulas in sequents in classical and intuitionistic logics, the claim that three occurrences (on the left) suffice may not be surprising.

The multiset view—which is applicable to the classical and intuitionistic sequent calculi—provides a framework in which it is easy to see that three is the greatest number of occurrences of a formula that can result. If there are at least two occurrences of $\mathscr{A}$ (either in the antecedent or in the succedent), then $n-1$ contractions can reduce the number of occurrences from n to 1. A scrutiny of the rules in LK reveals that, if an upper sequent of a single-premise connective rule contains 1 occurrence of $\mathscr{A}$, then the lower sequent contains 0, 1, or 2 occurrences. The number of occurrences depends on whether $\mathscr{A}$ in the upper sequent is a subaltern or a parametric formula, and whether $\mathscr{A}$ in the lower sequent is the principal formula. If the connective rule has two premises, then the multisets of parametric formulas in the two premises can differ only in the rule of implication introduction on the left. Then it may happen that $\mathscr{A}$ is parametric in the antecedent in both premises, and it is the principal formula of the rule. Then the number of occurrences adds up to three. In LK and LJ, any "missing" occurrences can be thinned back into a sequent—either on the left-hand or on the right-hand side of the turnstile.

Relevant implication does not have as a rule either thinning or expansion. This means that it may not be feasible to always contract all the occurrences into one occurrence, because afterward there is no way, in general, to reinstate the occurrences that have been contracted away. A way to ensure control over contraction without forcing too much contraction to be applied is to combine the introduction rule for implication on the left with a flexible contraction. In order to distinguish this sequent calculus from the previous one (and from the ones we introduce later), we denote it by $[LR_{\to}]$.[10]

The notion of a sequent remains as before. $[LR_{\to}]$ comprises an *axiom* and two *connective rules*:

$$\mathscr{A} \vdash \mathscr{A} \ \text{id},$$

$$\frac{\alpha \vdash \mathscr{A} \qquad \beta, \mathscr{B} \vdash \mathscr{C}}{[\alpha, \beta, \mathscr{A} \to \mathscr{B}] \vdash \mathscr{C}} \ [\to\vdash], \qquad \frac{\alpha, \mathscr{A} \vdash \mathscr{B}}{\alpha \vdash \mathscr{A} \to \mathscr{B}} \ \vdash\to.$$

The brackets in $[LR_{\to}]$ are intended to be a reminder of the [ ]'s in the $[\to\vdash]$ rule. The meaning of the brackets is that the lower sequent in the rule may include a limited amount of contraction—in lieu of the missing contraction rule.

$\alpha, \beta, \mathscr{A} \to \mathscr{B}$ is a multiset, whereas $[\alpha, \beta, \mathscr{A} \to \mathscr{B}]$ is a submultiset of that multiset, which has to satisfy conditions (1)–(2).

(1) $\mathscr{A} \to \mathscr{B}$ occurs at least once in $[\alpha, \beta, \mathscr{A} \to \mathscr{B}]$, but it may have 0, 1, or 2 fewer occurrences than in $\alpha, \beta, \mathscr{A} \to \mathscr{B}$.
(2) If $\mathscr{C}$, which is distinct from $\mathscr{A} \to \mathscr{B}$, occurs in α, β, then $\mathscr{C}$ occurs at least once in $[\alpha, \beta, \mathscr{A} \to \mathscr{B}]$, but it may have 0 or 1 fewer occurrences than in α, β.

In other words, the limited amount of contraction means that the $[\to\vdash]$ rule can be simulated by $\to\vdash$ in $LR_{\to}$ and $W\vdash$, where the number of applications of $W\vdash$ has a finite upper bound. If α, β consists of n formulas distinct from the principal formula of the rule, plus $m > 2$ occurrences of the principal formula of the rule, then $[\alpha, \beta, \mathscr{A} \to \mathscr{B}]$ is either $\alpha, \beta, \mathscr{A} \to \mathscr{B}$, or obtained from the latter by no more than $n+2$ contractions, where only the principal formula of the rule may be contracted more than once.

Another useful way to think about the new $[\to\vdash]$ rule is that particular instances of applications of the contraction rule are part of $[\to\vdash]$, namely, contractions that could not have been carried out on the premises. For instance, if $\mathcal{C}$ occurs once in both α and β, then α, β contains two occurrences of $\mathcal{C}$; hence, an application of the contraction rule can reduce the number of occurrences of $\mathcal{C}$ from two to one. However, neither in α nor in β can the single occurrence of $\mathcal{C}$ be contracted (to void).

The calculus $[LR_{\to}]$ may be used to show the decidability of $R_{\to}$. First of all, the axiom system $R_{\to}$ is equivalent to $LR_{\to}$ with respect to the set of theorems, which in turn is equivalent to $[LR_{\to}]$. In sum, if $\mathcal{A}$ is a theorem of $R_{\to}$, then there is a proof of $\vdash \mathcal{A}$ in $[LR_{\to}]$, and the other way around. But all proofs in $[LR_{\to}]$ can include only a bounded amount of contraction, so the search for a proof is bounded too.

We extend $[LR_{\to}]$ with $\boldsymbol{t}$ by adding the same axiom and rule involving the constant $\boldsymbol{t}$ as for $LR^{\boldsymbol{t}}_{\to}$.

Theorem 2.4 *The single cut rule is* admissible *in* $[LR^{\boldsymbol{t}}_{\to}]$.

Proof The proof is very similar to the proof of Lemma 2.2, except that the cases for $W\vdash$ are omitted, and the cases for $[\to\vdash]$ have to be scrutinized to ensure that the "permutation" of $[\to\vdash]$ and of the cut can be performed. □

Lemma 2.5 *The axiomatic calculus* $R^{\boldsymbol{t}}_{\to}$ *and the sequent calculi* $LR^{\boldsymbol{t}}_{\to}$ *and* $[LR^{\boldsymbol{t}}_{\to}]$ *have the* same set of theorems.

Proof

1. From left to right, we can easily prove that the axioms of $R^{\boldsymbol{t}}_{\to}$ are provable in the sequent calculi. Modus ponens is emulated using the cut rule: if $\vdash \mathcal{A}$ and $\vdash \mathcal{A} \to \mathcal{B}$ are provable, then the latter yields $\mathcal{A} \vdash \mathcal{B}$ by cut, and by one more application of the cut rule, we get that $\vdash \mathcal{B}$ is provable.

2.1. From right to left, we note that each axiom in $LR^{\boldsymbol{t}}_{\to}$ is a theorem of $R^{\boldsymbol{t}}_{\to}$ as it is, or after an application of the $\vdash\to$ rule.

Further, we may show by induction on the height of a proof in $LR^{\boldsymbol{t}}_{\to}$ that if $\alpha \vdash \mathcal{A}$, then $\alpha^{\to} \to \mathcal{A}$ is a theorem of $R^{\boldsymbol{t}}_{\to}$, where $\alpha^{\to} \to \mathcal{A}$ is an implicational formula that contains the elements of the multiset α as antecedents (together with $\mathcal{A}$ as the consequent). We assume that some metatheorems about $R^{\boldsymbol{t}}_{\to}$ are known, such as if $\mathcal{A}_1 \to \cdots \mathcal{A}_n \to \mathcal{B}$ is a theorem, then so is $\mathcal{A}_{\pi(1)} \to \cdots \mathcal{A}_{\pi(n)} \to \mathcal{B}$, where π is a permutation of $\langle 1, \ldots, n \rangle$ (see, e.g., [3, Sections 4 and 8]).

2.2. If the last rule used in proving $\alpha \vdash \mathcal{A}$ is $\vdash\to$, then $\alpha^{\to} \to \mathcal{B} \to \mathcal{C}$ is one of the formulas from $(\alpha, \mathcal{B})^{\to} \to \mathcal{C}$; if α is empty, then $\alpha^{\to} \to \mathcal{B} \to \mathcal{C}$ is $\mathcal{B} \to \mathcal{C}$. Either way, the claim is obviously true because of the hypothesis of induction.

2.3. If the last rule is $[\to\vdash]$, then $\alpha^{\to} \to \mathcal{B}$ and $(\beta, \mathcal{C})^{\to} \to \mathcal{D}$ are provable. However, from $\mathcal{A}_1 \to \cdots \mathcal{A}_n \to \mathcal{B}$ and $\mathcal{C} \to \mathcal{B}_1 \cdots \mathcal{B}_m \to \mathcal{D}$, the formula $\mathcal{A}_1 \to \cdots \mathcal{A}_n \to (\mathcal{B} \to \mathcal{C}) \to \mathcal{B}_1 \to \cdots \mathcal{B}_m \to \mathcal{D}$ is provable in $R^{\boldsymbol{t}}_{\to}$, which is one of the formulas that correspond to $(\alpha, \beta, \mathcal{B} \to \mathcal{C})^{\to} \to \mathcal{D}$.

2.4. Finally, if the last rule is $W\vdash$, then by hypothesis, $(\alpha, \mathcal{B}, \mathcal{B})^{\to} \to \mathcal{C}$. One of the formulas that correspond to the sequent is $\mathcal{B} \to \mathcal{B} \to \cdots \mathcal{C}$, where the elements of α fill in the ellipsis in the formula. Then, by an instance of contraction, $\mathcal{B} \to \cdots \to \mathcal{C}$, which corresponds to $(\alpha, \mathcal{B})^{\to} \to \mathcal{C}$.

3. The equivalence of $LR^{\boldsymbol{t}}_{\to}$ and $[LR^{\boldsymbol{t}}_{\to}]$ is straightforwardly provable. From right to left, any proof in $[LR^{\boldsymbol{t}}_{\to}]$ either is a proof in $LR^{\boldsymbol{t}}_{\to}$ as it is or, if an $[\to\vdash]$ step involves some contraction, then it can be turned into a proof in $LR^{\boldsymbol{t}}_{\to}$ by adding finitely many contraction steps after an $\to\vdash$ step. From left to right, we note that there are only two rules that can introduce multiplicity. Further, the axioms contain at most

one formula on the left of the $\vdash$. If the rule is $\boldsymbol{t}\vdash$, then any $\boldsymbol{t}$ so introduced that is contracted may simply not be introduced to start with. If the rule is $\to\vdash$, then we may appeal to the motivation of the new [$\to\vdash$] rule, which is that only contractions that could not have been carried out in the premises may be carried out within the rule. □

3 Consecution Calculi

Structurally free logics were introduced in Dunn and Meyer [23]. Those logics incorporate combinators as formulas, which is a further step in associating certain structural rules and combinators with each other. We do not go that far here. In the sequent calculi that we have mentioned so far, the structural connective was associative and commutative. The , could be even viewed as a *polyadic* connective creating "flat" structures. Structurally free logics take the structural connective to be binary and do not assign to it any other properties such as commutativity. We adopt this idea here, and we refer to calculi that have such a structural connective as *consecution calculi*. To further emphasize the difference between the previous calculi and the next ones, we use ; for the structural connective.

First, we consider a consecution calculus for $R^{\boldsymbol{t}}_{\to}$, which is obtained from a consecution calculus for $T^{\boldsymbol{t}}_{\to}$ by strengthening a structural rule that allows permutation on the left with association to the right to a full permutation rule.

Definition 3.1 (Structures and consecutions) The set of *structures*, denoted by str, is inductively defined as follows.

(1) If $\mathcal{A}$ is a wff, then $\mathcal{A} \in$ str.
(2) If $\mathfrak{A}, \mathfrak{B} \in$ str, then $(\mathfrak{A}; \mathfrak{B}) \in$ str.

A *consecution* is a structure followed by a turnstile and then a formula.

Note that there is *no empty structure* according to the definition; every structure contains at least one formula. The structures have a certain similarity to multisets, but they are not the same, which is why we use Gothic letters as variables for structures. Occasionally, we will omit parentheses from left associated structures; that is, $\mathfrak{A}; \mathfrak{B}; \mathfrak{C}$ with all the parentheses would be written as $((\mathfrak{A}; \mathfrak{B}); \mathfrak{C})$. Square brackets are used to localize, in upper consecutions, an occurrence of a structure in a structure and, in lower consecutions, the result of the replacement of an occurrence of a structure by a structure.

The consecution calculus $LT^{\boldsymbol{t}}_{\to}$ consists of an axiom, two connective rules, and five structural rules:[11]

$$\mathcal{A} \vdash \mathcal{A} \ \ \text{id},$$

$$\frac{\mathfrak{A} \vdash \mathcal{B} \quad \mathfrak{C}[\mathcal{D}] \vdash \mathcal{E}}{\mathfrak{C}[\mathcal{B} \to \mathcal{D}; \mathfrak{A}] \vdash \mathcal{E}} \ \to\vdash, \qquad \frac{\mathfrak{A}; \mathcal{B} \vdash \mathcal{C}}{\mathfrak{A} \vdash \mathcal{B} \to \mathcal{C}} \ \vdash\to,$$

$$\frac{\mathfrak{A}[\mathfrak{B}; (\mathfrak{C}; \mathfrak{D})] \vdash \mathcal{E}}{\mathfrak{A}[\mathfrak{B}; \mathfrak{C}; \mathfrak{D}] \vdash \mathcal{E}} \ \mathsf{B}\vdash, \qquad \frac{\mathfrak{A}[\mathfrak{B}; (\mathfrak{C}; \mathfrak{D})] \vdash \mathcal{E}}{\mathfrak{A}[\mathfrak{C}; \mathfrak{B}; \mathfrak{D}] \vdash \mathcal{E}} \ \mathsf{B}'\vdash, \qquad \frac{\mathfrak{A}[\mathfrak{B}; \mathfrak{C}; \mathfrak{C}] \vdash \mathcal{D}}{\mathfrak{A}[\mathfrak{B}; \mathfrak{C}] \vdash \mathcal{D}} \ \mathsf{W}\vdash,$$

$$\frac{\mathfrak{A}[\mathfrak{B}] \vdash \mathcal{C}}{\mathfrak{A}[\boldsymbol{t}; \mathfrak{B}] \vdash \mathcal{C}} \ \mathsf{KI}_{\boldsymbol{t}}\vdash, \qquad \frac{\mathfrak{A}[\boldsymbol{t}; \boldsymbol{t}] \vdash \mathcal{B}}{\mathfrak{A}[\boldsymbol{t}] \vdash \mathcal{B}} \ \mathsf{M}_{\boldsymbol{t}}\vdash.$$

The labels on the structural rules include well-known combinators, which have the following axioms:

$$\begin{array}{cc} \mathsf{B}xyz \rhd x(yz), & \mathsf{B}'xyz \rhd y(xz), \\ \mathsf{W}xy \rhd xyy, & \mathsf{M}x \rhd xx, \\ \mathsf{K}xy \rhd x, & \mathsf{I}x \rhd x. \end{array}$$

The last two structural rules are subscripted with $\boldsymbol{t}$ because at least one of the structures is not arbitrary; in particular, it has to be $\boldsymbol{t}$.

The *single cut rule* now takes the following form:

$$\frac{\mathfrak{A} \vdash \mathcal{B} \qquad \mathfrak{C}[\mathcal{B}] \vdash \mathcal{D}}{\mathfrak{C}[\mathfrak{A}] \vdash \mathcal{D}} \text{ cut.}$$

The cut rule is admissible in this calculus.[12] Then it can be proved that $\boldsymbol{t} \vdash \mathcal{A}$ in $LT^{\boldsymbol{t}}_{\to}$ if and only if $\mathcal{A}$ is provable in $T^{\boldsymbol{t}}_{\to}$, where the latter is the axiom system obtained from $R^{\boldsymbol{t}}_{\to}$ by omitting (A3) and adding instead (A5):

(A5) $(\mathcal{A} \to \mathcal{B}) \to (\mathcal{B} \to \mathcal{C}) \to \mathcal{A} \to \mathcal{C}$.

The difference in the axiomatizations of the implicational fragments of the logic of ticket entailment and the logic of relevant implication can be transposed into the consecution calculus. The rule labeled as $\mathsf{C}\vdash$ below corresponds to axiom (A3). This allows us to obtain a consecution calculus for $R^{\boldsymbol{t}}_{\to}$. We add a ; into the subscript—as in $LR^{\boldsymbol{t}}_{\to ;}$—to distinguish this consecution calculus from the sequent calculi above:

$$\frac{\mathfrak{A}[\mathfrak{B}; \mathfrak{D}; \mathfrak{C}] \vdash \mathcal{E}}{\mathfrak{A}[\mathfrak{B}; \mathfrak{C}; \mathfrak{D}] \vdash \mathcal{E}} \ \mathsf{C}\vdash.$$

We note that we can add (A5) to $R^{\boldsymbol{t}}_{\to}$, because it is a theorem. On the other hand, we could leave $\mathsf{B}'\vdash$ in $LR^{\boldsymbol{t}}_{\to ;}$. The rule $\mathsf{B}'\vdash$ is a derived rule in a proof of a theorem of $R^{\boldsymbol{t}}_{\to}$. This is shown by the next chunk of a derivation,

$$\cfrac{\cfrac{\cfrac{\mathfrak{A}[\mathfrak{C}; (\mathfrak{B}; \mathfrak{D})] \vdash \mathcal{E}}{\mathfrak{A}[\mathfrak{C}; \mathfrak{B}; \mathfrak{D}] \vdash \mathcal{E}}}{\mathfrak{A}[\boldsymbol{t}; \mathfrak{C}; \mathfrak{B}; \mathfrak{D}] \vdash \mathcal{E}}}{\mathfrak{A}[\boldsymbol{t}; \mathfrak{B}; \mathfrak{C}; \mathfrak{D}] \vdash \mathcal{E}}$$

Alternatively, we could add the rule $\mathsf{CI}\vdash$:

$$\frac{\mathfrak{A}[\mathfrak{C}; \mathfrak{B}] \vdash \mathcal{D}}{\mathfrak{A}[\mathfrak{B}; \mathfrak{C}] \vdash \mathcal{D}} \ \mathsf{CI}\vdash.$$

Given this rule, the derivability of $\mathsf{B}'\vdash$ in all contexts is obvious.

We have the following two claims for $LR^{\boldsymbol{t}}_{\to ;}$.

Theorem 3.2 *The single cut rule is* admissible *in* $LR^{\boldsymbol{t}}_{\to ;}$.

Proof The proof is by triple induction on ϱ, the rank of the cut, on δ, the degree of the cut formula, and on μ, the contraction measure of the cut, and by an induction on χ, the height of the proof tree of $\mathfrak{A} \vdash \boldsymbol{t}$. □

Lemma 3.3 *The consecution* $\boldsymbol{t} \vdash \mathcal{A}$ *is* provable *in* $LR^{\boldsymbol{t}}_{\to ;}$ *iff* $\mathcal{A}$ *is* provable *in* $R^{\boldsymbol{t}}_{\to}$.

Proof The proof is as usual—once the cut theorem is proved. (We omit the details.) □

The extension of $LT^{t}_{\to}$ to $LR^{t}_{\to;}$ is possible, but this does not lead to the decidability of $T_{\to}$—at least, we do not see how it would. Hence, we pursue a different strategy by developing, in the next section, the consecution calculus $LT^{\textcircled{t}}_{\to}$.

4 A Consecution Calculus Motivated by Dual Combinators

The family of logics that are nowadays called relevance logics emerged from the logic of entailment E (as we already mentioned), which was the focus of research for quite some time. However, later on it became clear that E and its implicational fragment $E_{\to}$ have certain peculiarities. Given a simple type assignment system for combinatory logic, it is easy to establish that (A1)–(A5) are principal type schemas of combinators. (A5) is pt(B'), as the label on the corresponding structural rule suggests. All the combinators mentioned so far are *proper*, that is, their axioms do not contain constants or new variables on the right-hand side of the $\rhd$. In contrast, all the known axiomatizations of $E_{\to}$ have at least one axiom that is the principal type schema of an *improper* combinator.

The difference between $T^{t}_{\to}$ and $R^{t}_{\to}$, on one hand, and $E^{t}_{\to}$, on the other, may be understood algebraically in terms of the commutativity of fusion and identities. Fusion, denoted by $\circ$, may be added conservatively to these logics. $T^{\circ t}_{\to}$ has as theorems $(\boldsymbol{t} \circ \mathcal{A}) \to \mathcal{A}$ and $\mathcal{A} \to (\boldsymbol{t} \circ \mathcal{A})$, whereas $R^{\circ t}_{\to}$ also proves $\mathcal{A} \to (\mathcal{A} \circ \boldsymbol{t})$ and $(\mathcal{A} \circ \boldsymbol{t}) \to \mathcal{A}$. $E^{\circ t}_{\to}$ does not prove the last formula, which means that $\boldsymbol{t}$ is a left and upper right identity for $\circ$. The differences in the behavior of $\boldsymbol{t}$ led to the formulation of the consecution calculus $LE^{t}_{\to}$ in [9].

The bracket abstraction that is definable using the combinators B, B', I, and W captures a class of terms that has been called "hereditary right maximal." The idea is that in each subterm the right-hand term has an index not smaller than the index of the left-hand term.[13]

The above observations suggest that we can obtain a consecution calculus for $R^{t}_{\to}$ by adding to $LT^{t}_{\to}$ some special structural rules that involve $\boldsymbol{t}$.

Definition 4.1 $\mathbf{A} = \langle A; \leq, \circ, e\rangle$ is an *ordered groupoid with right identity* iff (1)–(4) are true:

(1) $\langle A; \leq\rangle$ is a poset;
(2) $\circ$ is a binary operation;
(3) $a \circ e = a$;
(4) $a \leq b$ and $c \leq d$ imply $a \circ c \leq b \circ d$.

Lemma 4.2 *Let* $\mathbf{A}$ *be an ordered groupoid with right identity;* $(a \circ b) \circ c \leq b \circ (a \circ c)$ implies *that* $(a \circ b) \circ c = (a \circ c) \circ b$.

Proof We have

$$\begin{aligned}(a \circ b) \circ c &\leq b \circ (a \circ c)\\ &= \big(b \circ (a \circ c)\big) \circ e\\ &\leq (a \circ c) \circ (b \circ e)\\ &= (a \circ c) \circ b.\end{aligned}$$ □

The consecution calculus $LT^{\textcircled{t}}_{\to}$ is defined as an extension of $LT^{t}_{\to}$ by the following two rules:

$$\frac{\mathfrak{A}[\mathfrak{B}] \vdash \mathcal{C}}{\mathfrak{A}[\mathfrak{B}; \boldsymbol{t}] \vdash \mathcal{C}}\ \mathsf{K}_{\boldsymbol{t}} \vdash, \qquad \frac{\mathfrak{A}[\mathfrak{B}; \boldsymbol{t}] \vdash \mathcal{C}}{\mathfrak{A}[\boldsymbol{t}; \mathfrak{B}] \vdash \mathcal{C}}\ \mathsf{T}_{t} \vdash.$$

The combinator T is definable as CI in the presence of those combinators. The subscript $_{\boldsymbol{t}}$ indicates that this structural rule may be applied if the right-hand-side structure is of a special shape, namely, $\boldsymbol{t}$.

The calculus $LT_{\rightarrow}^{\textcircled{t}}$ does not have any rules that would not be included or would not be admissible in $LR_{\rightarrow;}^{\boldsymbol{t}}$. Thus we need to show that it is *sufficient* to prove all the theorems of $LR_{\rightarrow}^{\boldsymbol{t}}$. Of course, we also have to show that the cut rule is admissible. (We leave the latter theorem and its proof for the next section.)

Assertion is not a theorem of ticket entailment, and it is sometimes taken as one of the axioms for $R_{\rightarrow}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{K}_{\boldsymbol{t}}\vdash\ \dfrac{\mathcal{A}\vdash\mathcal{A}}{\mathcal{A};\boldsymbol{t}\vdash\mathcal{A}} \qquad \mathcal{B}\vdash\mathcal{B}}{\mathcal{A}\rightarrow\mathcal{B};(\mathcal{A};\boldsymbol{t})\vdash\mathcal{B}}}{\mathcal{A};\mathcal{A}\rightarrow\mathcal{B};\boldsymbol{t}\vdash\mathcal{B}}}{\boldsymbol{t};(\mathcal{A};\mathcal{A}\rightarrow\mathcal{B})\vdash\mathcal{B}}\ \mathsf{T}_{\boldsymbol{t}}\vdash}{\boldsymbol{t};\mathcal{A};\mathcal{A}\rightarrow\mathcal{B}\vdash\mathcal{B}}}{\boldsymbol{t}\vdash\mathcal{A}\rightarrow((\mathcal{A}\rightarrow\mathcal{B})\rightarrow\mathcal{B})}$$

We indicated where the two rules that are specific to $LT_{\rightarrow}^{\textcircled{t}}$ are applied. (The thick line compresses two applications of the rule $\vdash\rightarrow$.) The bottom consecution of the proof implies that assertion is a theorem of $LT_{\rightarrow}^{\textcircled{t}}$. Another characteristic wff is *permutation*, which is the wff (A3) above:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{A}\vdash\mathcal{A} \qquad \dfrac{\dfrac{\dfrac{\mathcal{B}\vdash\mathcal{B} \qquad \mathcal{C}\vdash\mathcal{C}}{\mathcal{B}\rightarrow\mathcal{C};\mathcal{B}\vdash\mathcal{C}}}{\mathcal{B}\rightarrow\mathcal{C};(\mathcal{B};\boldsymbol{t})\vdash\mathcal{C}}\ \mathsf{K}_{\boldsymbol{t}}\vdash}{\mathcal{B};\mathcal{B}\rightarrow\mathcal{C};\boldsymbol{t}\vdash\mathcal{C}}}{\mathcal{B};(\mathcal{A}\rightarrow(\mathcal{B}\rightarrow\mathcal{C});\mathcal{A});\boldsymbol{t}\vdash\mathcal{C}}}{\mathcal{A}\rightarrow(\mathcal{B}\rightarrow\mathcal{C});\mathcal{B};\mathcal{A};\boldsymbol{t}\vdash\mathcal{C}}}{\boldsymbol{t};(\mathcal{A}\rightarrow(\mathcal{B}\rightarrow\mathcal{C});\mathcal{B};\mathcal{A})\vdash\mathcal{C}}\ \mathsf{T}_{\boldsymbol{t}}\vdash}{\dfrac{\dfrac{\boldsymbol{t};(\mathcal{A}\rightarrow(\mathcal{B}\rightarrow\mathcal{C});\mathcal{B});\mathcal{A}\vdash\mathcal{C}}{\boldsymbol{t};\mathcal{A}\rightarrow(\mathcal{B}\rightarrow\mathcal{C});\mathcal{B};\mathcal{A}\vdash\mathcal{C}}}{\boldsymbol{t}\vdash(\mathcal{A}\rightarrow(\mathcal{B}\rightarrow\mathcal{C}))\rightarrow(\mathcal{B}\rightarrow(\mathcal{A}\rightarrow\mathcal{C}))}}$$

Once again, we have collapsed the last couple of steps into one; they are alike and obvious.

5 Cut Theorem for the Consecution Calculus $LT_{\rightarrow}^{\textcircled{t}}$

A crucial question about $LT_{\rightarrow}^{\textcircled{t}}$ is the admissibility of the single cut rule. We separate out a lemma, which guarantees that the cut is admissible when the cut formula is $\boldsymbol{t}$.

Lemma 5.1 *If the cut formula is* $\boldsymbol{t}$*, then the single cut rule is* admissible.

Proof The proof is by induction on χ, the height of the tree ending in the left premise of an application of the cut rule, which has no other applications of the cut rule above it. The $LT_{\rightarrow}^{\textcircled{t}}$ calculus is right singular. The only way to change the right-hand-side component of a consecution, let us say a wff $\mathcal{A}$, is by the implication introduction rule, each application of which creates a new formula with $\mathcal{A}$ a proper subformula. The typical form of the segment of the proof we are considering is

$$
*\vdash\ \dfrac{\dfrac{\overset{\vdots}{\mathfrak{A}\vdash \boldsymbol{t}}}{\mathfrak{A}'\vdash \boldsymbol{t}} \qquad \overset{\vdots}{\mathfrak{B}[\boldsymbol{t}]\vdash\mathcal{A}}}{\mathfrak{B}[\mathfrak{A}']\vdash\mathcal{A}},
$$

where $*\vdash$ must be $\to\vdash$ or a structural rule, because of the preceding remark. We include a few concrete steps and leave the rest of the steps for the reader's consideration.

1. If $\chi = 1$, then $\mathfrak{A}' \vdash \boldsymbol{t}$ is an instance of the axiom (and this consecution is the whole subtree above the left premise). The cut is immediately eliminable by retaining the subtree rooted in the right premise.

2.1. If $\chi > 1$, then the last rule may be $\to\vdash$. We start with

$$
\dfrac{\dfrac{\overset{\vdots}{\mathfrak{C}\vdash\mathcal{C}} \quad \overset{\vdots}{\mathfrak{D}[\mathcal{D}]\vdash \boldsymbol{t}}}{\mathfrak{D}[\mathcal{C}\to\mathcal{D};\mathfrak{C}]\vdash \boldsymbol{t}} \qquad \overset{\vdots}{\mathfrak{B}[\boldsymbol{t}]\vdash\mathcal{A}}}{\mathfrak{B}[\mathfrak{D}[\mathcal{C}\to\mathcal{D};\mathfrak{C}]]\vdash\mathcal{A}},
$$

and we modify the proof into

$$
\dfrac{\overset{\vdots}{\mathfrak{C}\vdash\mathcal{C}} \qquad \dfrac{\overset{\vdots}{\mathfrak{D}[\mathcal{D}]\vdash \boldsymbol{t}} \quad \overset{\vdots}{\mathfrak{B}[\boldsymbol{t}]\vdash\mathcal{A}}}{\mathfrak{B}[\mathfrak{D}[\mathcal{D}]]\vdash\mathcal{A}}}{\mathfrak{B}[\mathfrak{D}[\mathcal{C}\to\mathcal{D};\mathfrak{C}]]\vdash\mathcal{A}}.
$$

We consider two steps in which the new rules in $LT^{\textcircled{t}}_{\to}$ appear.

2.2. The last rule in the proof of the left premise may be $\mathsf{KI}_{\boldsymbol{t}}$. Then we have

$$
\dfrac{\dfrac{\overset{\vdots}{\mathfrak{A}[\mathfrak{C}]\vdash \boldsymbol{t}}}{\mathfrak{A}[\mathfrak{C};\boldsymbol{t}]\vdash \boldsymbol{t}} \qquad \overset{\vdots}{\mathfrak{B}[\boldsymbol{t}]\vdash\mathcal{A}}}{\mathfrak{B}[\mathfrak{A}[\mathfrak{C};\boldsymbol{t}]]\vdash\mathcal{A}},
$$

and we permute the cut upward to obtain

$$
\dfrac{\dfrac{\overset{\vdots}{\mathfrak{A}[\mathfrak{C}]\vdash \boldsymbol{t}} \qquad \overset{\vdots}{\mathfrak{B}[\boldsymbol{t}]\vdash\mathcal{A}}}{\mathfrak{B}[\mathfrak{A}[\mathfrak{C}]]\vdash\mathcal{A}}}{\mathfrak{B}[\mathfrak{A}[\mathfrak{C};\boldsymbol{t}]]\vdash\mathcal{A}}.
$$

2.3. If the last rule is $\mathsf{T}_{\boldsymbol{t}}\vdash$, then we have the following portion of a proof:

$$
\dfrac{\dfrac{\overset{\vdots}{\mathfrak{A}[\mathfrak{C};\boldsymbol{t}]\vdash \boldsymbol{t}}}{\mathfrak{A}[\boldsymbol{t};\mathfrak{C}]\vdash \boldsymbol{t}} \qquad \overset{\vdots}{\mathfrak{B}[\boldsymbol{t}]\vdash\mathcal{A}}}{\mathfrak{B}[\mathfrak{A}[\boldsymbol{t};\mathfrak{C}]]\vdash\mathcal{A}}.
$$

The transformation is similar to the one in the previous case; we move the cut toward the top:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \mathfrak{A}[\mathfrak{C};\boldsymbol{t}]\vdash\boldsymbol{t}\end{array} \qquad \begin{array}{c}\vdots\\ \mathfrak{B}[\boldsymbol{t}]\vdash\mathcal{A}\end{array}}{\mathfrak{B}[\mathfrak{A}[\mathfrak{C};\boldsymbol{t}]]\vdash\mathcal{A}}}{\mathfrak{B}[\mathfrak{A}[\boldsymbol{t};\mathfrak{C}]]\vdash\mathcal{A}}.$$

The remaining steps (2.4–2.8) are similar; we omit the details. □

Theorem 5.2 *The single cut rule is* admissible *in* $LT_{\to}^{\textcircled{t}}$.

Proof The proof is by *triple* induction on δ, the degree of the cut formula, on ϱ, the rank of the cut and on μ, the contraction measure of the cut. We assume that the cut formula is not $\boldsymbol{t}$, because we already know, by Lemma 5.1, that if the cut formula is $\boldsymbol{t}$, then the cut is eliminable.

The contraction measure of the cut is the number of the applications of the $\mathsf{W}\vdash$ rule in the subtree in the proof that is rooted in the lower consecution of the cut. A proof might be contraction free; hence, μ is a natural number given a proof.

The degree of a formula is the number of occurrences of $\to$; therefore, $\delta \geq 0$ and is an integer.

The rank of the cut is the sum of the left and right ranks. The left rank is simply the number of consecutions in which the cut formula appears on the right-hand side of the turnstile (without interruptions). The right rank is the number of consecutions in which a formula that is congruent with the cut formula appears (again, without interruptions). A formula in the lower consecution of a rule is congruent with a formula in the upper consecution if it is of the same shape and occurs in the same substructure (as the letters indicate in the formulations of the rules). The only complication to note is that the new (displayed) occurrence of an implicational formula in $\to\vdash$ and $\vdash\to$ is the principal wff of those rules, and it is not congruent with any formula in the upper consecutions. Further, if a wff occurs in the structure labeled by $\mathfrak{C}$ in the $\mathsf{W}\vdash$ rule, then both matching occurrences in the upper consecution are congruent with the occurrence in the lower consecution.

It is convenient to outline the steps of the triple induction anchored by rank, because the cases show more variation with respect to ϱ than μ or δ.

1. If $\varrho = 2$, then the only case, when neither premise is an axiom, is if both are by an $\to$ introduction rule. The rules are well formulated, and the transformation is justified by a reduction in δ:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \mathfrak{A};\mathcal{A}\vdash\mathcal{B}\end{array}}{\mathfrak{A}\vdash\mathcal{A}\to\mathcal{B}} \qquad \dfrac{\begin{array}{c}\vdots\\ \mathfrak{B}\vdash\mathcal{A}\end{array} \quad \begin{array}{c}\vdots\\ \mathfrak{C}[\mathcal{B}]\vdash\mathcal{C}\end{array}}{\mathfrak{C}[\mathcal{A}\to\mathcal{B};\mathfrak{B}]\vdash\mathcal{C}}}{\mathfrak{C}[\mathfrak{A};\mathfrak{B}]\vdash\mathcal{C}} \rightsquigarrow \dfrac{\begin{array}{c}\vdots\\ \mathfrak{B}\vdash\mathcal{A}\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ \mathfrak{A};\mathcal{A}\vdash\mathcal{B}\end{array} \quad \begin{array}{c}\vdots\\ \mathfrak{C}[\mathcal{B}]\vdash\mathcal{C}\end{array}}{\mathfrak{C}[\mathfrak{A};\mathcal{A}]\vdash\mathcal{C}}}{\mathfrak{C}[\mathfrak{A};\mathfrak{B}]\vdash\mathcal{C}}.$$

2. If $\varrho > 2$, then either $\varrho_l > 1$ or $\varrho_r > 1$. The previous situation means that we have a chunk of proof, which looks like

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \mathfrak{A}\vdash\mathcal{A}\end{array}}{\mathfrak{A}'\vdash\mathcal{A}} \qquad \begin{array}{c}\vdots\\ \mathfrak{B}[\mathcal{A}]\vdash\mathcal{B}\end{array}}{\mathfrak{B}[\mathfrak{A}']\vdash\mathcal{B}}.$$

There is at least one consecution with the same wff on the right of the turnstile above the left premise of the cut. The modification of the proof consists of first applying the cut rule and then applying the same rule that resulted in the left premise in the original proof. The rules of $LT^{\textcircled{t}}_{\to}$ are well formulated, that is, properly *contextualized*, which means that each rule is applicable within the context of $\mathfrak{B}$. As an example, we consider $\mathsf{M}_{\boldsymbol{t}}$:

$$\dfrac{\dfrac{\mathfrak{A}[\boldsymbol{t};\boldsymbol{t}] \vdash \mathcal{A}}{\mathfrak{A}[\boldsymbol{t}] \vdash \mathcal{A}} \qquad \mathfrak{B}[\mathcal{A}] \vdash \mathcal{B}}{\mathfrak{B}[\mathfrak{A}[\boldsymbol{t}]] \vdash \mathcal{B}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\mathfrak{A}[\boldsymbol{t};\boldsymbol{t}] \vdash \mathcal{A} \qquad \mathfrak{B}[\mathcal{A}] \vdash \mathcal{B}}{\mathfrak{B}[\mathfrak{A}[\boldsymbol{t};\boldsymbol{t}]] \vdash \mathcal{B}}}{\mathfrak{B}[\mathfrak{A}[\boldsymbol{t}]] \vdash \mathcal{B}}\,.$$

The details of the other subcases are omitted.

3. If $\varrho_r > 1$, then the shape of the part of the proof we are interested in is

$$\dfrac{\mathfrak{A} \vdash \mathcal{A} \qquad \dfrac{\mathfrak{B}[\mathcal{A}] \vdash \mathcal{B}}{\mathfrak{B}'[\mathcal{A}] \vdash \mathcal{B}}}{\mathfrak{B}'[\mathfrak{A}] \vdash \mathcal{B}}\,.$$

The cut formula is not the principal formula of the last rule above the right premise, and it is not $\boldsymbol{t}$ either. This implies that the cut formula is parametric in the rule. We consider the typical case, and then we consider the case where the contraction measure of the cut in the transformed proof is strictly less than the contraction measure of the cut in the original proof.

Let the last rule be $\mathsf{B}'\vdash$, and let us assume that the cut formula occurs in $\mathfrak{C}$:

$$\dfrac{\mathfrak{E} \vdash \mathcal{B} \qquad \dfrac{\mathfrak{A}[\mathfrak{C}[\mathcal{B}];(\mathfrak{B};\mathfrak{D})] \vdash \mathcal{A}}{\mathfrak{A}[\mathfrak{B};\mathfrak{C}[\mathcal{B}];\mathfrak{D}] \vdash \mathcal{A}}}{\mathfrak{A}[\mathfrak{B};\mathfrak{C}[\mathfrak{E}];\mathfrak{D}] \vdash \mathcal{A}} \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\mathfrak{E} \vdash \mathcal{B} \qquad \mathfrak{A}[\mathfrak{C}[\mathcal{B}];(\mathfrak{B};\mathfrak{D})] \vdash \mathcal{A}}{\mathfrak{A}[\mathfrak{C}[\mathfrak{E}];(\mathfrak{B};\mathfrak{D})] \vdash \mathcal{A}}}{\mathfrak{A}[\mathfrak{B};\mathfrak{C}[\mathfrak{E}];\mathfrak{D}] \vdash \mathcal{A}}\,.$$

Let us assume that the last rule above the right premise is $\mathsf{W}\vdash$ with $\mathcal{B}$ occurring in $\mathfrak{C}$. The original and the transformed proofs look as follows:

$$\dfrac{\mathfrak{D} \vdash \mathcal{B} \qquad \dfrac{\mathfrak{A}[\mathfrak{B};\mathfrak{C}[\mathcal{B}];\mathfrak{C}[\mathcal{B}]] \vdash \mathcal{A}}{\mathfrak{A}[\mathfrak{B};\mathfrak{C}[\mathcal{B}]] \vdash \mathcal{A}}}{\mathfrak{A}[\mathfrak{B};\mathfrak{C}[\mathfrak{D}]] \vdash \mathcal{A}} \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\mathfrak{D} \vdash \mathcal{B} \qquad \dfrac{\mathfrak{D} \vdash \mathcal{B} \qquad \mathfrak{A}[\mathfrak{B};\mathfrak{C}[\mathcal{B}];\mathfrak{C}[\mathcal{B}]] \vdash \mathcal{A}}{\mathfrak{A}[\mathfrak{B};\mathfrak{C}[\mathfrak{D}];\mathfrak{C}[\mathcal{B}]] \vdash \mathcal{A}}}{\mathfrak{A}[\mathfrak{B};\mathfrak{C}[\mathfrak{D}];\mathfrak{C}[\mathfrak{D}]] \vdash \mathcal{A}}}{\mathfrak{A}[\mathfrak{B};\mathfrak{C}[\mathfrak{D}]] \vdash \mathcal{A}}\,.$$

This completes the proof of the cut theorem. □

Notes

1. We consider the present paper to be the first part of the solution. The second half of the solution is Bimbó and Dunn [14]. For abstracts of two related conference talks, see Bimbó and Dunn [12] and Bimbó and Dunn [13].

   We had an opportunity to work together in Edmonton on the $T_{\to}$ problem for almost a week in November 2010. At that time, we were able to discuss the general strategy and some of the details of our solution. Various events and time commitments have delayed the completion of our second paper. However, since the time we submitted this paper in May 2011, we have completed and submitted the second paper mentioned above.

   Vincent Padovani has not long ago also announced a solution in the draft paper posted at www.pps.jussieu.fr/~padovani/te_mscs_draft.pdf, but his argument is complex, and we have not had a chance to read, let alone verify it.

2. As says Došen [19, p. 344]: "Though Church has introduced the implicational fragment of **R** only in 1951, the idea on which the implication of **R** is based is analogous to what is in his λ-I calculus of the thirties, where the term $\lambda x.t$ is not well formed if x does not occur free in the term t."

3. Došen [19] points out that relevance logic goes back at least as far as Orlov [36], who constructed an axiomatization of the implication-negation fragment of R. Church and Moh, however, provided a *deduction theorem*, which is absent from Orlov's treatment. They gave purely implicational axioms from which all the implicational theorems of R can be derived—without detours through negation as with Orlov's axiomatization.

4. Meyer and McRobbie [32], [33] investigate the use of multisets for understanding the grouping of premises in natural deductions within the system R and certain fragments and extensions.

5. This suggests a third kind of multiplicity, wherein the order but not the number of occurrences in a given "place" make a difference. Thus the first two would differ but not the second from the third. No applications come to mind.

6. There seems to be yet no standard notation for multisets, but square brackets seem to be the emerging standard, which is somewhat unfortunate, because we will be using square brackets in another standard sense later. However, no confusion is likely to be caused, since we will not rely on notation to differentiate between sets, multisets, and sequences—rather we just list their elements and let the context determine which data type is meant.

7. For further information about relevance logics, including their semantic interpretation, see Meyer and Routley [34], Routley et al. [37], Anderson, Belnap, and Dunn [4], and Bimbó and Dunn [11].

8. See Schönfinkel [38], Curry, Hindley, and Seldin [18], as well as Mares and Meyer [28] for information about combinators and their connection to relevance logics.

9. The number of inductions may be disheartening, but at this time seems necessary. The referee called to our attention Ciabattoni and Terui [16] as a potential simplification of our proof. Unfortunately, their theorem does not cover $LR^{\boldsymbol{t}}_{\to}$.

10. This calculus is also described in [22, Section 3.6], and represents Dunn's reconstruction of Belnap and Wallace's [5] (published as [6]) reconstruction of the argument behind [27].

11. $LT^{\boldsymbol{t}}_{\to}$ is the same calculus as the one with the identical label in [10, Section 3.2].

12. See [10] and Bimbó [7].

13. For a precise description and a simplification of this class of terms, see Bimbó [8].

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