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An Algebraic Approach to Subframe Logics. Modal Case

Guram Bezhanishvili, Silvio Ghilardi, and Mamuka Jibladze

Abstract We prove that if a modal formula is refuted on a wK4-algebra $(B, \Box)$, then it is refuted on a finite wK4-algebra which is isomorphic to a subalgebra of a relativization of $(B, \Box)$. As an immediate consequence, we obtain that each subframe and cofinal subframe logic over **wK4** has the finite model property. On the one hand, this provides a purely algebraic proof of the results of Fine and Zakharyaschev for **K4**. On the other hand, it extends the Fine-Zakharyaschev results to **wK4**.

1 Introduction

It is a well-known result of Fine [11] that each subframe logic over **K4** has the finite model property (FMP for short). This result was generalized by Zakharyaschev [21] to all cofinal subframe logics over **K4**. The results of Fine and Zakharyaschev imply that subframe and cofinal subframe superintuitionistic logics also have the FMP. In fact, subframe superintuitionistic logics are exactly the logics axiomatized by adding $(\neg, \vee)$-free formulas to the intuitionistic propositional calculus **IPC**, and cofinal subframe superintuitionistic logics are exactly the logics axiomatized by adding $\vee$-free formulas to **IPC** [22]. On the other hand, as was shown by Wolter [17], there are subframe logics over **K** which do not have the FMP.

The proofs of Fine and Zakharyaschev are model-theoretic. It is the goal of this paper to give a purely algebraic proof of their results. We will also be able to generalize their results to cover all subframe and cofinal subframe logics over *weak K4*,

$$\mathbf{wK4} = \mathbf{K} + \Diamond\Diamond p \to (p \vee \Diamond p).$$

It is well known that **K4** is the modal logic of transitive frames. The modal logic **wK4** is a subsystem of **K4**. As was shown by Esakia [9], **wK4** is the modal logic of

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weakly transitive frames, where a frame $\mathfrak{F} = (W, R)$ is weakly transitive if wRv and vRu imply $w = u$ or wRu. Therefore, the main difference between **K4**-frames and **wK4**-frames is in the behavior of clusters $[w] = \{w\} \cup \{v \in W : wRv \text{ and } vRw\}$. In a **K4**-frame, each point in a proper cluster (that is, a cluster consisting of more than one point) must be reflexive, while in a **wK4**-frame, points in clusters may or may not be reflexive. In fact, each **wK4**-frame can be obtained from a **K4**-frame by deleting reflexive arrows in proper clusters, and so weakly transitive frames appear to be a modest generalization of transitive frames. But as we will see, the existence of irreflexive points in proper clusters causes additional technical difficulties.

The main interest in **wK4** stems from the topological semantics of modal logic. McKinsey and Tarski [13] introduced two topological semantics for modal logic: one is interpreting $\Diamond$ as topological closure, and another is interpreting $\Diamond$ as topological derivative. They showed that if we interpret $\Diamond$ as topological closure, then the modal logic of all topological spaces is **S4**. On the other hand, Esakia [9] showed that if we interpret $\Diamond$ as topological derivative, then the modal logic of all topological spaces is **wK4** and that **K4** is the modal logic of all T_d-spaces. We recall that a topological space X is a *T_d-space* if it satisfies the *T_d-separation axiom*: each point is locally closed; that is, each point is open in its closure. The T_d-separation axiom is a mild separation axiom, situated strictly in between T_0 and T_1 (see, e.g., [1]). In a recent paper [2], it was shown that the modal logic of all T_0-spaces is

$$\mathbf{wK4T}_0 = \mathbf{wK4} + p \wedge \Diamond(q \wedge \Diamond p) \to \Diamond p \vee \Diamond(q \wedge \Diamond q),$$

thus providing a useful modal logic strictly in between **wK4** and **K4**. In fact, there are continuum many logics between **wK4** and **K4**.

It is relatively easy to prove the FMP for **K4** by using the standard (transitive) filtration argument. It was shown in [2] that both **wK4** and $\mathbf{wK4T}_0$ also have the FMP, but the proofs are much more involved than that for **K4** (the reason being the technical difficulty mentioned above that proper clusters of **wK4**-frames may contain irreflexive points). Note that both **wK4** and $\mathbf{WK4T}_0$ are subframe logics, which are outside of the realm of subframe logics over **K4**, and so Fine's theorem does not apply to them. In this paper we show that all subframe and cofinal subframe logics over **wK4** also have the FMP.

In [3] we showed that for a Heyting algebra A and its dual space X, subframes of X give a dual characterization of nuclei on A, and we gave a relatively easy algebraic proof that each subframe and cofinal subframe superintuitionistic logic has the FMP. Diego's Theorem that implicative meet-semilattices are locally finite played a prominent role in our proof. Since each superintuitionistic logic is a fragment of a logic over **wK4**, we view this paper as a sequel to [3]. Here too we will use Diego's Theorem as well as the well-known fact that Boolean algebras are locally finite. However, unlike the case of superintuitionistic logics, our proof that each subframe and cofinal subframe logic over **wK4** has the FMP is much more involved.

The paper is organized as follows. In Section 2 we briefly recall the well-known duality between modal algebras and modal spaces. In Section 3 we prove some basic facts about wK4-algebras and their dual weakly transitive spaces. In Section 4 we discuss subframe and cofinal subframe logics over **wK4**. In Section 5 we prove the Main Lemma of the paper, which implies that each subframe and cofinal subframe logic over **wK4** has the FMP. Finally, in Section 6 we compare the proofs and techniques developed in this paper to those of [3].

2 Modal Algebras and Modal Spaces

We assume the reader's familiarity with the algebraic and general frame semantics of modal logic [4; 5; 12], and with the basics of topology [6].

We recall that a *modal algebra* is a pair $(B, \Box)$ such that B is a Boolean algebra and $\Box : B \to B$ is a unary function on B satisfying

1. $\Box(a \wedge b) = \Box a \wedge \Box b$,
2. $\Box 1 = 1$.

As usual, we define $\Diamond : B \to B$ by $\Diamond a = \neg\Box\neg a$. Then $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$ and $\Diamond 0 = 0$. Let MA denote the category of modal algebras and modal algebra homomorphisms.

Let X be a topological space. We recall that a subset S of X is *clopen* if S is closed and open, that X is *zero-dimensional* if clopen subsets of X form a basis, and that X is a *Stone space* if X is compact, Hausdorff, and zero-dimensional.

Let R be a binary relation on X. For $x \in X$ and $S \subseteq X$, let

$$R(x) = \{y \in X : xRy\} \text{ and } R^{-1}(S) = \{x \in X : \exists y \in S : xRy\}.$$

Then (X, R) is a *modal space* (also known as a *descriptive frame*) if X is a Stone space, $R(x)$ is closed for each $x \in X$, and $R^{-1}(S)$ is clopen for each clopen subset S of X.

Given two modal spaces (X, R) and (Y, Q), a map $f : X \to Y$ is a *modal space morphism* (also known as a *p-morphism*) if f is continuous, xRz implies $f(x)Qf(z)$, and $f(x)Qy$ implies there exists $z \in X$ such that xRz and $f(z) = y$. Let MS denote the category of modal spaces and modal space morphisms.

The next theorem is well known and forms the core of duality between modal algebras and modal spaces. We only give a sketch of the proof. The missing details can be found in any of [4; 5; 12; 15; 16]. We recall that two categories $\mathfrak{C}$ and $\mathfrak{D}$ are *dually equivalent* if $\mathfrak{C}$ is equivalent to the dual $\mathfrak{D}^d$ of $\mathfrak{D}$ (where the arrows of $\mathfrak{D}$ are reversed).

Theorem 2.1 MA *is dually equivalent to* MS.

Proof First define the contravariant functor $(-)_* : \text{MA} \to \text{MS}$ as follows. If $(B, \Box)$ is a modal algebra, then $(B, \Box)_* = (X, R)$, where X is the set of ultrafilters of B,

$$\varphi(a) = \{x \in X : a \in x\},$$

$\{\varphi(a) : a \in B\}$ is a basis for the topology on X, and

$$xRy \text{ iff } (\forall a \in B)(\Box a \in x \text{ implies } a \in y)$$

(equivalently $a \in y$ implies $\Diamond a \in x$ for all $a \in B$). Also, if $f : A \to B$ is a modal algebra homomorphism, then $f_* = f^{-1}$.

Next define the contravariant functor $(-)^* : \text{MS} \to \text{MA}$ as follows. For a modal space (X, R), let $\text{Cp}(X)$ denote the Boolean algebra of clopen subsets of X. Also, for $S \subseteq X$, let

$$\Box_R S = X - R^{-1}(X - S) = \{x \in X : R(x) \subseteq S\} \text{ and } \Diamond_R S = R^{-1}(S).$$

Then $(X, R)^* = (\text{Cp}(X), \Box_R)$, and if $f : X \to Y$ is a modal space morphism, then $f^* = f^{-1}$.

Consequently, $(-)_*$ and $(-)^*$ are well-defined contravariant functors. Moreover, φ sets a natural isomorphism between $(B, \Box)$ and $(B, \Box)_*{}^* = (\mathrm{Cp}(X), \Box_R)$, and so $\varphi(\Box b) = \Box_R \varphi(b)$ and $\varphi(\Diamond b) = \Diamond_R \varphi(b)$. Furthermore, $\varepsilon : X \to X^*{}_*$, given by

$$\varepsilon(x) = \{S \in \mathrm{Cp}(X) : x \in S\},$$

sets a natural isomorphism between (X, R) and $(X, R)^*{}_* = (\mathrm{Cp}(X), \Box_R)_*$. This yields the desired dual equivalence of MA and MS. □

3 wK4-Algebras and Weakly Transitive Spaces

Definition 3.1 Let $(B, \Box)$ be a modal algebra.

1. We call $(B, \Box)$ a *wK4-algebra* if $a \wedge \Box a \leq \Box\Box a$.
2. We call $(B, \Box)$ a *K4-algebra* if $\Box a \leq \Box\Box a$.
3. We call $(B, \Box)$ an *S4-algebra* if $\Box a \leq a$ and $\Box a \leq \Box\Box a$.

Clearly, $(B, \Box)$ is a wK4-algebra if and only if $\Diamond\Diamond a \leq a \vee \Diamond a$, $(B, \Box)$ is a K4-algebra if and only if $\Diamond\Diamond a \leq \Diamond a$, and $(B, \Box)$ is an S4-algebra if and only if $a \leq \Diamond a$ and $\Diamond\Diamond a \leq \Diamond a$. Let wK4 denote the category of wK4-algebras, K4 denote the category of K4-algebras, and S4 denote the category of S4-algebras. Clearly, S4 ⊂ K4 ⊂ wK4.

Definition 3.2 Let (X, R) be a modal space.

1. We call (X, R) a *weakly transitive space* if R is weakly transitive; that is, xRy and yRz imply $x = z$ or xRz.
2. We call (X, R) a *transitive space* if R is transitive.
3. We call (X, R) a *reflexive and transitive space* if R is reflexive and transitive.

The next lemma is well known. For (1) see [9, Proposition 7], and for (2)–(3) see, for example, [5, Section 5.2].

Lemma 3.3 *Let $(B, \Box)$ be a modal algebra and let $(X, R) = (B, \Box)_*$ be the dual of $(B, \Box)$.*

1. *$(B, \Box)$ is a wK4-algebra iff (X, R) is a weakly transitive space.*
2. *$(B, \Box)$ is a K4-algebra iff (X, R) is a transitive space.*
3. *$(B, \Box)$ is an S4-algebra iff (X, R) is a reflexive and transitive space.*

Let wTS denote the category of weakly transitive spaces, TS denote the category of transitive spaces, and RTS denote the category of reflexive and transitive spaces. Clearly, RTS ⊂ TS ⊂ wTS. As an immediate consequence of Theorem 2.1 and Lemma 3.3 we obtain the following theorem.

Theorem 3.4

1. wK4 *is dually equivalent to* wTS.
2. K4 *is dually equivalent to* TS.
3. S4 *is dually equivalent to* RTS.

Definition 3.5 Let $(B, \Box)$ be a wK4-algebra. For each $b \in B$, set

$$\Box^+ b = b \wedge \Box b.$$

It follows that

$$\Diamond^+ b = \neg\Box^+\neg b = b \vee \Diamond b.$$

For a weakly transitive space (X, R), let R^+ denote the *reflexive closure* of R; that is,

$$R^+ = R \cup \{(x, x) : x \in X\}.$$

Lemma 3.6 *Let* $(B, \Box)$ *be a wK4-algebra with the dual weakly transitive space* (X, R). *Then* $(B, \Box^+)$ *is an S4-algebra and* (X, R^+) *is a reflexive and transitive space, which is the dual space of* $(B, \Box^+)$.

Proof That $(B, \Box^+)$ is an S4-algebra follows from [9, Proposition 11]. Clearly, R^+ is reflexive and transitive. Since $R^+(x) = R(x) \cup \{x\}$ and both $R(x)$ and $\{x\}$ are closed, it follows that so is $R^+(x)$. Let $S \in \mathrm{Cp}(X)$. As $(R^+)^{-1}(S) = S \cup R^{-1}(S)$ and $R^{-1}(S) \in \mathrm{Cp}(X)$, we obtain $(R^+)^{-1}(S) \in \mathrm{Cp}(X)$. Therefore, (X, R^+) is a reflexive and transitive space. Lastly, as

$$\varphi(\Diamond^+ a) = \varphi(a \vee \Diamond a) = \varphi(a) \cup \Diamond_R \varphi(a) = \Diamond_{R^+} \varphi(a),$$

it follows that (X, R^+) is the dual space of $(B, \Box^+)$. □

Definition 3.7 For a wK4-algebra $(B, \Box)$, let

$$H := \Box^+(B) = \{\Box^+ b \ : \ b \in B\}.$$

Since $(B, \Box^+)$ is an S4-algebra, it is well known (see, e.g., [14, Section IV.1]) that $H = \{h \in B \ : \ \Box^+ h = h\}$, that H is a sublattice of B, and that H is a Heyting algebra with the implication given by

$$h \underset{H}{\to} h' := \Box^+(h \to h').$$

Let $(B, \Box)$ be a wK4-algebra with the dual weakly transitive space (X, R). We recall that $U \subseteq X$ is an *upset* of (X, R) if $x \in U$ and xRy imply $y \in U$. As follows from [7], elements of H dually correspond to clopen upsets of (X, R^+). It is easy to see that upsets of (X, R) are the same as upsets of (X, R^+). Consequently, elements of H dually correspond to clopen upsets of (X, R).

Let (X, R) be a weakly transitive space. Following Fine [10], for $S \subseteq X$, we call $x \in S$ a *maximal point* of S if xRy and $y \in S$ imply yRx. Let $\max(S)$ denote the set of maximal points of S. Also, let

$$\mu(S) := \{x \in S : R(x) \cap S = \varnothing\}.$$

Evidently $\mu(S) \subseteq \max(S)$. We note that $\max(S)$ coincides with the set $\max_{R^+}(S)$ of maximal points of S with respect to the relation R^+. The only difference between $\max(S)$ and $\max_{R^+}(S)$ is that all maximal points of S are reflexive with respect to R^+.

Next lemma generalizes a similar result of Fine [10, Section 5] for the transitive case to the weakly transitive case. We note that if $x \in \max(S)$, xRy, and $y \in S$, then $y \in \max(S)$.

Lemma 3.8 *Let* (X, R) *be a weakly transitive space. If* $S \in \mathrm{Cp}(X)$, *then for each* $x \in S$, *either* $x \in \mu(S)$ *or there exists* $y \in \max(S)$ *such that* xRy.

Proof Let $S \in \mathrm{Cp}(X)$ and $x \in S$. If $x \in \mu(S)$, then there is nothing to prove. Otherwise, as (X, R^+) is a reflexive transitive space, by [8, Section III.2], there exists $y \in \max(S)$ such that xR^+y. If xRy, then we are done. Otherwise, $x = y$, and so $y \notin \mu(S)$. Therefore, there exists $z \in S$ such that $x = yRz$. Since $y \in \max(S)$, we have $z \in \max(S)$. Thus, $xRz \in \max(S)$, which completes the proof. □

Lemma 3.9 *Let* (X, R) *be a weakly transitive space and let* $S \in \mathrm{Cp}(X)$. *Then*

1. $\Diamond_R S = \Diamond_R \max(S)$,
2. $\mu(S) = S - \Diamond_R S$,

3. $\Diamond_R S = \Diamond_{R^+} S - \mu(S)$.

Proof (1) Since $\max(S) \subseteq S$, we have $\Diamond_R \max(S) \subseteq \Diamond_R S$. Conversely, let $x \in \Diamond_R S$. Then there exists $y \in S$ such that xRy. By Lemma 3.8, either $y \in \mu(S) \subseteq \max(S)$ or there exists $z \in \max(S)$ such that yRz. In the former case, $x \in \Diamond_R \max(S)$. In the latter case, since R is weakly transitive, either xRz, and so again $x \in \Diamond_R \max(S)$, or $x = z$, which implies that both $x, y \in \max(S)$, and so yet again $x \in \Diamond_R \max(S)$. Therefore, in all possible cases, $x \in \Diamond_R \max(S)$, so $\Diamond_R S \subseteq \Diamond_R \max(S)$, and so $\Diamond_R S = \Diamond_R \max(S)$.

(2) We have $x \in \mu(S)$ if and only if $x \in S$ and $R(x) \cap S = \varnothing$ if and only if $x \in S$ and $x \notin \Diamond_R S$ if and only if $x \in S - \Diamond_R S$.

(3) We have $\Diamond_{R^+} S - \mu(S) = (S \cup \Diamond_R S) - (S - \Diamond_R S) = (S \cup \Diamond_R S) \cap ((X - S) \cup \Diamond_R S) = \Diamond_R S$. □

We conclude this section by the following lemma, which will be useful in Section 5.

Lemma 3.10 *Let $(B, \Box)$ be a wK4-algebra, $H = \Box^+(B)$, $b \in B$, and $h \in H$. Then*

1. $h \underset{H}{\to} \Box^+ b = \Box^+(h \to b)$,
2. $b \wedge \Box\neg b = b \wedge \Box^+\neg(b \wedge \Diamond b)$.

Proof (1) We have

$$\begin{aligned} h \wedge \Box^+(h \to b) &= \Box^+ h \wedge \Box^+(h \to b) \\ &= \Box^+(h \wedge (h \to b)) \\ &= \Box^+(h \wedge b) \\ &\leqslant \Box^+ b, \end{aligned}$$

and so

$$\Box^+(h \to b) \leqslant h \to \Box^+ b.$$

Applying $\Box^+$ gives

$$\Box^+(h \to b) = \Box^+\Box^+(h \to b) \leqslant \Box^+(h \to \Box^+ b).$$

The reverse inequality is trivial, so

$$\Box^+(h \to b) = \Box^+(h \to \Box^+ b) = h \underset{H}{\to} \Box^+ b.$$

(2) We have

$$\begin{aligned} b \wedge \Box\neg b &= b \wedge \Box\neg b \wedge \Box(\neg b \vee \Box\neg b) \\ &= (b \wedge \Box\neg b \wedge \Box(\neg b \vee \Box\neg b)) \vee 0 \\ &= (b \wedge \Box\neg b \wedge \Box(\neg b \vee \Box\neg b)) \vee (b \wedge \neg b \wedge \Box(\neg b \vee \Box\neg b)) \\ &= b \wedge (\Box\neg b \vee \neg b) \wedge \Box(\neg b \vee \Box\neg b) \\ &= b \wedge \Box^+(\neg b \vee \Box\neg b) \\ &= b \wedge \Box^+\neg(b \wedge \Diamond b). \end{aligned}$$

□

4 Subframe Logics over wK4

Let (X, R) be a modal space. For $S \subseteq X$, let R_S denote the restriction of R to S. It is easy to see that if S is a clopen subset of X, then (S, R_S) is again a modal space.

Definition 4.1 Let (X, R) be a modal space.

1. We say that $S \subseteq X$ is a *subframe* of X if $S \in \mathrm{Cp}(X)$.
2. We say that a subframe S of X is a *cofinal subframe* of X if $R(S) \subseteq (R^+)^{-1}(S)$.

Let L be a modal logic over $\mathbf{K}$ and let (X, R) be a modal space. We say that (X, R) is an *L-space* if each theorem of L is true in (X, R) under any valuation assigning clopen subsets of X to propositional letters.

Definition 4.2 Let L be a modal logic over $\mathbf{K}$.

1. We say that L is a *subframe logic* if for each L-space (X, R) and each subframe S of X, we have (S, R_S) is an L-space.
2. We say that L is a *cofinal subframe logic* if for each L-space (X, R) and each cofinal subframe S of X, we have (S, R_S) is an L-space.

It is obvious that each subframe logic is a cofinal subframe logic. The converse is not true in general. In fact, there are continuum many cofinal subframe logics which are not subframe logics (see, e.g, [5, Corollary 11.23]).

Let $(B, \Box)$ be a modal algebra and (X, R) be the dual modal space of $(B, \Box)$. It is well known (see, e.g., [17; 20]) that subframes of X correspond to relativizations of B. For $s \in B$, let $B_s := [0, s] = \{a \in B : a \leqslant s\}$, and for each $a, b \in B_s$, let

$$\begin{aligned} a \vee_s b &= a \vee b, \\ \neg_s a &= s \wedge \neg a, \\ 0_s &= 0, \\ 1_s &= s, \\ \Box_s a &= s \wedge \Box(s \to a). \end{aligned}$$

Then, as $a \leqslant s$, it is easy to see that

$$\Diamond_s a = \neg_s \Box_s \neg_s a = s \wedge \Diamond a.$$

Lemma 4.3 *If $(B, \Box)$ is a modal algebra and $s \in B$, then $(B_s, \Box_s)$ is a modal algebra.*

Proof It is clear (see, e.g., [14, Section II.6]) that B_s is a Boolean algebra. Moreover, for $a, b \in B_s$, we have

$$\begin{aligned} \Box_s(a \wedge b) &= s \wedge \Box(s \to (a \wedge b)) \\ &= s \wedge \Box((s \to a) \wedge (s \to b)) \\ &= s \wedge \Box(s \to a) \wedge \Box(s \to b) \\ &= \Box_s a \wedge \Box_s b. \end{aligned}$$

Furthermore,

$$\Box_s(1_s) = \Box_s(s) = s \wedge \Box(s \to s) = s \wedge \Box 1 = s \wedge 1 = s = 1_s.$$

Thus, $(B_s, \Box_s)$ is a modal algebra. $\Box$

Definition 4.4 For a modal algebra $(B, \Box)$ and $s \in B$, we call the modal algebra $(B_s, \Box_s)$ the *relativization* of $(B, \Box)$ to s.

Proposition 4.5 *Let $(B, \Box)$ be a modal algebra and (X, R) be its dual modal space. Then*

1. *subframes of (X, R) correspond to relativizations of $(B, \Box)$,*
2. *cofinal subframes of (X, R) correspond to those relativizations $(B_s, \Box_s)$ of $(B, \Box)$ for which*
$$s \le \Box\Diamond^+ s.$$

Proof (1) Let $S \subseteq X$. Then S is a subframe of X if and only if $S \in \mathrm{Cp}(X)$ if and only if there exists $s \in B$ such that $S = \varphi(s)$. Clearly, S with the subspace topology is the Stone space of B_s. Moreover, for each $a \in B_s$, we have

$$\varphi(\Diamond_s a) = \varphi(s \wedge \Diamond a) = \varphi(s) \cap \Diamond_R \varphi(a) = S \cap \Diamond_R \varphi(a) = \Diamond_{R_S} \varphi(a).$$

Thus, (S, R_S) is the dual space of $(B_s, \Box_s)$.

(2) Let S be a subframe of X. Then $S = \varphi(s)$ for some $s \in B$. Therefore,

$$\begin{aligned} S \text{ is a cofinal subframe} \ \ &\text{iff} \ \ R(S) \subseteq (R^+)^{-1}(S) \\ &\text{iff} \ \ R(S) \subseteq \Diamond_{R^+} \varphi(s) \\ &\text{iff} \ \ S \subseteq \Box_R \Diamond_{R^+} \varphi(s) \\ &\text{iff} \ \ \varphi(s) \subseteq \varphi(\Box\Diamond^+ s) \\ &\text{iff} \ \ s \le \Box\Diamond^+ s. \end{aligned}$$

Consequently, cofinal subframes of (X, R) correspond to those relativizations $(B_s, \Box_s)$ of $(B, \Box)$ for which $s \le \Box\Diamond^+ s$. $\Box$

Definition 4.6 Let $(B, \Box)$ be a wK4-algebra. We call $s \in B$ *dense* if $\Diamond^+ s = 1$.

Lemma 4.7 *Let $(B, \Box)$ be a wK4-algebra with the dual weakly transitive space (X, R), and let $s \in B$.*

1. *If s is dense, then $s \le \Box\Diamond^+ s$. Consequently, $\varphi(s)$ is a cofinal subframe of X.*
2. *If $\max(X) \subseteq \varphi(s)$, then s is dense.*

Proof (1) If s is dense, then $\Box\Diamond^+ s = \Box 1 = 1$, and so $s \le \Box\Diamond^+ s$. Thus, by Proposition 4.5, $\varphi(s)$ is a cofinal subframe of X.

(2) If $\max(X) \subseteq \varphi(s)$, then $\Diamond_R^+ \varphi(s) = X$, so $\Diamond^+ s = 1$, and so s is dense. $\Box$

Lemma 4.8 *Let $(B, \Box)$ be a modal algebra, $s \in B$, and $(B_s, \Box_s)$ be the relativization of $(B, \Box)$ to s.*

1. *If $(B, \Box)$ is a wK4-algebra, then so is $(B_s, \Box_s)$.*
2. *If $(B, \Box)$ is a K4-algebra, then so is $(B_s, \Box_s)$.*
3. *If $(B, \Box)$ is an S4-algebra, then so is $(B_s, \Box_s)$.*

Proof (1) For $a \in B_s$, we have

$$\begin{aligned} \Diamond_s \Diamond_s a = \Diamond_s (s \wedge \Diamond a) = s \wedge \Diamond (s \wedge \Diamond a) \le s \wedge \Diamond\Diamond a \le s \wedge (a \vee \Diamond a) = \\ (s \wedge a) \vee (s \wedge \Diamond a) = a \vee \Diamond_s a. \end{aligned}$$

Thus, $(B_s, \Box_s)$ is a wK4-algebra.

(2) For $a \in B_s$, we have

$$\Diamond_s \Diamond_s a = s \wedge \Diamond(s \wedge \Diamond a) \leq s \wedge \Diamond\Diamond a \leq s \wedge \Diamond a = \Diamond_s a.$$

Thus, $(B_s, \Box_s)$ is a K4-algebra.

(3) For $a \in B_s$, we have

$$\Diamond_s a = s \wedge \Diamond a \geq s \wedge a = a.$$

By (2) we also have that $\Diamond_s \Diamond_s a \leq \Diamond_s a$. Thus, $(B_s, \Box_s)$ is an S4-algebra. □

Consequently, each of **wK4**, **K4**, and **S4** is a subframe logic. Well-known examples of subframe logics over **K4** include the provability logic $\mathbf{GL} = \mathbf{K} + \Box(\Box p \to p) \to \Box p$ and the Grzegorczyk logic $\mathbf{S4.Grz} = \mathbf{S4} + \Box(\Box(p \to \Box p) \to p) \to p$, as well as each logic over $\mathbf{S4.3} = \mathbf{S4} + \Box(\Box p \to q) \vee \Box(\Box q \to p)$ (including all logics over $\mathbf{S5} = \mathbf{S4} + p \to \Box\Diamond p$). An example of a cofinal subframe logic which is not a subframe logic is the well-known system $\mathbf{S4.2} = \mathbf{S4} + \Diamond\Box p \to \Box\Diamond p$. More examples of subframe logics over **K4** can be found in [20]. We only recall from the introduction that **wK4** and $\mathbf{wK4T_0}$ are examples of interesting subframe logics outside of the realm of subframe logics over **K4**.

5 FMP for Subframe and Cofinal Subframe Logics over wK4

In this section we prove that each subframe and cofinal subframe logic over **wK4** has the FMP, thus extending the results of Fine [11] and Zakharyaschev [21] for **K4** to **wK4**. In fact, we prove the following general result, which is much stronger and implies the FMP of subframe and cofinal subframe logics over **wK4**.

Lemma 5.1 (Main Lemma) *Let $(B, \Box)$ be a wK4-algebra and let $\alpha(p_1, \ldots, p_n)$ be a modal formula built from the propositional letters $p_1, \ldots, p_n$. If $(B, \Box) \not\models \alpha(p_1, \ldots, p_n)$, then there exist a dense $s \in B$ and a finite subalgebra $(A_s, \Box_s)$ of the relativization $(B_s, \Box_s)$ of $(B, \Box)$ such that $(A_s, \Box_s) \not\models \alpha(p_1, \ldots, p_n)$.*

Idea of Proof Before proving the Main Lemma, which will be done in several steps, we give a general outline of the idea behind the proof. If $\alpha(p_1, \ldots, p_n)$ is refuted on a wK4-algebra $(B, \Box)$, then there exist $b_1, \ldots, b_n \in B$ such that $\alpha(b_1, \ldots, b_n) \neq 1$. Clearly, the subterms of $\alpha(b_1, \ldots, b_n)$ form a finite subset of B, and as B is locally finite, they generate a finite Boolean subalgebra B_α of B. Observe that to refute $\alpha(p_1, \ldots, p_n)$ in B, we only need elements of B_α. Next we form $\Box^+(B_\alpha) := \{\Box^+ b : b \in B_\alpha\}$, which is a subset of H. Clearly, $\Box^+(B_\alpha)$ is finite and as H is locally finite as an implicative meet-semilattice, by Diego's Theorem, the $(\wedge, \underset{H}{\to})$-subalgebra H_α of H generated by $\Box^+(B_\alpha)$ is also finite. As our next step, we generate the Boolean subalgebra A of B by $B_\alpha \cup H_\alpha$. Once again using that B is locally finite, we obtain that A is finite. Now the idea is to pick a dense $s \in B$ in such a way that $(A_s, \Box_s)$ is a subalgebra of $(B_s, \Box_s)$, where $A_s = \{a \wedge s : a \in A\}$, and show that $\alpha(p_1, \ldots, p_n)$ is refuted on $(A_s, \Box_s)$.

This indeed works if we start with a K4-algebra. However, for a wK4-algebra, the step which is problematic is to show that $(A_s, \Box_s)$ is a subalgebra of $(B_s, \Box_s)$. Therefore, we need to make H_α slightly bigger, which can be done by adding to $\Box^+(B_\alpha)$ some special elements of H and then generating H_α.

Now that we described the idea behind the proof (and discussed an additional difficulty we face when working with wK4-algebras instead of K4-algebras, which stems exactly from the existence of irreflexive points in proper clusters of the dual space of a wK4-algebra), we can go ahead and give a proof of the Main Lemma. As we said, it will be done in several steps and will require some additional lemmas.

Proof Let $(B, \Box)$ be a wK4-algebra and let $(B, \Box) \not\models \alpha(p_1, \ldots, p_n)$. Then there exist $b_1, \ldots, b_n \in B$ such that

$$\alpha(b_1, \ldots, b_n) \neq 1.$$

We let B_α denote the Boolean subalgebra of B generated by all subterms of $\alpha(b_1, \ldots, b_n)$. Since B is locally finite, B_α is finite. We also let A_α denote the set of all atoms of B_α.

Let H_α be the $(\wedge, \underset{H}{\rightarrow})$-subalgebra of H generated by the set

$$\left\{\Box^+ b : b \in B_\alpha\right\} \cup \left\{\Box^+ \neg(a \wedge \Diamond a) : a \in A_\alpha\right\}.$$

As this set is a finite subset of H, Diego's Theorem implies that H_α is finite.

Finally, let A be the Boolean subalgebra of B generated by $A_\alpha \cup H_\alpha$. Again, as B is locally finite, A is finite. For $a, b \in B$, let $b_a = b \wedge a$ denote the relativization of b to a, and let

$$s = \bigvee_{a \in A_\alpha} \bigwedge_{h \in H_\alpha} h_a \vee \Box_a^+ \neg_a h_a.$$

Since elements of A_α are pairwise orthogonal (that is, $a, b \in A_\alpha$ and $a \neq b$ imply $a \wedge b = 0$) and $\bigwedge_{h \in H_\alpha} h_a \vee \Box_a^+ \neg_a h_a \in B_a = [0, a]$ for each $a \in A_\alpha$, we have

$$s_a = s \wedge a = \bigwedge_{h \in H_\alpha} h_a \vee \Box_a^+ \neg_a h_a$$

for each $a \in A_\alpha$. Moreover, the s_a are pairwise orthogonal and

$$s = \bigvee_{a \in A_\alpha} s_a.$$

A slightly more explicit description of these elements is provided by the following lemma.

Lemma 5.2 *For each $a, h \in B$, we have*

$$\Box_a^+ \neg_a h_a = a \wedge \Box^+(h \to \neg a).$$

Proof We have

$$\begin{aligned}
\Box_a^+ \neg_a h_a &= a \wedge \Box^+ (a \to \neg_a h_a) \\
&= a \wedge \Box^+ (a \to (a \wedge \neg(a \wedge h))) \\
&= a \wedge \Box^+ (\neg a \vee (a \wedge (\neg a \vee \neg h))) \\
&= a \wedge \Box^+ (\neg a \vee (a \wedge \neg h)) \\
&= a \wedge \Box^+ (\neg a \vee \neg h) \\
&= a \wedge \Box^+ (h \to \neg a) .
\end{aligned}$$

□

We show that s is dense in B. In fact, we show a stronger result that s_a is dense in B_a for each $a \in A_\alpha$, which implies that s is dense in B.

Lemma 5.3 *Let (X, R) be a weakly transitive space and let U be a clopen upset of X. Then*

$$\max(X) \subseteq U \cup \Box_{R^+}(X - U).$$

Proof Let $x \in \max(X)$ and $x \notin U$. Since $x \in \max(X)$, we have $R^+(x) \subseteq \max(X)$. Therefore, as U is an upset, $R^+(x) \cap U \neq \varnothing$ implies $R^+(x) \subseteq U$. Thus, $R^+(x) \cap U = \varnothing$, so $R^+(x) \subseteq X - U$, and so $x \in \Box_{R^+}(X - U)$. Consequently, $\max(X) \subseteq U \cup \Box_{R^+}(X - U)$. $\Box$

Lemma 5.4 *For each $a \in A_\alpha$, we have s_a is dense in $(B_a, \Box_a)$. In particular, s is dense in $(B, \Box)$.*

Proof Since $s_a = \bigwedge_{h \in H_\alpha} h_a \vee \Box_a^+ \neg_a h_a$, by Lemma 4.7, it is sufficient to show that $\max \varphi(a) \subseteq \varphi(h_a) \cup \Box_{R^+}(\varphi(a) - \varphi(h_a))$. But this follows from Lemma 5.3 because each $\varphi(h_a)$ is a clopen upset of $\varphi(a)$. Thus, s_a is dense in $(B_a, \Box_a)$, and so s is dense in $(B, \Box)$. $\Box$

Lemma 5.5 *For each $a \in A_\alpha$ and $h \in H$, we have*

$$\Diamond(a \wedge h) = \Diamond(s \wedge a \wedge h).$$

Proof Since $s \wedge a \wedge h \leqslant a \wedge h$, we have $\Diamond(s \wedge a \wedge h) \leqslant \Diamond(a \wedge h)$. Conversely, it is sufficient to show that $\Diamond_R(\varphi(a) \cap \varphi(h)) \subseteq \Diamond_R(\varphi(s) \cap \varphi(a) \cap \varphi(h))$. As $\varphi(h)$ is an upset,

$$\max(\varphi(a) \cap \varphi(h)) \subseteq \max \varphi(a).$$

Moreover, by the proof of Lemma 5.4, $\max \varphi(a) \subseteq \varphi(s)$. Therefore, by Lemma 3.9,

$$\begin{aligned}\Diamond_R(\varphi(a) \cap \varphi(h)) &= \Diamond_R \max(\varphi(a) \cap \varphi(h)) \\ &= \Diamond_R(\varphi(s) \cap \max(\varphi(a) \cap \varphi(h))) \\ &\subseteq \Diamond_R(\varphi(s) \cap \varphi(a) \cap \varphi(h)).\end{aligned}$$

Thus, $\Diamond(a \wedge h) \leqslant \Diamond(s \wedge a \wedge h)$, hence the equality. $\Box$

Lemma 5.6 *The Boolean subalgebra $A_s := \{b_s : b \in A\}$ of $B_s = [0, s]$ is closed under $\Diamond_s$.*

Proof Since each element of A is a join of meets of elements of $B_\alpha \cup H_\alpha$ or their complements, B_α is closed under meets and complements, each element of B_α is a join of elements of A_α, and H_α is closed under meets, we obtain that each element of A is a join of elements of the form

$$a \wedge h \wedge \neg h_1 \wedge \cdots \wedge \neg h_n$$

for some $a \in A_\alpha$ and $h, h_1, \ldots, h_n \in H_\alpha$. By construction of s, for any $a \in A_\alpha$ and $h \in H_\alpha$ we have

$$s \wedge a \leqslant h_a \vee \Box_a^+ \neg_a h_a.$$

Therefore, by Lemma 5.2,

$$s \wedge a \leqslant (a \wedge h) \vee (a \wedge \Box^+(h \to \neg a)) = a \wedge (h \vee \Box^+(h \to \neg a)) \leqslant h \vee \Box^+(h \to \neg a).$$

Thus,

$$s \wedge a \wedge \neg h \leqslant \Box^+(h \to \neg a),$$

and so

$$s \wedge a \wedge \neg h \leqslant s \wedge a \wedge \Box^+(h \to \neg a).$$

On the other hand,

$$s \wedge a \wedge \Box^+(h \to \neg a) \leqslant s \wedge a \wedge (h \to \neg a) = s \wedge a \wedge (\neg h \vee \neg a) = s \wedge a \wedge \neg h.$$

Consequently,

$$s \wedge a \wedge \neg h = s \wedge a \wedge \Box^+(h \to \neg a).$$

By Lemma 3.10, we have

$$\Box^+(h \to \neg a) = h \underset{H}{\to} \Box^+\neg a.$$

By construction, $\Box^+\neg a \in H_\alpha$. Therefore, $\Box^+(h \to \neg a) \in H_\alpha$. Thus, each element of A_s is actually a join of elements of the form

$$s \wedge a \wedge h$$

for $a \in A_\alpha$ and $h \in H_\alpha$. Now, by Lemma 5.5,

$$\Diamond_s(s \wedge a \wedge h) = s \wedge \Diamond(s \wedge a \wedge h) = s \wedge \Diamond(a \wedge h).$$

Since $\Diamond_s$ is additive, it thus suffices to show that $\Diamond(a \wedge h)$ is in A.

By Lemma 3.9,

$$\Diamond_R(\varphi(a) \cap \varphi(h)) = \Diamond_{R^+}(\varphi(a) \cap \varphi(h)) - \mu(\varphi(a) \cap \varphi(h)),$$

and using the fact that $\varphi(h)$ is an upset, it is easy to see that

$$\mu(\varphi(a) \cap \varphi(h)) = \mu\varphi(a) \cap \varphi(h).$$

Therefore, using Lemma 3.9 again, we obtain

$$\begin{aligned}\Diamond_R(\varphi(a) \cap \varphi(h)) &= \Diamond_{R^+}(\varphi(a) \cap \varphi(h)) - (\mu\varphi(a) \cap \varphi(h))\\ &= \Diamond_{R^+}(\varphi(a) \cap \varphi(h)) - ((\varphi(a) - \Diamond_R\varphi(a)) \cap \varphi(h))\\ &= \Diamond_{R^+}(\varphi(a) \cap \varphi(h)) - (\varphi(a) \cap \Box_R(X - \varphi(a)) \cap \varphi(h)).\end{aligned}$$

Thus,

$$\Diamond(a \wedge h) = \Diamond^+(a \wedge h) - (a \wedge \Box\neg a \wedge h).$$

By Lemma 3.10,

$$a \wedge \Box\neg a = a \wedge \Box^+\neg(a \wedge \Diamond a),$$

By construction, $\Box^+\neg(a \wedge \Diamond a) \in H_\alpha$. Thus, $a \wedge \Box\neg a \wedge h \in A$. Moreover, as

$$\Diamond^+(a \wedge h) = \neg\Box^+(h \to \neg a)$$

and $\Box^+(h \to \neg a) \in H_\alpha$, we have $\Diamond^+(a \wedge h) \in A$. Consequently, $\Diamond(a \wedge h) \in A$, and so A_s is closed under $\Diamond_s$. □

It remains to show that $\alpha((b_1)_s, \ldots, (b_n)_s) \neq 1_s$ in A_s.

Lemma 5.7 $\alpha((b_1)_s, \ldots, (b_n)_s) = s \wedge \alpha(b_1, \ldots, b_n)$.

Proof It clearly suffices to prove that for each $a \in B_\alpha$, we have

$$\Diamond_s a_s = s \wedge \Diamond a.$$

As $\Diamond$ and $\Diamond_s$ are both additive, it actually suffices to prove the latter equality for $a \in A_\alpha$. But a particular case of Lemma 5.5 (with $h = 1$) gives

$$\Diamond_s a_s = s \wedge \Diamond(s \wedge a) = s \wedge \Diamond a.$$ □

Now, by Lemma 5.4, s_a is dense in $(B_a, \Box_a)$ for each $a \in A_\alpha$. Therefore, $s \wedge a \neq 0$ for each $a \in A_\alpha$. Moreover, $1 \neq \alpha(b_1, \dots, b_n) \in B_\alpha$, so there is an atom $a \in A_\alpha$ with $a \wedge \alpha(b_1, \dots, b_n) = 0$. As $a \wedge s \neq 0$, we cannot have $s \leqslant \alpha(b_1, \dots, b_n)$, so $s \wedge \alpha(b_1, \dots, b_n) \neq s$, which by Lemma 5.7 means

$$\alpha((b_1)_s, \dots, (b_n)_s) \neq 1_s.$$

Thus, we found a dense $s \in B$ and a finite subalgebra $(A_s, \Box_s)$ of $(B_s, \Box_s)$ such that $(A_s, \Box_s) \not\models \alpha(p_1, \dots, p_n)$. Consequently, the Main Lemma is proved. □

Remark 5.8 We show how our construction of $(A_s, \Box_s)$ simplifies when $(B, \Box)$ is a K4-algebra. Note that $\Box a \leqslant \Box\Box a$ is equivalent to $\Box^+\Box a = \Box a$. Therefore, if $(B, \Box)$ is a K4-algebra, then $\Box(B) := \{\Box b : b \in B\} \subseteq H$. Thus, in the proof of the Main Lemma, instead of working with $\Box^+(B_\alpha)$, we can work with $\Box(B_\alpha)$. Moreover, we do not need to add additional elements $\{\Box^+\neg(a \wedge \Diamond a) : a \in A_\alpha\}$ to $\Box^+(B_\alpha)$ to generate H_α. Instead we set H_α to be the $(\wedge, \underset{H}{\rightarrow})$-subalgebra of H generated by $\Box(B_\alpha)$. The only reason we needed $\{\Box^+\neg(a \wedge \Diamond a) : a \in A_\alpha\}$ was at the end of Lemma 5.6, in justifying that $\Diamond(a \wedge h)$ is in A for $a \in A_\alpha$ and $h \in H_\alpha$. But if $(B, \Box)$ is a K4-algebra, this is already clear from the equality $\Diamond(a \wedge h) = \Diamond^+(a \wedge h) - (a \wedge \Box\neg a \wedge h)$ because now $\Box\neg a \in H_\alpha$ by definition.

If, in addition, $(B, \Box)$ is an S4-algebra, then $\Box a = \Box^+ a$, and so $\Box(B) = \Box^+(B) = H$. Therefore, in addition to being able to take the simplified version of H_α as in the case of K4-algebras, we can also omit the last part of the proof of Lemma 5.6 altogether because in this case we have $\Diamond(a \wedge h) = \Diamond^+(a \wedge h) \in A$.

It is an easy consequence of the Main Lemma that each subframe and cofinal subframe logic over **wK4** has the FMP.

Theorem 5.9 *All subframe and cofinal subframe logics over* **wK4** *have the* FMP.

Proof Since subframe logics are contained in cofinal subframe logics, it is sufficient to prove the result for cofinal subframe logics. Let L be a cofinal subframe logic over **wK4** and let $L \not\models \alpha$. Then there exists a wK4-algebra $(B, \Box)$ such that $(B, \Box) \models L$ and $(B, \Box) \not\models \alpha$. By the Main Lemma, there exists a dense $s \in B$ and a finite subalgebra $(A_s, \Box_s)$ of the relativization $(B_s, \Box_s)$ of $(B, \Box)$ such that $(A_s, \Box_s) \not\models \alpha$. Let (X, R) be the dual weakly transitive space of $(B, \Box)$. By Lemma 4.7, $\varphi(s)$ is a cofinal subframe of X. Therefore, by Proposition 4.5, $(B_s, \Box_s) \models L$. Since $(A_s, \Box_s)$ is a subalgebra of $(B_s, \Box_s)$, we obtain $(A_s, \Box_s) \models L$. Thus, there exists a finite L-algebra $(A_s, \Box_s)$ refuting α, and so L has the FMP. □

We conclude this section by mentioning two possible applications of our method, which we leave as open problems. The first one is to study the size of $(A_s, \Box_s)$ and investigate whether our method sheds some new light on the computational complexity of satisfiability for subframe and cofinal subframe logics over **wK4**. The second one is to try to generalize our method to handle subframe and cofinal subframe logics in modal languages with several modalities. The first step in this direction would be to examine tense logics closely related to logics over **K4**. In [18; 19] Wolter gave a model-theoretic analysis of extensions of the Fine-Zakharyaschev results to tense logics. A natural next step would be to provide such an analysis for the algebraic technique developed in this paper.

6 Comparison of Subframe Logics in Modal and Intuitionistic Cases

We conclude the paper by comparing the proofs and techniques developed here with the proofs and techniques developed in [3] for subframe and cofinal subframe superintuitionistic logics. Let H be a Heyting algebra and let X be its dual space. Then X is a reflexive and transitive modal space, which in addition is antisymmetric. We recall (see [5, p. 289] and [3, Lemma 2]) that $S \subseteq X$ is a subframe of X if S is closed and $C \in \mathrm{Cp}(S)$ implies $R^{-1}(C) \in \mathrm{Cp}(X)$. It follows that each clopen subset of X is a subframe of X, but there exist subframes of X which may not be clopen (see [3, Remark 3]). Therefore, we have an evident difference between subframes in the intuitionistic and modal settings. Below we give an explanation of why this is so.

It was shown in [3] that subframes of X give a dual characterization of nuclei on H. Therefore, the notion of subframe in the intuitionistic setting naturally arises when studying nuclei on Heyting algebras. For a Heyting algebra H, let $N(H)$ denote the set of all nuclei on H. If X is the dual of H, then those subframes of X that are clopen subsets of X exactly correspond to those elements of $N(H)$ that are complemented in $N(H)$ [3, Theorem 32]. Now if it happens that H is a Boolean algebra, then $N(H)$ is isomorphic to H, and so each subframe of X is clopen. Thus, the intuitionistic notion of subframe, which is more general, coincides with the modal notion of subframe whenever the Heyting algebra under consideration happens to be a Boolean algebra.

On the other hand, proving that all subframe logics have the FMP is simpler in the intuitionistic setting. This is mostly because, instead of worrying about the whole $(B, \Box) \in \mathsf{wK4}$, we only need to worry about the Heyting algebra $H = \Box^+(B)$. Therefore, we only need to apply Diego's Theorem to the set of subterms of $\alpha(h_1, \dots, h_n)$ to generate a finite $(\wedge, \underset{H}{\to})$-subalgebra H_α of H, which will refute $\alpha(p_1, \dots, p_n)$. Then, using H_α, we define a nucleus j on H and show that H_α is a Heyting subalgebra of H_j. Since the superintuitionistic logic L under consideration is a subframe logic, $H \models L$ implies $H_j \models L$. As H_α is a Heyting subalgebra of H_j, we also have $H_\alpha \models L$. Thus, H_α is a finite L-algebra refuting $\alpha(p_1, \dots, p_n)$, and the FMP of L follows (see [3, Section 7] for details).

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Department of Mathematical Sciences
New Mexico State University
Las Cruces NM 88003
USA
gbezhani@nmsu.edu

Department of Computer Science
Università degli Studi di Milano
via Comelico 39
20135 Milano
ITALY
ghilardi@dsi.unimi.it

Department of Mathematical Logic
Razmadze Mathematical Institute
M. Aleksidze Str. 1
Tbilisi 0193
GEORGIA
jib@rmi.acnet.ge