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A Simple Proof that Super-Consistency Implies Cut Elimination

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Abstract We give a simple and direct proof that super-consistency implies the cut-elimination property in deduction modulo. This proof can be seen as a simplification of the proof that super-consistency implies proof normalization. It also takes ideas from the semantic proofs of cut elimination that proceed by proving the completeness of the cut-free calculus. As an application, we compare our work with the cut-elimination theorems in higher-order logic that involve V-complexes.

1 Introduction

Deduction modulo is an extension of predicate logic where some axioms may be replaced by rewrite rules. For instance, the axiom $x + 0 = x$ may be replaced by the rewrite rule $x + 0 \longrightarrow x$, and the axiom $x \subseteq y \Leftrightarrow \forall z\ (z \in x \Rightarrow z \in y)$ may be replaced by the rewrite rule $x \subseteq y \longrightarrow \forall z\ (z \in x \Rightarrow z \in y)$.

In the model theory of deduction modulo, it is important to distinguish the fact that some propositions are computationally equivalent, that is, congruent (e.g., $x \subseteq y$ and $\forall z\ (z \in x \Rightarrow z \in y)$), in which case they should have the same value in a model, from the fact that they are provably equivalent, in which case they may have different values. This has led, in Dowek [4], to the introduction of a generalization of Heyting algebras called *truth values algebras* and a notion of $\mathcal{B}$-valued model, where $\mathcal{B}$ is a truth values algebra. We have called *super-consistent* the theories that have a $\mathcal{B}$-valued model for all truth values algebras $\mathcal{B}$, and we have given examples of super-consistent theories as well as examples of consistent theories that are not super-consistent.

In deduction modulo, there are theories for which there exist proofs that do not normalize. But, we have proved in [4] that all proofs normalize in super-consistent

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theories. This proof proceeds by observing that reducibility candidates (see Girard [9]) can be structured in a truth values algebra and thus that super-consistent theories have reducibility candidate-valued models. Then, the existence of such a model implies proof normalization (see Dowek and Werner [8]) and hence cut elimination. As many theories, in particular arithmetic and simple type theory, are super-consistent, we get Gentzen's and Girard's theorems as corollaries.

This paper is an attempt to simplify this proof replacing the algebra of reducibility candidates $\mathcal{C}$ by a simpler truth values algebra $\mathcal{S}$. Reducibility candidates are sets of proofs. We show that we can replace each proof of such a set by its conclusion, obtaining this way sets of sequents, rather than sets of proofs, for truth values.

Although the truth values of our model are sets of sequents, our cut-elimination proof uses another truth values algebra whose elements are sets of contexts: the algebra of contexts Ω, which happens to be a Heyting algebra. From any $\mathcal{S}$-valued model of a theory we build a second-level model that is Ω-valued and that we use to show cut elimination.

This technique gives a proof that uses ideas taken from both methods employed to prove cut elimination: proof-term normalization and completeness of the cut-free calculus. From the first come the ideas of truth values algebra and neutral proofs, and from the second, the idea of building a model such that sequents valid in this model have cut-free proofs.

This paper is an extended version of the conference paper of Dowek and Hermant [7]. Some technical inaccuracies of [7] have been corrected in this version. In Section 2 we recall the technical material that will be useful to understand Section 3, which is the core of the paper. At the end of the paper, we provide an analysis of the proof obtained in the case of higher-order logic and compare it with other semantic proofs.

2 Super-Consistency

To keep the paper self contained, we recall in this section the definition of deduction modulo, truth values algebras, $\mathcal{B}$-valued models, and super-consistency. A more detailed presentation can be found in [4].

2.1 Deduction modulo Deduction modulo (see Dowek, Hardin, and Kirchner [6], Dowek and Werner [8]) is an extension of predicate logic (either single-sorted or many-sorted predicate logic) where a theory is defined by a set of axioms Γ and a congruence $\equiv$, itself defined by a confluent rewrite system rewriting terms to terms and atomic propositions to propositions.

In this paper we consider natural deduction rules. These rules are modified to take the congruence $\equiv$ into account. For example, the elimination rule of the implication is not formulated as the usual

$$\frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$$

but as

$$\frac{\Gamma \vdash C \quad \Gamma \vdash A}{\Gamma \vdash B} C \equiv A \Rightarrow B.$$

All the deduction rules are modified in a similar way, as shown in Figure 1. Note that the usual proviso that x does not appear freely in the $\forall_i$- and $\exists_e$-rules holds, as informally reminded by the side condition (see [8] for a more thorough presentation).

$$\frac{}{\Gamma \vdash B} \text{ axiom } (B \equiv A \text{ for some } A \text{ in } \Gamma)$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash C} \Rightarrow_i, C \equiv A \Rightarrow B \qquad \frac{\Gamma \vdash C \qquad \Gamma \vdash A}{\Gamma \vdash B} \Rightarrow_e, C \equiv A \Rightarrow B$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash C} \wedge_i, C \equiv A \wedge B \qquad \frac{\Gamma \vdash C}{\Gamma \vdash A} \wedge_e\text{-1}, C \equiv A \wedge B$$

$$\frac{\Gamma \vdash C}{\Gamma \vdash B} \wedge_e\text{-2}, C \equiv A \wedge B$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash C} \vee_i\text{-1}, C \equiv A \vee B$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash C} \vee_i\text{-2}, C \equiv A \vee B \qquad \frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C \qquad \Gamma \vdash D}{\Gamma \vdash C} \vee_e, \; D \equiv A \vee B$$

$$\frac{}{\Gamma \vdash A} \top_i, A \equiv \top \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A} \bot_e, B \equiv \bot$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash B} \forall_i, B \equiv \forall x A, \text{ with } x \notin FV(\Gamma) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash C} \forall_e, B \equiv \forall x A \text{ and } C \equiv (t/x)A$$

$$\frac{\Gamma \vdash C}{\Gamma \vdash B} \exists_i, B \equiv \exists x A \text{ and } C \equiv (t/x)A \qquad \frac{\Gamma \vdash C \qquad \Gamma, A \vdash B}{\Gamma \vdash B} \exists_e, C \equiv \exists x A, \text{ with } x \notin FV(\Gamma, B)$$

Figure 1 Rules of natural deduction modulo

In deduction modulo, there are theories for which there exist proofs that do not normalize. For instance, in the theory formed with the rewrite rule $P \longrightarrow (P \Rightarrow Q)$, the proposition Q has a proof

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash P \Rightarrow Q}\text{axiom} \quad \dfrac{}{P \vdash P}\text{axiom}}{P \vdash Q} \Rightarrow\text{-elim}}{\vdash P \Rightarrow Q} \Rightarrow\text{-intro} \qquad \dfrac{\dfrac{\dfrac{}{P \vdash P \Rightarrow Q}\text{axiom} \quad \dfrac{}{P \vdash P}\text{axiom}}{P \vdash Q} \Rightarrow\text{-elim}}{\vdash P} \Rightarrow\text{-intro}}{\vdash Q} \Rightarrow\text{-elim}$$

that does not normalize. In some other theories, such as the theory formed with the rewrite rule $P \longrightarrow (Q \Rightarrow P)$, all proofs strongly normalize.

In deduction modulo, like in predicate logic, closed normal proofs always end with an introduction rule. Thus, if a theory can be expressed in deduction modulo with rewrite rules only, that is, with no axioms, in such a way that proofs modulo these rewrite rules strongly normalize, then the theory is consistent. It has the disjunction property and the witness property, and various proof search methods for this theory are complete.

Many theories can be expressed in deduction modulo with rewrite rules only, in particular arithmetic and simple type theory, and the notion of cut of deduction modulo subsumes the notions of cut defined for each of these theories. For instance, simple type theory can be defined as follows.

Definition 2.1 (Simple type theory; see [2], [5], [3]) The sorts are inductively defined:

- ι and o are sorts;
- if T and U are sorts, then $T \to U$ is a sort.

The language contains the constants $K_{T,U}$ of sort $T \to U \to T$, $S_{T,U,V}$ of sort $(T \to U \to V) \to (T \to U) \to T \to V$, $\dot{\top}$ of sort o and $\dot{\bot}$ of sort o, $\dot{\Rightarrow}$, $\dot{\wedge}$, and $\dot{\vee}$ of sort $o \to o \to o$, $\dot{\forall}_T$ and $\dot{\exists}_T$ of sort $(T \to o) \to o$, the function symbols $\alpha_{T,U}$ of rank $\langle T \to U, T, U\rangle$, and the predicate symbol ε of rank $\langle o\rangle$.

The rules are

$$\begin{aligned}
\alpha\big(\alpha(\alpha(S_{T,U,V}, x), y), z\big) &\longrightarrow \alpha\big(\alpha(x, z), \alpha(y, z)\big),\\
\alpha\big(\alpha(K_{T,U}, x), y\big) &\longrightarrow x,\\
\varepsilon(\dot{\top}) &\longrightarrow \top,\\
\varepsilon(\dot{\bot}) &\longrightarrow \bot,\\
\varepsilon\big(\alpha(\alpha(\dot{\Rightarrow}, x), y)\big) &\longrightarrow \varepsilon(x) \Rightarrow \varepsilon(y),\\
\varepsilon\big(\alpha(\alpha(\dot{\wedge}, x), y)\big) &\longrightarrow \varepsilon(x) \wedge \varepsilon(y),\\
\varepsilon\big(\alpha(\alpha(\dot{\vee}, x), y)\big) &\longrightarrow \varepsilon(x) \vee \varepsilon(y),\\
\varepsilon\big(\alpha(\dot{\forall}_T, x)\big) &\longrightarrow \forall y \varepsilon\big(\alpha(x, y)\big),\\
\varepsilon\big(\alpha(\dot{\exists}_T, x)\big) &\longrightarrow \exists y \varepsilon\big(\alpha(x, y)\big).
\end{aligned}$$

2.2 Truth values algebras

Definition 2.2 (Truth values algebra) Let B be a set whose elements are called *truth values*, let B^+ be a subset of B whose elements are called *positive* truth values, let $\mathcal{A}$ and $\mathcal{E}$ be subsets of $\wp(B)$, let $\tilde{\top}$ and $\tilde{\bot}$ be elements of B, let $\tilde{\Rightarrow}$, $\tilde{\wedge}$, and $\tilde{\vee}$ be functions from $B \times B$ to B, let $\tilde{\forall}$ be a function from $\mathcal{A}$ to B, and let $\tilde{\exists}$ be a function from $\mathcal{E}$ to B. The structure $\mathcal{B} = \langle B, B^+, \mathcal{A}, \mathcal{E}, \tilde{\top}, \tilde{\bot}, \tilde{\Rightarrow}, \tilde{\wedge}, \tilde{\vee}, \tilde{\forall}, \tilde{\exists}\rangle$ is said to be a *truth values algebra* if the set B^+ is *closed under the intuitionistic deduction rules*, that is, if for all a, b, c in B, A in $\mathcal{A}$, and E in $\mathcal{E}$, we have the following:

1. if $a \tilde{\Rightarrow} b \in B^+$ and $a \in B^+$, then $b \in B^+$;
2. $a \tilde{\Rightarrow} b \tilde{\Rightarrow} a \in B^+$;
3. $(a \tilde{\Rightarrow} b \tilde{\Rightarrow} c) \tilde{\Rightarrow} (a \tilde{\Rightarrow} b) \tilde{\Rightarrow} a \tilde{\Rightarrow} c \in B^+$;
4. $\tilde{\top} \in B^+$;
5. $\tilde{\bot} \tilde{\Rightarrow} a \in B^+$;
6. $a \tilde{\Rightarrow} b \tilde{\Rightarrow} (a \tilde{\wedge} b) \in B^+$;
7. $(a \tilde{\wedge} b) \tilde{\Rightarrow} a \in B^+$;
8. $(a \tilde{\wedge} b) \tilde{\Rightarrow} b \in B^+$;
9. $a \tilde{\Rightarrow} (a \tilde{\vee} b) \in B^+$;
10. $b \tilde{\Rightarrow} (a \tilde{\vee} b) \in B^+$;
11. $(a \tilde{\vee} b) \tilde{\Rightarrow} (a \tilde{\Rightarrow} c) \tilde{\Rightarrow} (b \tilde{\Rightarrow} c) \tilde{\Rightarrow} c \in B^+$;
12. the set $a \tilde{\Rightarrow} A = \{a \tilde{\Rightarrow} e \mid e \in A\}$ is in $\mathcal{A}$, and the set $E \tilde{\Rightarrow} a = \{e \tilde{\Rightarrow} a \mid e \in E\}$ is in $\mathcal{A}$;

13. if all elements of A are in B^+, then $\tilde{\forall} A \in B^+$;
14. $\tilde{\forall}(a \mathrel{\tilde{\Rightarrow}} A) \mathrel{\tilde{\Rightarrow}} a \mathrel{\tilde{\Rightarrow}} (\tilde{\forall} A) \in B^+$;
15. if $a \in A$, then $(\tilde{\forall} A) \mathrel{\tilde{\Rightarrow}} a \in B^+$;
16. if $a \in E$, then $a \mathrel{\tilde{\Rightarrow}} (\tilde{\exists} E) \in B^+$;
17. $(\tilde{\exists} E) \mathrel{\tilde{\Rightarrow}} \tilde{\forall}(E \mathrel{\tilde{\Rightarrow}} a) \mathrel{\tilde{\Rightarrow}} a \in B^+$.

Proposition 2.3 *Any Heyting algebra is a truth values algebra. The operations* $\tilde{\top}$, $\tilde{\wedge}$, $\tilde{\forall}$ *are greatest lower bounds, the operations* $\tilde{\bot}$, $\tilde{\vee}$, $\tilde{\exists}$ *are least upper bounds, the operation* $\tilde{\Rightarrow}$ *is the arrow of the Heyting algebra, and* $B^+ = \{\tilde{\top}\}$.

Proof See [4, Proposition 1]. □

Definition 2.4 (Fullness) A truth values algebra is said to be *full* if $\mathcal{A} = \mathcal{E} = \wp(B)$, that is, if $\tilde{\forall} A$ and $\tilde{\exists} A$ exist for all subsets A of B.

Definition 2.5 (Ordered truth values algebra) An *ordered truth values algebra* is a truth values algebra together with a relation $\sqsubseteq$ on B such that

- $\sqsubseteq$ is an order relation, that is, a reflexive, antisymmetric, and transitive relation,
- B^+ is upward closed,
- $\tilde{\top}$ and $\tilde{\bot}$ are maximal and minimal elements,
- $\tilde{\wedge}$, $\tilde{\vee}$, $\tilde{\forall}$, and $\tilde{\exists}$ are monotone, and $\tilde{\Rightarrow}$ is left antimonotone and right monotone.

Definition 2.6 (Complete ordered truth values algebra) An ordered truth values algebra is said to be *complete* if every subset of B has a greatest lower bound for $\sqsubseteq$.

2.3 Models

Definition 2.7 ($\mathcal{B}$-structure) Let $\mathcal{L} = \langle f_i, P_j \rangle$ be a language for predicate logic, and let $\mathcal{B}$ be a truth values algebra. A *$\mathcal{B}$-structure* $\mathcal{M} = \langle M, \mathcal{B}, \hat{f}_i, \hat{P}_j \rangle$, for the language $\mathcal{L}$, is a structure such that $\hat{f}_i$ is a function from M^n to M where n is the arity of the symbol f_i and $\hat{P}_j$ is a function from M^n to B where n is the arity of the symbol P_j.

This definition extends trivially to many-sorted languages.

Definition 2.8 (Denotation) Let $\mathcal{B}$ be a truth values algebra, let $\mathcal{M}$ be a $\mathcal{B}$-structure, and let φ be an assignment. The denotation of propositions and terms in $\mathcal{M}$ is defined inductively as follows:

- $[\![x]\!]_\varphi = \varphi(x)$,
- $[\![f(t_1, \ldots, t_n)]\!]_\varphi = \hat{f}([\![t_1]\!]_\varphi, \ldots, [\![t_n]\!]_\varphi)$,
- $[\![P(t_1, \ldots, t_n)]\!]_\varphi = \hat{P}([\![t_1]\!]_\varphi, \ldots, [\![t_n]\!]_\varphi)$,
- $[\![\top]\!]_\varphi = \tilde{\top}$,
- $[\![\bot]\!]_\varphi = \tilde{\bot}$,
- $[\![A \Rightarrow B]\!]_\varphi = [\![A]\!]_\varphi \mathrel{\tilde{\Rightarrow}} [\![B]\!]_\varphi$,
- $[\![A \wedge B]\!]_\varphi = [\![A]\!]_\varphi \mathrel{\tilde{\wedge}} [\![B]\!]_\varphi$,
- $[\![A \vee B]\!]_\varphi = [\![A]\!]_\varphi \mathrel{\tilde{\vee}} [\![B]\!]_\varphi$,
- $[\![\forall x A]\!]_\varphi = \tilde{\forall}\{[\![A]\!]_{\varphi+(d/x)} \mid d \in M\}$,
- $[\![\exists x A]\!]_\varphi = \tilde{\exists}\{[\![A]\!]_{\varphi+(d/x)} \mid d \in M\}$.

Notice that the denotation of a proposition containing quantifiers may be undefined, but it is always defined if the truth values algebra is full.

Definition 2.9 (Denotation of a context and of a sequent) The denotation $[\![A_1, \ldots, A_n]\!]_\varphi$ of a context $A_1, \ldots, A_n$ is that of the proposition $A_1 \wedge \cdots \wedge A_n$. The denotation $[\![A_1, \ldots, A_n \vdash B]\!]_\varphi$ of the sequent $A_1, \ldots, A_n \vdash B$ is that of the proposition $(A_1 \wedge \cdots \wedge A_n) \Rightarrow B$.

Definition 2.10 (Model) A proposition A is said to be *valid* in a $\mathcal{B}$-structure $\mathcal{M}$, and the $\mathcal{B}$-structure $\mathcal{M}$ is said to be a *model of* A if, for all assignments φ, $[\![A]\!]_\varphi$ is defined and is a positive truth value.

Consider a theory in deduction modulo defined by a set of axioms Γ and a congruence $\equiv$. The $\mathcal{B}$-structure $\mathcal{M}$ is said to be a *model of the theory* $\Gamma, \equiv$ if all axioms of Γ are valid in $\mathcal{M}$ and for all terms or propositions A and B such that $A \equiv B$ and assignments φ, $[\![A]\!]_\varphi$, and $[\![B]\!]_\varphi$ are defined and $[\![A]\!]_\varphi = [\![B]\!]_\varphi$.

Deduction modulo is sound and complete with respect to this notion of model.

Proposition 2.11 (Soundness and completeness) *The proposition A is provable in the theory formed with the axioms Γ and the congruence $\equiv$ if and only if it is valid in all the models of $\Gamma, \equiv$ where the truth values algebra is full, ordered, and complete.*

Proof See [4, Theorem 1]. □

2.4 Super-consistency

Definition 2.12 (Super-consistent) A theory in deduction modulo formed with the axioms Γ and the congruence $\equiv$ is *super-consistent* if it has a $\mathcal{B}$-valued model for all full, ordered, and complete truth values algebras $\mathcal{B}$.

Proposition 2.13 *Simple type theory is super-consistent.*

Proof Let $\mathcal{B}$ be a full truth values algebra. We build the model $\mathcal{M}$ as follows. The domain M_ι is any nonempty set, for instance, the singleton $\{0\}$, the domain M_o is B, and the domain $M_{T \to U}$ is the set $M_U^{M_T}$ of functions from M_T to M_U. The interpretation of the symbols of the language is $\hat{S}_{T,U,V} = a \mapsto \big(b \mapsto (c \mapsto a(c)(b(c)))\big)$, $\hat{K}_{T,U} = a \mapsto (b \mapsto a)$, $\hat{\alpha}(a,b) = a(b)$, $\hat{\varepsilon}(a) = a$, $\hat{\dot{\top}} = \tilde{\top}$, $\hat{\dot{\bot}} = \tilde{\bot}$, $\hat{\dot{\Rightarrow}} = \tilde{\Rightarrow}$, $\hat{\dot{\wedge}} = \tilde{\wedge}$, $\hat{\dot{\vee}} = \tilde{\vee}$, $\hat{\dot{\forall}}_T = a \mapsto \tilde{\forall}(\text{Range}(a))$, and $\hat{\dot{\exists}}_T = a \mapsto \tilde{\exists}(\text{Range}(a))$, where $\text{Range}(a)$ is the range of the function a. The model $\mathcal{M}$ is a $\mathcal{B}$-valued model of simple type theory. □

3 Cut Elimination

3.1 The algebra of sequents

Definition 3.1 (Neutral proof) A proof is said to be *neutral* if its last rule is the axiom rule or an elimination rule but not an introduction rule.

We now define the notion of cut-free proof. Instead of giving a syntactic definition (absence of cut) we give a positive inductive definition.

Definition 3.2 (Cut-free proofs) *Cut-free* proofs are defined inductively as follows:

- a proof that ends with the axiom rule is cut-free;
- a proof that ends with an introduction rule and where the premises of the last rule are proved with cut-free proofs is cut-free;

- a proof that ends with an elimination rule and where the major premise of the last rule is proved with a neutral cut-free proof and the other premises with cut-free proofs is cut-free.

Definition 3.3 (The algebra of sequents)

- $\tilde{\top}$ is the set of sequents $\Gamma \vdash C$ that have a neutral cut-free proof or are such that $C \equiv \top$.
- $\tilde{\bot}$ is the set of sequents $\Gamma \vdash C$ that have a neutral cut-free proof.
- $a \mathbin{\tilde{\wedge}} b$ is the set of sequents $\Gamma \vdash C$ that have a neutral cut-free proof or are such that $C \equiv (A \wedge B)$ with $(\Gamma \vdash A) \in a$ and $(\Gamma \vdash B) \in b$.
- $a \mathbin{\tilde{\vee}} b$ is the set of sequents $\Gamma \vdash C$ that have a neutral cut-free proof or are such that $C \equiv (A \vee B)$ with $(\Gamma \vdash A) \in a$ or $(\Gamma \vdash B) \in b$.
- $a \mathbin{\tilde{\Rightarrow}} b$ is the set of sequents $\Gamma \vdash C$ that have a neutral cut-free proof or are such that $C \equiv (A \Rightarrow B)$, and for all contexts Σ such that $(\Gamma, \Sigma \vdash A) \in a$, we have $(\Gamma, \Sigma \vdash B) \in b$.
- $\tilde{\forall} S$ is the set of sequents $\Gamma \vdash C$ that have a neutral cut-free proof or are such that $C \equiv (\forall x A)$ and for every term t and every a in S, $(\Gamma \vdash (t/x)A) \in a$.
- $\tilde{\exists} S$ is the set of sequents $\Gamma \vdash C$ that have a neutral cut-free proof or are such that $C \equiv (\exists x A)$ and for some term t and some a in S, $(\Gamma \vdash (t/x)A) \in a$.

Let S be the smallest set of sets of sequents closed under $\tilde{\top}$, $\tilde{\bot}$, $\tilde{\wedge}$, $\tilde{\vee}$, $\tilde{\Rightarrow}$, $\tilde{\forall}$, $\tilde{\exists}$ and by arbitrary intersections.

Proposition 3.4 *The structure $\mathcal{S} = \langle S, S, \wp(S), \wp(S), \tilde{\top}, \tilde{\bot}, \tilde{\Rightarrow}, \tilde{\wedge}, \tilde{\vee}, \tilde{\forall}, \tilde{\exists}, \subseteq \rangle$ is a full, ordered, and complete truth values algebra.*

Proof As all truth values are positive, the conditions of Definition 2.2 are obviously met. Thus $\mathcal{S}$ is a truth values algebra. As the domains of $\tilde{\forall}$ and $\tilde{\exists}$ are defined as $\wp(S)$, this algebra is full. As it is closed under arbitrary intersections, all subsets of S have a greatest lower bound; thus all subsets of S have a least upper bound, and the algebra is complete. □

Remark The algebra $\mathcal{S}$ is not a Heyting algebra. In particular, $\tilde{\top} \mathbin{\tilde{\wedge}} \tilde{\top}$ and $\tilde{\top}$ are different: the first set contains the sequent $\vdash \top \wedge \top$, but not the second.

Proposition 3.5 *For all elements a of S, contexts Γ, and propositions A and B:*

- $(\Gamma, A \vdash A) \in a$;
- *if* $(\Gamma \vdash B) \in a$, *then* $(\Gamma, A \vdash B) \in a$;
- *if* $(\Gamma, A, A \vdash B) \in a$, *then* $(\Gamma, A \vdash B) \in a$;
- *if* $(\Gamma \vdash A) \in a$ *and* $B \equiv A$, *then* $(\Gamma \vdash B) \in a$;
- *if* $(\Gamma \vdash A) \in a$, *then* $\Gamma \vdash A$ *has a cut-free proof.*

Proof The first proposition is proved by noticing that the sequent $\Gamma, A \vdash A$ has a neutral cut-free proof. The others are proved by a simple induction on the construction of a. For instance, if $a = c \mathbin{\tilde{\wedge}} d$, then

- if $\Gamma \vdash B$ has a neutral cut-free proof, so has $\Gamma, A \vdash B$; otherwise $B \equiv (C \wedge D)$, $(\Gamma \vdash C) \in c$, and $(\Gamma \vdash D) \in d$; by induction hypothesis, $(\Gamma, A \vdash C) \in c$ and $(\Gamma, A \vdash D) \in d$, so by definition $\Gamma, A \vdash B \in c \mathbin{\tilde{\wedge}} d$;
- if $\Gamma, A, A \vdash B$ has a neutral cut-free proof, so has $\Gamma, A \vdash B$; otherwise $B \equiv (C \wedge D)$, $(\Gamma, A, A \vdash C) \in c$, and $(\Gamma, A, A \vdash D) \in d$; by induction hypothesis, $(\Gamma, A \vdash C) \in c$ and $(\Gamma, A \vdash D) \in d$, so by definition $\Gamma, A \vdash B \in c \mathbin{\tilde{\wedge}} d$;

- if $\Gamma \vdash A$ has a neutral cut-free proof, so has $\Gamma \vdash B$; otherwise $A \equiv (C \wedge D) \equiv B$, $(\Gamma \vdash C) \in c$, and $(\Gamma \vdash D) \in d$, so by definition $\Gamma \vdash B \in c \tilde{\wedge} d$;
- if $\Gamma \vdash A$ has a neutral cut-free proof there is nothing to show; otherwise $A \equiv (C \wedge D)$, $(\Gamma \vdash C) \in c$, and $(\Gamma \vdash D) \in d$; by induction hypothesis $\Gamma \vdash C$ and $\Gamma \vdash D$ have cut-free proofs, and we can add a $\wedge$-intro rule to obtain a cut-free proof of $\Gamma \vdash A$. □

Consider a super-consistent theory $\Gamma, \equiv$. By definition, it has an $\mathcal{S}$-model $\mathcal{M}$. In the rest of the paper, $\mathcal{M}$ refers to this model. Its domain is written M.

Proposition 3.6 (Substitution) *Let A be a proposition, let φ be an assignment, let x be a variable, and let t, u be terms. Let $\varphi' = \varphi + (\llbracket t \rrbracket_\varphi / x)$. Then $\llbracket (t/x)u \rrbracket_\varphi = \llbracket u \rrbracket_{\varphi'}$ and $\llbracket (t/x)A \rrbracket_\varphi = \llbracket A \rrbracket_{\varphi'}$.*

Proof The proof follows by structural induction. □

3.2 The algebra of contexts

Definition 3.7 (Fiber) Let b be a set of sequents, let A be a proposition, let σ be a substitution, and let f be a function mapping propositions to sets of sequents. We define the parameterized fiber over A in b, $b \lhd_\sigma^f A$ as the set of contexts $\Gamma = A_1, \ldots, A_n$ such that for any Δ such that $(\Delta \vdash \sigma A_i) \in f(A_i)$ for any i, we have $(\Delta \vdash \sigma A) \in b$.

Definition 3.8 (Γ-adapted context) Let $\Gamma = A_1, \ldots, A_n$ be a set of propositions, let φ be an assignment, and let σ be a substitution. Let Δ be a set of propositions. We say that Δ is Γ*-adapted for* φ, σ (in short: Δ is Γ-adapted) if, for any i, $(\Delta \vdash \sigma A_i) \in \llbracket A_i \rrbracket_\varphi$.

Proposition 3.9 (Composition of adapted contexts) *Let Γ_1, Γ_2 be two sets of propositions. Let φ be an assignment, let σ be a substitution, and let Δ_1, Δ_2 be Γ_1-adapted (resp., Γ_2-adapted) contexts for φ, σ. Then*

- *Δ_1, Δ_2 is (Γ_1, Γ_2)-adapted for φ, σ;*
- *if $\Delta_1 = \Delta, B, B$, then Δ, B is Γ_1-adapted for φ, σ.*

Proof Let A be a member of Γ_1, and let A' be a member of Γ_2. $(\Delta_1 \vdash \sigma A) \in \llbracket A \rrbracket_\varphi$ and $(\Delta_2 \vdash \sigma A') \in \llbracket A' \rrbracket_\varphi$ by definition. Then

- $(\Delta_1, \Delta_2 \vdash \sigma A) \in \llbracket A \rrbracket_\varphi$ and $(\Delta_1, \Delta_2 \vdash \sigma A') \in \llbracket A' \rrbracket_\varphi$ by the second point of Proposition 3.5;
- if Δ_1 matches the hypothesis, then $(\Delta, B \vdash \sigma A) \in \llbracket A \rrbracket_\varphi$ by the third point of Proposition 3.5. □

Definition 3.10 (Outer value) Let A be a proposition. We define the set of contexts $[A]$ as the set of contexts $\Gamma = A_1, \ldots, A_n$ such that for any assignment φ, any substitution σ, and any context Δ, whenever $(\Delta \vdash \sigma A_i) \in \llbracket A_i \rrbracket_\varphi$ for any i (in other words, Δ is Γ-adapted), then $(\Delta \vdash \sigma A) \in \llbracket A \rrbracket_\varphi$. Note that $[A]$ is precisely $\llbracket A \rrbracket \lhd_\sigma^{\llbracket \_ \rrbracket} A$.

Proposition 3.11 *For any context Γ and any propositions A and B,*

- *$(\Gamma, A) \in [A]$;*
- *if $\Gamma \in [B]$, then $(\Gamma, A) \in [B]$;*

- *if* $(\Gamma, A, A) \in [B]$*, then* $(\Gamma, A) \in [B]$*;*
- *if* $\Gamma \in [A]$ *and* $B \equiv A$*, then* $\Gamma \in [B]$*;*
- *if* $\Gamma \in [A]$*, then* $\Gamma \vdash A$ *has a cut-free proof.*

Proof This follows directly from the definitions and Proposition 3.5. For the first point, consider some φ, σ and a Δ that is (Γ, A)-adapted for φ, σ. In particular, by Definition 3.10, $(\Delta \vdash \sigma A) \in [\![A]\!]_\varphi$, which was to be proved. The second point restricts the sets of contexts: if Δ is (Γ, A)-adapted then it is obviously Γ-adapted, and the conditions of Definition 3.10 are fulfilled. Similarly, for the third point, if Δ is (Γ, A, A)-adapted, then it is (Γ, A)-adapted. The fourth point follows from Proposition 3.5, since $[\![A]\!]_\varphi = [\![B]\!]_\varphi$ by definition of the model $\mathcal{M}$. The last point is a consequence of Proposition 3.5. □

Definition 3.12 (The algebra of contexts) Let Ω be the smallest set of sets of contexts containing all the $[A]$ for some proposition A and closed under arbitrary intersections.

Remark 3.13 Notice that an element c of Ω can always be written in the form

$$c = \bigcap_{i \in \Lambda_c} [A_i].$$

Proposition 3.14 *The set* Ω *ordered by inclusion is a complete Heyting algebra.*

Proof As Ω is ordered by inclusion and closed under arbitrary intersections, the greatest lower bound of any subset of Ω can be defined as the intersection of all its elements, and it always exists. Thus, all its subsets also have least upper bounds, namely, the greatest lower bound of its majorizers.

The operations $\check{\top}$, $\check{\wedge}$, and $\check{\forall}$ are defined as nullary, binary, and infinitary greatest lower bounds, and the operations $\check{\bot}$, $\check{\vee}$, and $\check{\exists}$ are defined as nullary, binary, and infinitary least upper bounds. Finally, the arrow $\check{\Rightarrow}$ of two elements a and b is the least upper bound of all the c in Ω such that $a \cap c \leq b$:

$$a \check{\Rightarrow} b = \check{\exists}\{c \in \Omega \mid a \cap c \leq b\}.$$

To prove that Ω is a Heyting algebra, $\check{\Rightarrow}$ must have some specific properties (see Troelstra and van Dalen [17]). The following condition is necessary and sufficient:

$$a \leq b \check{\Rightarrow} c \qquad \text{iff } a \cap b \leq c.$$

The reverse implication holds by elementary lattice theory from the very definition of $\check{\Rightarrow}$, but the direct one requires some work. Let $\Gamma = A_1, \ldots, A_n$, and assume that it belongs to $a \cap b$. Our aim is to show that $\Gamma \in c = \bigcap [C_i]$. Let C be one of the C_i; we show that $\Gamma \in [C]$.

Unfolding the assumption, we know that $a \leq \check{\exists}\mathsf{D}$, with $\mathsf{D} = \{d \mid b \cap d \leq c\}$, or, unfolding a step further (see (2)),

$$a \leq \bigcap \left\{ [E] \;\middle|\; \bigcup \mathsf{D} \subseteq [E] \right\}. \tag{1}$$

We first show that $\bigcup \mathsf{D} \subseteq [\Gamma \Rightarrow C]$, where $\Gamma \Rightarrow C$ denotes $A_1 \Rightarrow \cdots A_n \Rightarrow C$ and later take advantage of this in (1).

Let $d \in \mathsf{D}$, and let $\Delta \in d$. We have $(\Delta, \Gamma) \in d$ and $(\Delta, \Gamma) \in a \cap b \subseteq b$ by Proposition 3.11. So, $(\Delta, \Gamma) \in b \cap d$, and since $d \in \mathsf{D}$, $(\Delta, \Gamma) \in c \subseteq [C]$.

We now prove by induction on n (the cardinality of Γ) that if $(\Delta, \Gamma) \in [C]$, then $\Delta \in [\Gamma \Rightarrow C]$. This is immediate if $n = 0$. Otherwise, we apply induction hypothesis, and we get that $(\Delta, A_1) \in [\Gamma_1 \Rightarrow C]$. Let φ be an assignment, let σ be a substitution, and let Σ_Δ be a Δ-adapted context. We must show that $(\Sigma_\Delta \vdash \sigma\Gamma \Rightarrow \sigma C) = (\Sigma_\Delta \vdash \sigma A_1 \Rightarrow \sigma\Gamma_1 \Rightarrow \sigma C)$ belongs to $[\![\Gamma \Rightarrow C]\!]_\varphi = [\![A_1]\!]_\varphi \tilde{\Rightarrow} [\![\Gamma_1 \Rightarrow C]\!]_\varphi$.

Let Σ_{A_1} be such that $\Sigma_{A_1} \vdash \sigma A_1 \in [\![A_1]\!]_\varphi$. Σ_{A_1} is A_1-adapted, and from Proposition 3.9, $\Sigma_\Delta, \Sigma_{A_1}$ is (Δ, A_1)-adapted. So, $(\Sigma_\Delta, \Sigma_{A_1} \vdash \sigma\Gamma_1 \Rightarrow \sigma A) \in [\![\Gamma_1 \Rightarrow C]\!]_\varphi$. This exactly means that $\Sigma_\Delta \in [\![A_1]\!]_\varphi \tilde{\Rightarrow} [\![\Gamma_1 \Rightarrow C]\!]_\varphi$ from Definition 3.3 of $\tilde{\Rightarrow}$. Therefore, by the very Definition 3.10, $\Delta \in [A_1 \Rightarrow \Gamma_1 \Rightarrow C]$.

So $[\Gamma \Rightarrow C]$ is an upper bound of D and, by (1), of a. By hypothesis on Γ, $\Gamma \in a \cap b \subseteq a \subseteq [\Gamma \Rightarrow C]$.

Let us call Γ_i the context $A_{i+1}, \ldots, A_n$. Γ_i is a suffix of Γ, and we show, by induction on $i \leq n$, that $\Gamma \in [\Gamma_i \Rightarrow C]$. The base case has just been proved above.

For the inductive step, we assume that $\Gamma \in [\Gamma_i \Rightarrow C] = [A_{i+1} \Rightarrow \Gamma_{i+1} \Rightarrow C]$. Let φ be an assignment, let σ be a substitution, and let Δ_Γ be a Γ-adapted context. We show that $(\Delta_\Gamma \vdash \sigma\Gamma_{i+1} \Rightarrow \sigma C) \in [\![\Gamma_{i+1} \Rightarrow C]\!]_\varphi$. This will allow us to conclude that $\Gamma \in [\Gamma_{i+1} \Rightarrow C]$.

We have $(\Delta_\Gamma \vdash \sigma A_{i+1} \Rightarrow \sigma\Gamma_{i+1} \Rightarrow \sigma C) \in [\![A_{i+1} \Rightarrow \Gamma_{i+1} \Rightarrow C]\!]_\varphi$ by the induction hypothesis, so

$$(\Delta_\Gamma \vdash \sigma A_{i+1} \Rightarrow \sigma\Gamma_{i+1} \Rightarrow \sigma C) \in [\![A_{i+1}]\!]_\varphi \tilde{\Rightarrow} [\![\Gamma_{i+1} \Rightarrow C]\!]_\varphi.$$

If the sequent $\Delta_\Gamma \vdash \sigma A_{i+1} \Rightarrow \sigma\Gamma_{i+1} \Rightarrow \sigma C$ has a neutral cut-free proof, we add an elimination rule with a cut-free proof of $\Delta_\Gamma \vdash \sigma A_{i+1}$ (obtained by Proposition 3.5 since $(\Delta_\Gamma \vdash \sigma A_{i+1}) \in [\![A_{i+1}]\!]_\varphi$). This gives a neutral cut-free proof of the sequent $\Delta_\Gamma \vdash \sigma\Gamma_{i+1} \Rightarrow \sigma C$, and this sequent therefore belongs by Definition 3.3 to $[\![\Gamma_{i+1} \Rightarrow C]\!]_\varphi$.

Otherwise, following Definition 3.3 of $\tilde{\Rightarrow}$, since $(\Delta_\Gamma \vdash \sigma A_{i+1}) \in [\![A_{i+1}]\!]_\varphi$, we conclude directly that $(\Delta_\Gamma \vdash \sigma\Gamma_{i+1} \Rightarrow \sigma C) \in [\![\Gamma_{i+1} \Rightarrow C]\!]_\varphi$.

Consider now the nth case of the previous statement. It states $\Gamma \in [C]$, which was to be proved. This holds for any of the C_i such that $c = \bigcap_{i \in \Lambda_c} [C_i]$, so Γ belongs to their intersection c, and finally $a \cap b \subseteq c$ is proved. □

The binary least upper bound, $a \check{\vee} b$, of a and b is the intersection of all the elements of Ω that contain $a \cup b$. From Definition 3.12,

$$a \check{\vee} b = \bigcap_{(a \cup b) \subseteq c} c = \bigcap_{(a \cup b) \subseteq \bigcap [A_i]} \left(\bigcap [A_i] \right) = \bigcap_{(a \cup b) \subseteq [A]} [A].$$

The infinitary least upper bound $\check{\exists} E$ of the elements of a set E is the intersection of all the elements of Ω that contain the union of the elements of E. For the same reason as above,

$$\check{\exists} E = \bigcap_{(\bigcup E) \subseteq c} c = \bigcap_{(\bigcup E) \subseteq [A]} [A]. \tag{2}$$

Notice that the nullary least upper bound $\check{\bot}$ is $\bigcap \{a \mid d \leq a, \text{ for any } d \in \emptyset\}$, that is, the intersection of all the elements of Ω. Also, the nullary greatest lower bound $\check{\top}$ *is* the set of all contexts. We show in Proposition 3.15 below that it is equal to $[\top] \in \Omega$ and hence that this construction is well defined.

Finally, notice that Ω might be a nontrivial Heyting algebra, although the quotient Heyting algebra $\mathcal{S}/S^+$ is always trivial because $S^+ = S$. The construction of Definition 3.12 then does not boil down to this quotient and produces a more informative structure.

The next proposition, the key lemma of our proof, shows that the outer values of compound propositions can be obtained from the outer values of their components using the corresponding operation of the Heyting algebra Ω. Notice that, unlike most semantic cut-elimination proofs (see Okada [12], De Marco and Lipton [3], Hermant and Lipton [10]), we directly prove equalities in this lemma, and not just inclusions, although the cut-elimination proof is not completed yet.

Proposition 3.15 (Key lemma) *For all propositions A and B,*

- $[\top] = \check{\top}$,
- $[\bot] = \check{\bot}$,
- $[A \wedge B] = [A] \check{\wedge} [B]$,
- $[A \vee B] = [A] \check{\vee} [B]$,
- $[A \Rightarrow B] = [A] \check{\Rightarrow} [B]$,
- $[\forall x A] = \check{\forall}\{[(t/x)A] \mid t \in \mathcal{T}\}$,
- $[\exists x A] = \check{\exists}\{[(t/x)A] \mid t \in \mathcal{T}\}$,

where $\mathcal{T}$ is the set of open terms in the language of the theory.

Proof

- Let Γ be a context, and let Δ be a Γ-adapted context. By Definition 3.3, $(\Delta \vdash \top) \in \tilde{\top} = [\![\top]\!]$. Thus $\Gamma \in [\top]$, and $[\top] = \check{\top}$.
- The set $\check{\bot}$ is the intersection of all $[C]$. In particular, $\check{\bot} \subseteq [\bot]$. Conversely, let $\Gamma \in [\bot]$, let φ be an assignment, let σ be a substitution, and let Δ be Γ-adapted. Consider an arbitrary C. By Definition 3.3, $\Delta \vdash \bot$ has a neutral cut-free proof. So does $\Delta \vdash \sigma C$, and this sequent belongs to $[\![C]\!]_\varphi$; thus $\Gamma \in [C]$. Hence Γ is an element of all $[C]$ and therefore of their intersection $\check{\bot}$.
- Let $\Gamma \in [A] \check{\wedge} [B] = [A] \cap [B]$. Let φ be an assignment, let σ be a substitution, and let Δ be Γ-adapted. We have $\Gamma \in [A]$ and $\Gamma \in [B]$, and thus $(\Delta \vdash \sigma A) \in [\![A]\!]_\varphi$ and $(\Delta \vdash \sigma B) \in [\![B]\!]_\varphi$. From Definition 3.3, we get $(\Delta \vdash \sigma(A \wedge B)) \in [\![A \wedge B]\!]_\varphi$. Hence $\Gamma \in [A \wedge B]$.

  Conversely, let $\Gamma \in [A \wedge B]$, let φ be an assignment, let σ be a substitution, and consider a Γ-adapted context Δ. We have $(\Delta \vdash \sigma(A \wedge B)) \in ([\![A]\!]_\varphi \tilde{\wedge} [\![B]\!]_\varphi)$. If $\Delta \vdash \sigma(A \wedge B)$ has a neutral and cut-free proof, then so do $\Delta \vdash \sigma A$ and $\Delta \vdash \sigma B$ by the $\wedge$-elim rules, and this shows that $(\Delta \vdash \sigma A) \in [\![A]\!]_\varphi$ and $(\Delta \vdash \sigma B) \in [\![B]\!]_\varphi$. Otherwise, those last two statements follow directly from Definition 3.3. We conclude that $\Gamma \in [A]$ and $\Gamma \in [B]$, and hence that $\Gamma \in [A] \cap [B] = \Gamma \in [A] \check{\wedge} [B]$.
- To show $[A] \check{\vee} [B] \subseteq [A \vee B]$ it is sufficient to prove that $[A \vee B]$ is an upper bound of $[A]$ and $[B]$. Let $\Gamma \in [A]$, let φ be an assignment, let σ be a substitution, and let Δ be Γ-adapted. By hypothesis, $(\Delta \vdash \sigma A) \in [\![A]\!]_\varphi$, and by Definition 3.3 this means that $(\Delta \vdash \sigma(A \vee B)) \in ([\![A]\!]_\varphi \tilde{\vee} [\![B]\!]_\varphi) = [\![A \vee B]\!]_\varphi$. Thus $\Gamma \in [A \vee B]$. In a similar way, $[B] \subseteq [A \vee B]$.

  Conversely, let $\Gamma \in [A \vee B]$. Let C be such that $[A] \cup [B] \subseteq [C]$, let φ be an assignment, let σ be a substitution, and let Δ be Γ-adapted. By hypothesis, $(\Delta \vdash \sigma(A \vee B)) \in ([\![A]\!]_\varphi \tilde{\vee} [\![B]\!]_\varphi)$. Let us consider the three cases of Definition 3.3 for $\tilde{\vee}$. First, when $\Delta \vdash \sigma(A \vee B)$ has a neutral cut-free proof, σA is A-adapted for φ, σ by the first point of Proposition 3.9, so by Proposition 3.9 $(\Delta, \sigma A)$ is (Γ, A)-adapted. Since $(\Gamma, A) \in [A] \subseteq [C]$ by Proposition 3.11, the sequent $\Delta, \sigma A \vdash \sigma C$ has a cut-free proof by Proposition 3.11. By similar arguments, the sequent $\Delta, \sigma B \vdash \sigma C$ has a cut-free proof. Hence, we can apply the $\vee$-elim rule on those three premises and obtain a neutral cut-free proof of the sequent $\Delta \vdash \sigma C$, which belongs to $[\![C]\!]_\varphi$. Second, if $(\Delta \vdash \sigma A) \in [\![A]\!]_\varphi$, then by Definition 3.8, Δ is A-adapted, and since by Proposition 3.11, $A \in [A] \subseteq [C]$, we must have $\Delta \vdash \sigma C \in [\![C]\!]_\varphi$. The third and last case $(\Delta \vdash \sigma B) \in [\![B]\!]_\varphi$ is similar. In all three cases we have $\Delta \vdash \sigma C \in [\![C]\!]_\varphi$. Hence $\Gamma \subseteq [C]$ for any $[C]$ upper bound of $[A]$, $[B]$, and it is an element of their intersection, that is, of $[A] \check{\vee} [B]$.

- Let us show $[A \Rightarrow B] \subseteq [A] \check{\Rightarrow} [B]$, which is by definition equivalent to $[A] \cap [A \Rightarrow B] \subseteq [B]$. Let $\Gamma \in [A] \cap [A \Rightarrow B]$, and let Δ be Γ-adapted. Then $(\Delta \vdash \sigma A) \in [\![A]\!]_\varphi$ and $(\Delta \vdash \sigma A \Rightarrow \sigma B) \in [\![A \Rightarrow B]\!]_\varphi = [\![A]\!]_\varphi \tilde{\Rightarrow} [\![B]\!]_\varphi$. When $\Delta \vdash \sigma A \Rightarrow \sigma B$ has a neutral cut-free proof, since $\Delta \vdash \sigma A$ has a cut-free proof, $\Delta \vdash \sigma B$ has a neutral cut-free proof, and it belongs to $[\![B]\!]_\varphi$. Otherwise $(\Delta \vdash \sigma A) \in [\![A]\!]_\varphi$, and we apply Definition 3.3 of $\tilde{\Rightarrow}$ with an empty context Σ to get $(\Delta \vdash \sigma B) \in [\![B]\!]_\varphi$. Therefore, $\Gamma \in [B]$.

  Conversely let us show $[A] \check{\Rightarrow} [B] \subseteq [A \Rightarrow B]$. We have to prove that $[A \Rightarrow B]$ is an upper bound of the set of all the $c \in \Omega$ such that $c \cap [A] \subseteq [B]$. For such a c, let $\Gamma \in c$, let φ be an assignment, let σ be a substitution, and let Δ be a Γ-adapted context. We must show $(\Delta \vdash \sigma A \Rightarrow \sigma B) \in [\![A \Rightarrow B]\!]_\varphi = [\![A]\!]_\varphi \tilde{\Rightarrow} [\![B]\!]_\varphi$.

  For this, let Σ be such that $(\Delta, \Sigma \vdash \sigma A) \in [\![A]\!]_\varphi$. By Definition 3.8, (Δ, Σ) is A-adapted, and by Proposition 3.9, (Δ, Σ) is (Γ, A)-adapted. From Proposition 3.11, $[A]$ and c (by definition equal to some $\bigcap_{i \in \Lambda_c} [C_i]$) admit weakening, so $\Gamma, A \in c \cap [A] \subseteq [B]$. Therefore, by Definition 3.10 of $[B]$, $\Delta, \Sigma \vdash \sigma B \in [\![B]\!]_\varphi$, and the claim follows directly from Definition 3.3 of $\tilde{\Rightarrow}$.

- Let $\Gamma \in \bigcap\{[(t/x)A], t \in \mathcal{T}\}$. Let φ be an assignment, and let σ be a substitution. Let Δ be Γ-adapted; we show that $\Delta \vdash \sigma \forall x A \in [\![\forall x A]\!]_\varphi$. We assume without loss of generality that x does not appear in Δ, nor in Γ, nor in σ. Let $t \in \mathcal{T}$ and $d \in M$. By freshness of x, Δ is also Γ-adapted for $\varphi + (d/x)$, $\sigma + (t/x)$. Also, we have $(t/x)\sigma A = (\sigma + (t/x))A$, and by hypothesis, $\Gamma \in [(x/x)A] = [A]$. It means that $(\Delta \vdash (\sigma + (t/x))A) \in [\![(x/x)A]\!]_{\varphi+(d/x)}$. Hence, by Definition 3.3, $(\Delta \vdash \forall x(\sigma A)) \in \tilde{\forall}\{[\![A]\!]_{\varphi+(d/x)} \mid d \in M\}$.

  Conversely, let $\Gamma \in [\forall x A]$. Let $t \in \mathcal{T}$, φ be an assignment, let σ be a substitution, and let Δ be Γ-adapted. Assume without loss of generality that x does not appear in Δ nor in Γ, φ, σ.

  By hypothesis, $(\Delta \vdash \sigma \forall x A) \in [\![\forall x A]\!]_\varphi$. If $\Delta \vdash \sigma \forall x A$ has a neutral cut-free proof, then so does the sequent $\Delta \vdash (\sigma t/x)\sigma A)$. Since $\sigma(t/x)A = (\sigma t/x)\sigma A$, we have $(\Delta \vdash \sigma((t/x)A)) \in [\![A]\!]_{\varphi+([\![t]\!]_\varphi/x)}$, which is equal to $[\![(t/x)A]\!]_\varphi$ by Proposition 3.6.

  Otherwise, Definition 3.3 ensures that for any d, including $[\![t]\!]_\varphi$, $(\Delta \vdash \sigma((t/x)A)) \in [\![A]\!]_{\varphi+(d/x)}$. Thus $(\Delta \vdash \sigma((t/x)A)) \in [\![(t/x)A]\!]_\varphi$. So $\Gamma \in [(t/x)A]$ for any t, and it is then an element of the intersection.

- We first show that $[\exists x A]$ is an upper bound of the set $\{[(t/x)A] \mid t \in \mathcal{T}\}$. Consider some term t and a context $\Gamma \in [(t/x)A]$. Let φ be an assignment, let

σ be a substitution, and let Δ be a Γ-adapted context. Assume without loss of generality that x does not appear in Δ nor in σ. Since $\sigma(t/x)A = (\sigma t/x)\sigma A$, we have by hypothesis $(\Delta \vdash (\sigma t/x)(\sigma A)) \in [\![(t/x)A]\!]_\varphi$, which is equal to $[\![A]\!]_{\varphi+([\![t]\!]_\varphi/x)}$ by Proposition 3.6. This shows that $(\Delta \vdash \sigma\ \exists x A) \in \tilde{\exists}\{[\![A]\!]_{\varphi+(d/x)}, d \in M\}$ by Definition 3.3. Hence $\Gamma \in [\exists x A]$. So $\check{\exists}\{[(t/x)A] \mid t \in \mathcal{T}\} \subseteq [\exists x A]$.

Conversely, let $\Gamma \in [\exists x A]$. Let $c = \bigcap [C_i]$ be an upper bound of $\{[(t/x)A] \mid t \in \mathcal{T}\}$. We can choose $c = [C]$, since we need the intersection of the upper bounds.

Let φ be an assignment, and let σ be a substitution; let Δ be Γ-adapted, and assume without loss of generality that x does not appear in C, nor in Δ, nor in σ. Finally, notice that $A \in [(x/x)A] \subseteq [C]$ and that, by hypothesis on Γ, $(\Delta \vdash \sigma \exists x A) \in [\![\exists x A]\!]_\varphi$.

Assume $\Delta \vdash \sigma \exists x A$ has a neutral cut-free proof. Then, since σA is A-adapted, we have $(\sigma A \vdash \sigma C) \in [\![C]\!]_\varphi$. In particular, by Proposition 3.5, this sequent has a cut-free proof. Since x does not appear in C nor in σ, we can apply an $\exists$-elimination rule between a proof of this sequent and the neutral cut-free proof of $\Delta \vdash \exists x \sigma A$, yielding a neutral cut-free proof of $\Delta \vdash \sigma C$. Hence $(\Delta \vdash \sigma C) \in [\![C]\!]_\varphi$.

Otherwise, by Definition 3.3, $\Delta \vdash \exists x \sigma A$ is such that for some term t and element d, $(\Delta \vdash \sigma' A) \in [\![A]\!]_{\varphi'}$, calling $\sigma' = \sigma + (t/x)$ and $\varphi' = \varphi + (d/x)$. So, Δ is A-adapted for φ', σ'. Since $A \in [C]$, this implies that $(\Delta \vdash \sigma' C) \in [\![C]\!]_{\varphi'}$, but since x does not appear in C, this is the same as $(\Delta \vdash \sigma C) \in [\![C]\!]_\varphi$.

Therefore, $\Gamma \in [C]$. This is valid for any $[C]$ upper bound of $\{[(t/x)A] \mid t \in \mathcal{T}\}$. So, Γ is in their intersection, that is, $\tilde{\exists}\{[(t/x)A], t \in \mathcal{T}\}$. □

Proposition 3.16 *We have* $\Gamma \in [\Gamma]$.

Proof Let $\Gamma = A_1, \ldots, A_n$. By Definition 2.9 and Proposition 3.15, $[\Gamma] = [A_1 \wedge \cdots \wedge A_n] = [A_1] \check{\wedge} \cdots \check{\wedge} [A_n]$. Using Proposition 3.11, we have $\Gamma \in [A_1], \ldots, \Gamma \in [A_n]$; thus $\Gamma \in ([A_1] \check{\wedge} \cdots \check{\wedge} [A_n])$. □

Definition 3.17 (The model $\mathcal{D}$) Let $\mathcal{T}$ be the set of classes of open terms modulo $\equiv$. Let φ be a substitution with values in $\mathcal{T}$. For each function symbol f_i and each predicate symbol P_j of the language, we let

- $\hat{f_i} : t_1, \ldots, t_n \mapsto f_i(t_1, \ldots, t_n)$,
- $\hat{P_j} : t_1, \ldots, t_n \mapsto [P_j(t_1, \ldots, t_n)]$.

Let $\mathcal{D} = \langle \mathcal{T}, \Omega, \hat{f_i}, \hat{P_j} \rangle$.

Proposition 3.18 (The model $\mathcal{D}$)

- *$\mathcal{D}$ is an Ω-structure in the sense of Definition 2.7.*
- *The denotation $[\![\ ]\!]^{\mathcal{D}}$ is such that for any assignment (i.e., substitution) φ, any term t, and any proposition A:*

$$[\![t]\!]^{\mathcal{D}}_\varphi = \varphi t \qquad \textit{and} \qquad [\![A]\!]^{\mathcal{D}}_\varphi = [\varphi A].$$

- *$\mathcal{D}$ is a model for $\equiv$ in the sense of Definition 2.10.*

Proof The first point is immediate by Definition 2.7. The proof of $[\![t]\!]^{\mathcal{D}}_\varphi = \varphi t$ is a straightforward structural induction on t. So is the proof of $[\![A]\!]^{\mathcal{D}}_\varphi = [\varphi A]$: the base case follows from the definition of $\hat{P}_j$ and the inductive cases from Proposition 3.15. Let us consider, for instance, the $\exists$-case. First, assume that x does not appear in φ; otherwise rename it. Then, by definition, $[\![\exists x A]\!]^{\mathcal{D}}_{\varphi} = \check{\exists}\{[\![A]\!]^{\mathcal{D}}_{\varphi+(t/x)}, t \in \mathcal{T}\}$. By induction hypothesis, this is equal to $\check{\exists}\{[(\varphi + (t/x))A], t \in \mathcal{T}\} = \check{\exists}\{[(t/x)(\varphi A)], t \in \mathcal{T}\}$. By Proposition 3.15, this is equal to $[\exists x(\varphi A)] = [\varphi(\exists x A)]$.

The last point is a direct consequence of Proposition 3.11 and of the second point: if $A \equiv B$, then, for any substitution φ, $\varphi A \equiv \varphi B$ and $[\![A]\!]^{\mathcal{D}}_{\varphi} = [\varphi A] = [\varphi B] = [\![B]\!]^{\mathcal{D}}_{\varphi}$. □

3.3 Cut elimination First, with the help of the model $\mathcal{D}$ we conclude directly that the cut-free calculus is complete.

Proposition 3.19 (Completeness of the cut-free calculus) *If the sequent $\Gamma \vdash B$ is valid in the model $\mathcal{D}$ (i.e., $\check{\top} \subseteq [\![\Gamma \vdash B]\!]$ or, equivalently, $[\![\Gamma]\!] \subseteq [\![B]\!]$), then it has a cut-free proof.*

Proof Let φ be the identity assignment. By Proposition 3.16 and by hypothesis,

$$\Gamma \in [\Gamma] = [\![\Gamma]\!]_{\varphi} \subseteq [\![B]\!]_{\varphi} = [B].$$

By Proposition 3.11, the sequent $\Gamma \vdash B$ has a cut-free proof. □

Theorem 3.20 (Cut elimination) *If the sequent $\Gamma \vdash B$ is provable, then it has a cut-free proof.*

Proof From the soundness theorem (Proposition 2.11) and Proposition 2.3, if $\Gamma \vdash B$ is provable, then it is valid in all Heyting algebra-valued models of the congruence, in particular $\mathcal{D}$. Hence, by Proposition 3.19, it has a cut-free proof. □

Remark 3.21 In the previous proof, the induction is performed by the soundness theorem, while the inductive cases are performed by Proposition 3.15, which ensures that $[\_]$ is a model interpretation. So, we observe a split of the cut-elimination theorem in two parts. This has to be compared to proofs of cut elimination *via* normalization, that, given a proof of $\Gamma \vdash A$, would show directly $[\Gamma] \subseteq [A]$ or something similar ([8, Theorem 3.1] for instance). This split is essentially made possible by Definition 3.10.

4 Application to Simple Type Theory

As a particular case, we get a cut-elimination proof for simple type theory.

Let us inspect the model construction in more detail in this case. Based on the language of simple type theory, we first build the truth values algebra of sequents $\mathcal{S}$ of Definition 3.3. Then using the super-consistency of simple type theory, we build the model $\mathcal{M}$ as in Proposition 2.13. In particular $M_\iota = \{0\}$, $M_o = S$ (see Definition 3.3), and $M_{T \to U} = M_U^{M_T}$. Then, we build the model $\mathcal{D}$ as in Proposition 3.18, and we let $D_T = \mathcal{T}_T$, where $\mathcal{T}_T$ is the set of equivalence classes modulo $\equiv$ of terms of sort T. In particular, we have $D_\iota = \mathcal{T}_\iota$ and $D_o = \mathcal{T}_o$.

This construction differs from that of the V-complexes of Prawitz [13], Takahashi [16], Andrews [1], De Marco and Lipton [3], and Hermant and Lipton [10], [11] used to prove cut admissibility in higher-order logic. Let us analyze this further.

4.1 Principles of the proof with V-complexes We give here a sketch of a proof with V-complexes in the simpler case of classical logic, as given in [1], for instance, or, in a modern and intuitionistic version, as given in [3]. Notice that, in contrast with the presentation of Definition 2.1, the ε-symbol is absent so that the logical connectors merge with the associated "dotted" constant. For instance, in this section we shall consider $\dot\wedge$ to be the same as $\wedge$.

Let $\Gamma \vdash \Delta$ be a sequent that has no proof in the cut-free sequent calculus. We assume that we are given a semivaluation V (see Schütte [14]) compatible with this sequent, that is, a partial interpretation function from the propositions into $\{0, 1\}$ such that $V(\Gamma) = 1$ and $V(\Delta) = 0$. Such a semivaluation can be obtained by an (infinite) tableau procedure or as an abstract consistency property (see [1], [3]). It is weaker than a model interpretation, in the sense that it is partial, and consequently consistency conditions are weaker: if we know the truth value of a proposition A, enough must be known on the truth value of its immediate subpropositions. For instance, $V(A \wedge B) = 0$ *implies* $V(A) = 0$ or $V(B) = 0$, and the other value might be left undefined.

The goal, and the difficulty in the simple type theory case—or higher-order logic—as identified by Schütte [14], resides in the extension of V into a model interpretation. The answer, given independently by Takahashi [16] and Prawitz [13], is to construct a new interpretation domain, called V-complexes, as follows.

First, for every type T, the interpretation domain of the model is built by gluing together a syntactic and a semantic part:

$$V_T = \mathcal{T}_T \times \mathcal{M}_T,$$

where $\mathcal{T}_T$ is the set of terms of type T that are in normal form, $\mathcal{M}_\iota = \{\iota\}$, $\mathcal{M}_o = \{0, 1\}$, and $\mathcal{M}_{A\to B}$ is the function space $V_B^{V_A}$, that is, it is composed of functions of $\mathcal{T}_A \times \mathcal{M}_A \to \mathcal{T}_B \times \mathcal{M}_B$ that verify the following criterion. A pair $\langle t, f\rangle$ belongs to V_T if and only if t is in normal form and of type T, and

- when T is ι, f is equal to ι;
- when T is o and $V(t)$ is defined, f is equal to $V(t)$ (this way we enforce the adequacy with the semivaluation V); otherwise f can be either 0 or 1, and indeed both V-complexes $\langle t, 0\rangle$ and $\langle t, 1\rangle$ belong to the domain V_o;
- when T is a function type $A \to B$, $f \in \mathcal{M}_{A\to B}$ can be decomposed in a function f_1 from $\mathcal{T}_A \times \mathcal{M}_A$ to $\mathcal{T}_B$ and a function f_2 from $\mathcal{T}_A \times \mathcal{M}_A$ to $\mathcal{M}_B$. Given any V-complex $\langle t', a\rangle \in \mathcal{M}_A$, we require that $f_1(\langle t', a\rangle) = nf(tt')$ and $\langle nf(tt'), f_2(\langle t', a\rangle)\rangle \in V_B$, where $nf(tt')$ is the normal form of tt'.

V-complexes were introduced to deal with two main problems of higher-order logic: impredicativity and intensionality. Tait's method (see [15]) solves the first problem by performing an induction on the type, in this way avoiding an impossible induction on term size. This has to be improved to handle intensionality: logically speaking, $\top$ and $\top \wedge \top$ must have the same denotation, while we must still be able to make a semantic distinction between the denotations of $P(\top)$ and $P(\top \wedge \top)$ since the first propositions are equiprovable while the second are not. Moreover, the interpretation $\langle P, f\rangle$ of P must be such that the logical denotation (the second component of the interpretation) of $P(\top)$ and $P(\top \wedge \top)$ is different. This is possible only if f_2 uses both sides of its argument; in particular, we must have

$$f_2\big(\langle \top \wedge \top, [\![\top \wedge \top]\!]\rangle\big) \neq f_2\big(\langle \top, [\![\top]\!]\rangle\big),$$

although $[\![\top \wedge \top]\!] = [\![\top]\!] = 1$.

This is achieved by introducing a syntactical component into the semantic denotation of the terms. It then becomes possible to have different values and behaviors, depending on this syntactical component. This is reflected by the behavior of the function f_2: it crucially depends on both components.

As a consequence, we separate the logical denotation of terms (that equalize $\top$ and $\top \wedge \top$ in any Boolean algebra or Heyting algebra) from their interpretation in the model (that does not and that is more related to the meaning of the proposition that to its denotation) lying at a lower level, the level of V-complexes.

On the basis of V-complexes, we define an interpretation for any term t by induction on its structure. Let us see some key cases.

- If t is not a logical symbol, we interpret it by a default V-complex associated to t, of shape $\langle t, d\rangle$. (Of course, a lemma states that it exists.)
- If t is the logical symbol $\dot{\forall}_T$ we construct the V-complex $\langle \dot{\forall}_T, f\rangle$, where f is the following function: to any V-complex $\langle t, g\rangle$ of type $T \to o$ it associates the V-complex of type o $\langle \dot{\forall}_T t, v\rangle$, where v is equal to 1 if and only if for any V-complex d of type T, $g(d)$ is a V-complex that has 1 as second component. So we quantify over *all* the V-complexes of type T.
- The interpretation of the application symbol α applies a V-complex $\langle t, f\rangle$ to another one $\langle u, g\rangle$ as $f(\langle u, g\rangle)$. Notice that its first member is, by the conditions on f, the normal form of (tu).

It is a matter of technique to check that this construction really produces V-complexes. The last step is to consider a generalized notion of model, since now terms of type o have a denotation in $\mathcal{T} \times \{0, 1\}$, which is not a Boolean algebra. Then we can state that the interpretation we built is compatible with V, and the propositions of Γ are interpreted by 1 (as a second component) while those of Δ are interpreted by 0. Therefore the sequent $\Gamma \vdash \Delta$ is not valid if it has no cut-free proof. This yields a proof of a strong version of the completeness theorem from which we derive the cut-elimination theorem.

4.2 Comparison In contrast, in our construction, we have two separate models, the term model $\mathcal{D}$, which corresponds to the left-hand side of a V-complex, and the model $\mathcal{M}$, which corresponds to the right-hand part.

The novelty is that $\mathcal{M}_{A\to B}$ is just $\mathcal{M}_A \to \mathcal{M}_B$ and not $\mathcal{T}_A \times \mathcal{M}_A \to \mathcal{M}_B$. This is possible because when we build $\mathcal{M}$, instead of taking $\mathcal{M}_o = \{0, 1\}$, we have taken $\mathcal{M}_o = S$, which is a truth value algebra but not a Heyting algebra. Thus $[\![\dot{\top} \dot{\wedge} \dot{\top}]\!]^{\mathcal{M}}$ and $[\![\dot{\top}]\!]^{\mathcal{M}}$ need not be equal, the truth values containing more information, and we do not need to glue an extra syntactical argument $\dot{\top} \dot{\wedge} \dot{\top}$ or $\dot{\top}$ to have $f_2([\![\dot{\top} \dot{\wedge} \dot{\top}]\!]) \neq f_2([\![\dot{\top}]\!])$.

The same phenomenon arises in $\mathcal{D}$ since we choose a syntactic model: following Proposition 3.18, $[\![\dot{\top} \dot{\wedge} \dot{\top}]\!]^{\mathcal{M}}$ and $[\![\dot{\top}]\!]^{\mathcal{M}}$ are respectively equal to $\dot{\top} \dot{\wedge} \dot{\top}$ and $\dot{\top}$. In a similar way $[\![P(\dot{\top})]\!]^{\mathcal{D}} = [\![P(\dot{\top})]\!]^{\mathcal{M}} \lhd P(\dot{\top}) = [P(\dot{\top})]$, and from Proposition 3.11, $[\![P(\dot{\top})]\!]^{\mathcal{D}}$ contains $P(\dot{\top})$, while $[\![P(\dot{\top} \dot{\wedge} \dot{\top})]\!]^{\mathcal{D}}$ contains $P(\dot{\top} \dot{\wedge} \dot{\top})$. None of those interpretations, in the general case, contains the other proposition; therefore they are not equal, as required.

One may also wonder where the separation of the logical denotation 1 of $\dot{\top}$ from its interpretation in $\mathcal{D}$ appears in our proof. The expression $\dot{\top}$ has an existence only at the *term* level since at the *propositional* one it is replaced by $\varepsilon(\dot{\top})$. The interpretations in $\mathcal{D}$ of $\dot{\top}$ and $\varepsilon(\dot{\top})$, respectively, correspond to the interpretation in V_o and the denotation in a Heyting algebra of $\top$ in previous proofs.

The separation between denotation and interpretation, which had to be defined "by hand," introducing a new definition for models, in the earlier works with V-complexes (see [13], [16], [1], [3], [11], [10]), is automatically captured by the simple syntactical device ε.

4.3 Conclusion Thus the main difference between our model construction and that of the V-complexes is that we have broken this dependency on u of the right component of the pair obtained by applying $\langle t, f\rangle$ to $\langle u, g\rangle$. This leads to a two-stage construction where the very notion of V-complex has vanished and the second model is syntactical. The reason why we have been able to do so is that by starting with an underlying model of sequents S, our semantic objects $[A]$ are much sharper and do not require additional construction. Moreover, the presence of the symbol ε has simplified the dependency of the semantics on the syntax and allowed a purely syntactical model at the term level.

It has to be noticed that super-consistency allows us to construct a model on more usual Heyting algebras, such as the Lindenbaum–Heyting algebra or the context-based ones used for cut elimination (see [10], [12]), where $[A]$ is defined as the set of contexts Γ such that $\Gamma \vdash A$ has a cut-free proof. It gives us an interpretation on this algebra that satisfies the congruence $\equiv$. The pitfall is that if we build such a model in an ordinary way, we cannot prove that $[\![A]\!] = [A]$. To achieve this goal we have to proceed by first defining the algebra of sequents in an untyped way and then by extracting the needed contexts in order to force $[\![A]\!]$ to be equal to $[A]$. So a two-stage construction seems unavoidable when one uses super-consistency to show the admissibility of the cut rule.

It remains to be understood if such a construction can also be carried out for a normalization proof.

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