Notre Dame Journal of Formal Logic
Volume 52, Number 4, 2011

Quantifier Elimination and Other Model-Theoretic Properties of BL-Algebras

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Abstract This work presents a model-theoretic approach to the study of first-order theories of classes of BL-chains. Among other facts, we present several classes of BL-algebras, generating the whole variety of BL-algebras, whose first-order theory has quantifier elimination. Model-completeness and decision problems are also investigated. Then we investigate classes of BL-algebras having (or not having) the amalgamation property or the joint embedding property and we relate the above properties to the existence of ultrahomogeneous models.

1 Introduction

The aim of this paper is to investigate the algebraic semantics of Hájek's basic fuzzy logic BL (cf. [20]) by model-theoretic means. An investigation of this kind was first carried out for finitely and infinitely valued Łukasiewicz logics, product logic, and Gödel logic in [26; 27; 25; 4; 8; 28]. In particular, the research focus has been on the study of quantifier elimination and model completions for classes of algebras related to the above-mentioned logics.

Lacava and Saeli showed in [26; 27] that the first-order theory of the variety generated by a finite MV-algebra admits a model completion, and the same holds for the theory of the class of all linearly ordered MV-algebras. Lacava also showed that the theory of the whole variety of MV-algebras does not even have a model-companion. Caicedo proved in [8] quantifier elimination for the class of DMV-chains, that is, the linearly ordered members of the equivalent variety semantics of the Rational Łukasiewicz logic. Baaz and Veith studied the MV, product, and Gödel algebras over the real unit interval [0, 1] and provided quantifier elimination results by geometric means (see [4]). They also proved elimination of quantifiers for each theory of a finite MV-chain. The second author of this work showed in [28], by adopting

Received August 9, 2010; accepted February 13, 2011; printed October 5, 2011
2010 Mathematics Subject Classification: Primary, 03B50, 03C60
Keywords: BL-algebras, quantifier elimination, model-completeness, amalgamation property
 10.1215/00294527-1499336

a uniform approach, that certain theories of linearly ordered commutative residuated lattices do enjoy elimination of quantifiers. Those results were exploited in order to obtain a proof of the amalgamation property for varieties of representable commutative residuated lattices by purely model-theoretic means. In particular, it was proved in [28] that the theory of divisible MV-chains, divisible product algebras, and densely ordered Gödel chains (among other theories) admit quantifier elimination, and the varieties of MV, product, and Gödel algebras have the amalgamation property.[1]

As said above, the aim of this paper is to follow the model-theoretic investigation carried out in the previously mentioned works and focus on the class of BL-algebras, that is, the algebraic semantics of Hájek's logic BL. Although not as well known as Łukasiewicz logic or Gödel logic, BL has been the object of growing interest and deep investigation in mathematical logic. Indeed, not only is BL a common fragment of Łukasiewicz, product, and Gödel logics, but it also is the logic of continuous t-norms and their residua [10]; that is, the logic that is obtained when the conjunction is interpreted as a continuous, monotonic, commutative monoidal operation on the real unit interval [0, 1] and the implication is interpreted as its residuum [20]. BL is, then, a very natural logic for the treatment of intermediate truth values.

There is no need to explain here the importance of model theory in classical logic and algebra. However, the study of the connections between model theory and many-valued logics has begun very recently and deserves further investigation. In the papers [21; 13], for instance, model theory is applied to an investigation of the semantics of first-order fuzzy logics. Instead, in this paper, we will focus on first-order theories of classes of algebras that constitute an algebraic semantics for BL and for some of its extensions.

Clearly, we cannot expect the theory of BL-algebras to have quantifier elimination because otherwise it would be model-complete and, consequently, complete, since it has a minimal model, that is, the two element BL-chain. However, that is obviously not the case, since not every embedding between BL-chains is elementary.

Therefore, we will concentrate on some prominent classes of BL-algebras whose first-order theory is complete, and we will prove quantifier elimination for some finitely axiomatizable extensions by definitions of such theories. We will focus on classes of BL-chains that are ordinal sums of divisible MV-chains, and we will consider, in particular, the class of ordinal sums with a discretely ordered infinite set of components, with finitely many components and with a densely ordered set of components. We will show that all those classes have a finite extension by definitions that have quantifier elimination, and we will exhibit an explicit quantifier elimination algorithm. We will see that these theories behave in a different way; for instance, some of them are model-complete while some are not. Moreover, we will prove some consequences of quantifier elimination, including completeness and decidability (we will also exhibit an explicit decision algorithm).

Also, we will offer a model-theoretic investigation of the amalgamation property and the joint embeddability property using the above-mentioned quantifier elimination results. We will conclude our study with a representation theorem for BL-algebras in the style of Di Nola's theorem for MV-algebras, and we will also introduce a new possible line of research involving concepts like ultrahomogeneity, uniform local finiteness, and Fraïssé limits.

2 Preliminaries

BL-algebras constitute the equivalent algebraic semantics (in the sense of [7]) of Hájek's logic BL [20]. In order to introduce BL-algebras properly, we start from the following definitions.

Definition 2.1 (cf. [6]) A *hoop* is a structure $\mathbf{A} = \langle A, \cdot, \Rightarrow, 1\rangle$ where $\langle A, \cdot, 1\rangle$ is a commutative monoid, and $\Rightarrow$ is a binary operation such that

$$x \Rightarrow x = 1, \qquad x \Rightarrow (y \Rightarrow z) = (x \cdot y) \Rightarrow z \qquad \text{and} \qquad x \cdot (x \Rightarrow y) = y \cdot (y \Rightarrow x).$$

In any hoop, the operation $\Rightarrow$ induces a partial order $\leq$ defined by $x \leq y$ if and only if $x \Rightarrow y = 1$. Moreover, hoops are precisely the partially ordered commutative residuated integral monoids (pocrims) in which the meet operation $\sqcap$ is definable as $x \sqcap y = x \cdot (x \Rightarrow y)$. Finally, hoops satisfy the following divisibility condition:

(div) If $x \leq y$, then there is an element z such that $z \cdot y = x$.

Definition 2.2 A hoop is said to be *basic* if and only if it satisfies the identity,

(lin) $(x \Rightarrow y) \Rightarrow z \leq ((y \Rightarrow x) \Rightarrow z) \Rightarrow z$.

A *Wajsberg hoop* is a hoop satisfying

(W) $(x \Rightarrow y) \Rightarrow y = (y \Rightarrow x) \Rightarrow x$.

A *cancellative hoop* is a hoop satisfying

$(canc)$ $x \Rightarrow (x \cdot y) = y$.

A *bounded hoop* is a hoop with an additional constant 0 satisfying the equation $0 \leq x$. In a bounded hoop, we define $\sim x = x \Rightarrow 0$. A BL-*algebra* is a bounded basic hoop. A *Wajsberg algebra* is a bounded Wajsberg hoop. A *product algebra* (cf. [20]) is a BL-algebra satisfying the equations $x \sqcap \sim x = 0$ and $\sim\sim z \leq ((x \cdot z) \Rightarrow (y \cdot z)) \Rightarrow (x \Rightarrow y)$. A *Gödel algebra* (cf. [20]) is a BL-algebra satisfying the equation $x^2 = x$.

The varieties of BL-algebras, Wajsberg algebras, basic hoops, Wajsberg hoops, and cancellative hoops, product algebras, and Gödel algebras will be denoted by $\mathcal{BL}$, $\mathcal{W}$, $\mathcal{BH}$, $\mathcal{WH}$, $\mathcal{CH}$, $\mathcal{P}$, and $\mathcal{G}$, respectively.

In any BL-algebra, as well as in any basic hoop, the join operation $\sqcup$ is definable as $x \sqcup y = ((x \Rightarrow y) \Rightarrow y) \sqcap ((y \Rightarrow x) \Rightarrow x)$. Moreover, the identity,

(prl) $(x \Rightarrow y) \sqcup (y \Rightarrow x) = 1$,

holds in every BL-algebra and in every basic hoop. In particular, BL-algebras are commutative, integral, and bounded residuated lattices (cf. [32] or [24]) satisfying conditions (div) and (prl). BL-algebras can be also characterized as those bounded hoops that are isomorphic to a subdirect product of linearly ordered bounded hoops. We also recall that a cancellative hoop is a Wajsberg hoop, and a Wajsberg hoop is basic (cf. [17] and [1]). Moreover, a linearly ordered Wajsberg hoop is either cancellative or the reduct of a Wajsberg algebra (cf. [17]). Finally, Wajsberg algebras are term-wise equivalent to Chang's MV-algebras [12]: any Wajsberg algebra is an MV-algebra with respect to $\sim$ and to the operation $\oplus$ defined as $x \oplus y = (\sim x) \Rightarrow y$. Conversely, any MV-algebra is a Wajsberg algebra with respect to the operations $\cdot$ and $\Rightarrow$ defined as $x \cdot y = \sim(\sim x \oplus \sim y)$ and as $x \Rightarrow y = (\sim x) \oplus y$. Thus, in what follows, MV-algebras and Wajsberg algebras will be regarded as the same kind of structures. In every MV-algebra, we define $x \ominus y = x \cdot (\sim y)$ and

$x \Leftrightarrow y = (x \Rightarrow y) \cdot (y \Rightarrow x)$. Moreover, lattice operations are definable as in BL-algebras. However, the join has a simpler definition, namely, $x \sqcup y = (x \Rightarrow y) \Rightarrow y$. In every MV-algebra we define for every natural number n, the terms x^n and nx by induction as follows:

$$x^0 = 1\,,\; x^{n+1} = x^n \cdot x \qquad 0x = 0\,,\; (n+1)x = (nx) \oplus x\,.$$

With $[0,1]_{\mathrm{MV}}$ we denote the MV-algebra whose domain is the real unit interval $[0,1]$, equipped with the constants 0 and 1 and with the operations $\oplus$ and $\sim$ defined by $x \oplus y = \min\{x+y, 1\}$ and $\sim x = 1-x$, and with $[0,1]_{\mathrm{Q}}$ we denote the subalgebra of $[0,1]_{\mathrm{MV}}$ whose domain is the set of rational numbers in $[0,1]$. It is well known that $[0,1]_{\mathrm{MV}}$ generates the whole variety of MV-algebras. The algebra $[0,1]_{\mathrm{MV}}$ will be often identified with the Wajsberg algebra $[0,1]_{\mathrm{W}}$ on $[0,1]$ which is term-wise equivalent to $[0,1]_{\mathrm{MV}}$. In this algebra, the operations $\cdot$ and $\Rightarrow$ are defined as $x \cdot y = \max\{x+y-1, 0\}$ and $x \Rightarrow y = \min\{1-x+y, 1\}$.

Moreover, with $[0,1]_{\mathrm{P}}$ we denote the product algebra whose domain is the real unit interval $[0,1]$, equipped with the constants 0 and 1 and with the operations $\cdot_P$ (ordinary product) and $\Rightarrow_P$ defined by $x \Rightarrow_P y = 1$ if $x \leq y$, and $x \Rightarrow_P y = \frac{y}{x}$ otherwise. Finally, by $[0,1]_{\mathrm{G}}$ we denote the Gödel algebra with lattice reduct $[0,1]$ with operations $x \cdot_G y = \min\{x, y\}$ and $x \Rightarrow_G y = 1$ if $x \leq y$ and $x \Rightarrow_G y = y$ otherwise. $\mathcal{P}$ and $\mathcal{G}$ are generated as quasi varieties by $[0,1]_{\mathrm{P}}$ and by $[0,1]_{\mathrm{G}}$, respectively (see [20]).

Definition 2.3 Let $\langle I, \leq\rangle$ be a totally ordered set with minimum i_0. For all $i \in I$, let $\mathbf{A}_i$ be a hoop such that for $i \neq j$, $\mathbf{A}_i \cap \mathbf{A}_j = \{1\}$, and assume that $\mathbf{A}_{i_0}$ is bounded. Then $\bigoplus_{i\in I} \mathbf{A}_i$ (the *ordinal sum* of the family $(\mathbf{A}_i)_{i\in I}$) is the structure whose base set is $\bigcup_{i\in I} \mathbf{A}_i$, whose bottom is the minimum of $\mathbf{A}_{i_0}$, whose top is 1, and whose operations are

$$x \Rightarrow y = \begin{cases} x \Rightarrow^{\mathbf{A}_i} y & \text{if } x, y \in \mathbf{A}_i \\ y & \text{if } \exists i > j\,(x \in \mathbf{A}_i \text{ and } y \in \mathbf{A}_j) \\ 1 & \text{if } \exists i < j\,(x \in \mathbf{A}_i \setminus \{1\} \text{ and } y \in \mathbf{A}_j) \end{cases}$$

$$x \cdot y = \begin{cases} x \cdot^{\mathbf{A}_i} y & \text{if } x, y \in \mathbf{A}_i \\ x & \text{if } \exists i < j\,(y \in \mathbf{A}_j,\, x \in \mathbf{A}_i \setminus \{1\}) \\ y & \text{if } \exists i < j\,(x \in \mathbf{A}_j,\, x \in \mathbf{A}_i \setminus \{1\}) \end{cases}$$

In [2], the following theorem is proved.

Theorem 2.4 *Every linearly ordered* BL*-algebra* $\mathbf{A}$ *is the ordinal sum of an indexed family* $\langle \mathbf{W}_i : i \in I\rangle$ *of linearly ordered Wajsberg hoops, where* I *is a linearly ordered set with minimum* i_0*, and* $\mathbf{W}_{i_0}$ *is bounded.*

In what follows, the Wajsberg hoops $\mathbf{W}_i$ in Theorem 2.4 will be called *the Wajsberg components of* $\mathbf{A}$. Using the fact that the $\mathbf{W}_i$ are closed under hoop operations, it is easy to prove (cf. [2]) that with reference to Theorem 2.4, the subalgebras of $\mathbf{A} = \bigoplus_{i\in I} \mathbf{W}_i$ are those of the form $\mathbf{B} = \bigoplus_{i\in I} \mathbf{U}_i$, where for $i \in I$, $\mathbf{U}_i$ is a subhoop of $\mathbf{W}_i$ (possibly trivial if $i \neq i_0$), and $\mathbf{U}_{i_0}$ is a Wajsberg subalgebra of $\mathbf{W}_{i_0}$.

As shown in [1], $\mathcal{BL}$ is generated as a quasi variety by the ordinal sum $(\omega)[0,1]_{\mathrm{MV}}$ of ω copies of $[0,1]_{\mathrm{MV}}$.

Definition 2.5 Let $\mathbf{A}$ be a hoop, possibly with additional operators. A *congruence filter* of $\mathbf{A}$ is the congruence class of 1 with respect to some congruence of $\mathbf{A}$.

The congruence filters of a (bounded or unbounded) hoop are precisely its *implicative filters* (*filters*, for short), that is, the subsets F of $\mathbf{A}$ such that $1 \in F$, and whenever $x \in F$ and $x \Rightarrow y \in F$, then $y \in F$. The lattice of congruences and the lattice of congruence filters of a hoop are isomorphic under the isomorphism that associates to a congruence filter F the congruence θ_F defined by $x\theta_F y$ if and only if $x \Rightarrow y \in F$ and $y \Rightarrow x \in F$. The inverse isomorphism associates to every congruence θ the set $F_\theta = \{x : x\theta 1\}$. We will denote by $\mathbf{A}/F$ the quotient of $\mathbf{A}$ modulo θ_F, and for $a \in \mathbf{A}$, we will denote by a/F the equivalence class of a modulo θ_F.

We conclude this section reviewing some categorical equivalences between varieties of BL-algebras or hoops and lattice-ordered Abelian groups.

Definition 2.6 A *lattice-ordered Abelian group* (cf. [5] or [19]) (ℓ*-group*, for short) is an algebra $\mathbf{G} = \langle G, +, -, 0, \sqcup, \sqcap\rangle$ such that $\langle G, +, -, 0\rangle$ is an Abelian group, $\langle G, \sqcup, \sqcap\rangle$ is a lattice, and for all $x, y, z \in G$, we have $x + (y \sqcup z) = (x + y) \sqcup (x + z)$. For every natural number n, nx is inductively defined by $0x = 0$ and $(n + 1)x = nx + x$. A *strong unit* of a lattice-ordered Abelian group $\mathbf{G}$ is an element $u \in \mathbf{G}$ such that for all $g \in \mathbf{G}$ there is $n \in \mathbf{N}$ such that $g \leq nu$.

The variety of lattice-ordered Abelian groups will be denoted by $\mathcal{LG}$.

Definition 2.7 The *radical* of a Wajsberg algebra $\mathbf{A}$ (denoted by $\mathrm{Rad}(\mathbf{A})$) is the intersection of its maximal filters. A Wajsberg algebra $\mathbf{A}$ is said to be *perfect* if, for all $x \in \mathbf{A}$, either $x \in \mathrm{Rad}(\mathbf{A})$ or $\sim x \in \mathrm{Rad}(\mathbf{A})$. We define the functor Γ (cf. [31]) from the category of lattice-ordered Abelian groups with strong unit into the category of Wajsberg algebras as follows: if $\mathbf{G}$ is a lattice-ordered Abelian group and u is a strong unit of $\mathbf{G}$, then $\Gamma(\mathbf{G}, u)$ denotes the algebra $\mathbf{A}$ whose universe is $\{x \in \mathbf{G} : 0 \leq x \leq u\}$ and whose operations are $x \cdot y = (x + y - u) \sqcup 0$ and $x \Rightarrow y = ((u - x) + y) \sqcap u$. Moreover, if h is a morphism of lattice-ordered Abelian groups with strong unit from $(\mathbf{G}, u)$ into $(\mathbf{H}, w)$, then $\Gamma(h)$ denotes its restriction to $\Gamma(\mathbf{G}, u)$.

We define the functor Υ from the category of lattice-ordered Abelian groups into the category of cancellative hoops as follows: if $\mathbf{G}$ is a lattice-ordered Abelian group, then $\Upsilon(\mathbf{G})$ is the algebra whose universe is the negative cone $\{x \in \mathbf{G} : x \leq 0\}$ of $\mathbf{G}$ and whose operations are the restriction of $+$ to the negative cone of $\mathbf{G}$ and the operation $\Rightarrow$ defined by $x \Rightarrow y = (y - x) \sqcup 0$. Moreover, if h is a morphism of lattice-ordered Abelian groups from $\mathbf{G}$ into $\mathbf{H}$, then $\Upsilon(h)$ is the restriction of h to $\Upsilon(\mathbf{G})$.

Finally, we define a functor Λ from the category of perfect Wajsberg algebras into the category of cancellative hoops as follows: given a perfect Wajsberg algebra $\mathbf{A}$, $\Lambda(\mathbf{A})$ is the algebra $\mathbf{C}$ whose universe is $\mathrm{Rad}(\mathbf{A})$ and whose operations $\cdot$ and $\Rightarrow$ are the restrictions to $\mathbf{C}$ of the monoidal operation and its residuum (note that any filter is closed under such operations). Moreover, given a morphism h of perfect Wajsberg algebras from $\mathbf{W}$ into $\mathbf{U}$, $\Lambda(h)$ denotes the restriction of h to $\mathrm{Rad}(\mathbf{W})$.

It is well known that Γ, Υ, and Λ have an inverse functor Γ^{-1}, Υ^{-1}, and Λ^{-1}, respectively, and that the pairs (Γ, Γ^{-1}), $(\Upsilon, \Upsilon^{-1})$, and (Λ, Λ^{-1}) are equivalences of categories. These results can be found in [31], [17], and [15], respectively.

We now recall a few basic notions from model theory that will be used throughout the rest of the paper. The reader can find a detailed and extensive treatment of the subject in [9; 23; 29].

Definition 2.8 Let T be a first-order theory in some language L.

1. We say that T admits *elimination of quantifiers* (QE) in L if for every formula $\varphi(\overline{x})$ there is a quantifier-free formula $\psi(\overline{x})$ that is provably equivalent to $\varphi(\overline{x})$ in T.
2. T is said to be *model-complete* if every embedding between models of T is elementary; that is, for any $\mathbf{A}, \mathbf{B} \models T$, every embedding $f : \mathbf{A} \to \mathbf{B}$, every L-formula $\varphi(x_1, \ldots, x_m)$, and $a_1, \ldots, a_m \in \mathbf{A}$,
$$\mathbf{A} \models \varphi(a_1, \ldots, a_m) \text{ iff } \mathbf{B} \models \varphi(f(a_1), \ldots, f(a_m)).$$
3. Two L-structures $\mathbf{A}, \mathbf{B}$ are said to be *elementarily equivalent* if, for every L-sentence φ, $\mathbf{A} \models \varphi$ if and only if $\mathbf{B} \models \varphi$.
4. T is called *complete* if for every L-sentence φ, either $T \vdash \varphi$ or $T \vdash \neg\varphi$.
5. A model $\mathbf{A}$ is called a *minimal model* of T (*prime model*, respectively) whenever $\mathbf{A}$ is embeddable (elementary embeddable, respectively) into every $\mathbf{B} \models T$.
6. $T_\forall$ denotes the universal theory of T, that is, the set of universal sentences that are consequences of T.

Note that if T is model-complete and has a prime model, it also is complete, and all of its models are elementarily equivalent to each other.

Definition 2.9 Let $\mathcal{K}$ be a class of structures in the same signature. A *V-formation in* $\mathcal{K}$ is a finite sequence $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, and i, j are embeddings of $\mathbf{A}$ into $\mathbf{B}$ and into $\mathbf{C}$, respectively.

Given a V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{K}$, we say that $(\mathbf{D}, h, k)$ is an *amalgam* of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{K}$ if $\mathbf{D} \in \mathcal{K}$ and h, k are embeddings of $\mathbf{B}$ and of $\mathbf{C}$, respectively, into $\mathbf{D}$ such that the compositions $h \circ i$ and $k \circ j$ coincide. We say that $\mathcal{K}$ *has the amalgamation property* (AP) if every V-formation in $\mathcal{K}$ has an amalgam in $\mathcal{K}$. We say that $\mathcal{K}$ has the *strong amalgamation property* (SAP) if every V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{K}$ has an amalgam $(\mathbf{D}, h, k)$ in $\mathcal{K}$ such that $h(\mathbf{B}) \cap k(\mathbf{C}) = h(i(\mathbf{A})) = k(j(\mathbf{A}))$. We say that a theory T has the (strong) amalgamation property if so does its class of models.

Theorem 2.10 ([23])

(1) *A theory T admits* QE *in* L *if and only if T is model-complete and $T_\forall$, the universal fragment of T, has the* AP.

(2) *If a theory T is model-complete, then it enjoys the* SAP.[2]

3 Algorithms for Quantifier Elimination in Divisible Ordered Abelian Groups and in Divisible MV-Chains

In what follows, classical connectives are denoted by $\vee$ (disjunction), $\wedge$ (conjunction), $\to$ (implication), and $\neg$ (negation). In order to study quantifier elimination in classes of BL-algebras, we start from quantifier elimination in the theory of ordered divisible Abelian groups. It is a well-known fact that this theory has QE in the language of ordered groups (see [29]). We briefly review the proof of this fact because it contains some techniques that we will make use of in the rest of the paper. Moreover, as mentioned above, quantifier elimination for divisible MV-chains was shown in [28], but in this section we give a more constructive and algorithmic proof of the same fact.

We start with a review of quantifier elimination for the theory ODAG of ordered divisible Abelian groups. ODAG is axiomatized by the axioms of ordered groups plus the sentence $\forall x \exists y (ny = x)$ for all natural numbers $n > 1$. In order to obtain quantifier elimination we will first work in an extension by definitions of ODAG obtained by adding, for every $n > 1$, a function symbol d_n (division by n), together with the sentence $\forall x (nd_n(x) = x)$. Then the symbols d_n are eliminated by multiplying, so to speak, all terms by a suitable natural number (for instance, $d_n(t) < d_m(s)$ is equivalent to $mt < ns$). In general, it suffices to multiply all maximal terms in an atomic formula by the product of all n such that d_n occurs in the formula.

The quantifier elimination proceeds as follows:

(1) It suffices to eliminate $\exists$ in formulas of the form $\exists x C(x, y_1, \ldots, y_n)$, where C is a conjunction of literals containing the variable x.

(2) We can eliminate $\leq$: $s \leq t$ can be reduced to $s = t \vee s < t$.

(3) We can eliminate $\neg$: $\neg(s = t)$ can be reduced to $s < t \vee t < s$ and $\neg(s < t)$ can be reduced to $s = t \vee t < s$.

(4) After some obvious algebraic manipulations, we can write every literal of the form $s \lhd t$ (with $\lhd \in \{=, <\}$) as $s' + kx \lhd t'$, or as $s' \lhd kx + t'$, where s', t' are terms not containing x. Moreover, after multiplying both members of any literal by suitable constants, we can assume that the same k occurs in all expressions $s' + kx \lhd t'$, or $s' \lhd kx + t'$.

(5) If the conjunction C contains some literal of the form $s' + kx = t'$ or of the form $s' = kx + t'$, then just eliminate $\exists x$ and replace x by $d_k(t' - s')$ (by $d_k(s' - t')$, respectively).

(6) Otherwise, C can be reduced to a formula of one of the following forms:

(a) $\bigwedge_{i=1}^{n} kx < u_i$, or
(b) $\bigwedge_{j=1}^{m} v_j < kx$, or
(c) $\left(\bigwedge_{i=1}^{n} kx < u_i\right) \wedge \left(\bigwedge_{j=1}^{m} v_j < kx\right)$,

where u_i, v_j are terms. Then, in cases (a) and (b), the formula C can be reduced to $\top$ (symbol for truth), whereas in case (c) it can be reduced to $\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{m} v_i < u_i$.

Finally, one can eliminate the operators d_n by multiplying all terms by a suitable natural number as shown above. Therefore we have the following theorem.

Theorem 3.1 ODAG *has* QE *in the language* $\langle +, -, 0, < \rangle$.

The above-mentioned quantifier elimination result can be extended to the theory DMV of divisible MV-chains. This theory is obtained by adding to the first-order theory of MV-chains the sentence $\forall x \forall y (x \leq y \vee y \leq x)$ plus the sentences

$$\forall x \exists y \; x = py, \qquad \exists x \; (p-1)x = {\sim} x,$$

for each prime number p.[3] Note that these two sentences may be replaced by a single one: $\forall x \exists y ((p-1)y = x \ominus y)$, again for every prime number p.

Theorem 3.2 ([4; 8; 28]) DMV *has* QE *in the language* $\langle \oplus, \sim, 0, < \rangle$.

We will give an algorithmic proof of the above theorem in DMV expanded with an additional operator d_2 (division by 2), and by the axiom $d_2(x) = x \ominus d_2(x) = x$.

By a standard technique, we can reduce the quantifier elimination problem to the problem of eliminating $\exists x$ in formulas of the form $Q_1 x_1 \dots Q_n x_n C$ where $Q_1, \dots, Q_n$ are quantifiers and C is a disjunction of conjunctions of formulas of the form $x = y \oplus z$ or $x = \sim y$ or $x < y$ where x, y, z are variables. Now let C_1 be the formula obtained from C by replacing each subformula of the form $x = y \oplus z$ by $(y + z \leq u \wedge x = y + z) \vee (y + z > u \wedge x = u)$, each subformula of the form $x = \sim y$ by $x = u - y$, and 1 by u, where u is a new variable not occurring in ψ. Moreover, define inductively ψ_i $(i = 0, 1, \dots, n)$ by $\psi_0 = C_1$, $\psi_{i+1} = \exists x_{i+1}(0 \leq x_{i+1} \wedge x_{i+1} \leq u \wedge \psi_i)$ if $Q_{i+1} = \exists$, and $\psi_{i+1} = \forall x_{i+1}((0 \leq x_{i+1} \wedge x_{i+1} \leq u) \to \psi_i)$ if $Q_{i+1} = \forall$. Finally, let V be the set of free variables in ψ_n and let $\psi' = u > 0 \wedge \left(\bigwedge_{z \in V}(0 \leq z \wedge z \leq u)\right) \wedge \psi_n$. Now let $\mathbf{G}$ be an ordered Abelian group, let $u_0 \in \mathbf{G}$, $u_0 > 0$, and let $\mathbf{G}(u_0)$ be the ordered subgroup of $\mathbf{G}$ consisting of all $z \in \mathbf{G}$ for which there is a positive integer n such that $|z| \leq n u_0$. Note that u_0 is a strong unit of $\mathbf{G}(u_0)$. Then the following lemma is easy to prove.

Lemma 3.3 *Let v be a valuation of all variables into an ordered Abelian group $\mathbf{G}$, let $u_0 \in \mathbf{G}$, with $u_0 > 0$, such that for every variable z, $v(z) \in \Gamma(\mathbf{G}(u_0), u_0)$, and $v(u) = u_0$. Then $\mathbf{G}, v \models \psi'$ if and only if $\Gamma(\mathbf{G}(u_0), u_0), v \models \psi$.*

Now apply the quantifier elimination procedure for ordered divisible Abelian groups to ψ', thus getting a quantifier-free formula ψ'' equivalent to ψ'. Write ψ'' in disjunctive normal form, $\psi'' = \bigvee_{i=1}^{k} \bigwedge_{j=1}^{h_i} L_{ij}$, where, after some algebraic manipulation, we can assume without loss of generality that every L_{ij} has the form $\sum_{r=1}^{n_{ij}} k_r^{ij} x_r \lhd \sum_{s=1}^{m_{ij}} h_s^{ij} y_s$, where $\lhd \in \{=, <\}$, x_r, y_s are variables (the variable u might be one of them), and h_s^{ij}, k_r^{ij} are nonnegative integers. Note that if ψ'' is satisfied by a valuation v, then it must be $v(u) > 0$, and for every free variable x in ψ'' it must be $0 \leq v(x) \leq v(u)$. Let $M_{ij} = \sum_{r=1}^{n_{ij}} k_r^{ij} + \sum_{s=1}^{m_{ij}} h_s^{ij}$ and let $s_{i,j}$ be such that $2^{s_{i,j}} \geq M_{i,j}$. Define inductively $d_2^n(x)$ by $d_2^0(x) = x$ and $d_2^{n+1}(x) = d_2(d_2^n(x))$. Let for every variable x, $x^* = x$ if $x \neq u$ and $x^* = 1$ if $x = u$, and let $L_{ij}^* = \bigoplus_{r=1}^{n_{ij}} k_r^{ij} d_2^{s_{i,j}}(x_r^*) \lhd \bigoplus_{s=1}^{m_{ij}} h_s^{ij} d_2^{s_{i,j}}(y_s^*)$. Then, if v is a valuation in $\Gamma(\mathbf{G}(u_0), u_0)$, we have that

$$\begin{aligned}\Gamma(\mathbf{G}(u_0), u_0), v \models \psi \text{ iff } & \mathbf{G}, v \models \psi' \\ \text{iff } & \mathbf{G}, v \models \psi'' \\ \text{iff } & \Gamma(\mathbf{G}(u_0), u_0), v \models \bigvee_{i=1}^{k} \bigwedge_{j=1}^{h_i} L_{ij}^*.\end{aligned}$$

Hence, $\bigvee_{i=1}^{k} \bigwedge_{j=1}^{h_i} L_{ij}^*$ is a quantifier-free formula equivalent to ψ in all totally ordered divisible MV-algebras. Consequently, we have the following theorem.

Theorem 3.4 *There is an explicit and easy quantifier elimination algorithm for* DMV *in the language $\langle \oplus, \sim, 0, <, d_2 \rangle$.*

4 Reductions of Quantifier-Free Formulas in an Ordinal Sum of Totally Ordered Wajsberg Hoops

In the present section, we prepare the ground for a general investigation on quantifier elimination for classes of BL-algebras, possibly with additional operators, that will be carried out in the following sections. To this purpose, we will establish some general reductions of formulas in the language of BL-algebras that are valid for *any class of* BL*-chains*. We define

$x \leq y := x \Rightarrow y = 1$;

$x < y := x \leq y \wedge \neg(x = y)$;

$x \ll y := (y = 1 \wedge x < 1) \vee (y < 1 \wedge (y \Rightarrow x) \Rightarrow x = 1)$; this means $x < y$ and either $y = 1$ or x and y are in different Wajsberg components;

$x \equiv y := \neg(x \ll y \vee y \ll x)$; that means either $x = y = 1$ or $x, y < 1$ and x and y are in the same Wajsberg component;

$x \prec y := x < y \wedge x \equiv y$.

Note that if x, y are elements of a BL-chain, then exactly one of $x \ll y$, $y \ll x$, $x = y$, $x \prec y$, or $y \prec x$ holds. Moreover, in every BL-chain we have $x \leq y$ if and only if either $x = y$ or $x \prec y$ or $x \ll y$, and $x < y$ if either $x \prec y$ or $x \ll y$.

Let φ be any quantifier-free formula in the language of BL-algebras. Let $T(\varphi)$ be the set of all terms occurring in φ (including subterms) plus 0 and 1. Then φ is a Boolean combination of formulas of the form $t = s$ with $t, s \in T(\varphi)$. Since $\neg(u = t)$ is provably equivalent to $u \ll t \vee u \prec t \vee t \prec u \vee t \ll u$, we can write φ as a disjunction of conjunctions of formulas of the form $s \ll t$, $s \prec t$, and $s = t$ with $s, t \in T(\varphi)$. A conjunction C of formulas as shown above is said to be *φ-complete* if for every pair $s, t \in T(\varphi)$ exactly one of the formulas $s \ll t$, $s \prec t$, and $s = t$ is a subformula of C. Moreover, C is said to be *satisfiable* if it can be satisfied in some BL-chain. It is clear that φ is provably equivalent to the disjunction of all φ-complete and satisfiable conjunctions C that imply φ in the theory of BL-chains. A similar observation holds if we consider BL-chains with additional operators.

Theorem 4.1 *There is a nondeterministic polynomial time algorithm for deciding, given a φ-complete conjunction C, if it is satisfiable and if it implies φ in the class of* BL*-algebras that are ordinal sums of* MV*-chains.*

Proof First of all, we write every term in φ in a language with $\cdot, \Rightarrow, 0, 1$ only. Then we write φ as a disjunction of conjunctions $C_i : i = 1, \ldots, n$ of formulas of the form $t = u$ or $t \ll u$ or $t \prec u$, where t, u are terms. Clearly, C implies φ if and only if either C is unsatisfiable or there is an i such that every conjunct in C_i is also a conjunct in C. Hence, it remains to produce an NP-algorithm to check the satisfiability of C. The procedure is similar to the procedure used in [3] to check NP-completeness of BL. We first verify *external satisfiability*. This means the following: let, for $\lhd \in \{=, \ll, \prec, \equiv\}$, $t \lhd_C u$ if and only if $t \lhd u$ is a conjunct in C. Moreover, let $t \equiv_C u$ if and only if either $t =_C u$ or $t \prec_C u$ or $u \prec_C t$ holds, $t \leq_C u$ if and only if either $t =_C u$ or $t \prec_C u$ or $t \ll_C u$ holds, and $t <_C u$ if and only if either $t \ll_C u$ or $t \prec_C u$ holds. Then external satisfiability means that the following conditions hold.

(a) $\ll_C$ is transitive and irreflexive.

(b) The relations $\equiv_C$ and $=_C$ are equivalence relations on $T(\varphi)$. Moreover, if $t \equiv_C 1$, then $t =_C 1$.
(c) $\prec_C$ is a transitive and irreflexive relation on every equivalence class with respect to $\equiv_C$.
(d) If either $t \equiv_C u$ and $u \ll_C w$, or $u \equiv_C w$ and $t \ll_C u$, then $t \ll_C w$.
(e) For all $t, u \in T(\varphi)$, exactly one of $t \ll_C u$, $u \ll_C t$, $u =_C t$, $t \prec_C u$ or $u \prec_C t$ holds. Therefore, exactly one of $t \equiv_C u$, $u \ll_C t$, or $t \ll_C u$ holds, exactly one of $t <_C u$, $t =_C u$, or $u <_C t$ holds, and if $t \equiv_C u$, then exactly one of $t =_C u$, $t \prec_C u$, or $u \prec_C t$ holds.
(f) $=_C$ is a congruence relation; that is, if $t =_C u$ and $t' =_C u'$, then $t \cdot t' =_C u \cdot u'$ and $t \Rightarrow t' =_C u \Rightarrow u'$.
(g) If $t \equiv_C u$, then $t \cdot u \equiv_C u$. Moreover, $\equiv_C$ is compatible with $\cdot$; that is, if $t \equiv_C u$ and $t' \equiv_C u'$, then $t \cdot t' \equiv_C u \cdot u'$.
(h) For all $t \in T(\varphi)$, we have $0 \leq_C t \leq_C 1$.
(i) If $t \ll_C u$, then $t \cdot u =_C t$, $u \cdot t =_C t$, and $u \Rightarrow t =_C t$.
(j) If $t \leq_C u$, then $t \Rightarrow u =_C 1$, and if $u <_C t$, then $t \Rightarrow u \equiv_C u$.
(k) $w \cdot t \ll_C u$ iff at least one of $w \ll_C u$ or $t \ll_C u$ holds.
(l) $u \ll_C w \cdot t$ iff both $u \ll_C w$ and $u \ll_C t$ hold.
(m) $w \Rightarrow t \ll_C u$ iff $t <_C w$ and $t \ll_C u$.
(n) $u \ll_C w \Rightarrow t$ iff either $u \ll_C 1$ and $w \leq_C t$, or $u \ll_C t$ and $t <_C w$.
(o) $0 \cdot t =_C t \cdot 0 =_C 0$, $1 \cdot t =_C t \cdot 1 =_C t$, $0 \Rightarrow t =_C u \Rightarrow 1 =_C 1$, and $1 \Rightarrow t =_C t$.
(p) $0 \ll_C 1$.

If any condition among (a), . . . ,(p) fails, then C is unsatisfiable. Otherwise, we start reducing atomic formulas according to conditions (i), . . . ,(o) listed above. Each of these clauses says that whenever a certain condition is satisfied, then a relation of the form $w =_C t$ holds, where t is a proper subterm of w. Then we can reduce w to t. Consequently,

(i) if $t \ll_C u$, then we reduce $t \cdot u$ and $u \cdot t$ to t, and $u \Rightarrow t$ to t;
(j) if $t \leq_C u$, then we reduce $t \Rightarrow u$ to 1;
(k) if $w \cdot t \ll_C u$, then one of $w \ll_C u$ or $t \ll_C u$ (or both) must hold; accordingly, we reduce the formula $w \cdot t \ll u$ to $w \ll u$ or to $t \ll u$ (or to their conjunction);
(l) if $u \ll_C w \cdot t$, then we reduce the formula $u \ll w \cdot t$ to $u \ll w \wedge u \ll t$;
(m) if $w \Rightarrow t \ll_C u$, then one of $t \ll_C w$ or $t \prec_C w$ must hold, and $t \ll_C u$ must hold, too; accordingly, we reduce $w \Rightarrow t \ll u$ to $t \ll w \wedge t \ll u$ or to $t \prec w \wedge t \ll u$;
(n) if $u \ll_C w \Rightarrow t$, then at least one of the following pairs of conditions must hold: $\{u \ll_C 1, w \ll_C t\}$, or $\{u \ll_C 1, w \prec_C t\}$, or $\{u \ll_C 1, w =_C t\}$, or $\{u \ll_C t, t \ll_C w\}$, or $\{u \ll_C t, t \prec_C w\}$; accordingly, we reduce $u \ll w \Rightarrow t$ to $u \ll 1 \wedge w \ll t$, or to $u \ll 1 \wedge w \prec t$, or to $u \ll 1 \wedge w = t$, or to $u \ll t \wedge t \ll w$, or to $u \ll t \wedge t \prec w$.
(o) we reduce every term of the form $0 \cdot t$ or $t \cdot 0$ to 0, every term of the form $1 \cdot t$ or $t \cdot 1$ or $1 \Rightarrow t$ to t, and every term of the form $0 \Rightarrow t$ or $u \Rightarrow 1$ to 1.

We perform the above-mentioned reductions iteratively until no reduction is possible. Each reduction simplifies some term. Therefore, the reduction procedure eventually ends, and we obtain a conjunction C' that is equivalent to C and in which no term or subformula can be reduced. If C is a complete and satisfiable φ-conjunction and C implies φ, then we call such a reduced formula C' *an irreducible φ-conjunction*. Thus, φ is provably equivalent to the disjunction of all irreducible φ-conjunctions (if there is no irreducible φ-conjunction, then φ is equivalent to $\bot$).

Before we continue with the proof of the theorem, we need the following.

Lemma 4.2 *Every irreducible φ-conjunction C' satisfies the following conditions.*

1. *If $w \ll_{C'} t$, that is, if $w \ll t$ is a subformula of C', then w and t must be either variables or constants.*
2. *If $w \equiv_{C'} t$ and u, v are subterms of w or of t, then $u \equiv_C v$, and, therefore, exactly one of $u = v$ or $v \prec u$ or $u \prec v$ is a subformula of C'.*

Proof (1) If $w \ll_{C'} t$ and either w or t is not atomic, then the formula $w \ll t$ can be reduced according to one of conditions (j), (l), (m), and (n).

(2) If (2) does not hold, then there is either a subterm u of w that is not equivalent to w or a subterm v of t that is not equivalent to t with respect to $\equiv_{C'}$. Suppose, for instance, that u is a subterm of w that is not equivalent to w, and take u of maximum complexity with this property. Then, since u cannot be equal to w, u is a proper subterm of w, and so there is a subterm v of w which has one of the forms $v' \cdot u$, or $u \cdot v'$, or $v' \Rightarrow u$, or $u \Rightarrow v'$. Moreover, by the maximality of u, we have $v \equiv_{C'} w$.

If either $v = v' \cdot u$ or $v = u \cdot v'$, then, recalling that u is not equivalent to v, we must have $v \equiv_C v' \ll_{C'} u$, and $u \cdot v'$ (or $v' \cdot u$) can be reduced to v', which is impossible as C' is irreducible. If $v = v' \Rightarrow u$, then we must have $v' \leq_{C'} u$, because $u <_{C'} v'$ would imply $v \equiv_{C'} u$. But in this case $v' \Rightarrow u$ would be reduced to 1, and C' would not be irreducible. Finally, if $v = u \Rightarrow v'$, then we must have either $u \leq_{C'} v'$ or $v' \ll_C u$ because $v' \prec_{C'} u$ would imply $v \equiv_{C'} u$. But if $u \leq_{C'} v'$, then $u \Rightarrow v'$ would be reduced to 1 and if $v' \ll_{C'} u$, then $u \Rightarrow v'$ would be reduced to w. In any case, C' would not be irreducible. This ends the proof of Lemma 4.2. □

We conclude the proof of Theorem 4.1. By Lemma 4.2, we obtain a formula C' equivalent to C that is a conjunction of formulas of the form $w \ll t$ with w, t variables or constants and of formulas of the form $w \prec t$ or $w = t$ such that $u \equiv_{C'} v$ whenever u and v are subterms of t or of w. Note that C' can be computed from C in polynomial time. Let α be an equivalence class of the set of all terms in C' with respect to $\equiv_{C'}$. Then α is closed under taking subterms and represents a set of terms which are evaluated in the same component. Let C_α be the conjunction of all formulas $t \lhd u$ such that $t, u \in \alpha$, $\lhd \in \{\prec, =\}$ and $t \lhd_{C'} u$. Then, along the lines of [3], we can see that C is satisfiable if and only if C' is satisfiable if and only if every C_α is satisfiable in the class of MV-chains (with $<$ in place of $\prec$). Since satisfiability in the class of MV-chains is NP-complete, Theorem 4.1 follows. □

As mentioned in the introduction, the theory of BL-chains does not have quantifier elimination. Indeed, if such a theory had quantifier elimination, then it would be model-complete and hence complete, since it has a minimal model, the two element BL-chain. But there are many sentences which are neither provable nor disprovable in the theory of BL-chains. Here are some examples:

1. The formula $\exists x_1 \ldots \exists x_n (x_1 \ll x_2 \wedge \cdots \wedge x_{n-1} \ll x_n \wedge x_n \ll 1)$ expressing that there are at least n Wajsberg components.

2. The formula $\forall x \exists y(y \leq x \wedge y^2 = y \wedge ((x \Rightarrow y) \Rightarrow y) = x)$. We will see that this formula expresses that every Wajsberg component is an MV-algebra.
3. The formula $\forall x \forall y(x \ll y \to \exists z(x \ll z \wedge z \ll y))$, expressing the fact that the index set in the ordinal sum is dense.
4. The formula $\forall x \exists y(y = y^{n-1} \Rightarrow x)$. We will see that this formula expresses that every element has a largest nth root.

In the next sections we will consider some special theories of classes of BL-algebras, possibly with additional operators, which have quantifier elimination. These theories are suggested by the examples mentioned above.

5 Ordinal Sums of Divisible MV-Chains with Infinite and Discretely Ordered Index Set

We are going to investigate the theory, denoted DBL_∞, of ordinal sums of divisible MV-chains where the index set is discretely ordered with minimum, but without maximum; that is, every element of the index set has an immediate successor, and every element different from the minimum has an immediate predecessor.

Definition 5.1 DBL_∞ is the first-order theory in the language of BL-chains that is axiomatized as follows.

1. Axioms of BL-chains.
2. For every $n > 1$, the n-root axiom $\forall x \exists y \left(y^{n-1} \Rightarrow x = y\right)$.
3. The axiom $\forall x \exists y((y \leq x) \wedge (y^2 = y) \wedge ((x \Rightarrow y) \Rightarrow y = x))$.
4. The axiom $\forall x(x \ll 1 \to \exists y(x \ll y \wedge y \ll 1 \wedge \forall z \neg(x \ll z \wedge z \ll y)))$.
5. The axiom $\forall x((0 \ll x \wedge x \ll 1) \to \exists y(y \ll x \wedge \forall z \neg(y \ll z \wedge z \ll x))$.

The models of DBL_∞ will be called DBL_∞*-algebras*.

Axiom (3) says that every element of the algebra is in an MV-component. Indeed, if (3) holds, then for every x there is an idempotent $y \leq x$ such that $(x \Rightarrow y) \Rightarrow y = x$. Now, y must be in the same component as x. This is obvious if $x = 1$, because 1 belongs to every component. If $x < 1$, then y cannot be in a lower component; otherwise, $x \Rightarrow y = y$ and $(x \Rightarrow y) \Rightarrow y = 1$. If $x < 1$ were in a cancellative component, then such component would not have any idempotent except for 1, and since $y \leq x < 1$ and y is an idempotent, y would belong to a lower component, which has been excluded before.

Axioms (4) and (5) say that the order of components is discrete and has no maximum. Indeed, axiom (4) says that every element in the index set of the ordinal sum has a successor, and axiom (5) says that every element in the index set, except for the minimum, has a predecessor.

Finally, we discuss axiom (2). Let y be such that $y^{k-1} \Rightarrow x = y$. If $x = 1$, then $y = 1$. If $x < 1$, then we must have $y > x$; otherwise, $x \geq y = y^{k-1} \Rightarrow x = 1$. Moreover, x and y must belong to the same component; otherwise, $x \ll y$ and $y^{k-1} \Rightarrow x = x \ll y$. By residuation, $y^k = y \cdot y^{k-1} \leq x$ and if $z > y$, then $z^k \geq z \cdot y^{k-1} > x$. Finally, $y^k = y^{k-1} \cdot y = y^{k-1} \cdot (y^{k-1} \Rightarrow x) = y^{k-1} \sqcap x = x$. In other words, y is the greatest kth root of x.

We now prove that every component of a DBL_∞-algebra is a divisible MV-algebra. Since every component is an MV-algebra, and since every element and its kth root belong to the same component, the claim follows from the next lemma.

Lemma 5.2 *Let* $\mathbf{A}$ *be an* MV*-algebra such that for every* $k > 1$ *every element* x *of* A *has a maximum* k*th root* $r_k(x)$*. Then* $\mathbf{A}$ *is divisible, with division operators given by* $d_k(x) = r_k(x) \ominus r_k(0)$*. Conversely, if* $\mathbf{A}$ *is a divisible* MV*-algebra, then every element* x *of* $\mathbf{A}$ *has a maximum* k*th root given by* $r_k(x) = d_k(1) \Rightarrow d_k(x)$.

Proof Since being the maximum kth root of x and being equal to x divided by k can be expressed by equations, it suffices to verify the claims in $[0, 1]_{\mathrm{MV}}$. This can be checked by a straightforward computation which is left to the reader. □

Since the ordinal sum of ω copies of $[0, 1]_{\mathrm{MV}}$ is a DBL_∞-algebra and generates the variety of BL-algebras, we obtain the following proposition.

Proposition 5.3 *The class of* DBL_∞*-algebras generates the whole variety of* BL*-algebras.*

In fact, we can say even more: every DBL_∞-algebra has infinitely many components, and each component, being a divisible MV-algebra, contains any finite MV-algebra as a subalgebra. Moreover, every finite BL-chain is an ordinal sum of finitely many finite MV-chains [1], and each of them embeds into any divisible MV-algebra. Hence, every finite BL-chain embeds into any DBL_∞-algebra. Since the variety of BL-algebras is generated by the class of finite BL-chains, we obtain the following.

Proposition 5.4 *Any* DBL_∞*-algebra generates the whole variety of* BL*-algebras.*

We will now see that DBL_∞ does not enjoy quantifier elimination (it is not even model-complete), but we will show that some finitely axiomatizable extension by definitions of DBL_∞ does have quantifier elimination.

Lemma 5.5 DBL_∞ *is not model-complete.*

Proof Let, for every natural number n, $\mathbf{A}_n$ be an isomorphic copy of $[0, 1]_{\mathrm{MV}}$, and consider the ordinal sums $\mathbf{B} = \bigoplus_{n\in\omega} \mathbf{A}_n$ and $\mathbf{C} = \bigoplus_{n\in\omega, n\neq 1} \mathbf{A}_n$. Then $\mathbf{C}$ is a submodel of $\mathbf{B}$ and both are models of DBL_∞, but $\mathbf{C}$ is not an elementary substructure of $\mathbf{B}$: if c is the bottom element of $\mathbf{A}_2$, then the formula $\exists x(0 \ll x \wedge x \ll c)$ is true in $\mathbf{B}$ but not in $\mathbf{C}$. □

We are going to introduce a suitable extension by definitions of DBL_∞, called DBL_∞^+, that does have QE.

Definition 5.6 The theory DBL_∞^+ is the theory in the language of BL-chains, expanded with the unary function symbols s, p, r_2, and *, and whose axioms are those of DBL_∞ plus the following ones:

1. $(r_2(x) = y) \leftrightarrow (y \Rightarrow x = y)$;
2. $x^* = y \leftrightarrow (y^2 = y \wedge y \leq x \wedge ((x \Rightarrow y) \Rightarrow y = x))$;
3. $s(x) = y$
   $\leftrightarrow ((x = 1 \wedge y = 1) \vee (x \ll y \wedge y^2 = y \wedge \neg\exists z(x < z \wedge z < y \wedge z^2 = z))$;
4. $p(x) = y$
   $\leftrightarrow ((x \equiv 0 \wedge y = 0) \vee (x = 1 \wedge y = 1) \vee (y < x \wedge y^2 =$
   $y \wedge \neg\exists z(x < z \wedge z < y \wedge z^2 = z)))$.

The models of DBL_∞^+ will be called DBL_∞^+*-algebras*.

Theorem 5.7 *The theory* DBL_∞^+ *has* QE *in the language* $\langle\cdot, \Rightarrow, 0, 1, <, s, p, r_2, ^*\rangle$.

Proof The quantifier elimination procedure is the following: we eliminate $\exists x$ in formulas of the form $\exists x\varphi(x, y_1, \ldots, y_n)$ with φ quantifier-free. Let k be the maximum complexity of all terms in φ, and let $T^+(\varphi)$ be the set of all terms of complexity $\leq k$ whose variables occur in φ. Note that $T^+(\varphi)$ is finite. Let us say that a conjunction C of formulas of the form $t = u$ or $t \prec u$ or $t \ll u$ with $t, u \in T^+(\varphi)$ is φ^+*-complete* if, for all $t, u \in T^+(\varphi)$, exactly one of $t = u$ or $t \prec u$ or $t \ll u$ or $u \prec t$ or $u \ll t$ is a conjunct in C. Clearly, we can restrict ourselves to quantifier elimination in φ^+-complete and satisfiable conjunctions C which imply φ. Relations $=_C, \prec_C, \equiv_C$ and so on are defined as in Section 4.

First of all, if $x =_C 1$, then $\exists x C$ reduces to $C(1)$, and the procedure terminates. Hence, we will suppose that $x \ll_C 1$. We start reducing terms in $T^+(\varphi)$. Besides reductions (i), …,(o) as in the proof of Theorem 4.1, we perform the following reductions.

(q) We replace every term of the form $s(p(t))$ by t^* if $0 \ll_C t$ and by $s(0)$ otherwise.

(r) We replace every term of the form $s(t^*)$ or $s(r_2(t))$ by $s(t)$.

(s) We replace every term of the form $p(s(t))$ or $(t^*)^*$ by t^*, and every term of the form $p(r_2(t))$ or $p(t^*)$ by $p(t)$.

(t) We replace every term of the form $(s(t))^*$ ($(p(t))^*$, respectively) by $s(t)$ ($p(t)$, respectively) and every term of the form $(r_2(t))^*$ by t^*.

(u) We replace every term of the form $s(t * u)$ ($p(t * u)$, $(t * u)^*$, respectively) as follows: if $t \ll_C u$, then we replace it by $s(t)$ ($p(t)$, t^*, respectively); otherwise, we replace it by $s(u)$ ($p(u)$, u^*, respectively). Moreover, we replace $p(0)$ and 0^* by 0, and $p(1)$, $s(1)$ and 1^* by 1.

(v) We replace every term of the form $s(t \Rightarrow u)$ ($p(t \Rightarrow u)$, $(t \Rightarrow u)^*$, respectively) by 1 if $t \leq_C u$, and by $s(u)$ ($p(u)$, u^*, respectively) if $u <_C t$.

Note that the reduced terms are all in $T^+(\varphi)$. After these reductions are performed, every term of the form $s(t)$ is of the form $s^n(z)$, where z is a variable or 0, every term of the form $p(t)$ is of the form $p^n(z)$, where z is a variable, and every term of the form t^* has the form z^*, where z is variable. Terms of the form $s^n(z)$, $s^n(0)$, $p^n(z)$, or z^*, where z is a variable, will be called *generalized atoms*.

Now we simplify formulas of the form $t \ll u$ by means of the following reductions.

(w) Any formula of the form $r_2(u) \ll t$ or of the form $u \ll r_2(t)$ can be reduced to $u \ll t$.

(x) Any formula of the form $s^h(x) \ll u$ can be reduced to $x \ll p^h(u)$, and any formula of the form $u \ll s^h(x)$ can be reduced to

$$(u \ll s^h(0)) \vee (s^{h-1}(0) \ll u \wedge p^h(u) \ll x).$$

(y) Any formula of the form $p^h(x) \ll u$ can be reduced to

$$\big(x \ll s^{h+1}(0) \wedge 0 \ll u\big) \vee ((s^h(0) \ll x) \wedge (x \ll s^h(u))),$$

and any formula of the form $u \ll p^h(x)$ can be reduced to $s^h(u) \ll x$.

When doing reductions (k), …,(y) we may introduce disjunctions. In this case, we rewrite the obtained formula in disjunctive normal form and we eliminate $\exists x$ in each disjunct as explained below.

Along the lines of Lemma 4.2, and using in addition reductions (p), …,(y), we reduce the quantifier elimination problem to the elimination of $\exists x$ in formulas $\exists x C'$ where C' is a conjunction of formulas of one of the following forms.

1. $u \ll t$, where u and t are either generalized atoms or atomic terms, and one of them is x.
2. $u \prec t$, where the following conditions are satisfied:
   (i) at least one of u, t contains x;
   (ii) if v, w are subterms of u or of t which do not occur under the scope of s or of p or of *, then we must have $v \equiv_C w$. For instance, if $u = x \Rightarrow s(y)$, $t = s(y) \Rightarrow p(z)$ and $u \prec_{C'} t$, then we must have $x \equiv_{C'} s(y) \equiv_{C'} p(z) \equiv_{C'} x \Rightarrow s(y) \equiv_{C'} s(y) \Rightarrow p(z)$, but we need not (and we cannot) have $x \equiv_{C'} y$ or $y \equiv_{C'} z$.
3. $u = t$, where conditions (i) and (ii) in (2) are satisfied.

Let $C^{\ll}$ be the conjunction of all formulas of the form $u \ll t$ such that $u \ll_{C'} t$, and let $\alpha_1, \ldots, \alpha_k$ be the equivalence classes of terms in $T^+(\varphi)$ with respect to $\equiv_{C'}$. Let, for $i = 1, \ldots, k$, $C_i^{\equiv}$ be the conjunction of all formulas of the form $t \prec u$ ($t = u$, respectively) such that $t, u \in \alpha_i$ and $t \prec_{C'} u$ ($t =_{C'} u$, respectively), and let $C^{\equiv} = C_1^{\equiv} \wedge \cdots \wedge C_k^{\equiv}$. Then, since $C^{\ll}$ and $C^{\equiv}$ are mutually independent (the former is about the order of components, the latter is about the internal structure of each component), we can eliminate $\exists x$ in $\exists x C^{\equiv}$ and in $\exists x C^{\ll}$ independently.

We start from $\exists x C^{\equiv}$. Replace in $C^{\equiv}$ every occurrence of a term of the form t^* or $s(t)$ or $p(t)$ that is not under the scope of s or of p or of * by 0 (thus, e.g., in the formula $s^2(v) \prec w$ we replace $s^2(v)$ by 0, and we do *not* replace $s(v)$ by 0, and so the reduced formula is $0 \prec w$ and not $s(0) \prec w$). Moreover, replace $\prec$ by $<$, thus obtaining a formula $D_i^{\equiv}$ in the language of divisible MV-algebras in which $t \Rightarrow s$ is an abbreviation for $\sim t \oplus s$, $t \cdot s$ is an abbreviation for $\sim(\sim t \oplus \sim s)$, and $r_2(t)$ is an abbreviation for $d_2(1) \oplus d_2(t)$. Let D_i be the formula obtained from $C_i^{\equiv}$ by this replacement. We distinguish several cases.

Case 1 If D_i has no free variables, then it is equivalent to $\top$ or to $\perp$, and we replace it by $E_i = \top$ or by $E_i = \perp$ accordingly.

Case 2 If x occurs in D_i, then x does not occur under the scope of s or of p or of * in $C_i^{\equiv}$ and this is only possible if $x \in \alpha_i$. Then we eliminate $\exists x$ in $\exists x(D_i \wedge x < 1)$ (recall we are assuming $x \ll_C 1$) according to the quantifier elimination procedure in divisible MV-chains, thus obtaining a formula F_i without x. If F_i has no free variables, then it is equivalent to $\top$ or to $\perp$, and we replace F_i by $E_i = \top$ or by $E_i = \perp$ accordingly. If, say, y is free in F_i, then we replace 0 in F_i by y^*, $\sim t$ by $t \Rightarrow y^*$, $t \oplus u$ by $(t \Rightarrow y^*) \Rightarrow u$, and $d_2(t)$ by $(r_2(t) \Rightarrow r_2(y^*)) \Rightarrow y^*$, and we denote by E_i the formula obtained in this way.

Case 3 If x does not occur in D_i but D_i has at least one free variable, say y, then we denote by E_i the formula obtained by replacing in D_i the constant 0 by y^*, $\sim t$ by $t \Rightarrow y^*$, $t \oplus u$ by $(t \Rightarrow y^*) \Rightarrow u$, and $d_2(t)$ by $(r_2(t) \Rightarrow r_2(y^*)) \Rightarrow y^*$. Then it is easily seen that the formula $E^{\equiv} = \bigwedge_{i=1}^n E_i$ is quantifier-free and is equivalent to $\exists x C^{\equiv}$.

We now reduce $\exists x C^{\ll}(x)$. Note that $C^{\ll}(x)$ has either the form (a) $\bigwedge_{i=1}^n (x \ll t_i)$ or the form (b) $\bigwedge_{j=1}^m (u_j \ll x)$ or the form (c) $\bigwedge_{i=1}^n (x \ll t_i) \wedge \bigwedge_{j=1}^m (u_j \ll x)$. Then in case (a), $\exists x C^{\ll}$ reduces to $\bigwedge_{i=1}^n (0 \ll t_i)$; in case (b), $\exists x C^{\ll}$ reduces to

$\bigwedge_{j=1}^{m}(u_j \ll 1)$; and in case (c), $\exists x C^{\ll}$ reduces to $\bigwedge_{i=1}^{n}\bigwedge_{j=1}^{m}(s(u_j) \ll t_j)$. Let $E^{\ll}$ be the reduced formula. Then $\exists x C$ can be reduced to $E^{\equiv} \wedge E^{\ll}$, and the proof is finished. □

Remark 5.8 In order to prove that the procedure shown above is really an algorithm, we need to prove that even in the extended language, there is an algorithm to check satisfiability of a φ^+-complete conjunction. This is shown in the next theorem.

Theorem 5.9 *There is an* NP*-algorithm to check if a φ^+-complete conjunction is satisfiable or not.*

Proof The proof is similar to the proof of Theorem 4.1. The main differences are the following.

1. For external satisfiability we have to add the following clauses:
   (1a) if $t \ll_{C'} 1$, then we must have $t \ll_{C'} s(t)$ (if $s(t) \in T^+(\varphi)$, of course), and there cannot be a term u such that $t \ll_{C'} u \ll_{C'} s(t)$;
   (1b) if $0 \ll_{C'} t \ll_{C'} 1$, then we must have $p(t) \ll_{C'} t$ and there cannot be a term u such that $p(t) \ll_{C'} u \ll_{C'} s(t)$;
   (1c) if either $w =_{C'} s(t)$ or $w =_{C'} p(t)$ or $w =_{C'} t^*$, then for any $t \in T^+(\varphi)$, we cannot have $t \prec_{C'} w$;
   (1d) $t^* \equiv_{C'} t, r_2(t) \equiv_{C'} t$, and if $t \equiv_{C'} u$, then $s(t) =_{C'} s(u)$, $p(t) =_{C'} p(u)$, and $t^* =_{C'} u^*$.
2. To the reductions (i), . . . ,(o), we have to add reductions (q), . . . ,(v).
3. After we have constructed conjunctions C_α as in the proof of Theorem 4.1, we have to replace every term of the form $s(t)$ or $p(t)$ or t^* by 0 before checking satisfiability of C_α in the class of divisible MV-algebras. □

Corollary 5.10 *The theory* DBL_∞ *is complete and decidable.*

Proof Completeness of DBL_∞ is proved as follows. Let DBL_∞^+ be the extension of DBL_∞ by the operators s, p, r_2, and $*$ together with their defining axioms. Then DBL_∞^+ is model-complete. Moreover, in any model of DBL_∞^+ the order of components must contain an initial segment isomorphic to the ordered set ω of natural numbers. Also, every component, being a divisible MV-algebra, must contain the subalgebra $[0,1]_{\mathbb{Q}}$ of $[0,1]_{\mathrm{MV}}$ with domain the rational numbers in $[0, 1]$. Thus, the ordinal sum of ω copies of $[0,1]_{\mathbb{Q}}$ is a minimal model of DBL_∞^+, and since DBL_∞^+ is model-complete, it is also a prime model. It follows that DBL_∞^+ is complete. Finally, since DBL_∞^+ is an extension by definitions of DBL_∞, DBL_∞ is in turn complete.

The decidability of DBL_∞ follows from the fact that it is complete and recursively axiomatizable, but we will present a more algorithmic proof. After the quantifier elimination procedure, every sentence of DBL_∞^+ (then, a fortiori, every sentence of DBL_∞) becomes equivalent to a Boolean combination of equalities between closed terms of DBL_∞^+. These terms are either equal to 1 or have the form $qs^h(0)$ for some rational $q \in (0, 1]$ and for some natural number h, where we set $qs^0(0) = q0 = 0$. The operations are as follows: first of all, for every closed term t, $1 \cdot t = t \cdot 1 = 1 \Rightarrow t = t$ and $t \Rightarrow 1 = 1$. Moreover, for $h, k \in \omega$, if $h = k$, we have

$$qs^h(0) \cdot rs^k(0) = \begin{cases} 0 & \text{if } h = k = 0 \\ s^{h-1}(0) & \text{if } h = k > 0 \text{ and } q + r \le 1 \\ (q + r - 1)s^h(0) & \text{otherwise} \end{cases}$$

$$qs^h(0) \Rightarrow rs^k(0) = \begin{cases} 1 & \text{if } q \le r \\ (1-q+r)s^h(0) & \text{otherwise.} \end{cases}$$

If $h < k$, then $qs^h(0) \cdot rs^k(0) = qs^h(0)$, and $qs^h(0) \Rightarrow rs^k(0) = 1$. If $k < h$, then $qs^h(0) \cdot rs^k(0) = rs^k(0)$, and $qs^h(0) \Rightarrow rs^k(0) = rs^k(0)$. Moreover,

$$r_2(qs^h(0)) = \begin{cases} \frac{1}{2}s^{h+1}(0) & \text{if } q = 1 \\ \frac{q+1}{2}s^h(0) & \text{otherwise} \end{cases}$$

$$(qs^h(0))^* = \begin{cases} 0 & \text{if } h = 0 \\ s^{h-1}(0) & \text{otherwise} \end{cases}$$

$$s(qs^h(0)) = \begin{cases} s^{h+1}(0) & \text{if } q = 1 \\ s^h(0) & \text{otherwise} \end{cases}$$

$$p(qs^h(0)) = \begin{cases} 0 & \text{if } h \le 1 \\ s^{h-1}(0) & \text{if } h > 1 \text{ and } q = 1 \\ s^{h-2}(0) & \text{otherwise.} \end{cases}$$

It is clear that equality between terms of the form shown above is decidable, and the decidability of DBL_∞^+ and of DBL_∞ immediately follows. □

6 Ordinal Sums of Finitely Many Divisible MV-Chains

Definition 6.1 A divisible BL-algebra with n MV-components (nDBL-*algebra* for short) is a BL-chain which is the ordinal sum of n divisible MV-algebras. In the sequel, nDBL will denote the theory of nDBL-algebras, that is, the set of all first-order formulas valid in all nDBL-algebras.

The class of nDBL-algebras is not a variety, but it is an elementary class. Indeed, it is axiomatized by axioms (1), (2), and (3) of DBL_∞ plus the axiom

(4′) $\exists x_1 \ldots \exists x_n \left(\bigwedge_{i=1}^n (x_i \ll x_{i+1}) \wedge (x_n \ll 1) \wedge \forall z \left(\bigvee_{i=1}^n (z \equiv x_i) \vee z = 1\right)\right)$,

stating the existence of exactly n components.

The class of nDBL-algebras does not generate the whole variety of BL-algebras: from [2] it easily follows that the variety $\mathcal{BL}_n$ generated by the class of nDBL-algebras is the class of all subdirect products of BL-chains with n components at most. However, every n-generated BL-algebra belongs to $\mathcal{BL}_n$. It follows that a BL-formula in n propositional variables is provable if and only if it holds in all nDBL-algebras. Finally, again from [2], it follows that any nDBL-algebra generates the whole variety $\mathcal{BL}_n$.

We now investigate some model-theoretic properties of the class of nDBL-algebras. We start from the following theorem.

Theorem 6.2 nDBL *does not have* QE *in the language* $\langle \cdot, \Rightarrow, 0, 1, < \rangle$.

Proof It follows from [29] that a theory T has QE if and only if the following condition holds.

(◇) Suppose that $\mathbf{B}$, $\mathbf{C}$ are models of T and $\mathbf{A}$ is a substructure of $\mathbf{B}$ and of $\mathbf{C}$ (possibly, not a model of T). Then for every quantifier-free formula $\varphi(x_1, \ldots, x_n, y)$ and for every $a_1, \ldots, a_n \in \mathbf{A}$, if there is a $b \in \mathbf{B}$ such that $\mathbf{B} \models \varphi(a_1, \ldots, a_n, b)$, then there is a $c \in \mathbf{C}$ such that $\mathbf{C} \models \varphi(a_1, \ldots, a_n, c)$.

We prove that ($\diamond$) does not hold when T is nDBL with $n > 2$. Suppose for simplicity $n = 3$ (the proof can be easily extended for any $n > 2$). Let $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ be three copies of $[0, 1]_{\mathrm{MV}}$ such that for $i, j = 1, 2, 3$ and $i \neq j$, $\mathbf{A}_i \cap \mathbf{A}_j = \{1\}$. Let $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$, $\mathbf{B} = \mathbf{A}_1 \oplus \mathbf{A}_2 \oplus \mathbf{A}_3$, and $\mathbf{C} = \mathbf{A}_1 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$. Then $\mathbf{A}$ is a subalgebra of both $\mathbf{B}$ and $\mathbf{C}$. Now, let a be the minimum of $\mathbf{A}_2$, b the minimum of $\mathbf{A}_3$ and let $\varphi(x, y)$ be the formula $(x < y) \wedge (y < 1) \wedge \left(y^2 = y\right)$. Then $\mathbf{B} \models \varphi(a, b)$, but there is no $c \in \mathbf{C}$ such that $\mathbf{C} \models \varphi(a, c)$. A similar proof works if $n > 3$. Thus, for $n > 2$, nDBL does not enjoy QE. $\square$

We now consider an extension by definitions of nDBL that does have quantifier elimination. We add to the language of nDBL the symbol r_2, with the same intended meaning as in the case of DBL_∞ algebras, plus the constant symbols $c_1, \dots, c_{n-1}$ that denote the idempotent elements different from 0 and 1. The symbol r_2 is defined as in DBL_∞^+:

$$(r_2(x) = y) \leftrightarrow (y \Rightarrow x = y).$$

As for the new constants, even if they have a natural universal definition, for our purposes, it is more convenient to use the following (equivalent) existential definition:

$$c_i = y \leftrightarrow \exists x_1 \dots$$
$$\exists x_{n-1} \left(\bigwedge_{i=1}^{n-1} (x_i^2 = x_i) \wedge 0 \ll x_1 \wedge x_{n-1} \ll 1 \wedge \bigwedge_{i=1}^{n-2} (x_i \ll x_{i+1}) \wedge y = x_i \right).$$

Definition 6.3 The above-defined extension of nDBL will be denoted by $n\mathrm{DBL}^+$ and its models will be called $n\mathrm{DBL}^+$*-algebras*.

Theorem 6.4 *$n\mathrm{DBL}^+$ has QE in the language $\langle \cdot, \Rightarrow, 0, c_1, \dots, c_{n-1}, 1, <, r_2 \rangle$.*

Proof The proof is a simplification of the quantifier elimination proof for DBL_∞^+. The main differences are: (1) in the case of $n\mathrm{DBL}^+$, we do not have the symbols s, p, or $*$ that are replaced by the new constants, and (2) in the case of $n\mathrm{DBL}^+$ there are only n components, and so every idempotent element of an $n\mathrm{DBL}^+$-algebra belongs to the set $\{0, c_1, \dots, c_{n-1}, 1\}$. As in the case of DBL_∞^+, it is sufficient to eliminate $\exists x$ in $\exists x C$ where C is a φ^+-complete and satisfiable conjunction which implies φ (the concept of φ^+-complete conjunction is defined analogously). If $x =_C 1$, then $\exists x C$ can be reduced to $C(1)$, and the procedure terminates. If $x \ll_C 1$, then we start reducing atomic formulas. Apart from the reductions (i), ...,(o), we have to perform (w), and for the satisfiability of a φ^+-complete conjunction C, we have to verify, besides (a), ...,(o), the following conditions:

(a1) $r_2(t) \equiv_C t$,
(a2) $0 \ll_C c_1 \ll_C \cdots \ll_C c_{n-1} \ll_C 1$,
(a3) we cannot have $t \prec_C c_i$, and
(a4) for every term t, either $t =_C 1$, or $t \equiv_C 0$, or $t \equiv_C c_i$ for some i.

After all the reductions have been made, we have reduced the problem to the elimination of $\exists x$ in formulas $\exists x C'$ such that

1. if $u \ll_{C'} t$, then u and t must be either variables, or constants;
2. if $u \prec_{C'} t$ or $u =_{C'} t$, and if w, v are subterms of u or of t, then $w \equiv_C v$.

Now C' can be written as $C_0 \wedge C_1(x) \wedge C_2(x)$, where x does not occur in C_0, $C_1(x)$ has the form $\bigwedge_{j=1}^{k} w_j \lhd_j s_j$, where $\lhd_j$ is either $=$ or $\prec$ and for every subterm v of

w_j or of s_j, we have $v \equiv_C x$, and $C_2(x)$ has either the form (a) $\bigwedge_{j=1}^{r} t_j \ll x$ or the form (b) $\bigwedge_{j=1}^{s} x \ll u_j$ or the form (c) $\bigwedge_{j=1}^{r} t_j \ll x \wedge \bigwedge_{j=1}^{s} x \ll u_j$.

Let $z = 0$ if $x \equiv_{C'} 0$ and $z = c_i$ if $x \equiv_{C'} c_i$. Let $D_1(x)$ be the formula in the language of divisible MV-algebras obtained from $C_1(x)$ by replacing $t \Rightarrow s$ by $\sim t \oplus s$, $t \cdot s$ by $\sim(\sim t \oplus \sim s)$, $r_2(t)$ by $d_2(t) \oplus d_2(1)$, and z by 0. Let F_1 be the formula obtained by eliminating $\exists x$ in $\exists x(D_1(x) \wedge x < 1)$ according to the quantifier elimination procedure for divisible MV-chains. Finally, let E_1 be the result of replacing in F_1 0 by z, every term of the form $\sim t$ by $t \Rightarrow z$, every term of the form $t \oplus s$ by $(t \Rightarrow z) \Rightarrow s$, every term of the form $d_2(t)$ by $(r_2(t) \Rightarrow r_2(0)) \Rightarrow z$.

Now let E_2 be defined as follows: if $x \equiv_{C'} 0$, then just replace x by 0 in $C_2(x)$ and if $x \equiv_{C'} c_k$, then replace x by c_k in $C_2(x)$. It is readily seen that $\exists x C$ is equivalent to $C_0 \wedge E_1 \wedge E_2$. □

Corollary 6.5 *The theories nDBL$^+$ and nDBL are complete and decidable.*

Proof The theory nDBL$^+$ is model-complete and has a prime model, namely, the ordinal sum of n copies of $[0, 1]_{\mathbb{Q}}$. Hence, nDBL$^+$ is complete, and being recursively axiomatizable, it is decidable. A better decision algorithm can be obtained by means of quantifier elimination by a procedure very similar to that of Corollary 5.10 (of course, here we have to replace $s^i(0)$ by c_i). □

Remark 6.6 As in the previous cases, we have an NP-algorithm for the satisfiability of a φ-complete conjunction. The proof is similar to (and easier than) the proof for DBL$^+_\infty$.

We are going to prove that nDBL is model-complete. Note that this does not follow from Theorem 6.4: every theory has an extension by definitions that has quantifier elimination, but not all theories are model-complete.

Theorem 6.7 *nDBL is model-complete.*

Proof The claim follows from the following lemma.

Lemma 6.8 *Let T be a theory, and let T' be a finite extension by definitions of T that has* QE. *Suppose that the (finitely many) new symbols are all function symbols and that they have a definition in T by means of an existential formula. Then T is model-complete.*

Proof Let $f_1, \ldots, f_n$ be the new function symbols, and let $(f_1(x_1, \ldots, x_n) = y) \leftrightarrow \Phi_1(x_1, \ldots, x_n, y), \ldots, (f_n(x_1, \ldots, x_n) = y) \leftrightarrow \Phi_n(x_1, \ldots, x_n, y)$ be their existential definitions in T. Then $T \vdash \forall x_1 \ldots \forall x_n \exists! y \Phi_1(x_1, \ldots, x_n, y)$. Now let $\mathbf{B} \models T$ and let $\mathbf{A}$ be a substructure of $\mathbf{B}$ such that $\mathbf{A} \models T$. We need to prove that $\mathbf{A}$ is an elementary substructure of $\mathbf{B}$. Since T' is an extension by definitions of T, $\mathbf{A}$ and $\mathbf{B}$ can be uniquely extended to models $\mathbf{A}'$ and $\mathbf{B}'$ of T' in which the domains and the interpretations of symbols of T remain unchanged.

Claim A $\mathbf{A}'$ is a submodel of $\mathbf{B}'$.

Proof of Claim A Let $a_1, \ldots, a_n \in \mathbf{A}$, and let $b = f_i^{A'}(a_1, \ldots, a_n)$. Then $\mathbf{A} \models \Phi_i(a_1, \ldots, a_n, b)$, and we need to prove that $b = f_i^{B'}(a_1, \ldots, a_n)$. But $\Phi_i(x_1, \ldots, x_n, y)$ is an existential formula, and so it is preserved under extensions. Thus, $\mathbf{B} \models \Phi_i(a_1, \ldots, a_n, b)$, and $b = f_i^{B'}(a_1, \ldots, a_n)$. □

Continuing with the proof of Lemma 6.8, let $\psi(z_1, \dots, z_n)$ be a formula of T and let $a_1, \dots, a_n \in \mathbf{A}$. We need to prove that $\mathbf{A} \models \psi(a_1, \dots, a_n)$ if and only if $\mathbf{B} \models \psi(a_1, \dots, a_n)$. Now ψ is provably equivalent in T' to a quantifier-free formula ψ'. Thus, we have the following chain of equivalences: $\mathbf{A} \models \psi(a_1, \dots, a_n)$ if and only if $\mathbf{A}' \models \psi(a_1, \dots, a_n)$ if and only if $\mathbf{A}' \models \psi'(a_1, \dots, a_n)$ if and only if $\mathbf{B}' \models \psi'(a_1, \dots, a_n)$ (as $\mathbf{A}'$ is a submodel of $\mathbf{B}'$ and ψ' is quantifier-free) if and only if $\mathbf{B}' \models \psi(a_1, \dots, a_n)$ if and only if $\mathbf{B} \models \psi(a_1, \dots, a_n)$. This ends the proof of Lemma 6.8. □

Since nDBL satisfies the assumptions of Lemma 6.8, we obtain that nDBL is model-complete. □

7 BL-Chains with Dense Order of Components

In this section we present another extension of the first-order theory of BL-algebras that has quantifier elimination.

Definition 7.1 A BL-chain $\mathbf{B}$ is said to be *strongly dense* if

1. it is an ordinal sum $\bigoplus_{i \in I} \mathbf{B}_i$ of divisible MV-algebras $\mathbf{B}_i$,
2. the index set I is a densely ordered set with minimum and without a maximum.

The class of strongly dense BL-chains is less natural than the classes of DBL_∞-algebras or nDBL-algebras. Indeed, while there are standard DBL_∞-algebras (for instance, the ordinal sum of ω copies of $[0, 1]_{\mathrm{MV}}$) and standard nDBL-algebras (the ordinal sum of n copies of $[0, 1]_{\mathrm{MV}}$), there is no standard strongly dense BL-chain; that is, there is no strongly dense BL-chain having $[0, 1]$ as a lattice reduct, as shown in the next lemma. The lemma also shows that every BL-chain embeds into a strongly dense BL-chain, and, consequently, the variety generated by the class of SDBL-algebras is the whole variety of all BL-algebras.

Lemma 7.2

1. *Every* BL*-chain* $\mathbf{B}$ *embeds into a strongly dense* BL*-chain.*
2. *No standard* BL*-algebra is strongly dense.*
3. *There is a strongly dense* BL*-chain whose lattice reduct is a sublattice of* $[0, 1]$.

Proof (1) Every BL-chain $\mathbf{B}$ is the ordinal sum of a family of MV-chains and of totally ordered cancellative hoops. Every cancellative hoop embeds into the hoop reduct of a perfect MV-chain via the functor Λ^{-1}. Hence, every BL-chain embeds into an ordinal sum of a family of MV-chains.

Next, every MV-chain embeds into a divisible MV-chain. Hence, every BL-chain embeds into an ordinal sum $\bigoplus_{i \in I} \mathbf{B}_i$ of divisible MV-algebras. Now, let $I \times_{\mathrm{lex}} \mathbf{Q}^+$ be the Cartesian product of I and the nonnegative rationals $\mathbf{Q}^+$ endowed with the lexicographic order. Then $I \times_{\mathrm{lex}} \mathbf{Q}^+$ is densely ordered and has minimum $(i_0, 0)$, where $i_0 = \min(I)$, but no maximum. Now let, for all $(i, q) \in I \times \mathbf{Q}^+$,

$$\mathbf{B}_{(i,q)} = \begin{cases} \mathbf{B}_i & \text{if } q = 0 \\ [0, 1]_{\mathrm{MV}} & \text{otherwise.} \end{cases}$$

Then $\bigoplus_{(i,q) \in I \times_{\mathrm{lex}} \mathbf{Q}^+} \mathbf{B}_{(i,q)}$ is a strongly dense BL-chain in which $\mathbf{B}$ can be embedded.

(2) Let $\mathbf{C}$ be an MV-component of a standard BL-algebra, and let $m = \sup(\mathbf{C} \setminus \{1\})$. $\mathbf{C} \setminus \{1\}$ is order isomorphic to $[0, 1)$. Therefore, m is not in $\mathbf{C} \setminus \{1\}$; otherwise, it would be the maximum of a right-open interval. Then m must belong to a component $\mathbf{C}'$ above $\mathbf{C}$. Hence, $m = \sup(\mathbf{C} \setminus \{1\}) = \inf(\mathbf{C}')$. But then there cannot be a component between $\mathbf{C}$ and $\mathbf{C}'$, and the order of components cannot be dense.

(3) Let h be an order isomorphism from $[0, 1]$ onto $[\frac{1}{4}, 1]$, let C be the Cantor set, and let $C' = h(C)$. Then $[\frac{1}{4}, 1] \setminus C'$ is a union of a countable family of mutually disjoint open intervals $(a_i, b_i) : i \in I$. Let $m \notin I$, let $a_m = 0$ and $b_m = \frac{1}{4}$, and let $J = I \cup \{m\}$. We order J letting, for $h, k \in J$, $h < k$ if and only if $b_h \leq a_k$. Then J is a densely ordered set with minimum m and without maximum. Now let for all $j \in J$, $\mathbf{A}_j$ be an isomorphic copy of $[0, 1]_{\mathrm{MV}}$ with domain $[a_j, b_j) \cup \{1\}$. Then $\bigoplus_{j \in J} \mathbf{A}_j$ is a strongly dense BL-chain whose lattice reduct is a sublattice of $[0, 1]$. □

Remark 7.3 If $\mathbf{A}$ is a strongly dense BL-algebra, then every finite BL-chain $\mathbf{B}$ is embeddable into $\mathbf{A}$. Indeed, $\mathbf{B}$ is the ordinal sum of finitely many finite MV-algebras. Moreover, any component of $\mathbf{B}$ is embeddable into any component of $\mathbf{A}$ and the finite order of the components of $\mathbf{B}$ is embeddable into the order of components of $\mathbf{A}$. It follows that $\mathbf{B}$ can be embedded into $\mathbf{A}$. Since the variety of BL-algebras is generated by the class of finite BL-chains, we obtain that any strongly dense BL-algebra generates the whole variety of BL-algebras.

Strongly dense BL-chains constitute an elementary class that is axiomatized as follows.

1. Axioms of BL-chains.
2. For every $n > 1$, the n-root axiom $\forall x \exists y \left(y^{n-1} \Rightarrow x = y\right)$.
3. The axiom $\forall x \exists y (y \leq x \wedge y \cdot y = y \wedge (x \Rightarrow y) \Rightarrow y = x)$.
4. The density axiom $\forall x \forall y (x \ll y \to \exists z (x \ll z \wedge z \ll y))$.

Definition 7.4 The theory axiomatized by the axioms (1), (2), (3), and (4) above will be called *the first-order theory of strongly dense* BL-*chains* and it will be denoted by SDBL.

We will now construct an extension by definitions SDBL$^+$ of SDBL which has quantifier elimination. The theory SDBL$^+$ is obtained from SDBL by adding the n-root operation r_2 and an additional unary operation * together with their definitions:

1. $\forall x \forall y (r_2(x) = y \leftrightarrow (y \Rightarrow x = y))$, and
2. $\forall x \forall y (x^* = y \leftrightarrow (y \leq x \wedge y^2 = y \wedge ((x \Rightarrow y) \Rightarrow y = x)))$.

Theorem 7.5 *The theory* SDBL$^+$ *has* QE *in the language* $\langle \cdot, \Rightarrow, 0, 1, r_2, ^*, < \rangle$.

Proof As usual, in order to eliminate $\exists x$ in $\exists x \varphi$ with φ quantifier-free, it suffices to eliminate $\exists x$ in formulas of the form $\exists x C$ where C is a φ^+-complete satisfiable conjunction which implies φ. If $x =_C 1$, then $\exists x C$ can be reduced to $C(1)$ and the procedure terminates. Thus, suppose $x \ll 1$. Adapting the proof of Lemma 4.2, we obtain that $\exists x C$ is equivalent to $\exists x C'$ where C' is a conjunction of formulas having one of the following forms:

1. $u \ll t$ where u and t are either variables or constants;
2. $u \prec t$ or $u = t$, where for each pair v, w of subterms of u or of t we have $v \equiv_C w$.

Now C' can be written as $C_0 \wedge C_1(x) \wedge C_2(x)$, where

(a) x does not occur in C_0;
(b) $C_1(x)$ is a conjunction of formulas of the form $t \lhd u$, where $\lhd$ is either $=$ or $\prec$, x occurs either in t or in u (or in both), and for every subterm w of t or of u we have $w \equiv_C x$;
(c) $C_2(x)$ is a conjunction of formulas of the form $t \ll u$, where x is one of t or u.

Elimination of $\exists x$ in $\exists x C_1(x)$ is reduced to quantifier elimination in divisible MV-chains, as in Theorem 5.7. Let E_1 be the resulting formula.

Now note that $C_2(x)$ has one of the following forms: (a) $\bigwedge_{i=1}^{n}(x \ll t_i)$ or (b) $\bigwedge_{j=1}^{m}(s_j \ll x)$ or (c) $\bigwedge_{i=1}^{n}(x \ll t_i) \wedge \bigwedge_{j=1}^{m}(s_j \ll x)$. Then, in case (a), let E_2 be the formula $\bigwedge_{i=1}^{n}(0 \ll t_i)$; in case (b), let E_2 be the formula $\bigwedge_{j=1}^{m}(s_j \ll 1)$; and in case (c), let E_2 be the formula $\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{m}(s_j \ll t_j)$. Then $\exists x C$ can be reduced to $C_0 \wedge E_1 \wedge E_2$, and the proof is complete. □

As a consequence, we obtain the following corollary.

Corollary 7.6 *The theories* SDBL *and* SDBL$^+$ *are complete and decidable.*

Proof SDBL$^+$ is model-complete and it has a prime model that is constructed as follows: let $\mathbf{Q}^+$ be the ordered set of all nonnegative rational numbers and let, for all $q \in \mathbf{Q}^+$, $\mathbf{A}_q$ be an isomorphic copy of $[0, 1]_{\mathrm{Q}}$. Then the ordinal sum $\bigoplus_{q \in \mathbf{Q}^+} \mathbf{A}_q$, with the operations r_2 and * defined in the obvious way, is a prime model of SDBL$^+$. Hence, SDBL$^+$ is complete, and since SDBL$^+$ is an extension by definitions of SDBL, SDBL is in turn complete and decidable. As in the cases of DBL$^+_\infty$ and nDBL$^+$, there is also an algorithmic proof of the decidability of SDBL$^+$ or SDBL based on the quantifier elimination algorithm. □

Theorem 7.7 SDBL *is model-complete.*

Proof SDBL$^+$ has quantifier elimination, and hence it is model-complete. Moreover, the defining formulas of the symbols of SDBL$^+$ not in the language of SDBL are quantifier-free. Therefore, the claim follows from Lemma 6.8. □

8 Quantifier Elimination for BL-Chains with Finitely Many Finite MV-Components

Our aim is now to investigate quantifier elimination for the theories of finite BL-chains. We give a characterization of finite BL-chains whose theory has quantifier elimination in the language of BL-algebras with order, and we prove that, if constants for all idempotent elements are added to the language, quantifier elimination always holds. Notice that the first-order theory of *any* finite algebra $\mathbf{A}$ has quantifier elimination in the language of $\mathbf{A}$ added by a constant for each element of the algebra. Moreover, in this case, the quantifier elimination procedure is polynomial time: let $\varphi(x_1, \ldots, x_n)$ be any formula, and let D_φ be the set $\{(a_1, \ldots, a_n) : \mathbf{A} \models \varphi(a_1, \ldots, a_n)\}$. Then $\varphi(x_1, \ldots, x_n)$ is equivalent to $\bigvee_{(a_1,\ldots,a_n) \in D_\varphi} \bigwedge_{i=1}^{n}(x_i = a_i)$.

Also, the first-order theory of *any* finite algebra is categorical and hence it is model-complete. Consequently, the theory of any finite BL-chain is model-complete and, if the language is enriched by a constant for each element of the algebra, it has quantifier elimination. However, as we show below, to obtain quantifier elimination it suffices to introduce constants only for the idempotent elements.[4] Let $\mathbf{S}_n$ denote the MV-chain over

$$S_n = \left\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\right\},$$

and let $\mathrm{Th}(\mathbf{S}_n)$ denote the first-order theory of $\mathbf{S}_n$. An axiomatization of $\mathrm{Th}(\mathbf{S}_n)$ was first given by Lacava and Saeli in [26]. There are several ways to prove that $\mathrm{Th}(\mathbf{S}_n)$ admits elimination of quantifiers in the language $\langle \oplus, \sim, 0, < \rangle$. One follows by the results in [26]. Lacava and Saeli indirectly proved that the universal fragment $\mathrm{Th}_\forall(\mathbf{S}_n)$ of $\mathrm{Th}(\mathbf{S}_n)$ has the amalgamation property, and they also showed that $\mathrm{Th}(\mathbf{S}_n)$ is model-complete. Quantifier elimination is an immediate consequence of the above results by general model-theoretic properties (that was not pointed out in [26]).

Another way, first noticed by Baaz and Veith in [4], is to rely on the fact that each $\mathbf{S}_n$ is ultrahomogeneous. Recall that a structure $\mathbf{A}$ is *ultrahomogeneous* if every isomorphism between finitely generated substructures of $\mathbf{A}$ extends to an automorphism of $\mathbf{A}$. As shown in [23], a finite structure $\mathbf{A}$ in a language L admits quantifier elimination if and only if it is ultrahomogeneous. It is immediately seen that each $\mathbf{S}_n$ is indeed ultrahomogeneous (every isomorphism between subalgebras of $\mathbf{S}_n$ is just the identity isomorphism) and consequently admits elimination of quantifiers in the language $\langle \oplus, \sim, 0, < \rangle$.

Using the concept of ultrahomogeneity we can start studying quantifier elimination of finite BL-chains. Such chains are finite ordinal sums of finite MV-chains, hence of the form $\bigoplus_{i=0}^{m-1} \mathbf{S}_{n_i}$. In what follows, $\mathrm{Th}(\bigoplus_{i=0}^{m-1} \mathbf{S}_{n_i})$ denotes the first-order theory of $\bigoplus_{i=0}^{m-1} \mathbf{S}_{n_i}$ in the language $\langle \cdot, \Rightarrow, 0, 1, < \rangle$ (we do not give an axiomatization, but $\bigoplus_{i=0}^{m-1} \mathbf{S}_{n_i}$, being a finite structure with finite signature, has a finite categorical axiomatization). Notice that in the language $\langle \cdot, \Rightarrow, 0, 1, < \rangle$ the substructures of $\bigoplus_{i=0}^{m-1} \mathbf{S}_{n_i}$ correspond to its subalgebras. Then we have the following proposition.

Proposition 8.1 $\mathrm{Th}(\bigoplus_{i=0}^{m-1} \mathbf{S}_{n_i})$ *has* QE *in the language* $\langle \cdot, \Rightarrow, 0, 1, < \rangle$ *if and only if* $m \leq 2$*; that is,* $\bigoplus_{i=0}^{1} \mathbf{S}_{n_i}$ *is the ordinal sum of at most two* MV*-components.*

Proof Suppose first that $m \geq 3$. Since $\mathbf{A} = \bigoplus_{i=0}^{m-1} \mathbf{S}_{n_i}$ is a finite structure, to prove the claim it suffices to find an isomorphism between subalgebras of $\mathbf{A}$ that cannot be extended to an automorphism of $\bigoplus_{i=0}^{m-1} \mathbf{S}_{n_i}$. Let $\mathbf{B}_j$ and $\mathbf{B}_k$ be the two-element MV-subchain of $\mathbf{S}_{n_j}$ and $\mathbf{S}_{n_k}$, respectively, with $0 < j < k \leq m-1$, and let $\mathbf{C}$ be any subalgebra of $\mathbf{S}_{n_0}$. Both $\mathbf{C} \oplus \mathbf{B}_j$ and $\mathbf{C} \oplus \mathbf{B}_k$ are subalgebras of $\mathbf{A}$ and are mutually isomorphic. However, it is easily seen that the unique isomorphism between $\mathbf{C} \oplus \mathbf{B}_1$ and $\mathbf{C} \oplus \mathbf{B}_2$ cannot be extended to an automorphism of $\mathbf{A}$ since such an automorphism would not respect the order of the components. This means that $\mathbf{A}$ is not ultrahomogeneous and consequently does not have elimination of quantifiers in the language $\langle \cdot, \Rightarrow, 0, 1\ < \rangle$.

Suppose now that $m \leq 2$. If $m = 1$, the result immediately follows from the fact that each finite MV-chain is ultrahomogeneous, and the fact that an equivalent axiomatization for each $\mathrm{Th}(\mathbf{S}_n)$ can be given in the language $\langle \cdot, \Rightarrow, 0, 1, < \rangle$. If $m = 2$, notice that the subalgebras of $\mathbf{A} = \mathbf{S}_{n_0} \oplus \mathbf{S}_{n_1}$ are all the subalgebras of $\mathbf{S}_{n_0}$, and all the ordinal sums $\mathbf{A}_0 \oplus \mathbf{A}_1$ where $\mathbf{A}_0$ is a subalgebra of $\mathbf{S}_{n_0}$ and $\mathbf{A}_1$ is a subalgebra of $\mathbf{S}_{n_1}$. It is easily seen that there are no distinct subalgebras of $\mathbf{A} = \mathbf{S}_{n_0} \oplus \mathbf{S}_{n_1}$ isomorphic to each other: in other words, any isomorphism between subalgebras of $\mathbf{A}$

is just the identity mapping, which is trivially extended to the identity automorphism of $\mathbf{A}$. This concludes the proof. $\square$

By Proposition 8.1, in order to prove quantifier elimination for structures of the form $\bigoplus_{i=0}^{m-1} \mathbf{S}_{n_i}$ with $m > 2$, we need to expand the language. We do so by introducing a constant c_i for each idempotent element different from 0 and 1. That is, for each BL-chain of the form $\bigoplus_{i=0}^{m-1} \mathbf{S}_{n_i}$, $\mathrm{Th}^c(\bigoplus_{i=0}^{m-1} \mathbf{S}_{n_i})$ will denote the first-order theory of $\bigoplus_{i=0}^{m-1} \mathbf{S}_{n_i}$ in the language

$$\langle \cdot, \Rightarrow, 0, c_1, \ldots, c_{m-1}, 1, <\rangle$$

where $c_1, \ldots, c_{m-1}$ denote the idempotent elements of $\bigoplus_{i=0}^{m-1} \mathbf{S}_{n_i}$ different from 0 and 1, with $0 < c_1 < \cdots < c_{m-1} < 1$. Let $c_0 = 0$, $c_m = 1$, and let, for $i < m$, $\sim_{c_i} = x \Rightarrow c_i$. Then $\mathrm{Th}^c(\bigoplus_{i=0}^{m-1} \mathbf{S}_{n_i})$ is axiomatized as follows:

1. the axioms of linearly ordered BL-algebras;
2. $\forall x\ (x \cdot x = x) \leftrightarrow (\bigvee_{i=0}^{m}\ x = c_i)$;
3. $\forall x\ ((c_i \leq x) \wedge (x < c_{i+1})) \to (\sim_{c_i}\sim_{c_i} x = x)$, with $0 \leq i \leq m-1$;
4. $\exists x\ ((c_i \leq x) \wedge (x < c_{i+1})) \wedge (x^{n_i - 2} = \sim_{c_i} x)$, with $0 \leq i \leq m-1$.

Axiom (2) states that $0 = c_0, c_1, \ldots, c_{m-1}, c_m = 1$ are exactly the idempotent elements of $\bigoplus_{i=0}^{m-1} \mathbf{S}_{n_i}$. Axioms (3) and (4) state that each component i is an MV-algebra with n_i elements. Notice that both (3) and (4) are sets of axioms: for each $0 \leq i \leq m-1$ we have a different axiom.

Theorem 8.2 $\mathrm{Th}^c(\bigoplus_{i=0}^{m-1} \mathbf{S}_{n_i})$ *has* QE *in the language* $\langle \cdot, \Rightarrow, 0, c_1, \ldots, c_{m-1}, 1, <\rangle$.

Proof We just prove that $\bigoplus_{i=0}^{m-1} \mathbf{S}_{n_i}$ is ultrahomogeneous with respect to the language $\langle \cdot, \Rightarrow, 0, c_1, \ldots, c_{m-1}, 1, <\rangle$. This easily follows by noticing that the substructures of $\mathbf{A} = \bigoplus_{i=0}^{m-1} \mathbf{S}_{n_i}$ are only those BL-subalgebras of $\mathbf{A}$ of the form $\bigoplus_{i=0}^{m-1} \mathbf{B}_i$, where each $\mathbf{B}_i$ is a subalgebra of $\mathbf{S}_{n_i}$, and so any isomorphism between subalgebras of $\mathbf{A}$ is just the identity mapping. $\square$

9 Ordinal Sums of MV, Product, and Gödel Chains

In this section we investigate quantifier elimination for ordinal sums of MV-chains, product chains, and Gödel chains. The results are strictly connected to those offered in the previous sections, and for this reason, only the main differences and most important steps in the proofs will be pointed out.

For each ordered Abelian group $\mathbf{G} = \langle G, +, -, 0_G, \leq_G\rangle$, let $G^- = \{x \in G : x \leq 0\}$, let $\Pi(\mathbf{G}) = \langle G^- \cup \{\bot\}, \cdot, \Rightarrow, 0, 1\rangle$ be a structure where $\bot$ is a new element such that $\bot < x$ for all $x \in G^-$, and

1. $x \cdot y := x + y$, for all $x, y \in G^-$,
2. $\bot \cdot x = x \cdot \bot = \bot$, for all $x \in G^- \cup \{\bot\}$,
3. $x \Rightarrow y = 1$, for $x \leq y$, $x, y \in G^- \cup \{\bot\}$,
4. $x \Rightarrow y := y - x$, for $y < x$, $x, y \in G^-$,
5. $x \Rightarrow \bot = \bot$, for $x \in G^-$,
6. $0 := \bot$, $1 := 0_G$.

Then $\Pi(\mathbf{G})$ is a product chain (see [22]).

Conversely, let $\mathbf{A} = \langle A, \cdot, \Rightarrow, 0, 1\rangle$ be any linearly ordered product algebra. Then, as shown in [22], there exists a unique (up to isomorphism) ordered Abelian group $\mathbf{G} = \Pi^{-1}(\mathbf{A}) = \langle G, +, -, 0_G, \leq_G\rangle$ such that $\Pi(\mathbf{G})$ is isomorphic to $\mathbf{A}$.

Hence, $G^- = \{x \mid x \le 0_G, x \in G\}$ is order-isomorphic with $A \setminus \{0\}$, and for all $x, y \in A \setminus \{0\}$,

1. $0_G := 1$,
2. $x + y := x \cdot y$,
3. $x - y := y \Rightarrow x$, for $x \le y$.

$\Pi^{-1}(\mathbf{A})$ can be constructed from $\mathbf{A}$ mimicking the construction of the integers from the natural numbers. Moreover, Π and Π^{-1} can be extended to morphisms of product chains and ordered Abelian groups resulting in a categorical equivalence.[5]

As shown in [28], for every formula $\varphi(\overline{x})$ in the language $\langle \cdot, \Rightarrow, 0, 1, < \rangle$ and every product chain $\mathbf{A}$, there exists a formula $\varphi^\sharp(\overline{x})$ in the language of ordered groups $\langle +, -, 0_G, < \rangle$ such that for every product chain $\mathbf{A}$ and for all $\overline{b} \in A$, $\mathbf{A} \models \varphi(\overline{b})$ if and only if $\Pi^{-1}(\mathbf{A}) \models \varphi^\sharp(\overline{b})$. This allows to prove elimination of quantifiers for the class of divisible product chains in the language $\langle \cdot, \Rightarrow, 0, 1, < \rangle$ [28]. The same results can be obtained in a more constructive way by following the strategy adopted in Section 3 for MV-chains.

As for Gödel chains, it is easy to see that the class of densely ordered Gödel chains has quantifier elimination in the language $\langle \cdot, \Rightarrow, 0, 1, < \rangle$. Indeed, the theory of densely ordered Gödel chains can be equivalently axiomatized in the language $\langle 0, 1, < \rangle$, that is, the language of dense linear orders with endpoints DLOE, which is well known to have elimination of quantifiers in language $\langle 0, 1, < \rangle$, and can be interpreted into DLOE (see [28]).

Notice that ordinal sums of product, MV, and Gödel components can be seen in terms of Wajsberg hoops. Indeed, a product component corresponds to the ordinal sum of the two-element Wajsberg hoop and a cancellative Wajsberg hoop, while a Gödel component corresponds to the ordinal sum of two-element Wajsberg hoops.

Now let I be a finite sequence of labels of components among the types MV, P (product), and G (Gödel) such that the first label is MV and there are no consecutive labels equal to G. For instance, I may be the sequence $\langle$MV, G, P, G, MV$\rangle$.

Given a finite sequence $I = \langle i_0, \ldots, i_{n-1} \rangle$ as shown above, a BL(I)-*algebra* is a BL-chain that is the ordinal sum of n components $\mathbf{A}_0, \ldots \mathbf{A}_{n-1}$ such that for $k = 0, \ldots, n-1$, i_k = MV (i_k = P, $\mathrm{i_k}$ = G, respectively) if and only if $\mathbf{A}_k$ is an MV-algebra (a product algebra, a Gödel algebra, respectively). A *divisible* BL(I)-*algebra* (DBL(I)-*algebra*, for short) is a BL(I)-algebra whose MV and product components are divisible and whose Gödel components are densely ordered. In the sequel, DBL(I) will denote the theory of DBL(I)-algebras, that is, the set of all first-order formulas valid in all DBL(I)-algebras. Of course, each theory DBL(I) depends on I, but we will show that for any choice of the finite sequence I, the corresponding theory DBL(I) does admit elimination of quantifiers.

In what follows, we fix a sequence $I = \langle i_0, \ldots, i_{n-1} \rangle$ of length n, with $i_0, \ldots, i_k \in \{\mathrm{MV}, \mathrm{P}, \mathrm{G}\}$, with i_0 = MV and such that for $k = 0, \ldots, n-2$, if i_k = G, then $i_{k+1} \neq$ G. Given I, let $c_1, \ldots, c_{n-1}$ be constants denoting the idempotent elements separating each component in a DBL(I)-algebra. We assume that $0 < c_1 < \cdots < c_{n-1} < 1$.

DBL(I) is axiomatizable in the language

$$\langle \cdot, \Rightarrow, 0, c_1, \ldots, c_{n-1}, 1, < \rangle.$$

Indeed, setting $c_0 = 0$, $c_n = 1$, and $\sim_{c_i} x = x \Rightarrow c_i$, we just need to add to the axioms of linearly ordered BL-algebras the following axioms:

(1) an axiom stating that all the c_is are idempotent elements:

$$\bigwedge_{i=0}^{n}(c_i \cdot c_i = c_i);$$

(2) for $k = 0, \dots, n-1$, if $i_k = \text{MV}$, then we add axioms saying that the set $[c_k, c_{k+1}) \cup \{1\}$ is the domain of a divisible MV-algebra:

$$\forall x\ (((c_k \leq x) \wedge (x < c_{k+1})) \to ({\sim_{c_k}}{\sim_{c_k}} x = x)),$$

$$\forall x \exists y\ \big(((c_k \leq x) \wedge (x < c_{k+1})) \to \big(y^{m-1} \Rightarrow x = y\big)\big) \quad (m \text{ any integer } > 1);$$

(3) for $j = 0, \dots, n-1$, if $i_j = \text{P}$, then we add axioms saying that the set $[c_j, c_{j+1}) \cup \{1\}$ is the domain of a divisible product algebra:

$$\forall x \forall y (((c_j < x) \wedge (x < c_{j+1}) \wedge (c_j < y) \wedge (y < c_{j+1})) \to (x \Rightarrow (x \cdot y) = y)),$$

$$\forall x \exists y\ (((c_j \leq x) \wedge (x < c_{j+1})) \to \big(x = y^m\big)) \quad (m \text{ any integer } > 1);$$

(4) for $h = 0, \dots, n-1$, if $i_h = \text{G}$, then we add axioms saying that the set $[c_h, c_{h+1}) \cup \{1\}$ is the domain of a densely ordered Gödel-algebra:

$$\forall x\ (((c_h \leq x) \wedge (x < c_{h+1})) \to (x^2 = x)),$$

$$\forall x \forall y \exists z (((c_h \leq x) \wedge (y \leq c_{h+1}) \wedge (x < y)) \to (x < z \wedge z < y)).$$

We redefine the symbols $\ll$ and $\equiv$, since we are now working with a different notion of components.

$$x \ll y := \bigvee_{i=1}^{n}((x < c_i) \wedge (c_i \leq y)).$$

$$x \equiv y := \bigwedge_{i=1}^{n}((x < c_i) \leftrightarrow (y < c_i)).$$

$$x \prec y := (x < y) \wedge (x \equiv y).$$

Theorem 9.1 *For any finite index sequence I, the theory* DBL(I) *has* QE *in the language*

$$\langle \cdot, \Rightarrow, 0, c_1, \dots, c_{n-1}, 1, < \rangle.$$

Proof Again, it suffices to eliminate $\exists x$ in any formula $\exists x C(x)$ where $C(x)$ is a complete and satisfiable conjunction. After the usual reductions, we reduce the problem to the elimination of $\exists$ in $\exists x C'(x)$ where $C'(x)$ is a conjunction equivalent to $C(x)$ and such that, with reference to the notation used in Theorem 5.7,

1. if $u \ll_{C'} t$, then u and t are either variables, or constants;
2. if $u \prec_{C'} t$ or $u =_{C'} t$, and if w, v are subterms of u or of t, then $w \equiv_{C'} v$.

Thus, C' can be written as $C_0 \wedge C_1(x) \wedge C_2(x)$, where x does not occur in C_0, $C_1(x)$ has the form $\bigwedge_{j=1}^{k} t_j \lhd_j u_j$, where $\lhd_j$ is either $=$ or $\prec$ and for every subterm v of t_j or of u_j, we have $v \equiv_{C'} x$, and $C_2(x)$ has either the form (a) $\bigwedge_{j=1}^{r} t_j \ll x$ or the form (b) $\bigwedge_{j=1}^{s} x \ll u_j$ or the form (c) $\bigwedge_{j=1}^{r} t_j \ll x \wedge \bigwedge_{h=1}^{s} x \ll u_h$.

If $x =_{C'} 1$, then $\exists x C'(x)$ can be reduced to $C'(1)$ and we are done. Then suppose $x \ll_{C'} 1$. Let $k < n$ be such that $x \equiv_{C'} c_k$. If $i_k = \text{MV}$, ($i_k = \text{P}$, $i_k = \text{G}$, respectively), let $D_1(x)$ be the formula obtained from $C_1(x)$ by replacing c_k by 0, and let F_1 be the formula obtained by eliminating $\exists x$ in $\exists x(D_1(x) \wedge x < 1)$, according to the quantifier elimination procedure for divisible MV-chains (divisible product chains, densely ordered Gödel chains, respectively). Finally, let E_1 be the result of replacing in F_1 every occurrence of 0 by c_k if $x \equiv_{C'} c_k$ (and by itself if $x \equiv_{C'} 0$).

Now, let E_2 be defined as follows: if $x \equiv_{C'} 0$, then replace x by 0 in $C_2(x)$, and if $x \equiv_{C'} c_k$, then replace x by c_k in $C_2(x)$. It is readily seen that $\exists x C$ is equivalent to $C_0 \wedge E_1 \wedge E_2$. □

10 Amalgamation and Joint Embeddability Property in Classes of BL-Algebras

The amalgamation property of the algebraic semantics of an algebraizable logic is strongly related to the deductive interpolation and the Robinson property of the logic, while the joint embeddability property is strongly related to the Halldén completeness of the logic (cf. [18]). In this section we investigate the relationship between these properties and quantifier elimination. As a consequence we prove that some classes of BL-algebras introduced in the previous sections have the amalgamation property and other classes have the joint embeddability property.

In [30], the variety of all BL-algebras is shown to have the AP. We will present here an alternative proof of this fact. We recall the following result from [30].

Proposition 10.1 *Let $\mathcal{V}$ be any variety of* BL*-algebras. If every V-formation consisting of totally ordered elements of $\mathcal{V}$ has an amalgam in $\mathcal{V}$, then $\mathcal{V}$ itself has the* AP.

We will also use the following result.

Lemma 10.2 *Let $\mathcal{K} \subseteq \mathcal{H}$ be two classes of algebras, and suppose that $\mathcal{K}$ has the* AP *and for every V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{H}$ there are a V-formation $(\mathbf{A}', \mathbf{B}', \mathbf{C}', i', j')$ in $\mathcal{K}$ and embeddings h_1, h_2, h_3 of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ into $\mathbf{A}', \mathbf{B}', \mathbf{C}'$, respectively, such that $i' \circ h_1 = i \circ h_2$ and $j' \circ h_1 = j \circ h_3$. Then $\mathcal{H}$ has the* AP.

Proof Let $(\mathbf{D}, h, k)$ be an amalgam of $(\mathbf{A}', \mathbf{B}', \mathbf{C}', i', j')$ in $\mathcal{K}$. Then $(\mathbf{D}, h \circ h_2, k \circ h_3)$ is an amalgam of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{H}$. □

Theorem 10.3 *The variety of* BL*-algebras has the* AP.

Proof It suffices to prove that every V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are BL-chains has an amalgam in the variety of BL-algebras.

Lemma 10.4 *Given a V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ where $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are* BL*-chains, there are strongly dense* BL*-chains $\mathbf{A}'$, $\mathbf{B}'$, $\mathbf{C}'$, embeddings h_1, h_2, and h_3 of $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ into $\mathbf{A}', \mathbf{B}'$, and $\mathbf{C}'$, respectively, and embeddings i', j' of $\mathbf{A}'$ into $\mathbf{B}'$ and $\mathbf{C}'$, respectively, such that $i' \circ h_1 = h_2 \circ i$ and $j' \circ h_1 = h_3 \circ j$.*

Proof Define for every totally ordered Wajsberg hoop $\mathbf{W}$, a Wajsberg hoop $\mathbf{W}^*$ as follows: if $\mathbf{W}$ is bounded, then $\mathbf{W}^* = \mathbf{W}$; otherwise, $\mathbf{W}^* = \Lambda^{-1}(\mathbf{W})$ (cf. Section 2 for the definition of Λ and of Λ^{-1}). Moreover, let $\mathbf{W}^+$ be the *divisible hull* of $\mathbf{W}^*$, that is, the smallest (up to isomorphism) divisible MV-algebra containing $\mathbf{W}^*$. The algebra $\mathbf{W}^+$ is constructed as follows: let $(\mathbf{G}, u) = \Gamma^{-1}(\mathbf{W}^*)$, let $\mathbf{G}^+$ be the divisible hull of $\mathbf{G}$ (minimum divisible ordered Abelian group extending $\mathbf{G}$), and let $\mathbf{W}^+ = \Gamma(\mathbf{G}^+, u)$.

Note that $\mathbf{W}$ is a subhoop of $\mathbf{W}^+$. Moreover, every embedding h of $\mathbf{W}$ into (the hoop reduct of) a divisible MV-algebra $\mathbf{U}$ has a unique extension h^+ to an embedding of $\mathbf{W}^{+}$ into $\mathbf{U}$. Now let $\mathbf{A}'$ and h_1 be defined as follows: represent $\mathbf{A}$ as an ordinal sum $\bigoplus_{m\in M}\mathbf{W}_m$ where M is a totally ordered set with minimum m_0 and each $\mathbf{W}_m$ is a totally ordered Wajsberg hoop and $\mathbf{W}_{m_0}$ is bounded. Let $\mathbf{Q}^{+}$ be the totally ordered set of nonnegative rational numbers, and let $M_{\mathbf{Q}} = M \times_{\text{lex}} \mathbf{Q}^{+}$ with the lexicographic order, and define for all $(m, q) \in M_{\mathbf{Q}}$, $\mathbf{W}_{(m,q)}$ as follows.

(1) If $q = 0$, then $\mathbf{W}_{(m,q)}$ is an isomorphic copy of $\mathbf{W}_m^{+}$ with domain the set $\{(w, m, q) : w \in \mathbf{W}_m^{+}\setminus\{1\}\} \cup \{1\}$ and with operations defined so that the map $h_{m,q}$ defined by

$$h_{m,q}(w) = \begin{cases} 1 & \text{if } w = 1 \\ (w, m, q) & \text{otherwise} \end{cases}$$

is an isomorphism from $\mathbf{W}_i^{+}$ onto $\mathbf{W}_{(i,q)}$. (Thus, for instance, $(w, m, q)\cdot(w', m, q) = (w \cdot w', m, q)$).

(2) If $q \neq 0$, then $\mathbf{W}_{(m,q)}$ is an isomorphic copy of $[0, 1]_{\text{MV}}$ such that the domain of $\mathbf{W}_{(m,q)}$ is $\{(\alpha, m, q) : \alpha \in [0, 1)\} \cup \{1\}$ and the map $k_{m,q}$ defined by

$$k_{m,q}(\alpha) = \begin{cases} 1 & \text{if } \alpha = 1 \\ (\alpha, m, q) & \text{otherwise} \end{cases}$$

is an isomorphism from $[0, 1]_{\text{MV}}$ into $\mathbf{W}_{(m,q)}$.

Now let $\mathbf{A}' = \bigoplus_{(m,q)\in M_{\mathbf{Q}}}\mathbf{W}_{(i,q)}$, and let for $w \in \mathbf{A}$,

$$h_1(w) = \begin{cases} 1 & \text{if } w = 1 \\ (w, m, 0) & \text{if } w \in \mathbf{W}_m\setminus\{1\}\,. \end{cases}$$

In similar fashion, if $\mathbf{B} = \bigoplus_{r\in R}\mathbf{U}_r$ and $\mathbf{C} = \bigoplus_{s\in S}\mathbf{V}_s$, we construct the ordered sets $R_{\mathbf{Q}} = R \times_{\text{lex}} \mathbf{Q}^{+}$ and $S_{\mathbf{Q}} = S \times_{\text{lex}} \mathbf{Q}^{+}$ and the algebras $\mathbf{U}_{(r,q)}$ and $\mathbf{V}_{(s,q)}$ in analogy with the construction of the algebras $\mathbf{W}_{(m,q)}$. Then we set $\mathbf{B}' = \bigoplus_{(r,q)\in R_{\mathbf{Q}}}\mathbf{U}_{(r,q)}$, $\mathbf{C}' = \bigoplus_{(s,q)\in S_{\mathbf{Q}}}\mathbf{V}_{(s,q)}$, and for $x \in \mathbf{B}$ and for $y \in \mathbf{C}$ we define

$$h_2(x) = \begin{cases} 1 & \text{if } x = 1 \\ (x, r, 0) & \text{if } x \in \mathbf{U}_r\setminus\{1\} \end{cases} \qquad h_3(y) = \begin{cases} 1 & \text{if } y = 1 \\ (y, s, 0) & \text{if } y \in \mathbf{V}_s\setminus\{1\}\,. \end{cases}$$

Finally, we define i' and j' as follows.

(i) We set $i'(1) = j'(1) = 1$.

(ii) For all $(w, m, q) \in \mathbf{A}'\setminus\{1\}$, we define $i'(w, m, q) = (i^*(w), r, q)$ and $j'(w, m, q) = (j^*(w), s, q)$, where (a) if $q = 0$, then i^* and j^* are the unique embeddings of $\mathbf{W}_i^{+}$ extending the restriction of i (j, respectively) to $\mathbf{W}_i$; (b) if $q \neq 0$, then $i^*(w) = j^*(w) = w$; (c) r and s are the unique elements of R (of S, respectively) such that if $w \in \mathbf{W}_m\setminus\{1\}$, then $i(w) \in \mathbf{U}_r\setminus\{1\}$ ($j(w) \in \mathbf{V}_s\setminus\{1\}$, respectively).

It is easy to see that $\mathbf{A}'$, $\mathbf{B}'$, $\mathbf{C}'$, h_1, h_2, h_3, i', and j' have the desired properties. □

Continuing with the proof of Theorem 10.3, by Lemma 10.1, it suffices to verify the amalgamation property for the class of totally ordered BL-algebras. By Lemmas 10.2 and 10.4, it suffices to verify that the class of SDBL-algebras has the AP, which follows from the fact that SDBL is a model-complete theory. □

Remark 10.5 The proof of Theorem 10.3 also shows that the class of all BL-chains has the AP.

By a similar argument, we can prove the following theorem.

Theorem 10.6 *For every n, the class $n\mathcal{BL}$ of all* BL*-chains that are ordinal sum of precisely n nontrivial Wajsberg components has the* AP.

Proof The class of nDBL-algebras is model-complete, and hence it has the amalgamation property. By Lemma 10.2, it suffices to prove that given a V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ where $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are BL-chains with exactly n Wajsberg components, there are nDBL-chains $\mathbf{A}'$, $\mathbf{B}'$, $\mathbf{C}'$, embeddings h_1, h_2, and h_3 of $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ into $\mathbf{A}'$, $\mathbf{B}'$, and $\mathbf{C}'$, respectively, and embeddings i', j' of $\mathbf{A}'$ into $\mathbf{B}'$ and $\mathbf{C}'$, respectively, such that $i' \circ h_1 = h_2 \circ i$ and $j' \circ h_1 = h_3 \circ j$.

Now let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ be a V-formation in $n\mathcal{BL}$, where $\mathbf{A} = \bigoplus_{m=0}^{n-1} \mathbf{W}_m$, $\mathbf{B} = \bigoplus_{m=0}^{n-1} \mathbf{U}_m$, and $\mathbf{C} = \bigoplus_{m=0}^{n-1} \mathbf{V}_m$. Then i maps $\mathbf{W}_m$ into $\mathbf{U}_m$ and j maps $\mathbf{W}_m$ into $\mathbf{V}_m$ ($m = 0, \ldots, n-1$). Now let $\mathbf{W}_m^+$, $\mathbf{U}_m^+$, and $\mathbf{V}_m^+$ be divisible MV-algebras in which $\mathbf{W}_m$, $\mathbf{U}_m$, and $\mathbf{V}_m$, respectively, can be embedded (cf. the construction of Lemma 10.4). Let $\mathbf{A}' = \bigoplus_{m=0}^{n-1} \mathbf{W}_m^+$, $\mathbf{B}' = \bigoplus_{m=0}^{n-1} \mathbf{U}_m^+$, and $\mathbf{C}' = \bigoplus_{m=0}^{n-1} \mathbf{V}_m^+$. Let h_1, h_2, and h_3 be the embeddings of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ into $\mathbf{A}'$, $\mathbf{B}'$, and $\mathbf{C}'$, respectively, obtained by combining the embeddings of their Wajsberg components.

Finally, let i' and j' be defined as follows: let $u \in \mathbf{A}'$. If $u = 1$, then $i'(u) = j'(u) = 1$. Otherwise, there is a unique m such that $u \in \mathbf{W}_m^+$. Then, let i^* and j^* be the unique embeddings of $\mathbf{W}_m^+$ into $\mathbf{U}_m^+$ (into $\mathbf{V}_m^+$, respectively) extending the restriction of i (j, respectively) to $\mathbf{W}_m$, and let $i'(u) = i^*(u)$ and $j'(u) = j^*(u)$. It is readily seen that $\mathbf{A}'$, $\mathbf{B}'$, $\mathbf{C}'$, h_1, h_2, h_3, i', and j' meet our requirements. □

By contrast, for $n > 2$, the variety $\mathcal{BL}_n$ generated by the class of nDBL-algebras does not have the AP. We exhibit an example showing the failure of amalgamation in $\mathcal{BL}_3$. The example can be easily generalized to $\mathcal{BL}_n$ for any $n > 2$.

Example 10.7 Let $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$, $\mathbf{B} = \mathbf{B}_1 \oplus \mathbf{B}_2 \oplus \mathbf{B}_3$, $\mathbf{C} = \mathbf{C}_1 \oplus \mathbf{C}_2 \oplus \mathbf{C}_3$, where $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$ are isomorphic copies of $[0, 1]_{\mathrm{MV}}$. Let i and j be the embeddings which map $\mathbf{A}$ into $\mathbf{B}_1 \oplus \mathbf{B}_2$ and into $\mathbf{C}_1 \oplus \mathbf{C}_3$, respectively. Suppose that $(\mathbf{D}, h, k)$ is an amalgam of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ and that $\mathbf{D}$ is a BL-algebra. Let $b \in \mathbf{B}_3 \setminus \{1\}$, and let F be a filter of $\mathbf{D}$ which is maximal among the class of filters G such that $h(b) \notin G$. Let $a_1 \in \mathbf{A}_1 \setminus \{1\}$, $c_2 \in \mathbf{C}_2 \setminus \{1\}$, and $a_2 \in \mathbf{A}_2 \setminus \{1\}$. Then the quotient $\mathbf{D}/F$ is subdirectly irreducible, and hence it is totally ordered. Moreover, $a_1, a_2, c_2 \notin F$ and denoting by x/F the equivalence class of x modulo the congruence generated by F, we have

$$k(j(a_1))/F \ll k(c_2)/F \ll k(j(a_2))/F = h(i(a_2))/F \ll h(b)/F.$$

Hence, $\mathbf{D}/F \notin \mathcal{BL}_3$, and a fortiori $\mathbf{D} \notin \mathcal{BL}_3$.

We now investigate the relationship between amalgamation, quantifier elimination, and joint embeddability property in classes of BL-algebras.

Definition 10.8 Let $\mathcal{K}$ be a class of algebras of the same type. We say that $\mathcal{K}$ has the *joint embeddability property* (JEP) if for all $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there is an algebra $\mathbf{C} \in \mathcal{K}$ such that $\mathbf{A}$ and $\mathbf{B}$ embed into $\mathbf{C}$.

The following lemma is almost trivial.

Lemma 10.9 *Let $\mathcal{K}$ be a class of algebras with the* AP *such that there is a minimal model for $\mathcal{K}$. Then $\mathcal{K}$ has the* JEP.

Corollary 10.10 *The class of all* BL*-algebras, the class of all n*DBL*-algebras, and the class of all* SDBL*-algebras have the* JEP.

We conclude this section with a condition for the joint embeddability property in a variety of universal algebras, which will be used to show that there are varieties of BL-algebras generated by a single chain that have the joint embeddability property but not the amalgamation property.

Theorem 10.11 *Let $\mathcal{V}$ be a variety of universal algebras and let $\mathcal{K}$ be an elementary subclass of $\mathcal{V}$ such that*

1. *every subdirectly irreducible element of $\mathcal{V}$ embeds into an element of $\mathcal{K}$,*
2. *the theory of $\mathcal{K}$ is model-complete and has a prime model.*

Then $\mathcal{V}$ has the JEP.

Proof By Lemma 10.9, $\mathcal{K}$ has the joint embeddability property. Now let $\mathbf{A}$, $\mathbf{B}$ be elements of $\mathcal{V}$. Then $\mathbf{A}$ and $\mathbf{B}$ have subdirect embeddings i_A and i_B into $\prod_{i\in I}\mathbf{A}_i$ and into $\prod_{j\in J}\mathbf{B}_j$, respectively, where for all $i \in I$ and for all $j \in J$, $\mathbf{A}_i$ and $\mathbf{B}_j$ are subdirectly irreducible elements of $\mathcal{V}$. By our assumption, for all $i \in I$ and for all $j \in J$, there are embeddings h_i and k_j of $\mathbf{A}_i$ and $\mathbf{B}_j$ into some $\mathbf{C}_i$ and $\mathbf{D}_j$, respectively, such that $\mathbf{C}_i, \mathbf{D}_j \in \mathcal{K}$. Since $\mathcal{K}$ has the joint embeddability property, there are algebras $\mathbf{E}_{i,j} \in \mathcal{K}$ and maps $f_{i,j}$ and $g_{i,j}$ which embed $\mathbf{C}_i$ and $\mathbf{D}_j$, respectively, into $\mathbf{E}_{i,j}$. Now let $\mathbf{E} = \prod_{i\in I, j\in J}\mathbf{E}_{i,j}$; let h be the map from $\mathbf{A}$ into $\mathbf{E}$ defined, for all $a \in \mathbf{A}$, by $(h(a))_{i,j} = f_{i,j}(h_i(i_A(a)_i))$; and let k be the map from $\mathbf{B}$ into $\mathbf{E}$ defined, for all $b \in \mathbf{B}$, by $(k(b))_{i,j} = g_{i,j}(k_j(i_B(b)_j))$. Then h and k embed $\mathbf{A}$ and $\mathbf{B}$, respectively, into $\mathbf{E}$, and the claim follows. $\square$

Corollary 10.12 *For every n, the variety $\mathcal{BL}_n$ is a variety generated by a single chain (namely, the ordinal sum of n copies of* $[0, 1]_{\mathrm{MV}}$*) that has the* JEP*; however, for $n > 2$, it does not have the* AP.

11 Quantifier Elimination: A Model Theoretic Approach

In this section we investigate quantifier elimination from a more abstract and less algorithmic point of view. As a result, we prove that some additional operators used in the above proofs are not really necessary if we just want to establish quantifier elimination without exhibiting an explicit algorithm.

We start from the easier case of finite ordinal sums of divisible MV-chains. In this case, we have seen that without the constants for the idempotent elements we do not have quantifier elimination. However, the square root operation r_2 is not really necessary. This can be seen just observing that r_2 or d_2 is not necessary to get quantifier elimination for divisible MV-algebras, as shown in [28]. However, we will sketch two alternative proofs. In the sequel, $n\mathrm{DBL}^c$ will denote the theory nDBL formulated in a language with constants $0, c_1, \ldots, c_{n-1}, 1$ with the following axioms (where we set $c_0 = 0$ and $c_n = 1$, and where $\sim_{c_i}$ is defined as in the axiomatization of $n\mathrm{DBL}^+$):

1. $\forall x((c_i \leq x \wedge x < c_{i+1}) \to (\sim_{c_i}\sim_{c_i} x = x))$ and $c_i < c_{i+1}$ $(i = 0, \ldots, n-1)$;

2. $\bigwedge_{i=0}^{n}(c_i \cdot c_i = c_i)$;
3. $\forall x \exists y(y^{m-1} \Rightarrow x = y)$ (m any integer > 1).

Now observe that nDBL^c, as well as nDBL, is model-complete, as shown in Section 6.

Lemma 11.1 *The universal fragment of nDBL^c, in a language with the constants for all idempotent elements, is the theory nBL of ordinal sums of n MV-chains.*

Proof The axioms of BL-chains and the formulas $\forall x((c_i \leq x \wedge x < c_{i+1}) \rightarrow (\sim_{c_i}\sim_{c_i} x = x))$, $c_i < c_{i+1}$ $(i = 0, \ldots, n-1)$, and $\bigwedge_{i=0}^{n}(c_i \cdot c_i = c_i)$, which axiomatize nBL, are universal and hold in any nDBL^c-chain. Moreover, every ordinal sum of n MV-chains can be extended to an nDBL^c-algebra: just replace any MV-component by its divisible hull. Hence, if a universal formula fails in an nBL-chain, it also fails in an nDBL^c-chain. $\square$

Theorem 11.2 *nDBL^c has* QE *in the language $\langle \cdot, \Rightarrow, 0, c_1, \ldots, c_{n-1}, 1, < \rangle$.*

Proof By Theorem 10.6, the class of models of nBL has the AP. Hence, nDBL^c is model-complete and the class of all models of its universal fragment has the AP. By Theorem 2.10, it follows that nDBL^c has QE. $\square$

We briefly sketch yet another proof. We say that a theory T has *algebraically prime models* if, for any $\mathbf{A} \models T_\forall$, there is $\mathbf{M} \models T$ and an embedding $i : \mathbf{A} \rightarrow \mathbf{M}$ such that for all $\mathbf{N} \models T$ and embeddings $j : \mathbf{A} \rightarrow \mathbf{N}$ there is $h : \mathbf{M} \rightarrow \mathbf{N}$ such that $j = h \circ i$.

If $\mathbf{M}, \mathbf{N} \models T$ and $\mathbf{M} \subseteq \mathbf{N}$, we say that $\mathbf{M}$ is *simply closed* in $\mathbf{N}$, and write $\mathbf{M} \prec_s \mathbf{N}$, if for any quantifier-free formula $\varphi(\overline{x}, y)$ and any $\overline{a} \in M$, if $\mathbf{N} \models \exists y \varphi(\overline{a}, y)$, then $\mathbf{M} \models \exists y \varphi(\overline{a}, y)$.

Theorem 11.3 (Corollary 3.1.12 in [29]) *Suppose T is a theory such that*

1. *T has algebraically prime models and*
2. *$\mathbf{M} \prec_s \mathbf{N}$ whenever $\mathbf{M} \subseteq \mathbf{N}$ are models of T.*

Then T has QE.

Lemma 11.4 *Let $\mathbf{A} = \bigoplus_{i=0}^{n-1} \mathbf{A}_i$, $\mathbf{B} = \bigoplus_{i=0}^{n-1} \mathbf{B}_i$, with $\mathbf{A}_i, \mathbf{B}_i \models$ DMV. Then, if $\mathbf{A} \subseteq \mathbf{B}$, $\mathbf{A} \prec_s \mathbf{B}$.*

Sketch of proof If $\mathbf{A} \subseteq \mathbf{B}$, then $\mathbf{A}_i \subseteq \mathbf{B}_i$, $(i = 0, \ldots, n-1)$, and since DMV is model-complete, $\mathbf{A}_i \prec \mathbf{B}_i$. Now suppose $\mathbf{B} \models \exists y \varphi(\overline{a}, y)$, with φ quantifier-free and $\overline{a} \in A$. By the usual reductions, along the lines of the proof of Theorem 9.1, we can assume without loss of generality that $\varphi(\overline{a}, y) = C_0 \wedge C_1(y) \wedge C_2(y)$, where C_0 does not contain y, $C_1(y)$ is a conjunction of literals of the form $y \ll t(\overline{a})$ or $s(\overline{a}) \ll y$, and $C_2(y)$ is a conjunction of literals of the form $u(\overline{a}, y) = v(\overline{a}, y)$ or $u(\overline{a}, y) \prec v(\overline{a}, y)$. Since the number of components in $\mathbf{A}$ and in $\mathbf{B}$ is the same, and since $C_2(y)$ can be translated into a DMV-formula, using the fact that $\mathbf{A}_i \prec \mathbf{B}_i$ we get that if $\mathbf{B} \models \exists y(C_0 \wedge C_1(y) \wedge C_2(y))$, then $\mathbf{A} \models \exists y(C_0 \wedge C_1(y) \wedge C_2(y))$. $\square$

In order to prove quantifier elimination, we are just left to prove the following lemma.

Lemma 11.5 *nDBL^c has algebraically prime models.*

Sketch of proof The universal theory of nDBL^c is nBL, and every nBL-chain $\mathbf{A}$ embeds in the ordinal sum $\mathbf{A}^*$ of the divisible hulls of its Wajsberg components.

Moreover, $\mathbf{A}^*$ is the minimum, up to isomorphism, $n\mathrm{DBL}^c$-algebra in which $\mathbf{A}$ embeds. $\square$

Therefore, we can again conclude the following.

Theorem 11.6 *$n\mathrm{DBL}^c$ has* QE *in the language* $\langle \cdot, \Rightarrow, 0, c_1, \ldots, c_{n-1}, 1, < \rangle$.

We now concentrate our attention on ordinal sums of infinitely many DMV-algebras with a discrete index set. We will prove that the symbols s and p are sufficient for quantifier elimination, while the symbols * and r_2 are redundant. We denote by DBL^s_∞ the theory DBL_∞ added with the symbols s and p together with their defining axioms.

Lemma 11.7 *DBL^s_∞ is model-complete.*

Proof DBL^+_∞ has QE and is an extension by definitions of DBL^s_∞ whose defining axioms are quantifier-free formulas. The claim follows from Lemma 6.8. $\square$

Lemma 11.8 *The universal fragment $\mathrm{DBL}^s_{\infty\forall}$ of DBL^s_∞ is the theory of ordinal sums of* MV*-algebras with discretely ordered index set in the language of* DBL^s_∞.

Proof The above theory has a universal axiomatization. Indeed, we can express by universal formulas the following facts:

1. for every x, $s(x)$ and $p(x)$ are idempotent elements;
2. if $x = 1$, then $s(x) = p(x) = 1$; otherwise, $x \ll s(x)$ and for all y it is not the case that $x \ll y \ll s(x)$;
3. if $x \equiv 0$, then $p(x) = 0$, and if $0 \ll x \ll 1$, then $p(x) \ll x$ and $s(p(x)) \equiv x$;
4. letting $x^* = p(s(x))$ and $\sim^* x = x \Rightarrow x^*$, one has $\sim^*\sim^* x = x$.

Clearly, the above axioms are universal and hold in any DBL^s_∞-algebra. On the other hand, every model $\mathbf{A}$ of the universal fragment $\mathrm{DBL}^s_{\infty\forall}$ of DBL^s_∞ can be extended to a model of DBL^s_∞: just replace any MV-component in $\mathbf{A}$ by its divisible hull. Hence, any universal formula which is not a theorem of $\mathrm{DBL}^s_{\infty\forall}$ can be invalidated in a model of DBL^s_∞. $\square$

Theorem 11.9 *The class of all models of $\mathrm{DBL}^s_{\infty\forall}$ has the* AP.

Proof Let $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ be a model of $\mathrm{DBL}^s_{\infty\forall}$, and let i_0 be the minimum of I. Let for all $x \in \mathbf{A} \setminus \{1\}$, $\mathrm{Ind}(x)$ be the unique $i \in I$ such that $x \in \mathbf{A}_i$, and let m_i be the minimum of $\mathbf{A}_i$. Define $s^*(i) = \mathrm{Ind}(s(m_i))$ and $p^*(i) = \mathrm{Ind}(p(m_i))$. Then $\langle I, <, i_0, s^*, p^* \rangle$ is a discretely ordered set without maximum, with minimum i_0, with successor function s^*, and with predecessor function p^*. Note that the theory of discretely ordered sets with minimum and without maximum has quantifier elimination in a language with symbols for the order, for the minimum, for the successor function, and for the predecessor function. Hence, the class of such linear orderings has the SAP.

Lemma 11.10 *Let $\mathbf{A} = \bigoplus_{r \in R} \mathbf{A}_r$ and $\mathbf{B} = \bigoplus_{s \in S} \mathbf{B}_s$ be models of $\mathrm{DBL}^s_{\infty\forall}$, and let i be an embedding of $\mathbf{A}$ into $\mathbf{B}$. Let for $r \in R$, m_r be the minimum of $\mathbf{A}_r$, and let for all $r \in R$, $i^*(r) = \mathrm{Ind}(i(m_r))$. Then i^* is an order embedding of R into S which preserves the minimum, the successor s^*, and the predecessor p^*.*

Proof The proof is easy and it is left to the reader. $\square$

We continue the proof of Theorem 11.9. Let $\mathbf{A} = \bigoplus_{r \in R} \mathbf{A}_r$, $\mathbf{B} = \bigoplus_{s \in S} \mathbf{B}_s$, and $\mathbf{C} = \bigoplus_{t \in T} \mathbf{C}_t$ be models of $\mathrm{DBL}^s_{\infty\forall}$, and let i and j be embeddings of $\mathbf{A}$ into a $\mathbf{B}$ and into $\mathbf{C}$, respectively. With reference to the notation of Lemma 11.10, (R, S, T, i^*, j^*) is a V-formation in the class of discrete orders with minimum and without maximum equipped with the successor function and with the predecessor function. Such a V-formation has an amalgam (U, h^*, k^*). We can assume without loss of generality that $U = h^*(S) \cup k^*(T)$ and that $h^*(S) \cap k^*(T) = h^*(i^*(R)) = k^*(j^*(S))$. Moreover, for each $r \in R$, the V-formation $(\mathbf{A}_r, \mathbf{B}_{i^*(r)}, \mathbf{C}_{j^*(r)}), i^r, j^r)$, where i^r and j^r are the restrictions of i and j to $\mathbf{A}_r$, respectively, has an amalgam in the class of MV-chains, $(\mathbf{D}_r, h_r, k_r)$.

Now we define, for $u \in U$, an MV-chain $\mathbf{E}_u$ as follows: if there is an $r \in R$ such that $u = h^*(i^*(r)) = k^*(j^*(r))$, then $\mathbf{E}_u = \mathbf{D}_r$; otherwise, either $u \in h^*(S) \setminus k^*(T)$ or $u \in k^*(T) \setminus h^*(S)$. In the former case, let s be the unique element of S such that $h^*(s) = u$, and define $\mathbf{E}_u = \mathbf{B}_s$. In the latter case, let t be the unique element of T such that $k^*(t) = u$, and define $\mathbf{E}_u = \mathbf{C}_t$. Let $\mathbf{E} = \bigoplus_{u \in U} \mathbf{E}_u$, and let for $b \in \mathbf{B}$ and $c \in \mathbf{C}$, $h(b)$ and $k(c)$ be defined as follows.

1. If $b = 1$, then $h(b) = 1$ and if $c = 1$, then $k(c) = 1$.
2. If $b < 1$ and $\mathrm{Ind}(b) \in i^*(R)$, $h(b) = h_r(b)$, where r is the unique element of R such that $i^*(r) = \mathrm{Ind}(b)$.
3. If $b < 1$ and $\mathrm{Ind}(b) \notin i^*(R)$, then $h(b) = b$. (Note that in this case $\mathbf{E}_{h^*(\mathrm{Ind}(b))} = \mathbf{B}_{\mathrm{Ind}(b)}$).
4. If $c < 1$ and $\mathrm{Ind}(c) \in j^*(S)$, then $k(c) = k_s(c)$, where s is the unique element of S such that $j^*(s) = \mathrm{Ind}(c)$.
5. If $c < 1$ and $\mathrm{Ind}(c) \notin j^*(S)$, then $k(c) = c$. (Note that in this case $\mathbf{E}_{k^*(\mathrm{Ind}(c))} = \mathbf{C}_{\mathrm{Ind}(c)}$).

It is rather straightforward to check that $(\mathbf{E}, h, k)$ is an amalgam in $\mathrm{DBL}^s_{\infty\forall}$ of the V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$. □

By Theorem 2.10, we conclude the following.

Theorem 11.11 DBL^s_∞ *has* QE.

Remark 11.12 We already know that DBL_∞ is not model-complete and hence it does not have quantifier elimination in the language of BL-algebras. It remains to decide whether DBL_∞ added with the function symbol s and with its defining axioms (but without the symbol p for the predecessor function) has quantifier elimination or not.

We now discuss the case of strongly dense BL-chains. We will prove that SDBL has QE in the language of BL-chains without any new symbol. We already know that SDBL is model-complete. Moreover, we also need to prove the following lemma.

Lemma 11.13 *The universal fragment,* $\mathrm{SDBL}_\forall$, *of* SDBL *is the theory of* BL-*chains.*

Proof The class of BL-chains has a universal axiomatization, and every universal formula which is valid in all BL-chains is a theorem of SDBL. Conversely, every BL-chain can be extended to an SDBL-chain, and hence if a universal formula can be invalidated in any BL-chain, it can also be invalidated in an SDBL-chain. □

Theorem 11.14 SDBL *has* QE *in the language* $\langle\, \cdot\,, \Rightarrow, 0, 1, <\rangle$.

Proof SDBL is model-complete and by Theorem 10.3, the class of all models of its universal fragment, namely, the class of all BL-chains, has the amalgamation property. The claim follows from Theorem 2.10. □

12 Further Results

In this section, we discuss some additional applications of model theory to BL-algebras.

12.1 Representation theorems A famous representation theorem by Di Nola (see [14]) says that every MV-chain embeds into an ultraproduct of the standard MV-algebra $[0,1]_{\mathrm{MV}}$ and that every MV-algebra can be represented as an algebra of functions from the set of its prime filters into an ultraproduct of $[0,1]_{\mathrm{MV}}$, with operations defined pointwise. In this subsection, we find similar results for BL-algebras.

First, we need to recall two classics results in model theory we are going to use.

Proposition 12.1 **([9])** *Let $\mathfrak{F}$ be a nonempty set of elementarily equivalent models. Then there exists a model $\mathbf{B}$ such that every model $\mathbf{A} \in \mathfrak{F}$ is elementarily embeddable in $\mathbf{B}$.*

Theorem 12.2 (Frayne's Theorem [9]) *$\mathbf{A}$ is elementarily equivalent to $\mathbf{B}$ if and only if $\mathbf{A}$ is elementarily embeddable in some ultrapower $\mathbf{B}^\star$ of $\mathbf{B}$.*

Now let $\mathbf{A}_Q$ be the minimal model of SDBL. Then we have the following theorem.

Theorem 12.3 (Representation Theorem 1)

1. *Every* BL-*chain embeds into an ultrapower of* $\mathbf{A}_Q$.
2. *For every* BL-*algebra* $\mathbf{B}$, *there exists an ultrapower* $\mathbf{A}_Q^\star$ *of* $\mathbf{A}_Q$ *such that* $\mathbf{B}$ *embeds into* $(\mathbf{A}_Q^\star)^{F(\mathbf{B})}$, *where* $F(\mathbf{B})$ *is the set of prime filters of* $\mathbf{B}$.

Proof (1) Let $\mathbf{A}$ be any BL-chain. Then $\mathbf{A}$ is embeddable into an SDBL-chain $\mathbf{D}$. Now SDBL is complete and model-complete, and hence $\mathbf{D}$ is elementarily equivalent to $\mathbf{A}_Q$. By Frayne's Theorem there exists an elementary embedding of $\mathbf{D}$ into some ultrapower $\mathbf{A}_Q^\star$ of $\mathbf{A}_Q$.

(2) Let $\mathbf{B}$ be any BL-algebra, and let $F(\mathbf{B})$ be the set of its prime filters. Then $\mathbf{B}$ is embeddable into the product $\prod\{\mathbf{B}/F \mid F \in F(\mathbf{B})\}$. As shown in (1), each $\mathbf{B}/F$ is embeddable into an SDBL-chain $\mathbf{D}_F$. By Proposition 12.1, there exists an SDBL-chain $\mathbf{E}$ in which each $\mathbf{D}_F$ can be embedded. Using Frayne's Theorem again, we get an elementary embedding of $\mathbf{E}$ into some ultrapower $\mathbf{A}_Q^\star$ of $\mathbf{A}_Q$. Hence, $\mathbf{B}$ embeds into $(\mathbf{A}_Q^*)^{F(\mathbf{B})}$. This concludes the proof of the theorem. □

The algebra $\mathbf{A}_Q$ is not a standard BL-chain, and hence Theorem 12.3 is not completely of the same form as Di Nola's theorem. We are going to prove that a similar statement holds with $\mathbf{A}_Q$ replaced by a *standard* BL-chain $\mathbf{C}$. The BL-chain, which will be called the *Cantor* BL-*chain* is defined as follows: Let C' be a homeomorphic copy of Cantor's set on $[\frac{1}{4}, 1]$. Then $[0,1] \setminus C'$ is the union of $I_0 = [0, \frac{1}{4})$ and of a countable family of mutually disjoint open intervals $I_k = (a_k, b_k) : k \in K$. Let

$K' = K \cup \{0\}$, let, for $k \in K'$, J_k be the closure of I_k, and let $\cdot_k$ be an isomorphic copy of the Łukasiewicz t-norm $x \cdot_{Ł} y = \max\{x + y - 1, 0\}$ on J_k. Then the operation

$$x \cdot y = \begin{cases} x \cdot_k y & \text{if } x, y \in J_k \quad (k \in K') \\ \min\{x, y\} & \text{otherwise} \end{cases}$$

is a continuous t-norm which, together with its residuum, generates a standard BL-chain, which will be called the *Cantor* BL-*chain* and will be denoted by $\mathbf{C}$. Note that $\mathbf{C}$ has a countable and dense set of components isomorphic to $[0, 1]_{\mathrm{MV}}$. This follows directly from the construction of Cantor's set.

Theorem 12.4 (Representation Theorem 2)

1. *Every* BL-*chain embeds into an ultraproduct of Cantor's* BL-*chain* $\mathbf{C}$.
2. *Every* BL-*algebra embeds into a direct product whose factors are equal to a (fixed) ultraproduct of* $\mathbf{C}$.

Proof First of all, note that $\mathbf{C}$ contains the ordinal sum of copies of $[0, 1]_{\mathrm{MV}}$ (corresponding to each interval J_k, $k \in K'$), with respect to the index set K', ordered by $k \prec k'$ if and only if $\max(J_k) \leq \min(J_{k'})$. Moreover, the above-defined order on K' is dense and has a minimum (the interval J_0), but not a maximum. It follows that the set of nonnegative rational numbers is order isomorphic to K', and hence, embedding every component of $\mathbf{A}_Q$ into the corresponding component of $\mathbf{C}$, we obtain an embedding of $\mathbf{A}_Q$ into $\mathbf{C}$. The result now follows directly from Theorem 12.3. □

For specific varieties of BL-algebras, namely, for the variety $\mathcal{BL}_n$ generated by all ordinal sums of n MV-chains, we can get a representation theorem even closer to Di Nola's theorem. Indeed, from [2], Theorem 7.4, it follows that $\mathcal{BL}_n$ is generated by the ordinal sum $\bigoplus_{i=0}^{n}[0, 1]_{\mathrm{MV}_i}$ where if $i = 0$, then $[0, 1]_{\mathrm{MV}_i}$ is an isomorphic copy of $[0, 1]_{\mathrm{MV}}$ and if $i > 0$, then $[0, 1]_{\mathrm{MV}_i}$ is an isomorphic copy of its hoop reduct. Moreover, one moment's reflection shows that this is the unique (up to isomorphism) standard BL-chain which generates $\mathcal{BL}_n$. We also recall that $\mathcal{BL}_n$ is equationally axiomatized in [2], as well as in [16].

Theorem 12.5 (Representation Theorem 3)

1. *For any* BL-*chain* $\mathbf{B} \in \mathcal{BL}_n$, *there is an ultraproduct of* $\bigoplus_{i=0}^{n-1}[0, 1]_{\mathrm{MV}_i}$ *in which* $\mathbf{B}$ *embeds.*
2. *For each* BL-*algebra* $\mathbf{B} \in \mathcal{BL}_n$, *there exists an ultrapower* $\mathbf{A}^\star$ *of* $\bigoplus_{i=0}^{n-1}[0, 1]_{\mathrm{MV}_i}$ *such that* $\mathbf{B}$ *embeds into the product* $(\mathbf{A}^\star)^{F(\mathbf{B})}$, *where* $F(\mathbf{B})$ *is the set of prime filters of* $\mathbf{B}$.

Proof (1) Every chain in $\mathcal{BL}_n$ is the ordinal sum of n components at most, and hence it embeds into an nDBL-chain. Now any two nDBL-chains are elementarily equivalent, and hence any nDBL-chain is elementarily equivalent to $\bigoplus_{i=0}^{n-1}[0, 1]_{\mathrm{MV}_i}$. By Frayne's theorem, every nDBL-chain embeds into an ultrapower of $\bigoplus_{i=0}^{n-1}[0, 1]_{\mathrm{MV}_i}$.

(2) Let $\mathbf{B} \in \mathcal{BL}_n$. Then $\mathbf{B}$ embeds into $\prod_{F \in F(\mathbf{B})} \mathbf{B}/F$. Moreover, every $\mathbf{B}/F$ embeds into an nDBL-chain $\mathbf{D}_F$, and by Proposition 12.1, all $\mathbf{D}_F$ embed into a common nDBL-chain, $\mathbf{E}$. Finally, by Frayne's theorem, $\mathbf{E}$ embeds into an ultrapower $\mathbf{A}^*$ of $\bigoplus_{i=0}^{n-1}[0, 1]_{\mathrm{MV}_i}$, and hence $\mathbf{B}$ embeds into $(\mathbf{A}^*)^{F(\mathbf{B})}$. □

Notice that for $n = 1$, the above theorem coincides with Di Nola's Representation Theorem.

12.2 Fraïssé limits and ultrahomogeneous BL-chains In this subsection we investigate Fraïssé limits of some classes of finite BL-chains. First, we recall some basic definitions.

Let L be a signature and $\mathbf{D}$ be an L-structure. The *age* of $\mathbf{D}$ is the class $\mathcal{K}$ of all finitely generated structures that can be embedded in $\mathbf{D}$. If $\mathcal{K}$ is the age of some structure $\mathbf{D}$, then $\mathcal{K}$ has the joint embeddability property and the *hereditary property* (HP); that is, if $\mathbf{A} \in \mathcal{K}$ and $\mathbf{B}$ is a finitely generated substructure of $\mathbf{A}$, then $\mathbf{B}$ is isomorphic to some structure in $\mathcal{K}$.

A structure $\mathbf{A}$ is *locally finite* if every finitely generated substructure of $\mathbf{A}$ is finite, and *uniformly locally finite* if there is a function $f : \omega \to \omega$ such that for every substructure $\mathbf{B}$ of $\mathbf{A}$, if $\mathbf{B}$ has a generator set of cardinality $\leq n$, then $\mathbf{B}$ itself has cardinality $\leq f(n)$. A class $\mathcal{K}$ of structures is *uniformly locally finite* if there is a function $f : \omega \to \omega$ such that the above property holds for every structure $\mathbf{A} \in \mathcal{K}$.

Theorem 12.6 (See [23])

1. *(Fraïssé Theorem) Let* L *be a finite or countable signature and let* $\mathcal{K}$ *be a nonempty finite or countable set of finitely generated* L*-structures which has the* HP, JEP, *and* AP. *Then there is an* L*-structure* $\mathbf{D}$, *unique (up to isomorphism), called* the Fraïssé limit of $\mathcal{K}$ *such that (i)* $|\mathbf{D}| \leq \omega$, *(ii)* $\mathcal{K}$ *is the age of* $\mathbf{D}$, *and (iii)* $\mathbf{D}$ *is ultrahomogeneous.*
2. *Let* L *be a finite or countable signature and* $\mathbf{D}$ *a finite or countable structure that is ultrahomogeneous. Let* $\mathcal{K}$ *be the age of* $\mathbf{D}$. *Then* $\mathcal{K}$ *is nonempty, has at most countably many isomorphism types of structure, and satisfies the* HP, JEP, *and* AP.
3. *Let* L *be a finite signature, and* $\mathbf{M}$ *a countable* L*-structure. Then* $\mathbf{M}$ *is ultrahomogeneous and uniformly locally finite if and only if* $\mathrm{Th}(\mathbf{M})$ *is* ω*-categorical and has* QE.

We will construct the Fraïssé limit of the following classes of structures:

1. the class of ordinal sums of n finite MV-chains;
2. the class of all finite BL-chains.

Let $n\mathrm{DBL}(\mathbf{Q})$ be the ordinal sum of n copies of $[0, 1]_{\mathrm{Q}}$, in the signature $\langle \cdot, \Rightarrow, 0, c_1, \ldots, c_{n-1}, 1, <\rangle$. From the above, we immediately have the following proposition.

Proposition 12.7

1. *The age of* $n\mathrm{DBL}(\mathbf{Q})$ *is the set of the ordinal sums of n copies of finite* MV*-chains.*
2. $n\mathrm{DBL}(\mathbf{Q})$ *is the Fraïssé limit of the set of the ordinal sums of n copies of finite* MV*-chains.*
3. *The age of* $n\mathrm{DBL}(\mathbf{Q})$ *has the* HP, JEP, *and* AP.
4. $n\mathrm{DBL}^c$ *is the first-order theory of* $n\mathrm{DBL}(\mathbf{Q})$, *has* QE, *and is not* ω*-categorical.*
5. $n\mathrm{DBL}(\mathbf{Q})$ *is ultrahomogeneous but not uniformly locally finite.*

That $n\mathrm{DBL}(\mathbf{Q})$ is ultrahomogeneous may be verified directly, because the only isomorphism between two subalgebras of $n\mathrm{DBL}(\mathbf{Q})$ is the identity. (Note that this is no longer true if we work in the language of BL-algebras without the constants c_i).

Moreover, since Th(nDBL($\mathbf{Q}$)) has QE and nDBL($\mathbf{Q}$) is not uniformly locally finite, we obtain that Th(nDBL($\mathbf{Q}$)) is not ω-categorical. This can also be verified directly: the ordinal sum of n copies of the subalgebra of $[0, 1]_{\mathrm{MV}}$ with domain the algebraic real numbers in [0, 1] is a countable model of Th(nDBL($\mathbf{Q}$)) which is not isomorphic to nDBL($\mathbf{Q}$).

A slightly more difficult problem is to give a description of the Fraïssé limit of the class of all finite BL-chains. We will prove the following theorem.

Theorem 12.8 *The Fraïssé limit of the class of all finite* BL*-chains exists and coincides with* $\mathbf{A}_Q$.

Proof First of all, the class of finite BL-chains has the HP, JEP, and AP. The HP is trivial, the JEP follows from the AP and from the fact that there is a minimal model for this class, and the AP follows from the fact that (1) the class of BL-chains has the AP, (2) if $(\mathbf{D}, h, k)$ is an amalgam of a V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, h, k)$, consisting of finite BL-chains, we may replace in it $\mathbf{D}$ by its subalgebra generated by $h(\mathbf{B}) \cup k(\mathbf{C})$, and so we may assume without loss of generality that $\mathbf{D}$ has only finitely many components and that each component is generated by the union of two finite MV-chains, and, consequently, is finite.

It follows that the Fraïssé limit of the class of all finite BL-chains exists, is locally finite and ultrahomogeneous and every finite BL-chain embeds into it. We now prove that every BL-chain $\mathbf{A}$ having the above properties is isomorphic to $\mathbf{A}_Q$.

First of all, one moment's reflection shows that if every finite BL-chains embeds into a BL-chain $\mathbf{A}$, then $\mathbf{A}$ must have infinitely many components.

We now prove that every component of $\mathbf{A}$ must be isomorphic to $[0, 1]_{\mathrm{Q}}$. Since $\mathbf{A}$ is locally finite, then every component cannot have co-infinitesimals (i.e., elements b such that for every n, $b^{n+1} < b_n$), and consequently every component must embed into the standard MV-algebra $[0, 1]_{\mathrm{MV}}$ [12]. If some irrational α is in this component, then by the McNaughton theorem [12], the subalgebra generated by α would contain every element of [0, 1] which is a linear combination with integer coefficients of 1 and α, and then it would be infinite. Hence, every component of $\mathbf{A}$ must be a subalgebra of $[0, 1]_{\mathrm{Q}}$. Now suppose, by way of contradiction, that some component $\mathbf{H}$ of $\mathbf{A}$ is a proper subalgebra of $[0, 1]_{\mathrm{Q}}$. Then there is a positive natural number n such that $\mathbf{S}_n$ does not embed into $\mathbf{H}$. If $\mathbf{H}$ is the first component, we obtain a contradiction because the finite BL-chain $\mathbf{S}_n$ does not embed into $\mathbf{A}$. If $\mathbf{H}$ is not the first component, then, since the ordinal sum of two copies of $\mathbf{S}_n$ embeds into $\mathbf{A}$, there is a component $\mathbf{K}$ of $\mathbf{A}$, different from the first component, in which $\mathbf{S}_n$ embeds. But then the subalgebra of $\mathbf{A}$ consisting of 0, 1 and the minimum of $\mathbf{H}$ and the subalgebra consisting of 0, 1 and the minimum of $\mathbf{K}$ are isomorphic, but the unique isomorphism between them does not extend to an automorphism of $\mathbf{A}$, that is, a contradiction.

Now the index set I of all components of $\mathbf{A}$ is countably infinite and has a minimum, and it is left to prove that it is densely ordered and it has no maximum. Suppose first that I has a maximum m. Let m_0 be the minimum index and let r be an index with $m_0 < r < m$. Let a_r and a_m be the minimum of the components with index r and m, respectively. Then the subalgebras consisting of 0, 1 and a_r and of 0, 1 and a_m are isomorphic, but the unique isomorphism between them does not extend to an automorphism of $\mathbf{A}$.

Finally, suppose, by way of contradiction, that I is not dense. Let $i, j \in I$ be such that $i < j$ and there is no $k \in I$ with $i < k < j$. Let a_i and a_j be the minimum elements of the components with index i and j, respectively. Then the subalgebras consisting of $0, 1, a_i$ and of $0, 1, a_j$ are isomorphic, and hence there must be an automorphism φ of $\mathbf{A}$ which maps a_i into a_j. One moment's reflection shows that $\varphi^{-1}(a_i) < a_i$ and there is no element z with $\varphi^{-1}(a_i) \ll z \ll a_i$. Hence, there is a predecessor of i in I (call it k). Note that any automorphism of $\mathbf{A}$ must map the first component into itself, and hence k cannot be the minimum of I. Let a_k be the minimum of the component with index k. Then the subalgebras consisting of $0, 1, a_k, a_i$ and $0, 1, a_k, a_j$ are isomorphic, but the unique isomorphism ψ between them cannot be extended to an automorphism of $\mathbf{A}$ (a_i would not belong to the image of any monomorphism extending ψ). This concludes the proof. $\square$

13 Final Remarks

In this work, we have given a deep and detailed analysis of several model-theoretic properties of classes of BL-algebras. In particular, we have studied quantifier elimination, model-completeness, completeness, and decidability of different theories of BL-chains and applied those results to obtain other properties such as the amalgamation property, the joint embeddability property, representation in terms of ultrapowers, and the study of Fraïssé limits. We conclude with a table that summarizes the main results of this work.

	Language	MC	QE	Completeness	Decidability
DBL_∞	$\langle\cdot, \Rightarrow, 0, 1, <\rangle$	No	No	Yes	Yes
DBL_∞^+	$\langle\cdot, \Rightarrow, 0, 1, <, s, p, r_2, ^*\rangle$	Yes	Yes	Yes	Yes
$n\mathrm{DBL}$	$\langle\cdot, \Rightarrow, 0, 1, <\rangle$	Yes	No	Yes	Yes
$n\mathrm{DBL}^+$	$\langle\cdot, \Rightarrow, 0, 1, <, r_2, c_1, \ldots, c_{n-1}\rangle$	Yes	Yes	Yes	Yes
SDBL	$\langle\cdot, \Rightarrow, 0, 1, <\rangle$	Yes	Yes	Yes	Yes
SDBL^+	$\langle\cdot, \Rightarrow, 0, 1, <, r_2, ^*\rangle$	Yes	Yes	Yes	Yes

Table 1 Theories of BL-chains and their model-theoretic properties: Model-completeness (MC), Quantifier elimination (QE), Completeness and Decidability.

Notes

1. This result was already known, but it was obtained in [28] by model-theoretic techniques.

2. See Theorem 8.4.1 and Corollary 8.6.3 in [23].

3. This axiomatization of the theory of divisible MV-chains was first given by Lacava and Saeli in [27].

4. One might as well add constants for the co-atoms of each component, as any element of any component, except from 1, is a power of the co-atom of that component.

5. Notice that this connection is even deeper. In fact, let $\mathcal{P}$ be the category whose objects are product algebras (not necessarily product chains) satisfying $x \Rightarrow 0 = 0$ for all $x > 0$, and whose morphisms are the homomorphisms. As shown by Cignoli and Torrens in [11], $\mathcal{P}$ is equivalent to the category of Abelian ℓ-groups with homomorphisms.

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Acknowledgments

Marchioni acknowledges partial support from the Spanish projects TASSAT (TIN2010-20967-C04-01), *Agreement Technologies* (CONSOLIDER CSD2007-0022, INGENIO 2010), the Generalitat de Catalunya grant 2009-SGR-1434, and Juan de la Cierva Program of the Spanish MICINN, as well as the ESF Eurocores-LogICCC/MICINN project (FFI2008-03126-E/FILO), and the Marie Curie IRSES Project (FP7-PEOPLE-2009)

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