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On Milliken-Taylor Ultrafilters

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Abstract We show that there may be a Milliken-Taylor ultrafilter with infinitely many near coherence classes of ultrafilters in its projection to ω, answering a question by López-Abad. We show that k-colored Milliken-Taylor ultrafilters have at least $k + 1$ near coherence classes of ultrafilters in its projection to ω. We show that the Mathias forcing with a Milliken-Taylor ultrafilter destroys all Milliken-Taylor ultrafilters from the ground model.

1 Milliken-Taylor Ultrafilters and Their Projections

Ultrafilters on ω and on other countable sets are useful objects in Ramsey theory in both ways: they give rise to interesting colorings with the ultrafilter as a parameter in their definition and in other situations they help to find homogeneous sets for colorings. Here we investigate questions along these lines.

We answer a question of López-Abad whether there can be more than two near coherence classes of ultrafilters in the core of a Milliken-Taylor ultrafilter. We show that in Milliken-Taylor ultrafilter with k colors there are $k+1$ near coherence classes in its projection to ω, generalizing a result of Blass [7].

Ultrafilters with additional properties have served to construct powerful notions of forcing. Moreover, the investigation whether such ultrafilters exist instigates the development of new forcing methods. Along these lines we investigate whether a Milliken-Taylor ultrafilter is preserved by forcing with another Milliken-Taylor ultrafilter. The somewhat surprising answer is no, independently of the relationship of the two ultrafilters. From this we can conclude that in any iteration of forcings with Milliken-Taylor ultrafilters at any stage fresh ultrafilters must be used, as there are no ultrafilters in earlier iteration stages that are preserved. In the rest of this introductory section we review part of the relevant background.

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Our nomenclature follows [11] and [5]. We let $\mathbb{F}$ be the collection of all *nonempty* finite subsets of ω. For $a, b \in \mathbb{F}$ we write $a < b$ if $(\forall n \in a)(\forall m \in b)(n < m)$. We will work with proper filters on $\mathbb{F}$, that is, nonempty subsets of $\mathcal{P}(\mathbb{F})$ that are closed under binary intersections and supersets and do not contain the empty set. A filter on $\mathbb{F}$ is called nonprincipal if it does contain all sets of the form $\mathbb{F} \setminus E$, E finite. A sequence $\bar{c} = \langle c_n : n \in \omega\rangle$ of members of $\mathbb{F}$ is called *unmeshed* if for all n, $c_n < c_{n+1}$. Henceforth, barred lowercase variables stand for such sequences. For $n \leq \omega$, the set $(\mathbb{F})^n$ denotes the collection of all unmeshed sequences in $\mathbb{F}$ of length n. If $\bar{c}$ is a sequence in $(\mathbb{F})^\omega$, we write $(\mathrm{FU})^\omega(\bar{c})$ for the set of all unmeshed sequences whose members are finite unions of some of the c_ns and we write $\mathrm{FU}(\bar{c})$ for the set of all finite unions of members of $\bar{c}$.

Definition 1.1 Given $\bar{c}$ and $\bar{d}$ in $(\mathbb{F})^\omega$, we say that $\bar{d}$ is a *condensation or a block-subsequence of* $\bar{c}$ and we write $\bar{d} \sqsubseteq \bar{c}$ if $\bar{d} \in (\mathrm{FU})^\omega(\bar{c})$. We say $\bar{d}$ is *almost a condensation of* $\bar{c}$ and we write $\bar{d} \sqsubseteq^* \bar{c}$ if and only if there is an n such that $\langle d_t : t \geq n\rangle$ is a condensation of $\bar{c}$.

Definition 1.2 A nonprincipal filter $\mathcal{F}$ on $\mathbb{F}$ is said to be a *union filter* if it has a basis of sets of the form $\mathrm{FU}(D)$ for $D \subseteq \mathbb{F}$. A nonprincipal filter $\mathcal{F}$ on $\mathbb{F}$ is said to be an *ordered-union filter* if it has a basis of sets of the form $\mathrm{FU}(\bar{d})$ for $\bar{d} \in (\mathbb{F})^\omega$. Let μ be an uncountable regular cardinal. An ordered-union filter is said to be *μ-stable* if, whenever it contains $\mathrm{FU}(\bar{d}_\alpha)$ for $\bar{d}_\alpha \in (\mathbb{F})^\omega$, $\alpha < \mu$, then it also contains some $\mathrm{FU}(\bar{e})$ for some $\bar{e}$ that is almost a condensation of $\bar{d}_\alpha$ for $\alpha < \mu$. We write $< \kappa$-stable for μ-stable for all $\mu < \kappa$. For "$\aleph_0$-stable" we say "stable." Stable ordered-union ultrafilters are also called *Milliken-Taylor ultrafilters*.

We recall some more conventional types of ultrafilters: We say "A is almost a subset of B" and write $A \subseteq^* B$ if and only if $A \setminus B$ is finite. Similarly, the symbol $=^*$ denotes equality up to finitely many exceptions in $[\omega]^\omega$ or in ω^ω. Let κ be a regular cardinal. An ultrafilter $\mathcal{U}$ is called a *P_κ-point* if for every $\gamma < \kappa$, for every $A_i \in \mathcal{U}$, $i < \gamma$, there is some $A \in \mathcal{U}$ such that for all $i < \gamma$, $A \subseteq^* A_i$; such an A is called a *pseudointersection* or a *diagonalization* of the A_i, $i < \gamma$. A $P_{\aleph_1}$-point is just called P-point. An ultrafilter $\mathcal{V}$ on ω is called a *Q-point or a rare ultrafilter*, if given a strictly increasing sequence π_n, there is $A \in \mathcal{V}$ such that for all n, $|A \cap [\pi_n, \pi_{n+1})| = 1$. An ultrafilter is called rapid if the set of the enumerating functions of the members of the ultrafilter is a dominating family. A selective ultrafilter (also called Ramsey ultrafilter) is an ultrafilter that is a P-point and a Q-point.

Union ultrafilters need not exist, since their existence implies the existence of P-points [8]. Theorem 1.9 below shows another reason for nonexistence. Ordered-union ultrafilters need not exist, as their existence implies the existence of Q-points [5, Prop. 3.9]; namely, the minimum and the maximum projections (see Definition 1.8) are Q-points. There are models without Q-points, for example, the Laver model [19] and all models of NCF [9]. NCF implies that any filter is nearly coherent to a filter with $\mathfrak{u} < \mathfrak{d}$ generators and this filter cannot be rapid. Since rapidness is preserved under finite-to-one functions, under NCF there is no rapid filter. Since all Q-points are rapid, there is no Q-point under NCF. The existence of stable ordered union-ultrafilters is even harder: Blass [5] showed that the minimum and the maximum projections of Milliken-Taylor ultrafilters are selective.

It is not known how to construct an ordered union-ultrafilter from just a union ultrafilter nor how to construct a Milliken-Taylor ultrafilter from an ordered union-ultrafilter. This leads to the questions whether there is any model with a union-ultrafilter without an ordered-union ultrafilter or a model with an ordered-union ultrafilter and no Milliken-Taylor ultrafilter. This asks for more forcing theory. The near coherence of filters principle implies that there are P-points (see [4]) but no union-ultrafilters [7, Theorem 38], and only very few models of the near coherence principle of filters are known [9; 10; 18]. This seems to be one of the few separation results.

With the help of Hindman's theorem one shows that under MA(σ-centered) stable (even $< 2^\omega$-stable) ordered-union ultrafilters exist [5]. We recall Hindman's theorem.

Theorem 1.3 (Hindman [14], Corollary 3.3) *If the set $\mathbb{F}$ is partitioned into finitely many pieces then there is a set $\bar{d} \in (\mathbb{F})^\omega$ such that $\mathrm{FU}(\bar{d})$ is included in one piece.*

Indeed, for constructing a forcing $\mathbb{M}(\mathcal{U})$ we use only Hindman's theorem. However, for analyzing its behavior we also derive from the following finite-dimensional version.

Theorem 1.4 (Milliken [20] and Taylor [22]) *If the set $(\mathbb{F})^n$ is partitioned into finitely many pieces then there is a set $\bar{d} \in (\mathbb{F})^\omega$ such that $(\mathrm{FU}(\bar{d}))^n$ is included in one piece.*

As a motivation for the question we answer in this paper we consider also the following theorem for colorings with arbitrary many colors.

Theorem 1.5 (Taylor [22], The canonical partition theorem) *If f is a function defined on $\mathbb{F}$, then there is a set $\bar{d} \in (\mathbb{F})^\omega$ such that one of the following five statements holds for all $s, t \in \mathrm{FU}(\bar{d})$.*

(1) $f(s) = f(t)$,

(2) $f(s) = f(t)$ *iff* $\min(s) = \min(t)$,

(3) $f(s) = f(t)$ *iff* $\max(s) = \max(t)$,

(4) $f(s) = f(t)$ *iff* $(\min(s), \max(s)) = (\min(t) \max(t))$,

(5) $f(s) = f(t)$ *iff* $s = t$.

Instead of $(\mathbb{F})^\omega$ smaller domains for the colorings and for finding the homogeneous set can be considered.

Definition 1.6 An ordered union ultrafilter $\mathcal{U}$ on $\mathbb{F}$ is said to have the *Ramsey property* if, for any $\bar{c} \in \mathcal{U}$, the set $(\mathrm{FU}(\bar{c}))^n$ is partitioned into finitely many pieces then there is a set $\bar{d} \in \mathcal{U}$ such that $(\mathrm{FU}(\bar{d}))^n$ is included in one piece. An ordered union ultrafilter $\mathcal{U}$ on $\mathbb{F}$ is said to have the *canonical partition property* if for any $\bar{c} \in \mathcal{U}$ for any function defined on $\mathrm{FU}(\bar{c})$ there is $\bar{d} \in \mathcal{U}$ such that $f \restriction \mathrm{FU}(\bar{d})$ is canonized as above.

Now Milliken-Taylor ultrafilters are the reservoirs for the homogeneous sets from the previous theorems by the following theorem.

Theorem 1.7 (Blass [5], Theorem 4.2(a) to (c)) *For an ordered union ultrafilter $\mathcal{U}$ the following are equivalent:*

(a) *stability,*

(b) the canonical partition property,

(c) the Ramsey property.

The following notions relate Milliken-Taylor ultrafilters to ultrafilters on ω.

Definition 1.8 Let $\mathcal{U}$ be a filter on $\mathbb{F}$.

(1) The *core of* $\mathcal{U}$ is the filter $\Phi(\mathcal{U})$ such that

$$X \in \Phi(\mathcal{U}) \text{ iff } (\exists \operatorname{FU}(\bar{c}) \in \mathcal{U})(\bigcup_{n\in\omega} c_n \subseteq X).$$

(2) The *minimum projection of* $\mathcal{U}$ is the filter $\min(\mathcal{U})$ such that

$$X \in \min(\mathcal{U}) \text{ iff } (\exists \operatorname{FU}(\bar{c}) \in \mathcal{U})(\{\min(c_n) : n \in \omega\} \subseteq X).$$

(3) Analogously we define the *maximum projection of* $\mathcal{U}$, $\max(\mathcal{U})$.

So it follows from their definitions that $\min(\mathcal{U})$ and $\max(\mathcal{U})$ are in $[\Phi(\mathcal{U})] = \{\mathcal{V} \in \beta\omega : \mathcal{V} \supseteq \Phi(\mathcal{U})\}$. If $\mathcal{U}$ is an ultrafilter then $\min(\mathcal{U})$ and $\max(\mathcal{U})$ are ultrafilters. From Theorem 1.9 it follows that $\Phi(\mathcal{U})$ is not necessarily an ultrafilter if $\mathcal{U}$ is a union ultrafilter. However, in this case $\Phi(\mathcal{U})$ is not meager; this is proved as in [11, Prop. 2.3(3)]. The following is easy to see: if $\{X \subseteq \mathbb{F} : |X| = 1\} \in \mathcal{U}$, then $\Phi(\mathcal{U})$ is an ultrafilter. By Theorem 1.9, such a $\mathcal{U}$ is not a union ultrafilter. So there is always an ultrafilter $\mathcal{U}$ on $\mathbb{F}$ such that $\Phi(\mathcal{U})$ is an ultrafilter on ω. We do not know whether in ZFC there is always an ultrafilter $\mathcal{U}$ such that $\Phi(\mathcal{U})$ is not ultra. As mentioned, union ultrafilters need not exist.

We recall near coherence classes. For $B \subseteq \omega$ and $h\colon \omega \to \omega$, we let $h''B = \{h(b) : b \in B\}$ and $h^{-1''}B = \{n : h(n) \in B\}$. By a *filter* we mean a proper filter on ω. We call a filter *nonprincipal* if it contains all cofinite sets. Let $\mathcal{F}$ be a nonprincipal filter on ω and let $h\colon \omega \to \omega$ be finite-to-one (that means that the preimage of each natural number is finite). Then also $h(\mathcal{F}) = \{X : h^{-1''}X \in \mathcal{F}\}$ is a nonprincipal filter. It is the filter generated by $\{h''X : X \in \mathcal{F}\}$. Two filters $\mathcal{F}$ and $\mathcal{G}$ are *nearly coherent* if there is some finite-to-one $h\colon \omega \to \omega$ such that $h(\mathcal{F}) \cup h(\mathcal{G})$ generates a filter. On the set $\beta\omega^*$ of nonprincipal ultrafilters on ω, the near coherence relation is an equivalence relation whose classes are called near coherence classes. Models with just one near coherence classes are known [9; 10; 18], and a model with just two classes is in preparation by Blass and Shelah. Apropos the possible infinite numbers of near coherence classes, Banakh and Blass [1] showed the following: If there are infinitely many near coherence classes, then there are $2^{\mathfrak{c}}$ classes. Under CH or $\mathfrak{u} \geq \mathfrak{d}$, there are $2^{\mathfrak{d}}$ classes, and this and similar forms are early results on the spectrum of possible numbers of near coherence classes [4]; for more history, see [1].

This paper is concerned with the following question. Let $\mathcal{U}$ be a Milliken-Taylor ultrafilter. How many near coherence classes of ultrafilters are in $[\Phi(\mathcal{U})]$? For the wider class of union ultrafilters there is a recent result by Blass.

Theorem 1.9 ([7], Theorem 38) *Let $\mathcal{U}$ be a union ultrafilter. Then* $\min(\mathcal{U})$ *and* $\max(\mathcal{U})$ *are nonnearly coherent P-points.*

In the light of the canonical partition property Theorem 1.7 for $\mathcal{U}$ it may appear difficult to find more near coherence classes beyond the class of $\min(\mathcal{U})$ and $\max(\mathcal{U})$. So are there more near coherence classes in $[\Phi(\mathcal{U})]$? We will see that canonization for functions does not mean canonization for ultrafilters $\mathcal{V}$ on ω; after all such a $\mathcal{V}$ amounts to $\chi(\mathcal{V})$ many functions that should have a sort of a common canonization.

We recall some definitions. The set of functions from ω to ω/ the set of finite-to-one functions from ω to ω/ the set of infinite subsets of omega are denoted by ω^ω, $[\omega]^\omega$, $\omega^{\omega,\mathrm{fto}}$. A notion of forcing $\mathbb{P}$ preserves an ultrafilter $\mathcal{U}$ if and only if $\Vdash_{\mathbb{P}}$ "$(\forall X \in [\omega]^\omega)(\exists Y \in \mathcal{U})(Y \subseteq X \vee Y \subseteq \omega \setminus X)$" and in the contrary case we say "$\mathbb{P}$ destroys $\mathcal{U}$." If $\mathbb{P}$ is proper and preserves $\mathcal{U}$ and $\mathcal{U}$ is a P-point, then $\mathcal{U}$ stays a P-point [9, Lemma 3.2].

Let $\mathcal{F}$ be a filter. $\mathcal{B} \subseteq [\omega]^\omega$ is a *pseudobase* for $\mathcal{F}$ if for every $X \in \mathcal{F}$ there is some $Y \in \mathcal{B}$ such that $Y \subseteq X$. A pseudobase $\mathcal{B} \subseteq \mathcal{F}$ is called a *base*. The character/ π-character of $\mathcal{F}$, $\chi(\mathcal{F})$/ $\pi\chi(\mathcal{F})$ is the smallest cardinality of a base/ pseudobase of $\mathcal{F}$. The ultrafilter characteristic, $\mathfrak{u}$, is the smallest character of a nonprincipal ultrafilter.

A set $M \subseteq {}^\omega\omega$ is called a *meager set* if it is a countable union of nowhere dense sets. The covering number for the ideal $\mathcal{M}$ of meager sets, $\mathrm{cov}(\mathcal{M})$, is the smallest number of meager sets that cover together the real line.

In Section 2 we give two types of construction, one based on MA(σ-centered) and one on $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$, of a Milliken-Taylor ultrafilter with infinitely many near coherence classes of ultrafilters in its projection. In Section 3 we generalize Blass's result Theorem 1.9 to $k+1$ classes. In Section 4 we show that Milliken-Taylor ultrafilters are not preserved under any forcing with another or the same Milliken-Taylor ultrafilter.

2 A Milliken-Taylor Ultrafilter with Infinitely Many Pairwise Nonnearly Coherent Ultrafilters in Its Core

Two near coherence classes of ultrafilters that have representatives in the core of a Milliken-Taylor filter can be named, namely, its minimum and its maximum projection. So there is the natural question whether more classes are represented in the core. It might be possible that there are just two near coherence classes in the universe and that there is a Milliken-Taylor ultrafilter such that just the minimum and the maximum projection are representatives of these two classes. Then the question has a negative answer. Here we show that a positive answer is consistent.

Theorem 2.1 *Let κ be a regular cardinal. Under* MA(σ*-centered*) *and* $\mathfrak{c} = \kappa$ *there is a* $<\mathfrak{c}$*-stable Milliken-Taylor ultrafilter with* $2^{\mathfrak{c}}$ *near coherence classes of ultrafilters represented in* $[\Phi(\mathcal{U})]$.

Remark 2.2 We carry out the construction only for countably many near coherence classes represented by $\mathcal{V}^n$, $n \in \omega$. We can get $\mathfrak{c}$ classes in $[\Phi(\mathcal{U})]$ by an easy modification of the construction. In order to get the maximal number $2^{\mathfrak{c}}$ near coherence classes represented in $[\Phi(\mathcal{U})]$ we use that we construct the $\mathcal{V}^n$ such that the sequence $\langle \mathcal{V}^n : n < \omega \rangle$ is discrete; that is, for every n, there is $A \in \mathcal{V}^n$, $A \notin \mathcal{V}^m$ for $m \neq n$. This is automatically fulfilled in our construction. Then Banakh and Blass's technique [1] to generate $2^{\mathfrak{c}}$ classes in the closure of $\{\mathcal{V}^n : n \in \omega\}$ in $\beta\omega \setminus \omega$ works. A self-contained proof would mean to repeat good parts of their work, and therefore we refer the reader to [1].

For the proof we need the following definition.

Definition 2.3 For $\bar{c} \in (\mathbb{F})^{\leq\omega}$ we let $\mathrm{set}(\bar{c}) = \bigcup\{c_k : k \in |\bar{c}|\}$. For $\bar{c} = (\mathbb{F})^{\leq\omega}$ and $A \subseteq \omega$ with $A \cap \mathrm{set}(\bar{c}) \neq \emptyset$, we define $\bar{c} \restriction A = \langle c_k \cap A : k \in \omega, A \cap c_k \neq \emptyset \rangle$

and $(\bar{c}\,;\text{past } n) = \langle c_k : c_k \cap [n,\infty)\rangle$. The number of blocks in $\bar{c}$ is denoted by $|\bar{c}|$. Unmeshed sets of blocks $\{c_n : n \in \omega\}$ are identified with their increasing enumerations $\bar{c} = \langle c_n : n \in \omega\rangle$. We do this also for finite numbers instead of ω.

Proof Let B_ε, $\varepsilon < \kappa$, $\varepsilon = 0 \mod 3$ enumerate $\mathcal{P}(\mathbb{F})$, Y_ε, $\varepsilon < \kappa$, $\varepsilon = 1 \mod 3$ enumerate $\mathcal{P}(\omega)$ and let f_ε, $\varepsilon < \kappa$, $\varepsilon = 2 \mod 3$, enumerate $\omega^{\omega,\mathrm{fto}}$. We modify the usual construction of a Milliken-Taylor ultrafilter by having three kinds of successor steps: Hindman steps, ultrafilter steps, and "making nonnearly coherent" steps. Also in the limit steps, we have to be careful to take a somewhat fat almost condensation. By induction on $\varepsilon < \mathfrak{c}$, we choose $\bar{c}_\varepsilon \in (\mathbb{F})^\omega$ and $X^n_\varepsilon \in [\omega]^\omega$, $n < \omega$, with the following rules:

(1) $\bar{c}_0 = \langle\{k\} : k < \omega\rangle$, $\bar{c}_\varepsilon \sqsubseteq^* \bar{c}_\delta$ for $\delta < \varepsilon < \mathfrak{c}$, we write $\bar{c}_\varepsilon = \langle c_{\varepsilon,k} : k < \omega\rangle$.

(2) $X^n_\varepsilon \subseteq \mathrm{set}(\bar{c}_\varepsilon)$ for $n < \omega$.

(3) $X^n_\varepsilon \subseteq^* X^n_\delta$ for $\delta < \varepsilon < \bar{c}$ for every $n < \omega$.

(4) If $\varepsilon = 0 \mod 3$ then we let $\bar{c}'_{\varepsilon+1}$ be gotten by merging block from $\bar{c}_\varepsilon$ and not dropping anything such that for each $k \in \omega$, $c'_{\varepsilon+1,k}$ contains an element of $X^i_{\varepsilon+1}$ for all $i \le k$. Then we take Hindman's theorem to get $\bar{c}_{\varepsilon+1} \sqsubseteq \bar{c}'_{\varepsilon+1}$ such that $\mathrm{FU}(\bar{c}_{\varepsilon+1}) \subseteq B_\varepsilon$ or $\subseteq \mathbb{F} \setminus B_\varepsilon$. Still $\mathrm{set}(\bar{c}_{\varepsilon+1}) \cap X^n_\varepsilon$ is infinite for all n, and we let $X^n_{\varepsilon+1} = X^n_\varepsilon \cap \mathrm{set}(\bar{c}_{\varepsilon+1})$.

(5) If $\varepsilon = 1 \mod 3$ we choose $X^n_{\varepsilon+1} \subseteq Y_\varepsilon \cap X^n_\varepsilon$ or $\subseteq (\omega \setminus Y_\varepsilon) \cap X^n_\varepsilon$. We let $\bar{c}_{\varepsilon+1} = \bar{c}_\varepsilon$.

(6) If $\varepsilon = 2 \mod 3$ we choose $X^n_{\varepsilon+1} \subseteq X^n_\varepsilon$ such that for all $n \neq m$, $f_\varepsilon{}''X^n_{\varepsilon+1} \cap f_\varepsilon{}''X^m_{\varepsilon+1} = \emptyset$. We let $\bar{c}_{\varepsilon+1} = \bar{c}_\varepsilon$.

(7) In the limit steps, we take parallel almost condensations and pseudointersection.

We explain the limit steps. First we consider the case of countable cofinality. Let $\varepsilon = \lim_{k\to\omega}\varepsilon_k$, ε_k, $k \in \omega$ strictly increasing, $\varepsilon_k = 0 \mod 3$. Take $n(k) > k, n(k-1)$ so that $(\bar{c}_{\varepsilon_k}\,;\text{past } n(k)) \sqsubseteq \bar{c}_{\varepsilon_{k+1}}$ and such that $\bar{c}_{\varepsilon_k,n(k)}$ contains an element of each of $X^j_{\varepsilon_k}$, $j \le k$. Then let $\bar{c}_\varepsilon = \bar{c}_{\varepsilon_0} \restriction [0,n(0))^\frown\bar{c}_{\varepsilon_1} \restriction [n(0),n(1))\ldots$. Here $^\frown$ means concatenation. Let $X^n_\varepsilon = \bigcup_{i<\omega} X^n_{\varepsilon_i} \cap [n(i-1),n(i))$. Then for every n, for every $k \ge n$ we take a block into the condensation that contains points of $X^n_{\varepsilon_k} \cap [n(k-1),n(k))$, so $X^n_\varepsilon \cap \mathrm{set}(\bar{c}_\varepsilon)$ is infinite.

Now we consider a general limit $\varepsilon < \mathfrak{c}$. We choose $n(k)$, $k \in \omega$ with the help of a σ-centered forcing as follows: $\mathbb{P} = \{(c,\bar{x},F) : c \in (\mathbb{F})^{<\omega}, F \subseteq \varepsilon \text{ finite}\}$, $c \in (\mathbb{F})^<\omega$, $\bar{x} = (x^i : i \le |c|)$, $x^i \in \mathbb{F}$, with $(c,\bar{x},F) \le_{\mathbb{P}} (d,\bar{y},F')$ if and only if $c \trianglelefteq d$, $x^i \subseteq y^i$ for $i \le |\bar{x}|$, $F \subseteq F'$ and $d \restriction (\max(\mathrm{set}(c),\max(\mathrm{set}(d)) \sqsubseteq \bar{c}_\zeta$ for all $\zeta \in F$ and $(y^i \setminus x^i) \subseteq (\mathrm{set}(d) \setminus \mathrm{set}(c)) \cap X^i_\zeta$ for $i \le |d|$ and $\zeta \in F$. By MA(σ-centered), there is a generic filter G and there is a generic real $\bar{c}_\varepsilon = \bigcup\{s : (s,F) \in G\}$ and $X^n_\varepsilon = \bigcup\{x^n : \exists s, F(s,\bar{x},F) \in G\}$.

After the inductive choices we let $\mathcal{V}^n = \{X \in [\omega]^\omega : \exists\varepsilon < \mathfrak{c}, X \supseteq X^n_\varepsilon\}$. Let $\mathcal{U} = \{B \subseteq \mathbb{F} : \exists\varepsilon < \mathfrak{c}, \mathrm{FU}(\bar{c}_\varepsilon) \subseteq B\}$. This is Milliken-Taylor ultrafilter. $\mathcal{V}^n$ is not nearly coherent to $\mathcal{V}^m$ for $m \neq n$ and $\mathcal{V}^n \in \Phi(\mathcal{U})$. □

Now we improve the theorem from MA(σ-centered) (which is equivalent to $\mathfrak{p} = \mathfrak{c}$ [3]) to the weaker hypothesis $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$. However, there is one price: we get less stability. There will be an inductive construction of length $\mathfrak{c}$; again, however, it is not $\sqsubseteq^*$-descending anymore. We call an arbitrary initial segment of the construction

$(\mathcal{F}, (\mathcal{G}^n)_n)$ and do not bother about indexing the stage. By enumerating all descending ω-sequences and adding almost condensations to them to the filter we ensure stability.

Theorem 2.4 *Let* $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$. *Then there is a Milliken-Taylor ultrafilter* $\mathcal{U}$ *with* $2^{\mathfrak{c}}$ *near coherence classes of ultrafilters represented in* $[\Phi(\mathcal{U})]$.

Proof We will do an inductive construction along $\mathfrak{c}$. On the way we need a form of "generic existence" over initial segments of the construction. There will be steps $\alpha < \mathfrak{c}$ where we have to take an almost condensation of an ω-sequence $\bar{d}^n$ of members of the first component $\mathcal{F}$, an ordered-union filter, of the initial segment $(\mathcal{F}, (\mathcal{G}^n)_n)$ of the construction.

We will use a Cohen real (which we have by $\mathrm{cov}(\mathcal{M}) > |\alpha|$) to find a fat enough almost condensation so that none of our initial segments to the countably many pairwise nonnearly coherent ultrafilter will get lost. In addition, there will be steps where we have to take a pseudointersection over an ω-sequence $(X_k)_k$ of members of $\mathcal{G}^n$ of the initial segment $(\mathcal{F}, (\mathcal{G}^n)_n)$ of the construction for some n. In the former proof the almost condensation step and the pseudointersection step for the filters on ω were carried out simultaneously. Again a Cohen real is used to show that there is a pseudointersection that has infinite intersection with each element of $\mathcal{G}^n$. (This time an unbounded real would suffice.) Moreover, we have Hindman steps. We need a form of Hindman's theorem that takes care of the initial segments $\mathcal{G}^n$ of the ultrafilters on ω. We will modify the following.

Theorem 2.5 ([11], Theorem 5) *Let* $\mathcal{F}$ *be an ordered union filter generated by* $< \mathrm{cov}(\mathcal{M})$ *sets. Suppose that* $\mathbb{F}$ *is partitioned into finitely many pieces. Then there is* $D \in (\mathbb{F})^\omega$ *such that* $\mathrm{FU}(D)$ *is contained in one piece of the partition and* $D \cap X$ *is infinite for each* $X \in \mathcal{F}$.

We remark that by [16, Proposition 6.2] and [11, Theorem 6] the cardinal $\mathrm{cov}(\mathcal{M})$ cannot be replaced by anything smaller.

Definition 2.6 Let $\mathcal{F}$ be an ordered union filter and let $\mathcal{G}^n$, $n \in \omega$ be filters on ω. $\bar{d} \in (\mathbb{F})^\omega$ is *good for* $(\mathcal{F}, (\mathcal{G}^n)_n)$ if, for all $\mathrm{FU}(\bar{c}) \in \mathcal{F}$, the following holds:

$$\begin{aligned} &(\forall k \in \omega)(\mathrm{FU}(\bar{c}) \cap \mathrm{FU}(\bar{d}\,;\text{past } k) \text{ is infinite}), \text{ and} \\ &(\forall n)(\forall X \in \mathcal{G}^n)(X \cap \mathrm{set}(\bar{c}) \cap \mathrm{set}(\bar{d}) \text{ is infinite}). \end{aligned} \tag{1}$$

At many stages in the construction we add sets to the filters $\mathcal{G}^n$ to get nonnearly coherent ultrafilters. Using the Cohen reals again, we get that the $\mathcal{G}^n$ will grow into P-points $\mathcal{V}^n$. Then the sequence $\langle \mathcal{V}^n : n \in \omega \rangle$ is automatically discrete and hence ensures by Banakh and Blass's result that there are $2^{\mathfrak{c}}$ near coherence classes in $[\Phi(\mathcal{U})]$. Since every filter can be completed to an ultrafilter, the steps corresponding to item (5) in the previous proof do not need the condition that the initial segment has size $< \mathrm{cov}(\mathcal{M})$. For getting the nonnear coherence we use that $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{u}$ and Blass's construction from [4] and thus we can have item (6) from the previous proof also in our current construction. The Hindman steps and the steps to get stability require more care.

The following ensures that the Hindman tasks in the construction can be performed.

Theorem 2.7 *Let $\mathcal{F}$ be an ordered union filter generated by $< \mathrm{cov}(\mathcal{M})$ sets, and let $\mathcal{G}^n$, $n \in \omega$, be filters on ω, generated by $\kappa < \mathrm{cov}(\mathcal{M})$ sets and $\mathcal{G}^n \subseteq \Phi(\mathcal{F})$. Suppose that $\mathbb{F}$ is partitioned into finitely many pieces. Then there is $\bar{d} \in (\mathbb{F})^\omega$ such that $\mathrm{FU}(\bar{d})$ is contained in one piece of the partition and $\mathrm{FU}(\bar{d}\ ;\ \mathrm{past}\ k) \cap X$ is infinite for each $X \in \mathcal{F}$ and $\mathrm{set}(\bar{d}) \cap X \cap Y$ is infinite for every $Y \in \bigcup_n \mathcal{G}^n$.*

Proof We could go for a modification of Eisworth's proof, using the Ellis-Numakura Theorem [12; 21] and Galvin and Glazer's proof of Hindman's theorem (see [15]) and strengthen them in order to show that no filter $\mathcal{G}^n$ gets lost. We use a more direct way, with Baumgartner's short proof of Hindman's theorem. In the course of the proof we argue thrice with the inequality $|\mathcal{F}|, |\mathcal{G}^n| < \mathrm{cov}(\mathcal{M})$.

Given a good $\bar{d}$ for $(\mathcal{F}, (\mathcal{G}^n)_n))$ and $B \subseteq \mathbb{F}$, we produce a B-homogeneous $\bar{e}$ and a $\bar{Z}$ such that $\bar{e} \sqsubseteq^* \bar{d}$ and $\bar{e}$ is good for $(\mathcal{F}, (\mathcal{G}^n)_n)$. We look at Baumgartner's short proof of Hindman's theorem [2] and rework it step for step in order to see that the proof can be carried out within the set of $\bar{e} \in \mathbb{F}^\omega$ such that $\bar{e}$ is good for $(\mathcal{F}, (\mathcal{G}^n)_n)$.

Modifying the notion of largeness from Baumgartner' proof, we say $X \subseteq \mathbb{F}$ is large for $\bar{d} \in (\mathbb{F})^\omega$ if and only if $\bar{d}$ is good for $(\mathcal{F}, (\mathcal{G}^n)_n)$ and for every $\bar{d}' \sqsubseteq \bar{d}$, $\mathrm{FU}(\bar{d}') \cap X \neq \emptyset$ or $\bar{d}'$ is not good for $(\mathcal{F}, (\mathcal{G}^n)_n)$.

Lemma 1(a) of Baumgartner: If X is large for $\bar{d}$ and $X = Y \cup Z$, then there is $\bar{d}' \sqsubseteq \bar{d}$ such that $\bar{d}'$ is good for $(\mathcal{F}, (\mathcal{G}^n)_n)$ and either Y is large for $\bar{d}'$ or Z is large for $\bar{d}'$.

We fill in Baumgartner's proof: Suppose it is false. Since Y is not large for $\bar{d}$, there is $\bar{d}' \sqsubseteq \bar{d}$, $\bar{d}'$ is good for $(\mathcal{F}, (\mathcal{G}^n)_n)$ and $\mathrm{FU}(\bar{d}') \cap Y = \emptyset$. Since by the assumption that the lemma is false, Z is not large for $\bar{d}'$ there is $\bar{d}'' \sqsubseteq \bar{d}'$ such that $\mathrm{FU}(\bar{d}'') \cap Z = \emptyset$ and $\bar{d}''$ is good for $(\mathcal{F}, (\mathcal{G}^n)_n)$. But now $\mathrm{FU}(\bar{d}'') \cap X = \emptyset$, contradicting the assumption that X is large for $\bar{d}$.

Lemma 1(b) of Baumgartner: If X is large for $\bar{d}$ then for every $n \geq 0$, $\{x \in X : \min(x) > n\}$ is large for $\bar{d}$. Clear, we take off only finitely many blocks and only finitely many points.

Lemma 2 of Baumgartner: Suppose X is large for $\bar{d}$. Then there is a finite set $E \subseteq \mathrm{FU}(\bar{d})$ such that for every $x \in \mathrm{FU}(\bar{d})$ if $x \cap \bigcup E = \emptyset$, then there exists $d \in \mathrm{FU}(E)$ such that $x \cup d \subseteq X$.

We recall and modify Baumgartner's proof: We let M be a model of ZFC*, a sufficiently rich finite fragment of ZFC that has cardinality κ such that $\mathcal{F}$ and each of its generators, $\mathcal{G}^n$, and each of its generators, B, and an enumeration of $\mathbb{F}$ are elements of M. Since $\kappa < \mathrm{cov}(\mathcal{M})$ there is a Cohen real c over M.

Suppose that the lemma is false. Then we may choose an arbitrary large $x_0 \in \mathrm{FU}(\bar{d})$, and we take as x_0 the first $c(0)$ blocks of $\bar{d}$. x_0 is also conceived as $E(x_0) \subset \mathrm{FU}(\bar{d})$ before taking the unions. Since the lemma is false there is $x_1 \in \mathrm{FU}(\bar{d})$, $E(x_1) \subset \mathrm{FU}(\bar{d})$ $x_1 > x_0$ and there is a $d \in \mathrm{FU}(E(x_1))$ such that $x_1 \cup d \not\subseteq X$. Again x_1 can be chosen arbitrarily large, since we assume that the lemma is false. We take for x_1 the next $c(1)$ blocks of $\bar{d}$. So we go on with the inductive choice of the x_i. Then we let $y_n = x_{2n} \cup x_{2n+1}$. We show $\bar{y}$ is good for $(\mathcal{F}, (\mathcal{G}^n)_n)$.

Suppose that there is some $\bar{c} \in \mathcal{F}$ and there is some k such that $\mathrm{FU}(\bar{y}\ ;\ \mathrm{past}\ k) \cap \mathrm{FU}(\bar{c}) = \emptyset$ or there are some n and some $Z \in \mathcal{G}^n$ and a $\bar{c} \in \mathcal{F}$ such that $\mathrm{set}(\bar{y}\ ;\ \mathrm{past}\ k) \cap \mathrm{set}(\bar{c}) \cap Z = \emptyset$. Since the choice of $\bar{y}$ was done with the generic c

in the generic extension $M[c]$, there is a Cohen condition

$$p \Vdash \mathrm{FU}(\bar{y}\ ;\text{past } k) \cap \mathrm{FU}(\bar{c}) = \varnothing,$$

or there is a condition forcing the second fact. We show how to derive a contradiction in the second case, the first case is similar. So fix p such that

$$p \Vdash \mathrm{FU}(\bar{y}\ ;\text{past } k) \cap \mathrm{set}(\bar{c}) \cap Z = \varnothing.$$

First let $m = \max\{\mathrm{dom}(p), r\} + 1$ and extend p to an arbitrary $p' \colon 2m + 1 \to \omega$. Then p' decides the first $2m + 1$ values of c and also the first $m + 1$ values of $\bar{y}$ as we set $y_{m+1} = x_{2m} \cup x_{2m+1}$. The set $\mathrm{set}(\bar{d}\ ;\text{past } k) \cap \mathrm{set}(\bar{c}) \cap Z$ is infinite, since $\bar{d}$ is good for $(\mathscr{F}, (\mathscr{G}^n)_n)$. So we fix an element t in this set, which appeared in construction stage $2m + 2$ in block k after x_{2m+1}, and let $q(2m + 2) = k$ then $q \geq p$ and $q \Vdash \mathrm{set}(\bar{y}\ ;\text{past } k) \cap \mathrm{set}(\bar{c}) \cap X \neq \varnothing$. Contradiction. So we showed that $\bar{y}$ is good for $(\mathscr{F}, (\mathscr{G}^n)_n)$. But by our choice of y_i, $\mathrm{FU}(\bar{y}) \cap X = \varnothing$, contradiction.

Lemma 3 of Baumgartner: Suppose X is large for $\bar{d}$. Then there is $e' \in \mathrm{FU}(\bar{d})$ and there is some $\bar{d}' \sqsubseteq \bar{d}$, $\bar{d}'$ is good for $(\mathscr{F}, (\mathscr{G}^n)_n)$ such that $\{x \in X\ :\ x \cup e' \in X\}$ is large for $\mathrm{FU}(\bar{d}')$.

This is proved literally as in Baumgartner, with large instead of large: Let E be as in Lemma 2 and let $\bar{d}^1 = \bar{d} \restriction [\max(E)+1, \infty)$. Then $\bar{d}^1$ is good for $(\mathscr{F}, (\mathscr{G}^n)_n)$, since we took off only finitely many blocks. So X is large for $\bar{d}^1$. For each $e' \in \mathrm{FU}(E)$ let $X_{e'} = \{x \in X\ :\ x \cup e' \in X\}$. So

$$X \cap \mathrm{FU}(\bar{d}^1) \subseteq \bigcup \{X_{e'}\ :\ e' \in \mathrm{FU}(E)\}.$$

By finitely many repeated applications of Lemma 1(a) there is $\bar{d}' \sqsubseteq \bar{d}^1$ good for $(\mathscr{F}, (\mathscr{G}^n)_n)$ and there is $e' \in \mathrm{FU}(E)$ such that $X_{e'}$ is large for $\bar{d}'$.

Lemma 4 of Baumgartner: If X is large for $\bar{d}$ then there is $\bar{d}' \sqsubseteq \bar{d}$ good for $(\mathscr{F}, (\mathscr{G}^n)_n)$ such that $\mathrm{FU}(\bar{d}') \subseteq X$.

Proof of Lemma 4: By Lemma 3 there are sequences $\bar{d}_n$, e'_n, X_n such that

(1) $\bar{d}_0 = \bar{d}$, $X_0 = X$,

(2) $e'_n \in \mathrm{FU}(\bar{d}_n)$ and $\bar{d}_n$ is good for $(\mathscr{F}, (\mathscr{G}^n)_n)$ and e'_n is the union of the first $c(n)$ elements of $\bar{d}_n$,

(3) $X_{n+1} \subseteq X_n$ and $\bar{d}_{n+1} \sqsubseteq \bar{d}_n$,

(4) X_n is large for $\bar{d}_n$,

(5) if $x \in X_{n+1}$, then $x \cup e'_n \in X_n$,

(6) $e'_n \cap e'_m = \varnothing$ if $m \neq n$.

Again we use the properties of the Cohen real to show that $\bar{e}'$ is good for $(\mathscr{F}, (\mathscr{G}^n)_n)$.

Now we choose $\bar{e} \sqsubseteq \bar{e}'$ such that e_0 is the union of the first $c(0)$ elements of $\bar{e}'$. Then we let for $n \geq 1$,

$$k_n = \max\{k\ :\ e'_k \subseteq \bigcup_{0 \leq i < n} e_i\},$$

and choose $e_n \in X_{k_{n+1}}$ such that e_n comprises $c(n)$ blocks of $\bar{e}'$. Again we use the properties of the Cohen real to give $\bar{e}$ is good for $(\mathscr{F}, (\mathscr{G}^n)_n)$.

If $\mathbb{F} = B_0 \cup \cdots \cup B_k$, then one of the B_i is large for any $\bar{d}$, since the set of requirements for being good for $(\mathscr{F}, (\mathscr{G}^n)_n)$ is directed. In the above formulas we take $B' = B$ or $B' = \mathbb{F} \setminus B$, so that B' is large for $\bar{d}$. $\square$

For the stability steps in the construction we use the following theorem, which is similar but easier than Theorem 2.7.

Theorem 2.8 *Let $\mathcal{F}$ be an ordered union filter generated by $< \mathrm{cov}(\mathcal{M})$ sets, and let $\mathcal{G}^n$, $n \in \omega$ be filters on ω, generated by $< \mathrm{cov}(\mathcal{M})$ sets and $\mathcal{G}^n \subseteq \Phi(\mathcal{F})$. Suppose that there is a $\sqsubseteq^*$-descending sequence $\bar{d}^n$, $n < \omega$, of $(\mathcal{F}, (\mathcal{G}^n)_n)$-good sequences. Then it has a lower bound that is good for $(\mathcal{F}, (\mathcal{G}^n)_n)$.*

Proof This is similar to just the last step of the previous proof. Take the Cohen real to pick a sufficiently large almost condensation of the $\bar{d}^n$. $\square$

3 Replacing $(\mathbb{F}, \cup)$ by k-Colored Block Sequences $(\mathbb{F}^k, {}^\frown)$

A Milliken-Taylor ultrafilter $\mathcal{U}^k$ for block sequences with $k + 1$ values is a generalization of an ordinary Milliken-Taylor ultrafilter in the following direction: In a Milliken-Taylor ultrafilter, we color block sequences with just one value $k = 1$ or $k = 1$, corresponding to "in" and "not in." The ultrafilters $\mathcal{U}^1$ are just the Milliken-Taylor ultrafilters. Now we use more than two colors to get more varied ultrafilters. We explain this.

Definition 3.1 $\mathbb{F}^k = \{s : \mathrm{dom}(s) \in F, s\colon \mathrm{dom}(s) \to \{1, \dots, k\}, s \text{ is onto}\}$. $S \subseteq \mathbb{F}^k$ is called unmeshed if $S = \{s_n : n \in |S|\}$ and $\mathrm{dom}(s_0) < \mathrm{dom}(s_1) \dots$. For unmeshed infinite sets $S \subseteq \mathbb{F}^k$ we use also our former barred lowercase letters $\bar{c}$ and so on. Let $(\mathbb{F}^k)^{\leq\omega} = \{T \subseteq \mathbb{F}^k : T \text{ unmeshed}\}$. For $\bar{c} \in (\mathbb{F}^k)^{\omega}$, let $\mathrm{FU}(\bar{c}) = \{c_{i_0}{}^\frown c_{i_1}{}^\frown \dots {}^\frown c_{i_n} : n \in \omega, \text{ for } j \leq n, c_{i_j} \in \bar{c}, \mathrm{dom}(c_{i_0}) < \mathrm{dom}(c_{i_1}) < \dots < \mathrm{dom}(c_{i_n})\}$.

Definition 3.2 $\mathcal{U}^k$ is called a *Milliken-Taylor ultrafilter (on $\mathbb{F}^k$)* if it is stable and if it has a basis of sets of the form $\mathrm{FU}(S)$ for unmeshed $S \subseteq \mathbb{F}^k$.

Under CH or MA(σ-centered) or $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ Milliken-Taylor ultrafilters on $\mathbb{F}^k$ exist. This is proved with the following strengthening of Hindman's theorem.

Theorem 3.3 (Hindman [14], Corollary 3.3) *If the set $\mathbb{F}^k$ is partitioned into finitely many pieces then there is a set $S \in (\mathbb{F}^k)^{\omega}$ such that $\mathrm{FU}(S)$ is included in one piece.*

$\mathcal{U}^k$ has the Ramsey property for n-tupels.

Theorem 3.4 *Let $\mathcal{U}^k$ be a Milliken-Taylor ultrafilter. For any unmeshed $S \in \mathcal{U}^k$ the set $(\mathrm{FU}(S))^n$ is partitioned into finitely many pieces then there is an unmeshed set $T \in \mathcal{U}$ such that $(\mathrm{FU}(T))^n$ is included in one piece.*

Definition 3.5 Let $\mathcal{U}^k$ be a Milliken-Taylor ultrafilter on $\mathbb{F}^k$. Then we define its core by

$$X \in \Phi(\mathcal{U}^k) \text{ iff } (\exists S \in \mathcal{U}^k)(X \supseteq \bigcup\{\mathrm{dom}(s) : s \in S\}),$$

and its color j core for $1 \leq j \leq k$ by

$$X \in \Phi_j(\mathcal{U}^k) \text{ iff } (\exists S \in \mathcal{U}^k)(X \supseteq \bigcup\{s^{-1}{}''\{j\} : s \in S\}).$$

For more on Ramsey theoretic ultrafilters on richer spaces see [13].

Theorem 3.6 *Every k-valued Milliken-Taylor has at least $k + 1$-near coherence classes of ultrafilters in $[\Phi(\mathcal{U})]$.*

Proof Then there are

$$\min_{\text{value } i}(\mathcal{U}^k) = \{\{\min(s^{-1\prime\prime}\{i\}) : s \in S\} : S \in \mathcal{U}^k\}$$

and $\max_{\text{value } i}(\mathcal{U}^k)$ for $1 \le i \le k$ and they assume $k+1$ near coherence classes, since

$$\begin{aligned}&\text{for some order } i_1 < \cdots < i_k \text{ of } \{1, \dots, k\},\\ &\text{for } \mathcal{U}^k \text{ many block pieces } s,\\ &\max(s^{-1\prime\prime}\{i_j\}) = \min(s^{-1\prime\prime}\{i_{j+1}\}) \text{ for } 1 \le j \le k.\end{aligned} \tag{2}$$

Blass's parity argument in his proof of Theorem 1.9 works in this situation and shows that there are at least $k+1$ near coherence classes among the minimum projections and one maximum projection. We show that $\min_{\text{value } i}(\mathcal{U}^k)$ and $\max_{\text{value } i}(\mathcal{U}^k)$ are not nearly coherent for $i = 1, \dots k$. By the above identities this gives $k+1$ near coherence classes. For completeness we repeat the proof here. Suppose that f is finite-to-one and monotone and onto, $f(\max_{\text{value } i}(\mathcal{U}^k)) = f(\max_{\text{value } i}(\mathcal{U}^k))$ and let $I_n = f^{-1\prime\prime}\{n\}$, $n \in \omega$, be adjacent increasing intervals. Now let

$$E = \{s \in \mathbb{F}^k : |I_k \cap s^{-1\prime\prime}\{i\}| \text{ is even}\}.$$

Since $\mathcal{U}$ is an ultrafilter, it must contain E or $\mathbb{F} \setminus E$. Since $\mathcal{U}^k$ is a union (well, rather concatenation) ultrafilter, by Theorem 3.4 there is an infinite family A of pairwise disjoint members of $\mathbb{F}^k$ such that $\mathrm{FU}(A) \subseteq E$ or $\mathrm{FU}(A) \cap E = \emptyset$. Since $\mathrm{FU}(A)$ is closed under concatenation, and the sum of two odd numbers is even, we have $\mathrm{FU}(A) \subseteq E$.

$A \in \mathcal{U}$, therefore, $\max''_{\text{value } i} A = \{\max(s^{-1\prime\prime}\{i\}) : s \in A\} \in \max_{\text{value } i}(\mathcal{U}^k)$ and $\min''_{\text{value } i} A = \{\min(s^{-1\prime\prime}\{i\}) : s \in A\} \in \min_{\text{value } i}(\mathcal{U}^k)$. Since the two ultrafilters are nearly coherent by our assumption, the two sets have a nonempty (indeed, an infinite) intersection. So there is an interval I_n with $\min(s^{-1\prime\prime}\{i\}), \max(t^{-1\prime\prime}\{i\}) \in I_n$. t meets besides I_n only earlier intervals, before I_n, if at all, and s meets besides I_n only later intervals after I_n, if at all. So $s \cup t \notin E$, but $s \cup t \in A$, contradiction to $\mathrm{FU}(A) \subseteq E$. $\square$

Since by identifying colors from a k-colored Milliken-Taylor ultrafilter we get an ℓ-colored for $\ell < k$, we see that this proof gives an alternative way to show that under $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ (otherwise, k-valued Milliken-Taylor ultrafilters need not exist) there are ordinary Milliken-Taylor ultrafilters with $k+1$ near coherence classes in their core for any $k \ge 1$.

4 Destroying Milliken Taylor Ultrafilters with $\mathbb{M}(\mathcal{U})$

We review Matet forcing $\mathbb{M}$ [6; 17] and its σ-centered suborders $\mathbb{M}(\mathcal{U})$ for a stable ordered-union ultrafilter $\mathcal{U}$.

Definition 4.1 In the *Matet forcing,* $\mathbb{M}$, the conditions are pairs $(s, \bar{c})$ such that $s \in \mathbb{F}$ and $\bar{c} \in (\mathbb{F})^\omega$ and $s < c_0$. The forcing order $(s', \bar{c}')$ is stronger than $(s, \bar{c})$, in symbols $(s', \bar{c}') \ge (s, \bar{c})$, if and only if $s \subseteq s'$ and $s' \setminus s$ is a union of finitely many of the c_n and $\bar{c}'$ is a condensation of $\bar{c}$. *The stronger condition is the larger one*. This is in contrast to the order of almost condensation.

In [6] it is shown that $\mathbb{M}$ is proper. In unpublished work, Blass and Laflamme independently have shown that $\mathbb{M}$ preserves P-points. Eisworth's work ([11, Theorem 4] or Theorem 3.4 below) implies this result, as we shall explain below.

Definition 4.2 Given an ordered-union ultrafilter $\mathcal{U}$ on $\mathbb{F}$ we let $\mathbb{M}(\mathcal{U})$ consist of all pairs $(s, \bar{c}) \in \mathbb{M}$ such that $s \in \mathbb{F}$ and $\mathrm{FU}(\bar{c}) \in \mathcal{U}$ and $s < \min(c_0)$. The forcing order is the same as in the Matet forcing.

It is well known [17; 6] that Matet forcing $\mathbb{M}$ can be decomposed into two steps $\mathbb{P}' * \mathbb{M}(\underset{\sim}{\mathcal{U}})$ such that $\mathbb{P}'$ is ω_1-closed (that is, every descending sequence of conditions of countable length has a lower bound) and adds a stable ordered-union ultrafilter $\mathcal{U}$ on the set $\mathbb{F}$ and that $\mathbb{M}(\mathcal{U})$ is the Matet forcing with sequences from the ultrafilter (and hence it is σ-centered).

If $\mathcal{U}$ is ultra on $\mathbb{F}$, then $\Phi(\mathcal{U})$ is not diagonalized (see [11, Prop. 2.3]) and also all finite-to-one images of $\Phi(\mathcal{U})$ are not diagonalized (same proof). So $\Phi(\mathcal{U})$ is not meager.

Definition 4.3 The *weak Rudin-Blass ordering* on filters on ω is defined as follows: Let $\mathcal{F} \leq_{\omega RB} \mathcal{G}$ if and only if there is a finite-to-one h such that $h(\mathcal{F}) \subseteq h(\mathcal{G})$.

The more common version of the Rudin-Blass ordering requires just $h(\mathcal{F}) \subseteq \mathcal{G}$ instead of $h(\mathcal{F}) \subseteq h(\mathcal{G})$. The above version is more suitable to describe which ultrafilters are preserved. Note that $\leq_{\omega RB}$ is called $\leq_{RB}$ in [11]. The following property of stable ordered-union ultrafilters $\mathcal{U}$ builds on the Ramsey property of $\mathcal{U}$ (Theorem 1.7) and will be important for our proof.

Theorem 4.4 (Eisworth [11], "→" Theorem 4, "←" Cor. 2.5, this direction works also with non-P ultrafilters) *Let $\mathcal{U}$ be a stable ordered-union ultrafilter on $\mathbb{F}$ and let $\mathcal{V}$ be a P-point. If and only if $\mathcal{V} \not\geq_{\omega RB} \Phi(\mathcal{U})$, then $\mathcal{V}$ continues to generate an ultrafilter after we force with $\mathbb{M}(\mathcal{U})$.*

Now we show that for Milliken-Taylor ultrafilters there is no analogue.

Theorem 4.5 *Forcing with $\mathbb{M}(\mathcal{U})$ destroys any Milliken-Taylor ultrafilter $\mathcal{V}$.*

Proof First case $\min(\mathcal{V}) \geq_{\omega RB} \Phi(\mathcal{U})$, then by [11, Corollary 2.5], $\min(\mathcal{V})$ is destroyed and hence $\mathcal{V}$ is destroyed. Second case: Same with $\max(\mathcal{V})$.

Third case: $\min(\mathcal{V})$ and $\max(\mathcal{V})$ are P-points that are $\not\leq_{\omega RB} \Phi(\mathcal{U})$. Then both are preserved by Theorem 3.4. However, we show in $\mathbf{V}^{\mathbb{M}(\mathcal{U})}$, $\min(\mathcal{V})$ and $\max(\mathcal{V})$ are nearly coherent. So by Blass's result, $\mathcal{V}$ is not a union ultrafilter anymore. The generic real is

$$r = \bigcup \{s \,:\, \exists \bar{c}(s, \bar{c}) \in G\}.$$

From r we get a finite-to-one function r^-, by letting $r^-(n) = |r \cap n|$. Then

$$\Vdash_{\mathbb{Q}_\alpha} \text{“}\underset{\sim}{r}^-(\min(\mathcal{V})) = \underset{\sim}{r}^-(\max(\mathcal{V})).\text{”}$$

Given $(s, \bar{c}) \in \mathbb{M}(\mathcal{U})$ and $E \in \min(\mathcal{V})$ and $F \in \max(\mathcal{V})$, there is some $\bar{d} \sqsubseteq^* \bar{c}$, $\bar{d} \in \mathcal{U}$, such that $E \cap \mathrm{set}(\bar{d}) = \emptyset$ and $F \cap \mathrm{set}(\bar{d}) = \emptyset$ (this is possible since $\min(\mathcal{V}), \max(\mathcal{V}) \not\geq_{\omega RB} \Phi(\mathcal{U})$ by the hypothesis). Now, for two suitable $k < k'$, we have $[\max(d_k), \min(d_{k'})) \cap F \neq \emptyset$ and $[\max(d_k), \min(d_{k'})) \cap E \neq \emptyset$. So $(s \cup d_{k'}, \bar{d} \restriction [k'+1, \infty))$ is stronger than $(s, \bar{c})$ and it forces that $r^-(E) \cap r^-(F) \neq \emptyset$. Since this works for any two sets, we have $r^-(\min(\mathcal{V}))$ is coherent with $r^-(\max(\mathcal{V}))$.

□

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