Notre Dame Journal of Formal Logic
Volume 53, Number 3, 2012

Bounds on the Strength of Ordinal Definable Determinacy in Small Admissible Sets

Diego Rojas-Rebolledo

Abstract We give upper and lower bounds for the strength of ordinal definable determinacy in a small admissible set. The upper bound is roughly a premouse with a measurable cardinal κ of Mitchell order κ^{++} and ω successors. The lower bound are models of ZFC with sequences of measurable cardinals, extending the work of Lewis, below a regular limit of measurable cardinals.

1 Introduction

In this paper, we present upper and lower bounds for the large cardinal strength of the following statement: In a small admissible M, any class of reals which is ordinal definable in M is determined. (Denote the statement by $\mathrm{Det}(*)$.)

For Section 2 we give a short overview of some of the background results that motivate this paper. Section 3 gives the basic definitions involved, some basic results about the structures M_x (least admissible sets), the theory of Kripke and Platek (KP), and a description of the complexity of the terms involved in the definition of $\mathrm{Det}(*)$.

In Section 4, we deal with the upper bound, that is, a large cardinal assumption from which one can get $\mathrm{Det}(*)$. We use methods of Kechris and Solovay [5] and results of Neeman [11, p. 204] to show that from a large cardinal assumption just below, "there is a premouse-like structure with a measurable cardinal κ of Mitchell order κ^{++} and ω successors," one can get not just $\mathrm{Det}(*)$ but a Turing cone of least admissible sets satisfying ordinal definable (OD) determinacy. In order to do this, we show how the determinacy of the point class of Boolean combinations of Π^1_1-sets (denote this class by $\mathscr{B}(\Pi^1_1)$) implies $\mathrm{Det}(*)$ in a nice way:[1] each level of the hierarchy of this class implies a level of $\mathrm{Det}(*)$ in terms of the complexity of the classes involved in the definition of $\mathrm{Det}(*)$. Then, we apply the results of Neeman [11], which state a correspondence between large cardinals and the levels of the class $\mathscr{B}(\boldsymbol{\Pi}^1_1)$.

Received March 25, 2011; accepted September 30, 2011
2010 Mathematics Subject Classification: Primary 03E45; Secondary 03E55, 03E60
Keywords: determinacy, admissible sets, large cardinals
 10.1215/00294527-1716766

In Section 5 we give a lower bound for the cardinal strength of the statement Det($*$). That is to say, we describe how to obtain models satisfying certain kind of large cardinals, from the assumption of ordinal definable determinacy in a least admissible set. In [7], Lewis adapted games of Martin in order to produce (from the assumption of Det($*$)) a countable model of ZFC with λ-many measurable cardinals (with λ recursive); the construction of such a model is carried inside the least admissible set, given by the determinacy assumption, and so it is restricted to KP. What we do is modify these games in order to obtain, from the same determinacy assumption, stronger sequences of measures (though still below a sequence of measurable cardinals with a regular limit), still under KP alone. The article is divided in five sections. The first two give a general description of the type of games that we will be using, the type of structures that will be obtained by these games, and a more formal approach of the games, but still thinking of the games as ordinal games, in which the players are thought of as playing sequences of ordinals instead of reals. In Section 3 we establish how these games can be coded into integer games, with ordinal definable payoff in admissible sets. Section 4 presents the bounding lemma that allows us to ensure that the loser in the game can make sure to lose for reasons other than not encoding the required structure. Finally, the last section is dedicated to the proof of the main theorem.

2 Background

During the last three decades, plenty of work has been done towards establishing the cardinal strength for determinacy statements. The first result obtained in this direction might be the one by Martin [8] and Harrington [4], which states that the cardinal strength for $\Pi^1_1(a)$-determinacy is the existence of $a^\#$. After this, there were several attempts to obtain (from large cardinals) the determinacy of $\boldsymbol{\Pi}^1_2$-sets. It seems that in 1978, D. Martin was able to do so but with a very large cardinal: an *ω-huge* cardinal. It was not until 1989 that Martin and Steel [9] settled the strength for the determinacy of the projective hierarchy. Since then (and before), several results concerning the strength of classes that lay below $\boldsymbol{\Pi}^1_2$ have been settled. The proofs, for direct implication or (mainly) relative consistency, are carried under the assumption of full ZFC. Some examples are Simms (Ph.D. thesis), which computed the strength of $\Sigma^0_1(\boldsymbol{\Pi}^1_1)$-determinacy: an inner model with a class of measurable cardinals. Later, using the core model theory developed by Mitchell [10], Neeman [11], and Steel [14] settled the strength for $\mathcal{A}(\boldsymbol{\Pi}^1_1)$ and $\bigcup_{\alpha<\omega_1}\Sigma^0_\alpha(\boldsymbol{\Pi}^1_1)$. For the latter, the results are gauged to the point that the strength of $\Delta^0_{\alpha+4}(\boldsymbol{\Pi}^1_1)$ corresponds to the existence of a mouse with a measurable cardinal of maximal Mitchell order and with $(\alpha+1)$ successors in it. Surprisingly, as Kechris and Solovay [5] showed, full lightface determinacy collapses at the level of Δ^1_2.

The determinacy statements that we consider in this paper (Det($*$)) fall somewhere below $\Sigma^0_\omega(\Pi^1_1)$ (see Corollary 4.4). Although from Det($*$) we can obtain models of ZFC with sequences of measures, which would make Det($*$) stronger than Π^1_1, one should not consider Det($*$) as a full level of determinacy between $\Sigma^0_\omega(\Pi^1_1)$ and Π^1_1. This is because Det($*$), if true, does not imply Π^1_1-determinacy (see Corollary 4.8). In contradistinction to the results just mentioned, in this paper the models of large cardinals obtained from Det($*$) are "constructed" mainly in KP: the models are produced from determined games played within an admissible set (the witness of $\mathrm{Det}(*)$).

3 Basics

The notion of determinacy is related to infinite games. For a set $A \subset {}^{\omega}\omega$ of real numbers, to say that A is *determined* means that in the game G_A associated to A, there is a winning strategy for either of the two players. In the associated game G_A, two players, **I** and **II**, interchange plays to throw a natural number at each time. A play of the game will look like this:

$$\begin{array}{llllll} \mathbf{I}: & a_0 & a_1 & a_2 & a_3 & a_4 \ldots, \\ \mathbf{II}: & b_0 & b_1 & b_2 & b_3 \ldots & (\text{where } a_i, b_i \in \omega). \end{array}$$

The play is thought of as the real number $z = \langle a_0, b_0, a_1, b_1, a_2, b_2, a_3, \ldots \rangle$. It is said that player **I** wins this play if $z \in A$; otherwise, **II** wins. The set A is called the *payoff set*. A *strategy* for a player, say player **I**, is a function $\sigma : \omega^{<\omega} \to \omega$ that tells **I** what to play. If, for **I**'s $(n+1)$th turn, **II** has played $\langle b_0, b_1, \ldots, b_n \rangle$ and **I** the sequence $\langle a_0, a_1, \ldots, a_n \rangle$, then $a_{n+1} = \sigma(\langle a_0, b_0, \ldots, a_n, b_n \rangle)$. The strategy σ is a *winning strategy* for **I** if this player can always win by simply following the strategy. If σ is a winning strategy for player **I** and **II** plays c, the play produced by c and σ is denoted by $\sigma * c$. (If player **II** has winning strategy σ, the play is denoted by $c * \sigma$.) If Γ is a family of sets of reals, by Γ-determinacy is meant the statement, "for all $A \in \Gamma$, A is determined," denoted by $\mathrm{Det}(\Gamma)$.

Definition 3.1 Let $\mathrm{Det}(*)$ denote the following determinacy statement:

$$\mathrm{Det}(*) \equiv_{\mathrm{def}} (\exists z \in {}^{\omega}\omega)(\forall n \in \omega)\big(M_z \models \mathrm{Det}(\mathrm{OD}_{\Sigma_n})\big),$$

where we have the following.

- By $M_z \models \mathrm{Det}(\mathrm{OD}_{\Sigma_n})$ is meant the following statement: "for every Σ_n-formula $\varphi(x, \vec{\alpha})$ with ordinal parameters, the class of reals X_φ defined in M_z by φ is determined in M_z" (for a formal definition, see Lemma 3.7).
- $M_z = L_{\omega_1^z}[z]$ is the *least admissible set containing the real* z. The set $L_{\omega_1^z}[z]$ is the ω_1^zth level of Gödel's constructible universe relative to z (see, e.g., Keisler and Knight [6], Devlin [3]).
- ω_1^x is the least countable ordinal not recursive in x; that is,

  $$\omega_1^x = \sup\big\{otp(y) : y \in WO \wedge y \leq_T x\big\}.$$

  The set WO is defined as the set of reals x such that the relation $<_x$ on ω, defined by $n <_x m \Leftrightarrow x(\langle n, m \rangle) = 0$, is well ordered. And the relation $\leq_T$ denotes Turing reducibility (see Soare [13, Chapter III]).

This way, the statement $\mathrm{Det}(*)$ can be written as

$$\mathrm{Det}(*) \equiv_{\mathrm{def}} (\exists x \in {}^{\omega}\omega)\big(M_x \models \mathrm{Det}(\mathrm{OD})\big),$$

which is formally a Σ^1_2-statement (look at Definition 3.3 and Lemma 3.7).

Next, we define a hierarchy of large cardinal statements.

Definition 3.2 Given a mouse in Steel's [14] sense,[2] for each $1 \leq \alpha < \omega_1$, define the large cardinal axiom $\mathrm{LC}_M^{\alpha+1}$ by

$$\mathrm{LC}_M^{\alpha+1} \equiv_{\mathrm{def}} (M \text{ is a mouse}) \wedge \big(M \models \exists\kappa(o(\kappa) = \kappa^{++} \wedge \kappa^{+(\alpha+1)} \text{ exists})\big),$$

where $\kappa^{+(\alpha+1)}$ denotes the $(\alpha+1)$st successor of κ and $o(\kappa)$ represents the Mitchell order of κ (see Mitchell [10]).

A formula φ is said to be Δ_0 if all its quantifiers are bounded. Set $\Sigma_0 = \Delta_0 = \Pi_0$; a formula φ is Σ_{n+1} (Π_{n+1}) if φ is of the form $\exists u\theta$ ($\forall u\theta$), with θ a Π_n-formula (Σ_n-formula).

We use the representation for the σ-algebra of Π^1_1-sets $\mathscr{B}(\Pi^1_1)$ given in Steel [14, p. 121] as an equivalent definition for the point classes $\mathscr{B}(\Pi^1_1)$ and $\mathscr{B}(\boldsymbol{\Pi}^1_1)$.

Definition 3.3 Let A be a subset of ${}^\omega\omega$.

- The set A is $\Sigma^0_n(\Pi^1_1)$ if there is a Σ_{n+1}-formula $\Psi(u)$ such that
$$A = \{z \in {}^\omega\omega : M_z \models \Psi(z)\}.$$
- For a fixed parameter $x_0 \in {}^\omega\omega$, A is $\Sigma^0_n(\Pi^1_1(x_0))$ if there is a Σ_{n+1}-formula $\Psi(u,v)$ such that $A = \{z \in {}^\omega\omega : M_{\langle z,x_0\rangle} \models \Psi(z,x_0)\}$.
- A is $\Sigma^0_n(\boldsymbol{\Pi}^1_1)$ if there is $x \in {}^\omega\omega$ such that A is $\Sigma^0_n(\Pi^1_1(x))$.
- $\mathscr{B}(\Pi^1_1) = \bigcup_{n\in\omega} \Sigma^0_n(\Pi^1_1)$ and $\mathscr{B}(\boldsymbol{\Pi}^1_1) = \bigcup_{n\in\omega} \Sigma^0_n(\boldsymbol{\Pi}^1_1)$.

Theorem 3.4 (Neeman [11, Corollary 8.9]) *For each $n \geq 1$,*
$$\mathrm{LC}^{n+1}_M \Rightarrow \big(M \models \mathrm{Det}(\Delta^0_{n+4}(\boldsymbol{\Pi}^1_1))\big).$$

The large cardinal that will give the upper bound to the determinacy statement $\mathrm{Det}(*)$ is not exactly one of those in the hierarchy described above. It will be proved, though, that for the determinacy statement $\mathrm{Det}_n(*)$ (obtained by restricting $\mathrm{Det}(*)$ to Σ_n-formulas) there is such a correspondence: $\mathrm{Det}_n(*)$ will correspond to LC^n_M, for $n > 1$ (see Theorem 4.6).

The upper bound for the strength of $\mathrm{Det}(*)$ will be given by the cardinal property $\mathrm{LC}(*)$ defined as follows: *There is a sequence of measures $\mathcal{U}$ such that*
$$(\forall n \geq 2)(\exists \delta_n \geq \omega_1)\big(L_{\delta_n}[\mathcal{U}] \models \exists \kappa_n(o(\kappa_n) = \kappa_n^{++} \wedge \kappa_n^{+n} \text{ exists})\big).$$
This large cardinal puts together an ω-sequence of those mice corresponding to the statements $\mathrm{Det}_n(*)$ just mentioned.

We proceed now to the notion of *admissible set,* and some basic known results that will be used. Following the definition of Barwise, Gandy, and Moschovakis [1], we say that a transitive set M is *admissible* if it is closed under pair and union, and $\langle M, \in\rangle$ satisfies the following axioms.

(1) Δ_0-*separation*: for every Δ_0-formula $\varphi(v)$ with parameters in M and for every $a \in M$,
$$\exists x \forall y\big(y \in x \leftrightarrow y \in a \wedge \varphi(y)\big).$$
(2) Δ_0-*collection*: for every Δ_0-formula $\varphi(u,v)$ with parameters in M,
$$\forall x \exists y\big(\varphi(x,y) \rightarrow \forall u \exists v(\forall x \in u)(\exists y \in v)\varphi(x,y)\big).$$

The theory of Kripke and Platek is the fragment of set theory obtained from the previous two axioms plus extensionality, ϵ-induction, empty set, pair, and union. This way, admissible sets are models of KP.

Let M be a transitive set; a formula φ is Σ_1 *in-M* (Π_1 *in-M*) if there is a Σ_1-formula (Π_1-formula) which is equivalent to φ inside M. A formula φ is Δ_1 *in-M* if it is both Σ_1 in-M and Π_1 in-M.

Let Σ be the smallest class of formulas containing the Δ_0-formulas; closed under conjunction, disjunction, bounded quantification, and existential unbounded quantification.

An important feature about admissible sets is that if M is admissible and φ is a formula in Σ, then φ is Σ_1 in-M. For proofs of these facts and the ones stated below the reader is referred to Devlin [3] and Barwise [2]. Admissible sets satisfy stronger principles than those stated; for example:

If φ is Δ_1 in-M, then $\langle M, \in\rangle$ satisfies $\exists x \forall y (y \in x \leftrightarrow y \in a \wedge \varphi(y))$.

This principle is known as Δ_1-*separation*. Also, admissible sets satisfy Σ_1-*collection*. Other principles that are satisfied in admissible sets are: The Σ_1-*bounding* principle, which states that for every formula $\varphi(u, v)$ which is Σ_1 in-M and has parameters inside M,

$$(\forall x \in u)\exists y\big(\varphi(x, y) \to \exists v(\forall x \in u)(\exists y \in v)\varphi(x, y)\big).$$

Also, the Σ_1-*replacement* principle, stated below, is satisfied. This principle is of particular interest to us as it will be used often.

Proposition 3.5 (Σ_1-replacement principle) *Let M be an admissible set, and let f be a function which is definable in M by a Σ_1-formula with parameters in M. If $a \in M$ is a subset of the domain of f, then $f \restriction a \in M$ and $f''a \in M$.*

We earlier mentioned the sets $M_x = L_{\omega_1^x}[x]$ as the least admissible set containing the real x. This is indeed an admissible set, and it is the least one containing x (see, e.g., Barwise, Gandy, and Moschovakis [1]). Some other useful properties about least admissible sets are condensed in the following proposition (the reader is referred to Barwise, Gandy, and Moschovakis [1] and Sacks [12] for proofs).

Proposition 3.6 *For all $x, y \in {}^{\omega}\omega$, $\Delta_1^1(x) = {}^{\omega}\omega \cap M_x$; if $x \in M_y$, then $\omega_1^x \leq \omega_1^y$ and $M_x \subseteq M_y$.*

The next lemma calculates the complexity of the expression of the sentences $\mathrm{Det}(\mathrm{OD}_{\Sigma_n})$ (introduced in Definition 3.1). For each n the sentence $\mathrm{Det}(\mathrm{OD}_{\Sigma_n})$ asserts that any ordinal definable class of reals, which is defined by a Σ_n-formula, is determined. We do not use the regular definition of OD in terms of reflection, as we are interested in defining $\mathrm{Det}(\mathrm{OD}_{\Sigma_n})$ in the admissible sets M_x. Instead, we formalize syntax and satisfaction following Devlin [3]. For any terms and notation involved in the proof, which are not defined in this paper, we refer the reader to Devlin [3, Sections I.9-11, II.6-7]. However, we present the argument as formally as possible to avoid major detours.

Lemma 3.7 *For $n \geq 1$, the formula $\mathrm{Det}(\mathrm{OD}_{\Sigma_n})$ is a Π_{n+3}-formula.*

Proof Using the notation of Devlin [3], $\mathrm{Det}(\mathrm{OD}_{\Sigma_n})$ is roughly defined as the following predicate ($\forall^0, \exists^0$ denote quantification over ω, and $\forall^1, \exists^1$ denote quantification over ${}^{\omega}\omega$):

$$\forall^0 l \forall\vec{\alpha}\forall\varphi\big(\mathrm{Finseq}(\vec{\alpha}) \wedge \mathrm{dom}(\vec{\alpha}) = l \wedge \mathrm{Fml}_{\Sigma_n}(\varphi) \\ \to \text{“}\{a \in {}^{\omega}\omega : \varphi[a, \vec{\alpha}]\} \text{ is determined”}\big).$$

The statement: “$\{a \in {}^{\omega}\omega : \varphi[a, \vec{\alpha}]\}$ is determined” can be defined as

$$(\exists^1\tau\forall^1 b\, \mathrm{Sat}_{\Sigma_n}(\varphi[\tau * b, \vec{\alpha}]) \vee (\exists^1\tau\forall^1 a \neg\, \mathrm{Sat}_{\Sigma_n}(\varphi[a * \tau, \vec{\alpha}])),$$

where $\mathrm{Sat}_{\Sigma_n}(\varphi[c,\vec{\alpha}])$ is roughly the Σ_n-formula of the language of set theory (LST) (in the case when n is even):

$$\begin{aligned}
&\exists y_1 \forall y_2 \cdots \exists y_n \exists \psi \exists f \exists \theta \exists u \exists m \exists w \big[\mathrm{Trans}(w) \\
&\quad \wedge (\omega, f, \theta, \vec{\alpha} \in w) \wedge (m = l + n) \\
&\quad \wedge \mathrm{Fr}(\psi, u) \wedge \mathrm{Fml}(\psi) \wedge (f : m \leftrightarrow u) \wedge \big(\mathring{\varphi} = \exists f(0) \forall f(1) \ldots \exists f(n-1) \\
&\quad \psi(f(0), \ldots, f(m-1))\big) \wedge \mathrm{Finseq}(\theta) \wedge \big(\mathrm{dom}(\theta) = m + 1\big) \\
&\quad \wedge (\theta_0 = \psi) \wedge (\forall i \in n) \wedge \big(\mathrm{Sub}(\theta_{i+1}, \theta_i, f(i), \mathring{x}_{i+1})\big) \\
&\quad \wedge (\forall n \leq i < m-1)\big(\mathrm{Sub}(\theta_{i+1}, \theta_i, f(i), \mathring{\vec{\alpha}}_{i-n})\big) \\
&\quad \wedge \mathrm{Sub}\big(\theta_m, \theta_{m-1}, f(m-1), \mathring{c}\big)\big] \rightarrow [\mathrm{Sat}(w, \theta_m)].
\end{aligned}$$

The expression "$\mathring{\varphi} = \exists f(0) \forall f(1) \ldots \exists f(n-1) \psi(f(0), \ldots, f(m-1))$" is in fact notation for the Σ_0-formula of LST that states that φ is the $\mathcal{L}_V$-formula constructed from ψ by alternated quantification (of $\exists$ and $\forall$) over the first n free variables of ψ. All the formulas, $\mathrm{Finseq}(\vec{\alpha})$, $\mathrm{dom}(\vec{\alpha}) = l$, $\mathrm{Trans}(w)$, $\mathrm{Fr}(\psi, u)$, $\mathrm{Sub}(\theta_{i+1}\theta_i, f(i), \mathring{x}_{i+1}))$, and $\mathrm{Sat}(w, \theta_m)$, are either Σ_0 or Δ_1^{KP} (see Devlin [3]). Hence, the above formulas ($\mathrm{Sat}_{\Sigma_n}(\varphi[\tau * b, \vec{\alpha}])$ and $\neg\, \mathrm{Sat}_{\Sigma_n}(\varphi[a * \tau, \vec{\alpha}])$) are Σ_n. This shows that $\mathrm{Det}(\mathrm{OD}_{\Sigma_n})$ is Π_{n+3}. When n is odd, just change the quantifiers $\exists y_n \exists \psi \exists f \exists \theta \exists u \exists m \exists w$ for the quantifiers $\forall y_n \forall \psi \forall f \forall \theta \forall u \forall m \forall w$. □

4 Upper Bound

We start this section with a basic known fact that relates the notions of determinacy and Turing invariance; this is also used in the proof of the main lemma of this section.

Lemma 4.1 *Let $B \subseteq {}^{\omega}\omega$ be a Turing invariant set (i.e., $((x \in B \wedge y \equiv_T x) \Rightarrow y \in B)$) such that $\forall x (\exists y \geq_T x)(y \in B)$. If B is determined, then $\exists x_0 (\forall y \geq_T x_0) \times (y \in B)$, that is, the cone of Turing degrees $\mathcal{C} = \{y : y \geq_T x_0\}$ is included in B.*

Proof Assume, on the contrary, that B is Turing invariant, determined, and assume that

$$\forall x (\exists y_1, y_2 \geq_T x)(y_1 \in B \wedge y_2 \notin B).$$

Let $v_0 \in {}^{\omega}\omega$ be a winning strategy for the set B.

(1) If v_0 is winning for player **I**, by assumption, there is $y_2 \geq_T v_0$ such that $y_2 \notin B$. Let $v_0 * y$ be the play of **I** when **II** plays y. Then, the play of the game $z = \langle v_0 * y_2, y_2 \rangle$ is in B ($z \geq_T y_2$). But this is not possible, since $z \leq_T y_2$ as well and B is Turing invariant, which gives $y_2 \in B$.

(2) If v_0 is winning for **II**, take $y_1 \geq v_0$ such that $y_1 \in B$. Then, as above, we conclude that $y_1 \notin B$. □

The next lemma gives a correspondence between the levels of the σ-algebra of co-analytical sets and the "levels" of $\mathrm{Det}(*)$. For simplicity we might be using the expression $\mathrm{Det}_n^y(*)$ instead of $M_y \models \mathrm{Det}(\mathrm{OD}_{\Sigma_n})$.

Lemma 4.2 *For every $n \in \omega$,*

$$\mathrm{Det}\big(\Sigma^0_{n+2}(\Pi^1_1)\big) \Rightarrow \exists x (\forall y \geq_T x)\, \mathrm{Det}_n^y(*).$$

Proof Assume, on the contrary, that $\forall x (\exists y \geq_T x) \neg\, \mathrm{Det}_n^y(*)$. Let B be the set $\{y \in {}^{\omega}\omega : \neg\, \mathrm{Det}_n^y(*)\}$, that is, $y \in B \Leftrightarrow M_y \models \neg\, \mathrm{Det}(\mathrm{OD}_{\Sigma_n})$. So $B \in \Sigma^0_{n+2}(\Pi^1_1)$

(by Definition 3.3 and Lemma 3.7), and so B is determined; also, B is Turing invariant. (If $x \equiv_T y$, then, by Proposition 3.6, $M_x = M_y$.) Hence, by the previous lemma, we may conclude that

$$\exists x_0(\forall y \geq_T x_0)\big(\neg\,\mathrm{Det}_n^y(*)\big).$$

We will use this real x_0 in the last part of the proof. Now, let $\Phi_n^0(z)$ be the formula that defines the following expression:

> "There is a Σ_n-formula $\varphi(u,\vec{v})$ with $r+1$ free variables, and there is an r-sequence of ordinals $\vec{\alpha}$ such that the set $X_\varphi^{\vec{\alpha}} = \{c \in {}^\omega\omega : \varphi[c,\vec{\alpha}]\}$ is not determined, and $\varphi(u,v)$ and $\vec{\alpha}$ are least (with respect to the Gödel ordering of formulas and the lexicographical $\in$-ordering, respectively) with this property, and $z \in X_\varphi^{\vec{\alpha}}$."

Observe that $\Phi_n^0(z)$ is a Σ_{n+3}-formula: as we pointed out in the proof of Lemma 3.7, the formula $\delta(\varphi,\vec{\alpha})$ that says, "$X_\varphi^{\vec{\alpha}}$ is determined" is Σ_{n+2}. Hence, if we write $\Phi_n^0(z)$ more carefully (still using the notation of Devlin [3]):

$$\begin{aligned}&\exists\varphi\exists\vec{\alpha}\big[\big(\mathrm{Fml}_{\Sigma_n}(\varphi)\wedge\vec{\alpha}\in\mathrm{On}^{<\omega}\wedge\delta[\varphi,\vec{\alpha}]\big)\\&\quad\wedge(\forall\psi\forall\vec{\beta}\big[\big(\mathrm{Fml}_{\Sigma_n}(\psi)\wedge\vec{\beta}\in\mathrm{On}^{<\omega}\wedge\delta[\psi,\vec{\beta}]\big)\to(\langle\varphi,\vec{\alpha}\rangle\leq\langle\psi,\vec{\beta}\rangle)\big]\\&\quad\wedge(z\in X_\varphi^{\vec{\alpha}})\big],\end{aligned}$$

it is clear that $\Phi_n^0(z)$ is a Σ_{n+3}-formula. Next, define the formula $\Phi_n(z)$ as

$$\neg\,\mathrm{Det}(\mathrm{OD}_{\Sigma_n})\wedge(\exists a,b,u,v\in{}^\omega\omega)\big(z=\langle a,b,u,v\rangle\wedge\Phi_n^0(\langle a,b\rangle)\big).$$

By Lemma 3.7 and the observation above, $\Phi_n(z)$ is Σ_{n+3}. Finally, let the set A_n be defined as $A_n = \{z \in {}^\omega\omega : M_z \models \Phi_n[z]\}$, so by Definition 3.3, the set A_n belongs to $\Sigma^0_{n+2}(\Pi^1_1)$. By the determinacy assumption, the set A_n is determined.

Now, consider the two-player game G_{A_n} with payoff A_n, in which each player plays two reals; player **I** plays $\langle a,u\rangle \in {}^\omega\omega$, and **II** plays $\langle b,v\rangle \in {}^\omega\omega$. They produce the real $z = \langle a,b,u,v\rangle$. Player **I** wins iff $z \in A_n$.

Now, since A_n is determined, and A_n is the payoff set of G_{A_n}, the game G_{A_n} is determined. Then, there must be a winning strategy $v_0 \in {}^\omega\omega$.

(1) If v_0 is winning for **I**, then, if **II** plays $\langle v_0,v_0\rangle$, the play z of the game is in A_n and Turing equivalent to v_0; hence $M_z = M_{v_0}$. Therefore, we must have $\neg\,\mathrm{Det}_n^{v_0}(*)$, that is, $M_{v_0} \models \neg\,\mathrm{Det}(\mathrm{OD}_{\Sigma_n})$. Let φ_0 and $\vec{\alpha}_0$ be the least (with respect to the ordering of formulas and the lexicographical $\in$-order, resp.) witness to the failure of $\mathrm{Det}(\mathrm{OD}_{\Sigma_n})$ in M_{v_0}, and let $X_0 = \{c \in {}^\omega\omega : \varphi_0[c,\vec{\alpha}_0]\}$.

Claim 4.3 $M_{v_0} \models$ (X_0 *is determined*).

Proof In fact, v_0 gives a winning strategy for **I** in $(G_{X_0})^{M_{v_0}}$ (the integer game with payoff X_0 played inside of M_{v_0}). If **II** plays $b \in ({}^\omega\omega)^{M_{v_0}}$, let $\langle a,u\rangle \in {}^\omega\omega$ be the output of v_0 when **II** plays $\langle b,v_0\rangle$ in G_{A_n}. Then, since v_0 is winning for **I**, $z = \langle a,b,u,v_0\rangle \in A_n$. Now, observe that $M_z = M_{v_0}$. Clearly $v_0 \leq_T z$; hence $v_0 \in \Delta^1_1(z)$, and so $M_{v_0} \subseteq M_z$ (see Proposition 3.6). To get $z \in \Delta^1_1(v_0)$, observe that $\langle a,u\rangle \in \Delta^1_1(b,v_0)$, but $b \in M_{v_0}$ (hence $b \in \Delta^1_1(v_0)$). So $\langle a,u\rangle \in \Delta^1_1(v_0)$ and $z \in \Delta^1_1(v_0)$; therefore $M_z = M_{v_0}$. But the definition of A_n states that $\langle a,b\rangle$ must be an element of the least witness of the failure of $\mathrm{Det}(\mathrm{OD}_{\Sigma_n})$ in M_z, therefore $\langle a, b\rangle \in X_0$. This proves the claim. □

The claim leads to the contradiction:

$$M_{v_0} \models (X_0 \text{ is determined}) \wedge (X_0 \text{ is not determined}).$$

Therefore the lemma is true, or v_0 is a winning strategy for player **II** instead.

(2) If v_0 is winning for **II**, consider u_0 to be $\langle v_0, x_0\rangle$ where x_0 is the real mentioned at the beginning of the proof. Then, since $u_0 \geq_T x_0$, $M_{u_0} \models \neg\,\mathrm{Det}(\mathrm{OD}_{\Sigma_n})$.

Let X_0 be the nondetermined class in M_{u_0} defined by the least formula (with respect to the Gödel ordering of formulas) and the least sequence of ordinal parameters (with respect to the lexicographical $\in$-ordering). In this case we say that X_0 is the "least" nondetermined class in M_{u_0}. The claim, as before, is that X_0 is in fact determined in M_{u_0} via $v_0 \in M_{u_0}$ (which is a contradiction). As before, v_0 works as a winning strategy, now for **II**, in the game $G_{X_0}^{M_{u_0}}$. In this case, for each possible play $a \in M_{u_0}$ of player **I** in $G_{X_0}^{M_{u_0}}$, consider the play $\langle a, u_0\rangle$ of **I** in G_{A_n}.

As both cases lead to a contradiction, the lemma is true. □

Corollary 4.4 $\mathrm{Det}(\Sigma^0_\omega(\Pi^1_1)) \Rightarrow \mathrm{Det}(*)$.

Proof For each $n \in \omega$, let z_n given by Lemma 4.2, and let z be recursively coded by the sequence $\langle z_n : n \in \omega\rangle$. Clearly z is a witness for $\mathrm{Det}(*)$. □

Remark 4.5 The contrary of the corollary does not hold (when $\mathrm{Det}(*)$ is true). This is because $\mathrm{Det}(*)$ is consistent with $V = L$ (see Corollary 4.8), so $\mathrm{Det}(*)$ (if true) does not even imply $\mathrm{Det}(\Pi^1_1)$.

It seems natural now to state a level-by-level implication from the large cardinals LC^n_M (see Definition 3.2) to the levels of $\mathrm{Det}(*)$. Let $\hat{\mathrm{LC}}^n_M$ be the statement LC^n_M plus the expression $(\exists\delta > \omega_1)(M = L_\delta[\mathcal{U}])$.

Theorem 4.6 *Assume* $\hat{\mathrm{LC}}^n_M$ *(for* $n \geq 2$*); then there is a cone of Turing degrees* $\mathcal{C}$ *such that* $(\forall y \in \mathcal{C})(M_y \models \mathrm{Det}(\mathrm{OD}_{\Sigma_n}))$.

Proof Let $n \geq 2$, and let $M = L_\delta[\mathcal{U}]$ as stated by $\hat{\mathrm{LC}}^n_M$. If κ is the measurable cardinal in M of Mitchell order κ^{++}, since κ^+ and κ^{++} exist in M and M satisfies the generalized continuum hypothesis (GCH), V_κ exists in M. Moreover, $V^M_\kappa \models \mathrm{ZFC}$. To show that, for example, *replacement* holds in V^M_κ, observe that from a witness of the failure of *replacement* in V^M_κ, that is, from a function $f : a \to V^M_\kappa$ ($a \in V^M_\kappa$) definable in V^M_κ, one can define in V^M_κ a function $g \in^{|a|} \kappa$ cofinal in κ. This function would be an element of $L_{\kappa^+}[\mathcal{U}]$ and also in M. Therefore, g would contradict the inaccessibility of κ in M. That the other ZFC axioms hold in V^M_κ follow trivially, by *replacement*$^{V^M_\kappa}$ and/or using an argument as above.

By Theorem 3.4, $M \models \mathrm{Det}(\Sigma^0_{n+2}(\Pi^1_1))$. But $\delta > \omega_1$ and all the reals of M are in V^M_κ, so every Δ^1_2-set of reals which is determined in M is also determined in V^M_κ. This way, $V^M_\kappa \models \mathrm{ZFC} + \mathrm{Det}(\Sigma^0_{n+2}(\Pi^1_1))$. Moreover, if we let $\Psi(x)$ be the statement $(\forall y \geq_T x)\,\mathrm{Det}^y_n(*)$, then, by Lemma 4.2: $V^M_\kappa \models \exists x \Psi(x)$. This gives a model of ZFC in which the determinacy statement holds. But again, as $\Psi(x)$ is Π^1_2, by Shoenfield's absoluteness, $\exists x \Psi(x)$ must hold in V as well. □

The next step is to obtain the whole of $\mathrm{Det}(*)$ from a single large cardinal axiom. As we said previously, this is the large cardinal $\mathrm{LC}(*)$ defined as follows: *There is a sequence of measures* $\mathcal{U}$ *such that*

$$(\forall n \geq 2)(\exists \delta_n \geq \omega_1)\big(L_{\delta_n}[\mathcal{U}] \models \exists \kappa_n (o(\kappa_n) = \kappa_n^{++} \wedge \kappa_n^{+n} \text{ exists})\big).$$

In fact, not just $\mathrm{Det}(*)$ for one real y can be obtained from $\mathrm{LC}(*)$, but a cone of Turing degrees of such y's.

Theorem 4.7 ($\mathrm{LC}(*)$) *There is a cone of Turing degrees* $\mathcal{C}$ *such that*

$$(\forall y \in \mathcal{C})\big(M_y \models \mathrm{Det}(\mathrm{OD})\big).$$

Proof For ease of notation, for each $n \in \omega$, let $N_n = L_{\delta_n}[\mathcal{U}]$ (where δ_n is as given in the definition of $\mathrm{LC}(*)$), and let κ_n be the corresponding cardinal in N_n of Mitchell order κ_n^{++}. Then (as before) for every $n \geq 2$, the statement $\exists z_n (\forall y \geq_T z_n) M_y \models \mathrm{Det}(\mathrm{OD}_{\Sigma_n})$ holds in N_n. Again, this last assertion is absolute for models containing the reals of N_n. So, for each $n \geq 2$ there is $z_n \in {}^\omega\omega$ such that for every y in the Turing cone determined by z_n, $M_y \models \mathrm{Det}(\mathrm{OD}_{\Sigma_n})$ holds in V. Let z be recursively coded by the sequence $\langle z_n : n \in \omega\rangle$, and let $\mathcal{C} = \{y \in {}^\omega\omega : y \geq_T z\}$. Then, if $y \in \mathcal{C}$, $y \geq_T z_n$ for every n. Hence, $M_y \models \mathrm{Det}(\mathrm{OD}_{\Sigma_n})$ holds for every n. This proves the theorem. □

As we mentioned earlier, $\mathrm{Det}(*)$ is a Σ^1_2-statement, and so (V, L)-absolute. We conclude then with the following.

Corollary 4.8 ($\mathrm{LC}(*)$) *The determinacy statement* $\mathrm{Det}(*)$ *is consistent with* $V = L$.

5 Lower Bound

In this section we give a lower bound for the cardinal strength of the statement $\mathrm{Det}(*)$ defined in Section 3. By the work of A. Lewis [7] we know that, under the assumption of $\mathrm{Det}(*)$, it is possible to build (inside M_x) a model of ZFC with λ measurable cardinals (for each $\lambda < \omega_1^{ck}$). Our goal is to extend these methods in order to obtain, from the same hypothesis, models of ZFC with stronger sequences of measures: with measurably many measurable cardinals, with a cardinal λ with λ measurable cardinals below (a 1-*fixed cardinal*), with a cardinal λ with λ 1-*fixed cardinals* below, and so on.

5.1 The games: General description If one intends to extend the methods of [7] in order to obtain a larger cardinal, one may start by trying to obtain models with stronger sequences of measures. One way of doing this could be by trying to obtain a cardinal λ, not as a given recursive ordinal but from the game itself, and then use λ for the construction of a λ-sequence of measures. This game would stay ordinal definable in M. A first attempt would be that of making λ measurable, so measurably many measurable cardinals would be obtained. The game for this would be something like the result of playing Lewis's game twice, plus a well order for λ, just being careful about the possible collapse of the cardinals during the recursive process described in [7]. Next, to obtain a model with a cardinal λ and λ measurable cardinals below it, extending the games just mentioned is not a clear option. We would need to play ω copies of these games (plus ω well orders). The problem is that when using the bounding argument, the play constructed by the loser would have to be recursive in σ (where σ is the winning strategy), but this is not clear as the sequence of well orders need not be recursive in σ; also one would have to deal with the recursive definition and the collapsing of the measures.

First, we will code larger sequences of "indiscernibles" for the measures and for the supremum of the ordinals of the model, as large as the ordinal to be measured itself. Then, instead of coding a sequence of ordinals by giving a parameter, a coding for the length of the sequence, and a definition (index of a Σ_1-formula) for each of the ordinals in the sequence, we will code the sequences of ordinals using a single definition and parameter, that is to say, using a parameter a and a single index that will correspond to a Σ_1-formula that defines in M_a the entire sequence.

The idea underlying the coding is that from each increasing sequence of ordinals $T : \omega_1^a \to \omega_1^a$ which is $\Sigma_1(a)$-definable in M_a, it is possible to define a sequence of integers. What we are looking for are sequences with "fixed points," something like sequences of ordinals that concentrate in their limits. So the supremum of the ordinals of the models would be a "fixed point," but so would each large cardinal that we will define in the model. Doing this also makes our games simpler, in the sense that we do not have to give the recursive definition of models as in [7] (although we keep some of the ideas for the definition of the rules). It also makes it simpler to obtain the large cardinals because there is no recursive construction of models, and so there are no collapsing cardinals. The model is obtained at once.

The bounding argument will follow, as in Martin's and Lewis' games, from the fact that the collection B of responses from the winner to what has been defined so far is in fact a set in M. This is true mainly because the set of possible plays for the loser at that level corresponds to a definable (in some level γ of M_σ) subset of ω, which is a set of $M_\sigma \subseteq M$.

5.2 *g*-structures and games of finite type In this section we define the large cardinals that we are intending to obtain from the determinacy hypothesis Det$(*)$, and also, we define the playful means to obtain them. We do this step by step, first defining the notion of *g-ordinals*, a notion that is involved in both the definition of the large cardinals and the definition of the games.

Definition 5.1 We define the notion *g-ordinal of type t* by finite induction on $t \in \omega$.

(1) If δ is a countable ordinal, then the pair $\langle \delta, \emptyset \rangle$ is said to be a *g-ordinal of type zero*. Keeping this in mind we abuse notation and say that δ is an ordinal of *g-type zero*.
(2) Given δ a countable ordinal, the pair $\langle \delta, I_\delta^t \rangle$ is said to be a *g-ordinal of type $t+1$* if $I_\delta^t = \langle \rho_\eta : \eta \in \delta \rangle$ is an increasing sequence of ordinals of g-type t cofinal in δ. In this case we also abuse notation to say that δ is an ordinal of *g-type $t+1$*.

If δ is of g-type $(t+1)$, the set I_δ^t that witnesses this will be called a *set of indiscernibles of type t* for δ. For each $1 \le k \le t$, if ξ is in a set of indiscernibles of *type k* for δ, then I_ξ^{k-1} is a *set of indiscernibles of type $k-1$* for ξ.

Definition 5.2 Let $t \in \omega$. A *g-structure of type* t is defined as a set of the form $\langle t, \lambda, \delta, I_\lambda \rangle$, where λ is an ordinal of g-type t, δ is of g-type $(t+1)$, I_λ is formed by all the sets of indiscernibles of type zero for λ, and λ is the first indiscernible of type t for δ in I^t_δ.

Let $\langle t, \lambda, \delta, I_\lambda \rangle$ be a g-structure, and let $\mathcal{U}_{I_\lambda}$ be the sequence of filters given by the sets of indiscernibles in I_λ; that is to say, $F \in \mathcal{U}_{I_\lambda}$ if and only if there is $I^0_\xi \in I_\lambda$ such that F is the tail filter (for ξ) on I^0_ξ in $N = L_\delta[\mathcal{U}_{I_\lambda}]$ (i.e., $x \in F$ iff $x \subset \xi \cap N$ and for some $\beta \in \xi$, $I^0_\xi \setminus \beta \subset x$). This way, we can think of $\langle t, \lambda, \delta, I_\lambda \rangle$ as coding the structure $\langle N = L_\delta[\mathcal{U}_{I_\lambda}], \in, N \cap \mathcal{U}_{I_\lambda} \rangle$. When $t = 0$ the g-structure is thought of as coding the structure $\langle L_\delta, \in \rangle$.

The hierarchy that we will be studying is the hierarchy of *finitely fixed cardinals*. A cardinal λ is said to be *zero-fixed* if λ is a measurable cardinal. For $t \geq 0$, we say that λ is $(t+1)$-*fixed* if the set $\{\kappa < \lambda : \kappa \text{ is a } t\text{-fixed cardinal}\}$ has cardinality λ.

Definition 5.3 A g-structure $\langle t, \lambda, \delta, I_\lambda \rangle$ is said to be *good* if

$$L_\delta[\mathcal{U}_{I_\lambda}] \models \mathrm{ZFC} + \lambda \text{ is } (t-1)\text{-fixed}.$$

We will obtain these g-structures from integer games ordinal definable in least admissible sets. In order to make the definition of the integer games more digestible, we give an "ordinal" version first. That is to say, we define a set of ordinal games that describe how to produce the structures. These games are two-player games in which the length of the plays is not relevant. The definition for these games will be given in parts using the notions of *partial ordinal game* and *iteration*. We define these notions by recursion in the next paragraphs.

The partial ordinal game of type zero, denoted by O^0_1, is a two-player game in which each player plays an ω-sequence of ordinals, $\langle \alpha_n : n \in \omega \rangle$ and $\langle \beta_n : n \in \omega \rangle$, respectively. The rule is that for each n, $\alpha_n < \beta_n < \alpha_{n+1}$. The first player who fails to satisfy this rule loses. Otherwise, if both players satisfy the rule we say that they have reached a *tie*. So a *tie play* of O^o_1 produces an ordinal $\lambda = \sup_n(\alpha_n) = \sup_n(\beta_n)$ of g-type zero.

Let λ be a countable limit ordinal. A λ-*iteration* of O^0_1 (denoted by O^0_λ) is a partial game in which a tie play produces an increasing λ-sequence of ordinals of type zero above λ. So in order not to lose, each player produces an increasing $(\omega \cdot \lambda)$-sequence $\langle \alpha_{\omega\cdot\eta+j} : \eta \in \lambda \wedge j \in \omega \rangle$ (and $\langle \beta_{\omega\cdot\eta+j} : \eta \in \lambda \wedge j \in \omega \rangle$, respectively) such that $\lambda \leq \alpha_0$, and $\alpha_\xi < \beta_\xi$ (for $\xi \in \omega \cdot \lambda$). The ordinals $\mu_\eta = \sup_j(\alpha_{\omega\cdot\eta+j}) = \sup_j(\beta_{\omega\cdot\eta+j})$ are the λ many g-type zero ordinals produced by the tie play (in this iteration O^0_λ).

The *ordinal game of type zero*, denoted by $\mathcal{O}^0$, is a two-player game in which the players must produce an ordinal δ and interleaving sequences of ordinals $T_I = \langle \alpha_\eta : \eta < \delta \rangle$ and $T_{II} = \langle \beta_\eta : \eta < \delta \rangle$ with supremum equal to δ in the following way: $T_I, T_{II} \restriction \omega$ should be a tie play for O^0_1 with supremum λ_0 (an ordinal of g-type zero). Then, the next segment of the sequence (as defined from ω to $\omega \cdot \lambda_0$) should be a tie play for $O^0_{\lambda_0}$; this play defines a sequence of g-type zero ordinals $\langle \mu_\eta : \eta < \lambda_0 \rangle$. Setting λ_1 to be the supremum of these ordinals, the players are asked to produce the next segment of the sequence (defined from $\omega \cdot \lambda_0$ to $\omega \cdot \lambda_1$) as a tie play for $O^0_{\lambda_1}$. They have to continue producing these tie plays for each of the partial games $O^0_{\lambda_n}$ so defined. If these rules are satisfied, by the end the players have produced $\lambda = \lambda_0$ (an ordinal of g-type zero) and a g-ordinal of type 1, namely, $\langle \delta, I^0_\delta \rangle$, where $\delta = \sup_n(\lambda_n)$ and $I^0_\delta = \langle \mu_\eta : \eta \in \delta \rangle$ ($\langle \mu_\eta : \eta < \lambda_n \rangle$ is produced by the corresponding tie play of $O^0_{\lambda_n}$).

In other words, if these basic rules are satisfied by a play, this play gives a g-structure of type zero, namely, $\langle 0, \lambda, \delta, \emptyset \rangle$. At this point we can define the *partial ordinal game of type* 1 (denoted as O^1_1) as the two-player game in which a tie play produces, following the description above, a g-structure of type zero. In order to finish the description of the game $\mathcal{O}^0$ we must define the "winning rules;" that is, we have to state what will happen when the g-structure of type zero has been produced. We will call these the *strong rules* of the game $\mathcal{O}^0$. To decide which player wins, we look at the model $N = L_\delta$ and decide a winner depending on the following cases.

(1) $N \models$ ZFC.
- In this case player **I** wins.

(2) $N \not\models$ "Collection," and Collection fails at ν (ν least). Let h be the "least" witness, that is, $h : \nu \to \delta$ cofinal and definable in N. Set $\gamma_0 = \min\{\gamma \in \nu : h(\gamma) \geq \mu_1\}$ and $\gamma_1 = \min\{\gamma \in \nu : h(\gamma) \geq \mu_2\}$.
- Player **I** wins if $\gamma_0 < \gamma_1$.

(3) $N \models$ "Collection $\wedge \neg$ Power Set," and Power Set fails at ν; $P(\nu) \cap N$ has order type δ. Set $\gamma_0 = \inf(Y_{\mu_1} \triangle Y_{\mu_2})$ and $\gamma_1 = \inf(Y_{\mu_2} \triangle Y_{\mu_3})$, where $\{Y_{\mu\eta} : \eta \in \delta\}$ is the canonical enumeration of $P(\nu)$ in N.
- Player **I** wins if $\gamma_0 < \gamma_1$.

A *successful play* for $\mathcal{O}^0$ would be a play in which player **I** wins because of condition (1), that is to say, a play that produces the good g-structure of type zero, $N = L_\delta \models$ ZFC.

This is the first step, we may well say the zero step, in our attempt to produce games for good g-structures for all finite types.

In general, to define the ordinal games for any type $t > 0$, assume that the *partial ordinal game of type t* O^t_1 and its iterations O^t_λ have been defined. So a tie play on O^t_1 produces a g-ordinal of type t $\langle \lambda_0, I^{t-1}_{\lambda_0} \rangle$. And a tie play on the λ-iteration of O^t_1 produces λ g-ordinals of type t above λ.

Similarly as before, in the *ordinal game of type t* $\mathcal{O}^t$ the players must produce a δ-sequence of g-type ordinals below δ $I^t_\delta = \langle \rho_\eta : \eta < \delta \rangle$. This must be done in the following way: First play a tie in O^t_1; if λ_0 is the ordinal of g-type t produced by this play, then they have to produce a tie play for $O^t_{\lambda_0}$, and so on. Just as before, we define the *partial game O^{t+1}_1 of type* $t + 1$ by taking plays in the iterations $O^t_{\lambda_n}$. The basic rule for the game $\mathcal{O}^t$ states that players must play tie plays of O^{t+1}_1. Observe again that a play in $\mathcal{O}^t$ that satisfies the basic rules produces a g-structure $\langle t, \lambda, \delta, I_\lambda \rangle$. That is to say, it produces the structure $N = L_\delta[\mathcal{U}_{I_\lambda}]$, where $\mathcal{U}_{I_\lambda} = \langle F(\eta) : \eta < \lambda \rangle$, $F(\eta)$ is the tail filter on $I^0_{\kappa_\eta}$, and κ_η is the ηth ordinal of g-type 1 below $\lambda(= \lambda_0 = \rho_0)$. Also, $I^t_\delta = \langle \rho_\eta : \eta < \delta \rangle$. To finish with the description of the game we state the strong rules; these rules are given in terms of the theory of N:

(1) $N \models ZF +$ "λ is a $(t-1)$-fixed cardinal from I_λ."
- In this case player **I** wins.

(2) $N \not\models$ Collection, and Collection fails at ν (ν least). Let h be the "least" witness; that is, $h : \nu \to \delta$ cofinal and definable in N. Define $\gamma_0 = \min\{\gamma \in \nu : h(\gamma) \geq \rho_1\}$ and $\gamma_1 = \min\{\gamma \in \nu : h(\gamma) \geq \rho_2\}$.
- Player **I** wins if $\gamma_0 < \gamma_1$.

(3) $N \models$ " Collection $\wedge \neg$ Power Set," and Power Set fails at ν (ν least); $P(\nu) \cap N$ has order type δ. Set $\gamma_0 = \inf(Y_{\rho_1} \triangle Y_{\rho_2})$ and $\gamma_1 = \inf(Y_{\rho_2} \triangle Y_{\rho_3})$, where $\{Y_{\rho_\eta} : \eta \in \delta\}$ is the canonical enumeration of $P(\nu)$ in N.
- Player **I** wins if $\gamma_0 < \gamma_1$.

(4) $N \models$ ZFC $\wedge$ "$F(\eta_0)$ is not an ultrafilter," for $\eta_0 \in \lambda$ least. Let X be the $<_N$-least unmeasured set. And let $\mu_1^{\kappa_{\eta_0}}$ be the second element in $I^0_{\kappa_{\eta_0}}$.
- Player **I** wins if $\mu_1^{\kappa_{\eta_0}} \notin X$.

(5) $N \models$ ZFC $\wedge$ "$\mathcal{U}_I$ is a sequence of ultrafilters $\wedge$ $(\exists \eta_0 \in \lambda)(F(\eta_0)$ is not κ_{η_0}-complete)" for η_0 least. Let $\nu = \min\{\nu < \kappa_{\eta_0} : F(\eta_0)$ is not $(\nu + 1)$-complete$\}$, and let $\langle X_\xi : \xi < \nu \rangle$ be the $<_N$-least witness of the κ_{η_0}-incompleteness of $F(\eta_0)$. (One can assume that the sequence is decreasing and continuous.) Then, let $g : \nu \to \kappa_{\eta_0}$ be defined by $g(\xi) = \min\{0 < \gamma < \kappa_{\eta_0} : (\forall \gamma' \geq \gamma)(\mu^{\eta_0}_{\gamma'} \in X_\xi)\}$ (where $\mu^{\eta_0}_{\gamma'}$ is the γ'th indiscernible in $I^0_{\kappa_{\eta_0}}$); observe that in this case g is cofinal and nondecreasing and continuous. Finally, let $\xi_0 = \min\{\xi < \nu : g(\xi_0) \neq g(\xi_0 + 1)\}$.
- Player **I** wins if $g(\xi_0 + 1) = g(\xi_0) + 1$.

To complete the inductive definition, we still have to define the iterations O^{t+1}_λ of O^{t+1}_1. But just as before, in O^{t+1}_λ each player must play λ many increasing sequences of ordinals each satisfying the rules of O^{t+1}_1, one on top of the next one, and the first one above λ. That is to say, a play of O^{t+1}_λ is the result of playing O^{t+1}_1 λ many times.

This finishes the description of the ordinal games of type t, games that produce g-structures of type t. Remember that what we want is to obtain good g-structures from the assumption of Det$(*)$.

Now we move to the next step. Since what we want is to obtain g-structures from the assumption Det$(*)$ (the assumption that states the determinacy of all integer games which are ordinal definable in some admissible M), we have to figure out a way of encoding the ordinal games just described from integer games in M. That is to say, we must state a way in which a single real codes a full play of $\mathcal{O}^t$, keeping in mind that the definitions involved in these integer games must be given in M without further parameters.

5.3 Coding games Let us start by fixing a least admissible set $M = M_x$ (for some $x \in \omega$). As we just said, throughout this section we will define an integer game $\mathcal{C}^t$ (for each $t \in \omega$) that is ordinal definable in M and that "codes" $\mathcal{O}^t$.

We start with the description of $\mathcal{C}^0$, for simplicity of course. First consider a fixed recursive enumeration of the Σ_1-formulas of LST, say, $\{\varphi_e\}_e$. The game $\mathcal{C}^0$ is defined as a two-player integer game in which players **I** and **II** alternate integers as follows:

$$\begin{array}{llll} \mathbf{I}: & e & a_0 & a_1 \ldots, \\ \mathbf{II}: & f & b_0 & b_1 \ldots \end{array}$$

with $e, f, a_i, b_i \in \omega$, so we can say that **I** plays $\langle e, a\rangle$ and **II** plays $\langle f, b\rangle$ (with $a, b \in {}^{\omega}\omega$). The idea is that with $\langle e, a\rangle$, **I** is playing the formula $\varphi_e(x, y, a)$, which will define in M_a an increasing sequence of ordinals $T_I = \langle \alpha_\eta : \eta \in \omega_1^a\rangle$; and player **II** is coding $\varphi_f(x, y, b)$, which defines in M_b a sequence $T_{II} = \langle \beta_\eta : \eta \in \omega_1^b\rangle$. Both do this in such a way that, up to some $\delta \in \omega_1^a \cap \omega_1^b$, T_I and T_{II} produce a play in the game $\mathcal{O}^0$. Formally, we enumerate the basic rules for $\mathcal{C}^0$ as follows: The integers e, f and the reals a, b must be played in such a way that

(1) $\omega_1^a = \omega_1^b$; otherwise, the one producing the smallest one loses;
(2) for each $\eta < \omega_1^a = \omega_1^b$: "($\varphi_e \restriction \eta$ defines an increasing sequence of ordinals below ω_1^a) $\Leftrightarrow$ ($\varphi_f \restriction \eta$ defines an increasing sequence of ordinals below ω_1^b)"; the first one failing loses;
(3) the formula $\varphi_e(x, y, a)$ must define in M_a a $\Sigma_1(a)$-increasing ordinal function $T_I = \langle \alpha_\eta : \eta \in \omega_1^b\rangle$; respectively, $\varphi_f(x, y, b)$ must define a corresponding T_{II}; the first one failing this loses;
(4) the sequences T_I and T_{II} must be interleaving, in the sense that $\alpha_\eta < \beta_\eta < \alpha_{\eta+1}$; otherwise, the first one failing to do so loses.

Observe that, for example, rule (1) can be expressed in M by

$$\forall \xi (WO_\xi^a = \emptyset \leftrightarrow WO_\xi^b = \emptyset),$$

where WO_ξ^a is defined as

$$r \in WO_\xi^a \Leftrightarrow (r \leq_T a) \wedge (\exists f : \langle \omega, <_r\rangle \to \langle \xi, \in\rangle \text{ order preserving}).$$

It should be clear then that $WO_\xi^a \neq \emptyset$ in M if and only if $\xi < \omega_1^a$. This being said, it should also be clear that the rest of the rules can be expressed in M without any further parameters.

If the basic rules are satisfied, then T_I and T_{II} have produced a $\Sigma_1(a, b)$-increasing sequence $T : \omega_1^a \to \omega_1^a$ in $M_a = M_b$. Let $T(\eta) = \bigcup T_I''\omega \cdot \eta = \bigcup T_{II}''\omega \cdot \eta$, for every $\eta < \omega_1^a$. This $\Sigma_1(\langle a, b\rangle)$-definition makes perfect sense, since for each $\eta < \omega_1^a$, the ordinal $\omega \cdot \eta$ also belongs to ω_1^a.

The strong rules for $\mathcal{C}^t$ will be given just as in the case of $\mathcal{O}^t$, in terms of the sequence of ordinals T played. The strong rules will be the same in both cases. In order to make sense of the strong rules of $\mathcal{O}^t$ in $\mathcal{C}^t$, we need to be able to "extract" a play of $\mathcal{O}^t$ from each play T of $\mathcal{C}^t$. That is to say, we need to be able to obtain the sequences of g-ordinals of type k ($k \leq t + 1$) that form a play in $\mathcal{O}^t$, and so obtain a g-structure, from the play T in $\mathcal{C}^t$. The next definition will help to understand how this will be done.

Definition 5.4 Let $\alpha < \omega_1$, let $T : \alpha \to \alpha$ be an increasing sequence, and let $\langle \delta, I_\delta^t\rangle$ be a g-ordinal of type $t + 1$ (with $\delta \leq \alpha$, and $t \geq 0$). To say that I *comes from* T means: If $t = 0$ and $I_\delta^0 = \langle \mu_\nu : \nu < \delta\rangle$, then for all $\nu < \delta$, $\mu_\nu = T(\nu)$. And if $t > 0$, then for each $\rho \in I_\delta^t$, $\rho = \bigcup T''\rho$.

A play T in $\mathcal{C}^t$ will "code" a play in $\mathcal{O}^t$ in the sense that from T it is possible to extract the g-ordinals as sequences that "come from" T. The next lemma explains the whole idea about the coding.

Lemma 5.5 *Let M_a be the corresponding least admissible set for a fixed real $a \in {}^{\omega}\omega$, and let $T : \omega_1^a \to \omega_1^a$ be an increasing sequence which is $\Sigma_1(a)$-definable in M_a and such that $T''\omega_1^a \subseteq \mathrm{Lim}(\omega_1^a)$. Then, for each $t \in \omega$ and $\zeta \in \omega_1^a$, there is* *a g-ordinal $\langle \delta, I^t_\delta \rangle$ of type $t+1$ above ζ that comes from T such that $I^t_\delta : \delta \to \delta$ is $\Sigma_1(a)$-definable in M_a, so it corresponds to a tie play in O^{t+1}_1.*

Proof We will construct these g-ordinals, with the properties specified, by finite induction on t.

Fix $\zeta < \omega^a_1$. For $t = 0$, in order to define $\langle \delta, I^0_\delta = \langle \mu_\nu : \nu < \delta \rangle \rangle$, first define the sequence $\Lambda_{\zeta,0} = \langle \lambda_n : n \in \omega \rangle$ by taking $\lambda_0 = T(\zeta+1)$ and $\lambda_{n+1} = \bigcup T''\lambda_n$. This definition by Σ_1-induction shows that $\Lambda_{\zeta,0} \in M$, then we set $\delta = \bigcup_n \lambda_n$ and $I^0_\delta = T \restriction (\delta \setminus \zeta + 1)$, so $\mu_\nu = T(\zeta + \nu)$.

Each of the objects just defined are $\Sigma_1(a)$-definable in M_a, including I^0_δ of course. It is also clear that $\delta = \bigcup T''\delta$.

For $t > 0$, assume that g-ordinals of type t $\langle \lambda, I^{t-1}_\lambda \rangle$ can be constructed from T in a $\Sigma_1(a)$-fashion. Define $\delta_{(\zeta,t-1)}$ to be the g-ordinal of type $t-1$ above ζ given by this inductive hypothesis. Construct $\langle \delta, I^t_\delta \rangle$, a g-ordinal of type $t+1$ above ζ as follows.

As in the basic step of the induction, first define a sequence $\Lambda_{\zeta,t} = \langle \lambda_n : n \in \omega \rangle$ by induction in the following way. For $n = 0$, define $\lambda_0 = \delta_{(\zeta,t-1)}$; then define a continuous sequence $\Upsilon_{\lambda_0} = \langle \rho_\nu : \nu < \lambda_0 \rangle$ by taking $\rho_0 = \lambda_0$ and $\rho_{\nu+1} = \delta_{(\rho_\nu,t-1)}$. Then set $\lambda_{n+1} = \bigcup \Upsilon_{\lambda_n}$, and define $\Upsilon_{\lambda_{n+1}}$ by taking $\rho_{\nu+1} = \delta_{(\rho_\nu,t-1)}$, with $\nu \in [\lambda_n, \lambda_{n+1})$. Just as before, set $\delta = \sup_n \lambda_n$ and $I^t_\delta = \bigcup_n \Upsilon_{\lambda_n} = \langle \rho_\nu : \nu < \delta \rangle$. Each of these definitions is given by Σ_1-recursion. □

Observe that the method used in the proof to build the g-ordinals takes into account each fixed point of the function T as they appear in $\in$-increasing order. That is to say, each g-ordinal constructed from T corresponds to a fixed point of T (i.e., to an ordinal α such that $\alpha = \bigcup T''\alpha$). But also, if δ is a g-ordinal constructed from T and $\alpha < \delta$ is a fixed point of T, then α was taken into account previously as a g-ordinal from T.

Now it makes sense to define the strong rules for $\mathcal{C}^t$ as the set of strong rules for $\mathcal{O}^t$ (given in Section 5.2), given T a play produced by **I** and **II** in $\mathcal{C}^t$ from T_I and T_{II}, respectively, when they play (say, $\langle e, a \rangle$ and $\langle f, b \rangle$) and the basic rules have been respected. Apply Lemma 5.5 to obtain a g-ordinal $\langle \delta, I^t_\delta \rangle$ that comes from T (this is $\Sigma_1(\langle a, b \rangle)$-definable in M_a); then look at $N = L_\delta[\mathcal{U}_\lambda]$ (where λ is the first element in I^t_δ), and depending on the theory of N, as defined for the strong rules of $\mathcal{O}^t$ in Section 5.2, decide a winner for the game $\mathcal{C}^t$.

5.4 Bounding lemma So far we have set up an integer game $\mathcal{C}^t$ (for $t \in \omega$) which is ordinal definable in admissible sets, and such that any play in this game that satisfies the basic rules produces a g-model. Our next major task is to show that when the game $\mathcal{C}^t$ is determined, there is a play that produces a good g-model. In order to achieve this, we make sure that the loser of the game $\mathcal{C}^t$ can make sure not to lose when playing against a winning strategy σ because of any of the basic rules. A way of doing this is by bounding all possible responses (in the corresponding ordinal game $\mathcal{O}^t$) of the winner at a given level.

On the other hand, the play for the loser built by this bounding argument can be carried out in such a way that the loser loses because of the first of the strong rules, and therefore a good g-model is produced. The following lemma summarizes the bounding argument.

Lemma 5.6 *Let $x \in {}^{\omega}\omega$ be such that $M = M_x \models \mathrm{Det(OD)}$. There is an index e for a Σ_1-formula $\varphi_e(x, y, z)$ such that*

(1) if $\sigma \in M$ is a winning strategy for a player in the game $\mathcal{C}^t$, then $\langle e, \sigma\rangle$ is a play for the loser that satisfies the basic rules of $\mathcal{C}^t$;

(2) any $\Sigma_1(\sigma)$-definable cofinal subsequence of the sequence defined by the play $\langle e, \sigma\rangle * \sigma$ also satisfies the basic rules of the game $\mathcal{C}^t$.

Proof Before we begin with the actual proof of the lemma, we will make some definitions and remarks.

Consider the game $\mathcal{C}^t$ as defined in M, and let $\sigma \in M$ be a winning strategy in this game, say, that σ is a winning strategy for player **II**. Let $\eta < \omega_1^\sigma$, let S be an increasing $(\eta+1)$-sequence of ordinals $S = \langle \alpha_\xi : \xi \le \eta\rangle$ that belongs to M_σ, and let $\nu < \omega_1^\sigma$ be such that $S \in L_\nu[\sigma]$. For $e' \in \omega$, let "$\varphi_{e'} \lhd_{\langle \eta', \eta\rangle} S$" stand for

$$\big((\forall \xi' < \eta')(\exists! \alpha)\varphi_{e'}(\xi', \alpha, \sigma)\big) \wedge (\varphi_{e'} \restriction \eta' \text{ is increasing}) \wedge \big((\forall \xi' < \eta')(\exists \xi < \eta) \\ \varphi_{e'}(\xi', \alpha_\xi, \sigma)\big) \wedge \varphi_{e'}(\eta', \alpha_\eta, \sigma).$$

Now, define the set $P_\nu^\eta(S) \subset \omega$ as follows:

$$e' \in P_\nu^\eta(S) \iff L_\nu[\sigma] \models \big[\exists \nu' \exists \eta' \exists w \big((\eta' \le \eta) \wedge (w = L_{\nu'}[\sigma]) \\ \wedge \mathrm{Sat}(w, \varphi_{e'} \lhd_{\langle \eta', \eta\rangle} S)\big)\big].$$

Again, as in Section 3, for the Δ_1 (in admissible sets) formulas, "$w = L_{\nu'}[\sigma]$" and "$\mathrm{Sat}(w, \psi)$," and for the formalization of syntax involved, we refer to Devlin [3]. This being said, in what follows of this proof and the proof of the next lemma, for expressions like "$\exists w(w = L_\gamma[\sigma] \wedge \mathrm{Sat}(w, \psi))$," we might simply write $L_\gamma[\sigma] \models \psi$.

Since the set $P_\nu^\eta(S)$ is definable in $L_\nu[\sigma]$ from σ (for $\nu \in \omega_1^\sigma$), it belongs to $L_{\nu+1}[\sigma] \subset M_\sigma$; hence $P_\nu^\eta(S) \in M_\sigma$. The set $P_\nu^\eta(S)$ corresponds to the set of all indexes $e' \in \omega$ such that, in $L_\nu[\sigma]$, it is true that $\varphi_{e'}$ defines (up to some $\eta' \le \eta$) a subsequence of S with largest element equal to α_η.

Now, define the set $B_\nu^\eta(S)$ in terms of $P_\nu^\eta(S)$ in the following way:

$$B_\nu^\eta(S) = \big\{\beta : \big(\exists e' \in P_\nu^\eta(S)\big)\exists \gamma_{e'}\big(L_{\gamma_{e'}}[\sigma] \models \text{“}\exists \langle f, b\rangle(\langle f, b\rangle = \langle e', \sigma\rangle * \sigma) \\ \wedge\ (\exists \eta' \le \eta)(\varphi_{e'} \lhd_{\langle \eta', \eta\rangle} S) \wedge \varphi_f(\eta', \beta, b)\text{”}\big)\big\}.$$

The set $B_\nu^\eta(S)$ is the image of the set $P_\nu^\eta(S)$ (which belongs to M_σ) under a $\Sigma_1(\sigma)$-definable function in M_σ. So it belongs to M_σ. Observe that this set $B_\nu^\eta(S)$ roughly corresponds to the set of all ordinals that the winner can play at the ηth level of the corresponding ordinal game, when the loser has played a subsequence S' of S such that (in $L_\nu[\sigma]$) S' is coded by a real $\langle e', \sigma\rangle$. The next lemma is still part of the preparations towards the proof of Lemma 5.6.

Lemma 5.7 *Fixing S and η ($S = \langle \alpha_\xi : \xi \le \eta\rangle$), there is an ordinal $\hat{\alpha} \in M_\sigma$, which is $\Sigma_1(\sigma)$-definable in M_σ, such that $B_{\hat{\alpha}}^\eta(S)$ is well defined and such that $\hat{\alpha} \ge \sup B_{\hat{\alpha}}^\eta(S)$.*

Proof Let α_0 be the rank of S, that is to say, $S \in L_{\alpha_0+1}[\sigma] \setminus L_{\alpha_0}[\sigma]$. For each $e' \in P_{\alpha_0}^\eta(S)$, let $\beta_{e'}$ be the image of e' under the $\Sigma_1(\sigma)$-function defined in $B_{\alpha_0}^\eta(S)$, and let $\gamma_{e'}$ be the least witness of this fact. Then, define $\alpha_1 = \sup\{\gamma_{e'} : e' \in P_{\alpha_0}^\eta(S)\}$. Inductively following the same idea, define α_{n+1} from α_n as the supremum of all the ordinals that witness that $\beta_{e'} \in B_{\alpha_n}^\eta(S)$ for each $e' \in P_{\alpha_n}^\eta(S)$. The sequence

$\langle \alpha_n : n \in \omega \rangle$ is defined by Σ_1-recursion in M_σ. So the sequences $\langle \alpha_n : n \in \omega \rangle$ and $\hat{\alpha} = \sup_n \alpha_n$ belong to M_σ. Finally, observe that in fact $\hat{\alpha}$ has the property that $\hat{\alpha} \geq \sup B^\eta_{\hat{\alpha}}(S)$: If $\hat{\alpha} < \sup B^\eta_{\hat{\alpha}}(S)$, there must be $\beta_{e'} \in B^\eta_{\hat{\alpha}}(S)$ such that $\hat{\alpha} < \beta_{e'}$, but the definition of $P^\eta_{\hat{\alpha}}(S)$ implies that $e' \in P^\eta_{\alpha_n}(S)$ for some $n \in \omega$. And the least witness $\gamma_{e'}$ of the fact that $\beta_{e'} \in B^\eta_{\alpha_n}(S)$ is the same that witnesses that $\beta_{e'} \in B^\eta_{\hat{\alpha}}(S)$. This leads to a contradiction $\hat{\alpha} < \beta_{e'} < \gamma_{e'} < \alpha_{n+1} < \hat{\alpha}$. □

We finish the preparations for the proof of Lemma 5.6 with the following remarks.

Remarks 5.8 Let $\hat{\alpha}$ be the ordinal defined in the proof of Lemma 5.7.

(1) The ordinal $\hat{\alpha}$ is least with the property that $\hat{\alpha} \geq \sup B^\eta_{\hat{\alpha}}(S)$: If α is such that $\alpha \geq B^\eta_\alpha(S)$, then $\alpha \geq \alpha_0$. (Otherwise $B^\eta_\alpha(S)$ makes no sense.) Also, observe that it is impossible that $\alpha_n < \alpha < \alpha_{n+1}$ (for some $n \in \omega$) since $B^\eta_{\alpha_n}(S) \subset B^\eta_\alpha(S)$. Finally, if $\alpha = \alpha_n$ for some n, then $\hat{\alpha} = \alpha_n = \alpha$.
(2) In the definition of $B^\eta_{\hat{\alpha}}(S)$ the existential quantifier $\exists \gamma_{e'}$ can be bounded by $\hat{\alpha}$: this is clear from the construction of $\hat{\alpha}$.
(3) The ordinal $\hat{\alpha}$ is Σ_1-definable in M_σ by the following formula $\psi(\alpha, S, \eta, \sigma)$:

$$\begin{aligned}\psi(\alpha, S, \eta, \sigma) \equiv (\forall \alpha' < \alpha)\exists \beta\big(\alpha' < \beta \wedge \beta \in B^\eta_{\alpha'}(S)\big) \wedge \big(\forall e' \in P_\alpha(t)\big)\\ \cdots (\exists \gamma < \alpha)\big(L_\gamma[\sigma] \models \text{“}\exists \langle f, b\rangle(\langle f, b\rangle = \langle e', \sigma\rangle * \sigma)\\ \wedge \cdots (\exists \eta' \leq \eta)(\varphi_{e'} \lhd_{\langle \eta', \eta\rangle} S) \wedge \varphi_f(\eta', \beta, b)\text{”}\big).\end{aligned}$$

Back to the proof of Lemma 5.6. The idea of this proof is to define e as the index of a Σ_1-formula $\Phi(x, y, z)$ which defines in M_σ a continuous sequence T_I, such that the $(\eta + 1)$th member of the sequence (say, $\alpha_{\eta+1}$) is the supremum of the set $B^\eta_{\alpha(\eta+1)}(T_I \upharpoonright \eta + 1)$. Since the set $B^\eta_{\alpha(\eta+1)}(T_I \upharpoonright \eta + 1)$ corresponds to the set of all responses of the winner (following the strategy) to any possible play of the loser that gives a subsequence of the sequence $T_I \upharpoonright \eta + 1$ with same last element, the lemma will follow.

We take e to be the index of the following $\Sigma_1(\sigma)$-formula $\Phi(\eta, \alpha, \sigma)$:

$$\begin{aligned}\exists S = \langle \alpha_\xi : \xi < \eta\rangle\Big((\alpha_0 = \omega) \wedge (\forall \xi < \eta)\Big((\mathrm{Succ}(\xi) \to \psi(\alpha_\xi, S \upharpoonright \xi, \xi, \sigma))\\ \cdots \Big(\mathrm{Lim}(\xi) \to \alpha_\xi = \bigcup_{\varsigma < \xi} \alpha_\varsigma\Big)\Big) \wedge \big(\mathrm{Succ}(\eta) \to \psi(\alpha, S, \eta - 1, \sigma)\big)\\ \wedge \cdots \Big(\mathrm{Lim}(\eta) \to \alpha = \bigcup_{\xi < \eta} \alpha_\xi\Big)\Big).\end{aligned}$$

This formula is defining, by recursion in M_σ, an increasing continuous sequence of ordinals $T_I = \langle \alpha_\eta : \eta < \omega_1^\sigma\rangle$ in the following way: If $T_I \upharpoonright \eta + 1 = \langle \alpha_\xi : \xi \leq \eta\rangle$ has been defined, then $\alpha_{\eta+1} = \alpha$ if and only if $\psi(\alpha, S, \alpha, \sigma)^{M_\sigma}$ (if and only if α is the least ordinal such that $\alpha \geq B^\eta_\alpha(T_I \upharpoonright \eta)$).

Observe that for any η, $T_I \upharpoonright \eta + 1 \in L_{\alpha(\eta+1)}[\sigma]$. By definition of $\alpha_{(\eta+1)}$, the rank of $T_I \upharpoonright \eta + 1$ is below $\alpha_{(\eta+1)}$. Therefore, the set $P^\eta_{\alpha_{\eta+1}}(T \upharpoonright \eta + 1)$ is well defined and $e \in P^\eta_{\alpha_{\eta+1}}(T \upharpoonright \eta + 1)$.

This last observation implies that $\langle e, \sigma\rangle$ is a play of the loser that satisfies the basic rules, because $\alpha_{\eta+1}$ is bounding all the possible plays β_η of the winner following the strategy σ, when the loser has played $\langle \alpha_\xi : \xi \leq \eta\rangle$ in the ordinal game.

Finally, since the definition of $P_\alpha^\eta(S)$ also considers the indices of those formulas that (in $L_\alpha[\sigma]$) define subsequences of S, the second part of the lemma also follows.

Now, when the winning strategy σ belongs to player **I** instead, we have to construct the first play of the loser (in the ordinal game) using sets $P_\nu(\emptyset)$ and $B_\nu(\emptyset)$, where

$$P_\nu(\emptyset) = \left\{e' \in \omega : L_\nu[\sigma] \models \left(\exists \nu' L_{\nu'}[\sigma] \models \exists! \alpha \varphi_{e'}(0, \alpha, \sigma)\right)\right\}$$

and $B_\nu(\emptyset)$ is the corresponding set. So the first ordinal of the sequence will correspond to $\hat{\alpha}$ built as in Lemma 5.7, but starting with ω instead. After that, the construction is the same.

Remark 5.9 Let σ be a winning strategy in $\mathcal{C}^t$, and let $\langle e, \sigma\rangle$ be the play as given in Lemma 5.6. Let T be the play of $\mathcal{O}^t$ produced by $\langle e, \sigma\rangle$ and $\langle e, \sigma\rangle * \sigma$, and let $N = L_\delta[\mathcal{U}_\lambda]$ to be the g-structure corresponding to T. If T' is obtained from T by removing intervals of the form $[\rho_1, \rho_\eta]$, $[\rho_1, \rho_\eta)$, or of the form $[\mu_0, \mu_\eta]$, $(\mu_0, \mu_\eta]$, $[\mu_0, \mu_\eta)$, or (μ_0, μ_η), where $\langle \rho_\eta : \eta < \delta\rangle$ is the sequence of ordinals of type t, and μ_0, μ_η belong to the same sequence of indiscernibles of type zero, then T' is $\Sigma_1(\sigma)$-definable in M_σ (say, by $\varphi_{e'}$), and $\langle e', \sigma\rangle$ is a play for the loser that satisfies the basic rules of the game $\mathcal{C}^t$ that produces the same g-model N. Just as we saw in the proof of Lemma 5.5 the sequences of indiscernibles of type t, $\langle \rho_\nu : \nu < \delta\rangle$ or $\langle \mu_\nu : \nu < \zeta\rangle$, are $\Sigma_1(\sigma)$-definable in M_σ. This way T' can be defined by $T' \restriction \rho_0 = T \restriction \rho_0$, and $T'(\rho_0 + \nu) = T(\rho_\eta + \nu)$ (for $\nu \in \omega_1^a$), and so it is a $\Sigma_1(a)$-definition. It should be clear that for each $t' \leq t$, each ordinal of g-type t that comes from T' is a g-ordinal of type t' that comes from T.

That the structure N is unchanged follows from the fact that by omitting finitely many ordinals of g-type t above $\lambda = \lambda_0 = \rho_0$, the height of the model obtained is unchanged, is still $\delta = \bigcup\{\rho_\nu : \nu < \delta\}$. On the other hand, the sequence of measures $\mathcal{U}_\lambda$ is also unchanged, since each measure U_ξ is unchanged when bounded subsets of indiscernibles for the measure are removed. This concludes the proof of Lemma 5.6. □

5.5 Obtaining *t*-fixed cardinals In this section we apply Lemma 5.6 to obtain good g-structures of type t for every $t \in \omega$, from the assumption of Det(∗). Remember that a good g-structure of type t has been defined as a structure of the form $L_\delta[\mathcal{U}_{I_\lambda}]$ that satisfies ZFC + "λ is $(t-1)$-fixed" and where λ is an ordinal of g-type t, δ is of g-type $(t+1)$, the set I_λ is formed by all the sets of indiscernibles of type zero for λ, and $\mathcal{U}_{I_\lambda}$ is the sequence of filters given by the sets of indiscernibles in I_λ. We summarize these results in the next theorem.

Theorem 5.10 *Assume* Det(∗)*, and let M be a witness for this. Then, for every $t \in \omega \setminus \{0\}$ there is a good g-structure of type t; that is to say, there is a model of* ZFC *with a $(t-1)$-fixed cardinal.*

Proof For each $t \in \omega$, consider the integer game $\mathcal{C}^t$ as defined in M. Since $M \models$ Det(OD), $\mathcal{C}^t$ is determined (for each t). From now on, we will work with fixed $t \in \omega$ and $\sigma \in M$, where σ is a winning strategy in the game $\mathcal{C}^t$.

Let $\langle e, \sigma\rangle$ be the play of the loser given by Lemma 5.6. This produces a play $T : \omega_1^\sigma \to \omega_1^\sigma$ for the loser in the ordinal game $\mathcal{O}^t$ satisfying the basic rules, so a g-model $N = L_\delta[\mathcal{U}_{I_\lambda}]$ of type t is produced. We claim that the g-structure N is good. This must be true because otherwise we could argue a contradiction. The argument is the following. If the model N is not good, then the winner of the game $\mathcal{C}^t$ must have won because of one of the winning conditions (2)-(5) (described in the set of strong rules; see Section 5.2), but in each of these cases there is a play for the loser, obtained by omitting an interval of either form as described in Remark 5.9, that produces the same model N. But the omission is made in order to force the winning condition for that case to change so that the loser wins now, and this is of course a contradiction. We verify this in each of the cases (2)-(5) and recall the definition of the strong winning conditions given in Section 5.2.

(1) If the winner is player **I** and he won because of condition (2), then $\gamma_0 < \gamma_1$, where $N \not\models$ "Collection," collection fails at ν (ν least), and $h : \nu \to \delta$ is cofinal, minimal, and definable in N. And γ_0, γ_1 are defined by $\gamma_0 = \min\{\gamma \in \nu : h(\gamma) \geq \rho_1\}$ and $\gamma_1 = \min\{\gamma \in \nu : h(\gamma) \geq \rho_2\}$. (Remember that $\rho_0 = \lambda < \rho_1$.) Now, since $\nu < \delta$ there must be $0 < \eta_0 < \delta$ such that $h''\nu \cap [\rho_{\eta_0+1}, \rho_{\eta_0+2}) = \emptyset$, so if we set $\gamma_{\eta_0} = \min\{\gamma \in \nu : h(\gamma) \geq \rho_{\eta_0+1}\}$ and $\gamma_{\eta_0+1} = \min\{\gamma \in \nu : h(\gamma) \geq \rho_{\eta_0+2}\}$, it is true that $\gamma_{\eta_0+1} = \gamma_{\eta_0+2}$. If φ' is the formula that defines (in M_σ) the sequence $T'_I : \omega_1^\sigma \to \omega_1^\sigma$ obtained from T by removing all ordinals between ρ_0 and ρ_{η_0}, Lemma 5.6 and Remark 5.9 imply that φ' is a $\Sigma_1(\sigma)$-formula $\varphi_{e'}$, that the play $\langle e', \sigma\rangle$ by **I** satisfies the basic rules, and that T' produces the same g-model N. So when the loser plays $\langle e', \sigma\rangle$, **II** wins because of condition (2). This is a contradiction. If, on the other hand, **II** is the winner, it is because of this same condition; that is, because $\gamma_0 = \gamma_1$, then there must be $0 < \eta_0 < \delta$ least such that $h(\gamma_0) > \rho_{\eta_0}$. Remember that δ was built in such a way that for any $\nu < \delta$, $\nu + \delta = \delta$. So, take e' as before, but by removing all ordinals between ρ_1 and ρ_{η_0}. The same argument as before brings up a contradiction.

(2) In this case $N \models$ "Collection $\wedge\, \neg$ Power Set." Say that Power Set fails at ν, so $P(\nu) \cap N$ has order-type δ. Set $\gamma_0 = \inf(Y_{\rho_1} \triangle Y_{\rho_2})$ and $\gamma_1 = \inf(Y_{\rho_2} \triangle Y_{\rho_3})$, where $\{Y_{\rho_\eta} : \eta \in \delta\}$ is the canonical enumeration of $P(\nu)$ in N. Player **I** wins if $\gamma_0 < \gamma_1$. When this is the case, since $\nu < \delta$ there must be $\eta_0 < \delta$ such that $\inf(Y_{\rho_{\eta_0}} \triangle Y_{\rho_{(\eta_0+1)}}) \geq \inf(Y_{\rho_{(\eta_0+1)}} \triangle Y_{\rho_{(\eta_0+2)}})$. And as before the sequence can be refined by omitting indiscernibles from ρ_1 to ρ_{η_0}. When **II** wins, we recall Lewis's observation in [7], which in our notation can be stated as follows. *Let $\gamma_n = \inf(Y_{\rho_n} \triangle Y_{\rho_{(n+1)}})$. Then, there is a subsequence $\langle \rho_{n_k} : k \in \omega\rangle$ of $\langle \rho_n : n \in \omega\rangle$ such that no three consecutive elements of $\langle \gamma_{n_k} : k \in \omega\rangle$ are the same.* This lemma implies that for some $\hat{k} \in \omega$, $\gamma_{n_{\hat{k}}} < \gamma_{n_{\hat{k}+1}}$; otherwise, there would be an infinite $\in$-descending sequence of ordinals. So one can define T' from T by removing finitely many g-ordinals of type t from ρ_1 up to $\rho_{n_{\hat{k}}}$, from $\rho_{n_{\hat{k}}}$ up to $\rho_{n_{\hat{k}+1}}$, and from $\rho_{n_{\hat{k}+1}}$ up to $\rho_{n_{\hat{k}+2}}$. This way we may conclude a contradiction just as in the previous cases.

(3) There is a winner because $N \models$ ZFC $\wedge$ "$F(\eta_0)$ is not an ultrafilter," for $\eta_0 \in \lambda$ least. If player **I** is the winner, $\mu_1^{\kappa_{\eta_0}} \notin X$, where X is the $<_N$-least unmeasured set, and $\mu_1^{\kappa_{\eta_0}}$ is the second element of $I^0_{\kappa_{\eta_0}}$. Since $X \setminus \kappa_{\eta_0} \notin F(\eta_0)$, there is a least $\nu_0 < \kappa_{\eta_0}$ such that $\mu_{\nu_0}^{\xi_{\eta_0}} \in X$. We can obtain a definition $\varphi_{e'}$ for the subsequences T' of T obtained by removing all ordinals between $\mu_1^{\xi_{\eta_0}}$

and $\mu_{\nu_0}^{\xi_{\eta_0}}$ to obtain a contradiction as before. If **II** wins because $\mu_1^{\xi_{\eta_0}} \in X$, since $X \notin F(\eta_0)$ we can do exactly the same to conclude the contradiction.

(4) The last case is when the winner wins because of $N \models$ ZFC $\wedge$ "$\mathcal{U}_I$ is a sequence of ultrafilters $\wedge$ $(\exists \eta_0 \in \lambda)(F(\eta_0)$ is not κ_{η_0}-complete)." If **I** is the winner in this case, that must be because $g(\xi_0 + 1) = g(\xi_0) + 1$, where $g : \nu \to \kappa_{\eta_0}$ is defined as $g(\xi) = \min\{0 < \gamma < \kappa_{\eta_0} : (\forall \gamma' \geq \gamma)(\mu_{\gamma'}^{\eta_0} \in X_\xi)\}$, and the sets X_ξ witness the κ_{η_0}-incompleteness of $F(\eta_0)$ (as described in rule (5) on page 363). Since $\nu < \kappa_{\eta_0}$ and g is cofinal, there must be a least $\xi_1 < \nu$ such that $g(\xi_1 + 1) > g(\xi_1) + 1$. So, we can obtain T' from T by removing all ordinals between $\mu_0^{\eta_0}$ to $\mu_{g(\xi_1)}^{\eta_0}$ and conclude a contradiction just as in item (2). If, on the other hand, **II** is the winner, then $g(\xi_0 + 1) > g(\xi_0) + 1$. In this case just define T' by removing all ordinals between $\mu_{g(\xi_0)}^{\eta_0}$ and $\mu_{g(\xi_0+1)}^{\eta_0}$.

□

The natural questions to ask would refer to the possibility of using these methods in order to obtain models with stronger sequences of measures, say, to begin with, a model of ZFC with a proper class of measurable cardinals. It seems that there is an intrinsic limitation for these methods in order to concentrate a sequence of measurable cardinals in a regular one. So other methods must be devised. Using some version of the games of [14] seems to be a good candidate; however, the author has not succeeded with this approach so far.

Some natural questions which arise at this point, and for which the author tried unsuccessfully to find an answer, are the following.

(1) Can the methods used in Section 5 be further extended in order to obtain countable models with stronger sequences of measures, say, with a proper class of measurable cardinals at least?
(2) How to obtain models with sequences of measures from Det$(*)$ without the restriction of working inside KP?
(3) Is the assumption of Det$(*)$ on a cone of Turing degrees actually "stronger" than Det$(*)$ in a single real?
(4) Corollary 4.4 and Remark 4.5 may suggest that the upper bound given in Section 4 might not be the optimal one. Is it possible to obtain Det$(*)$ from a weaker assumption?

Notes

1. As Lewis [7] already points out, "The standard arguments on OD by Kechris and Solovay show that if the point class $\Sigma_n^0(\Pi_1^1)$ is determined, then there is a cone of Turing degrees C such that for any real $x \in C$, $M_x \models$ Det(OD)." To make full sense out of this statement one would have to change the expression $\Sigma_n^0(\Pi_1^1)$ to $\mathcal{B}(\Pi_1^1)$ or to $\bigcup_n \Sigma_n^0(\Pi_1^1)$, or change the expression Det(OD) to Det(OD_{Σ_n}), as we do.

2. In this sense a mouse is not required to be iterable.

References

[1] Barwise, K. J., R. O. Gandy, and Y. N. Moschovakis, "The next admissible set," *Journal of Symbolic Logic*, vol. 36 (1971), pp. 108–120. Zbl 0236.02033. MR 0300876. 354, 355

[2] Barwise, K. J. *Admissible Sets and Structures*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1975. Zbl 0316.02047. MR 0424560. 355

[3] Devlin, K. J., *Constructibility*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1984. Zbl 0542.03029. MR 0750828. 353, 355, 356, 357, 366

[4] Harrington, L., "Analytic determinacy and $0^{\#}$," *Journal of Symbolic Logic*, vol. 43 (1978), pp. 685–693. Zbl 0398.03039. MR 0518675. 352

[5] Kechris, A. S., and R. M. Solovay, "On the relative consistency strength of determinacy hypothesis," *Transactions of the American Mathematical Society*, vol. 290 (1985), pp. 179–211. Zbl 0586.03045. MR 0787961. 351, 352

[6] Keisler, H. J., and Knight, F., "Barwise: Infinitary logic and admissible sets," *Bulletin of Symbolic Logic*, vol. 10 (2004), pp. 4–36. Zbl 1080.03026. MR 2062240. 353

[7] Lewis, A., "Large Dilatators and Large Cardinals; and Determinacy in Small Admissible Sets," Ph.D. dissertation, University of California, Berkeley, 1993. 352, 359, 360, 369, 370

[8] Martin, D. A., "Measurable cardinals and analytic games," *Fundamenta Mathematicae*, vol. 66 (1970), pp. 287–291. Zbl 0216.01401. MR 0258637. 352

[9] Martin, D. A., and J. R. Steel, "A proof of projective determinacy," *Journal of the American Mathematical Society*, vol. 2 (1989), pp. 71–125. Zbl 0668.03021. MR 0955605. 352

[10] Mitchell, W. J., "Sets constructible from sequences of ultrafilters," *Journal of Symbolic Logic*, vol. 39 (1974), pp. 57–66. Zbl 0295.02040. MR 0344123. 352, 354

[11] Neeman, I., "Unraveling Π^1_1 sets," *Annals of Pure and Applied Logic*, vol. 106 (2000), pp. 151–205. Zbl 0979.03040. MR 1785759. 351, 352, 354

[12] Sacks, G. E., "Countable admissible ordinals and hyperdegrees," *Advances in Mathematics*, vol. 20 (1976), pp. 213–262. Zbl 0439.03027. MR 0429523. 355

[13] Soare, R. I., *Recursively Enumerable Sets and Degrees: A Study of Computable Functions and Computably Generated Sets*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1987. Zbl 066703030. MR 0882921. 353

[14] Steel, J. R., "Determinacy in the Mitchell models," *Annals of Mathematical Logic*, vol. 22 (1982), pp. 109–125. Zbl 0573.03026. MR 0667223. 352, 353, 354, 370

Acknowledgments

The author thanks William Mitchell and the referees for valuable comments. Research was partially supported by Consejo Nacional de Ciencia y Tecnología (Mexico) and Consejo de Ciencia y Tecnología grant CHIS-2006-C06-457770 and by Programa de Mejoramiento del Profesorado grant UNACH-PTC-073.

Department of Mathematics and Computing Science
Saint Mary's University
Halifax, Nova Scotia B3H 3C3
Canada
rrdiego@gmail.com, drojas@cs.smu.ca