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ON CERTAIN MEAN VALUES OF THE DOUBLE ZETA-FUNCTION

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Abstract. In this article we discuss three types of mean values of the Euler double zeta-function. To get the results, we introduce three approximate formulas for this function.

§1. Introduction

Let $s_1 = \sigma_1 + it_1$, and let $s_2 = \sigma_2 + it_2$, with $\sigma_1, \sigma_2, t_1, t_2 \in \mathbb{R}$. The Euler double zeta-function is defined by

$$\zeta_2(s_1, s_2) = \sum_{m=1}^{\infty} \frac{1}{m^{s_1}} \sum_{n=1}^{\infty} \frac{1}{(m+n)^{s_2}}. \tag{1.1}$$

This series is absolutely convergent for $\sigma_2 > 1$ and $\sigma_1 + \sigma_2 > 2$ (see [8, p. 419]). We can continue $\zeta_2(s_1, s_2)$ meromorphically to $\mathbb{C}^2$, which is holomorphic in

$$\{(s_1, s_2) \in \mathbb{C}^2 \mid s_2 \neq 1, s_1 + s_2 \notin \{2, 1, 0, -2, -4, -6, \dots\}\},$$

as was proved in [1]. (The first study of the analytic continuation of $\zeta_2(s_1, s_2)$ is Atkinson's work [2]. Akiyama, Egami, and Tanigawa in [1] studied the analytic continuation of not only $\zeta_2(s_1, s_2)$ but also more general multiple zeta-functions. Zhao in [15] also obtained the continuation independently.) The special values of $\zeta_2(s_1, s_2)$ have been studied by various authors (see, e.g., [4], [14]).

The analytic properties of $\zeta_2(s_1, s_2)$ were studied by various authors (e.g., [7]–[9]). Recently, Matsumoto and Tsumura [10] studied the mean values

$$\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_2, \tag{1.2}$$

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where s_1 is a fixed complex number. Theirs was the first study of the mean values of $\zeta_2(s_1,s_2)$. In our article we study (1.2) in the regions which are not covered in the work of Matsumoto and Tsumura and we introduce new types of mean values of $\zeta_2(s_1,s_2)$.

In the following we prove the next three theorems.

THEOREM 1.1. *Let* $s_1=\sigma_1+it_1$, *let* $s_2=\sigma_2+it_2\in\mathbb{C}$, *let* $T\geq 2$, *and let*

$$I^{[1]}(T)=\int_2^T \left|\zeta_2(s_1,s_2)\right|^2 dt_1.$$

Assume that when t_1 *moves from* 2 *to* T, *the point* $(s_1,s_2)\in\mathbb{C}^2$ *does not encounter the singularities of* $\zeta_2(s_1,s_2)$. *In the case* $\sigma_1+\sigma_2>2$, *we have*

$$I^{[1]}(T)=\zeta_2^{[1]}(2\sigma_1,s_2)T+O(1),$$

where, here and below, the implied constants depend on σ_1, σ_2, *and* t_2, *and* $\zeta_2^{[1]}(2\sigma_1,s_2)$ *is a series which converges on* $\sigma_1+\sigma_2>3/2$. *(We define* $\zeta_2^{[1]}(\sigma_1,s_2)$ *in the next section.) In the case* $3/2<\sigma_1+\sigma_2\leq 2$, *we have*

$$I^{[1]}(T)=\zeta_2^{[1]}(2\sigma_1,s_2)T+\begin{cases} O(T^{4-2\sigma_1-2\sigma_2}) & (3/2<\sigma_1+\sigma_2<2),\\ O((\log T)^2) & (\sigma_1+\sigma_2=2).\end{cases}$$

In the case $\sigma_1+\sigma_2=3/2$, *we have*

$$I^{[1]}(T)=|s_2-1|^{-2}T\log T+O(T).$$

THEOREM 1.2. *Let* $s_1=\sigma_1+it_1$, *let* $s_2=\sigma_2+it_2\in\mathbb{C}$, *let* $T\geq 2$, *and let*

$$I^{[2]}(T)=\int_2^T \left|\zeta_2(s_1,s_2)\right|^2 dt_2.$$

Assume that when t_2 *moves from* 2 *to* T, *the point* $(s_1,s_2)\in\mathbb{C}^2$ *does not encounter the singularities of* $\zeta_2(s_1,s_2)$. *In the case* $\sigma_2>1$ *and* $\sigma_1+\sigma_2>2$, *we have*

$$I^{[2]}(T)=\zeta_2^{[2]}(s_1,2\sigma_2)T+O(1),$$

where, here and below, the implied constants depend on σ_1, σ_2, *and* t_1, *and* $\zeta_2^{[2]}(s_1,2\sigma_2)$ *is a series which converges on* $\sigma_1+\sigma_2>3/2$ *and* $\sigma_2>1/2$.

($\zeta_2^{[2]}(s_1,\sigma_2)$ is used in [10], and we show the definition of $\zeta_2^{[2]}(s_1,\sigma_2)$ in the next section.) In the case $\sigma_1>1$ and $1/2<\sigma_2\le 1$, we have

$$I^{[2]}(T)=\zeta_2^{[2]}(s_1,2\sigma_2)T+\begin{cases}O(T^{2-2\sigma_2}) & (\sigma_2\neq 1),\\ O((\log T)^2) & (\sigma_2=1).\end{cases}$$

In the case $\sigma_1\le 1$, $3/2<\sigma_1+\sigma_2\le 2$, and $s_1\neq 1$, we have

$$I^{[2]}(T)=\zeta_2^{[2]}(s_1,2\sigma_2)T+\begin{cases}O(T^{4-2\sigma_1-2\sigma_2}) & (\sigma_1+\sigma_2\neq 2),\\ O((\log T)^2) & (\sigma_1+\sigma_2=2).\end{cases}$$

In the case $s_1=1$ and $1/2<\sigma_2\le 1$, we have

$$I^{[2]}(T)=\zeta_2^{[2]}(s_1,2\sigma_2)T+\begin{cases}O(T^{2-2\sigma_2}(\log T)^2) & (\sigma_2\neq 1),\\ O((\log T)^4) & (\sigma_2=1).\end{cases}$$

In the case $\sigma_1>1$ and $\sigma_2=1/2$, we have

$$I^{[2]}(T)=\left|\zeta(s_1)\right|^2 T\log T+O(T).$$

In the case $\sigma_1+\sigma_2=3/2$ and $\sigma_2>1/2$, we have

$$I^{[2]}(T)=|s_1-1|^{-2}T\log T+O(T).$$

In the case $\sigma_2=1/2$, $\sigma_1=1$, and $s_1\neq 1$, we have

$$I^{[2]}(T)=\left(|s_1-1|^{-2}+\left|\zeta(s_1)\right|^2\right)T\log T+O(T).$$

In the case $\sigma_2=1/2$ and $s_1=1$, we have

$$I^{[2]}(T)=\frac{T(\log T)^3}{3}+O\big(T(\log T)^2\big).$$

THEOREM 1.3. *Let $s_1=\sigma_1+it$, let $s_2=\sigma_2+it\in\mathbb{C}$, let $T\ge 2$, and let*

$$I^{\square}(T)=\int_2^T\left|\zeta_2(s_1,s_2)\right|^2dt.$$

In the case $\sigma_2>1$ and $\sigma_1+\sigma_2>2$, we have

$$I^{\square}(T)=\zeta_2^{\square}(\sigma_1,\sigma_2)T+O(1),$$

where, here and below, the implied constants depend on σ_1 *and* σ_2*, and* $\zeta_2^{\square}(\sigma_1,\sigma_2)$ *is a series which converges if and only if* $\sigma_2 > 1/2$ *and* $\sigma_1+\sigma_2>1$*. (We define* $\zeta_2^{\square}(\sigma_1,\sigma_2)$ *in the next section.) In the case* $\sigma_1>1$ *and* $1/2<\sigma_2\leq 1$*, we have*

$$I^{\square}(T)=\zeta_2^{\square}(\sigma_1,\sigma_2)T+O(T^{2-2\sigma_2+\epsilon})+O(T^{1/2})$$

for sufficiently small $\epsilon>0$*. In the case* $\sigma_1\leq 1$ *and* $3/2<\sigma_1+\sigma_2\leq 2$*, we have*

$$I^{\square}(T)=\zeta_2^{\square}(\sigma_1,\sigma_2)T+O(T^{4-2\sigma_1-2\sigma_2+\epsilon})+O(T^{1/2})$$

for sufficiently small $\epsilon>0$*. In the case* $\sigma_1>1$ *and* $\sigma_2=1/2$*, we have*

$$I^{\square}(T)\sim\frac{\zeta(2\sigma_1)\zeta(\sigma_1+1/2)^2}{\zeta(2\sigma_1+1)}T\log T.$$

Note that we can obtain $I^{[2]}(T)\sim|\zeta(s_1)|^2T\log T$ $(\sigma_1>1,\ \sigma_2=1/2)$ by

$$\zeta_2(s_1,s_2)+\zeta_2(s_2,s_1)=\zeta(s_1)\zeta(s_2)-\zeta(s_1+s_2) \tag{1.3}$$

and Theorem 1.1.

Matsumoto and Tsumura [10] introduced $I^{[2]}(T)$ and studied the cases

(1) $\sigma_1>1$ and $\sigma_2>1$ in [10, Theorem 1.1],
(2) $\sigma_1+\sigma_2>2$ and $1/2<\sigma_2\leq 1$ in [10, Theorem 1.2],
(3) $1/2<\sigma_1<3/2$, $1/2<\sigma_2\leq 1$ and $3/2<\sigma_1+\sigma_2\leq 2$ in [10, Theorem 1.3].

They conjectured that when $\sigma_1+\sigma_2=3/2$, the form of the main term of the mean square formula would not be CT (with a constant C; most probably, some log-factor would appear) (see [10, conjecture (ii), p. 385]). Our results include the regions which Matsumoto and Tsumura did not study and give an improvement on the error estimate. Moreover, by Theorem 1.2 we see that their conjecture (ii) is true.

Outlines of the proof of our theorems are as follows. We can obtain Theorems 1.1 and 1.2 by using the mean value theorems for Dirichlet polynomials and suitable approximate formulas in each theorem (see [10, Theorems 3.1 and 6.3]). The approximate formulas used in the proofs of Theorems 1.1 and 1.2 are derived from the Euler–Maclaurin formula and the simplest approximate formula to $\zeta(s)$ due to Hardy and Littlewood (see [13, p. 77]). On the other hand, we need a more elaborate method to get the proof of Theorem 1.3. To obtain the suitable approximate formula for $\zeta_2(\sigma_1+it,\sigma_2+it)$

we need the technique of Kiuchi and Tanigawa [6], which enables us to get good estimates of the error terms in the Euler–Maclaurin formula.

In Theorem 1.1 (resp., Theorem 1.2) we regard s_2 (resp., s_1) as a constant term. On the other hand, from the study of Kiuchi, Tanigawa, and Zhai [7], we know that the behavior of $|\zeta_2(s_1, s_2)|$ depends on both s_1 and s_2 strongly. Therefore, it is also important to consider a mean value which depends on both s_1 and s_2.

From Theorems 1.1 and 1.2 we may expect that the behavior of $\zeta_2(s_1, s_2)$ in the region $\sigma_1 + \sigma_2 = 3/2$ is special. (Matsumoto and Tsumura in [10, Remark 1.6] conjectured that $\sigma_1 + \sigma_2 = 3/2$ might be the double analogue of the critical line of the Riemann zeta-function. The error terms in Theorem 1.3 support their conjecture.) However, we can take a different point of view. For the Riemann zeta-function $\zeta(\sigma + it)$, we know that

$$\int_2^T \left|\zeta(\sigma + it)\right|^2 dt \sim \zeta(2\sigma)T$$

(for $\sigma > 1/2$) and

$$\int_2^T \left|\zeta(1/2 + it)\right|^2 dt \sim T \log T$$

hold (see, e.g., [13, Theorems 7.2 and 7.3]). The line $\sigma = 1/2$ is the critical line for $\zeta(\sigma + it)$, and the series

$$\zeta(2\sigma) = \sum_{n=1}^{\infty} \frac{1}{n^{2\sigma}}$$

diverges on $\sigma = 1/2$. On the other hand, $\zeta_2^{\square}(\sigma_1, \sigma_2)$ converges if and only if $\sigma_2 > 1/2$ and $\sigma_1 + \sigma_2 > 1$. Moreover, if $\sigma_1 = \sigma_2 > 1/2$, then $I^{\square}(T) \sim \zeta_2^{\square}(\sigma_1, \sigma_2)T$ holds by

$$\int_2^T \left|\zeta(\sigma + it)\right|^4 dt = O(T)$$

for $\sigma > 1/2$ (see [13, Theorem 7.5]) and Carlson's mean value theorem (see [12, p. 304]). Hence, we can expect that $I^{\square}(T) \sim \zeta_2^{\square}(\sigma_1, \sigma_2)T$ holds for $\sigma_2 > 1/2$ and $\sigma_1 + \sigma_2 > 1$ and that the boundary of the region $\sigma_2 > 1/2$ and $\sigma_1 + \sigma_2 > 1$ is an analogue of the critical line for $\zeta_2(\sigma_1 + it, \sigma_2 + it)$.

§2. Lemmas for the proof of theorems

In this section, we collect some auxiliary results and definitions.

First, we give the definition of $\zeta_2^{[1]}(\sigma_1, s_2)$, $\zeta_2^{[2]}(s_1, \sigma_2)$, and $\zeta_2^{\square}(\sigma_1, \sigma_2)$.

We define

$$\zeta_2^{[1]}(\sigma_1, s_2) = \sum_{m=1}^{\infty} \frac{1}{m^{\sigma_1}} \left| \zeta(s_2) - \sum_{n=1}^{m} \frac{1}{n^{s_2}} \right|^2$$

for $s_2 \neq 1$. Since we have

$$\zeta_2^{[1]}(2\sigma_1, s_2) \ll \sum_{m=1}^{\infty} \begin{cases} m^{2-2\sigma_1-2\sigma_2} & (\sigma_2 > 1), \\ m^{-2\sigma_1}(\log m)^2 & (\sigma_2 = 1), \\ m^{2-2\sigma_1-2\sigma_2} & (\sigma_2 < 1), \end{cases} \tag{2.1}$$

the series $\zeta_2^{[1]}(2\sigma_1, s_2)$ converges in the region $\sigma_1 + \sigma_2 > 3/2$.

We define

$$\zeta_2^{[2]}(s_1, \sigma_2) = \sum_{n=2}^{\infty} \left| \sum_{m=1}^{n-1} \frac{1}{m^{s_1}} \right|^2 \frac{1}{n^{\sigma_2}}.$$

(This definition is the same as [10, (1.2)].) Since we have

$$\zeta_2^{[2]}(s_1, 2\sigma_2) \ll \sum_{n=2}^{\infty} \begin{cases} n^{-2\sigma_2} & (\sigma_1 > 1), \\ n^{-2\sigma_2}(\log n)^2 & (\sigma_1 = 1), \\ n^{2-2\sigma_1-2\sigma_2} & (\sigma_1 < 1), \end{cases} \tag{2.2}$$

the series $\zeta_2^{[2]}(s_1, 2\sigma_2)$ converges in the region $\sigma_2 > 1/2$ and $\sigma_1 + \sigma_2 > 3/2$.

We define

$$\zeta_2^{\square}(\sigma_1, \sigma_2) = \sum_{k=2}^{\infty} \Bigg(\sum_{\substack{mn=k \\ m<n}} \frac{1}{m^{\sigma_1} n^{\sigma_2}} \Bigg)^2.$$

We note that $\#\{(m,n) \mid mn = k, m < n\} \ll k^{\epsilon}$ for any $\epsilon > 0$. Since

$$\begin{aligned} \zeta_2(2\sigma_1, 2\sigma_2) &< \zeta_2^{\square}(\sigma_1, \sigma_2) \\ &= \sum_{k=2}^{\infty} k^{-2\sigma_2} \Bigg(\sum_{\substack{m \mid k \\ m<\sqrt{k}}} \frac{1}{m^{\sigma_1-\sigma_2}} \Bigg)^2 \\ &\ll \sum_{k=2}^{\infty} \begin{cases} k^{-2\sigma_2+\epsilon} & (\sigma_1 \geq \sigma_2), \\ k^{-\sigma_1-\sigma_2+\epsilon} & (\sigma_1 < \sigma_2) \end{cases} \end{aligned} \tag{2.3}$$

for any $\epsilon > 0$, the series $\zeta_2^{\square}(\sigma_1, \sigma_2)$ converges if and only if $\sigma_2 > 1/2$ and $\sigma_1 + \sigma_2 > 1$.

LEMMA 2.1 ([5, Theorem 5.2]). *Let $a_1, \ldots, a_N$ be arbitrary complex numbers. Then*

$$\int_0^T \Big|\sum_{n\le N} a_n n^{it}\Big|^2 dt = T\sum_{n\le N}|a_n|^2 + O\Big(\sum_{n\le N} n|a_n|^2\Big), \tag{2.4}$$

which also remains valid if $N=\infty$, provided that the series on the right-hand side of (2.4) converges.

The following lemmas are well-known results for $\zeta(s)$ (see [3, p. 114] and [13, Theorem 4.11]).

LEMMA 2.2. *Let $s = \sigma + it \in \mathbb{C}$, let $m \in \mathbb{N}\cup\{0\}$, let $N \in \mathbb{N}$, and let $M = 2m+1$. For $\sigma > -2m$ we have*

$$\zeta(s) = \sum_{n\le N}\frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{N^{-s}}{2} + \sum_{k=1}^{2m}\frac{B_{k+1}}{(k+1)!}(s)_k N^{-(s+k)} + R_{M,N}(s),$$

where $(s)_k = s(s+1)\cdots(s+k-1)$ and

$$R_{M,N}(s) = -\frac{(s)_M}{M!}\int_N^\infty B_M\big(x-[x]\big)x^{-s-M}\,dx.$$

COROLLARY 2.1. *Let $s = 1+it$. For fixed $t>0$ we have*

$$\zeta(s) - \sum_{n\le N}\frac{1}{n^s} = \frac{N^{1-s}}{s-1} + O(N^{-1}) = O(1),$$

where the implied constants do not depend on N.

LEMMA 2.3. *Let $s = \sigma + it \in \mathbb{C}$. We have*

$$\zeta(s) = \sum_{1\le n\le x}\frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma})$$

uniformly for $\sigma \ge \sigma_0 > 0$, $x \ge 1$, and $|t| \le 2\pi x/C$, where C is a given constant greater than 1.

We use the following evaluations in this paper.

Remark 2.1. Let $T \geq 1$, and let $M \geq 1$, with $M \ll \log T$. For fixed $\alpha, \beta \geq 0$ we have

$$\sum_{k \leq M} \left(\frac{T}{2^k}\right)^{\alpha} \left(\log\left(\frac{T}{2^k}\right)\right)^{\beta} \ll T^{\alpha} \sum_{k \leq M} \left(\frac{1}{2^{\alpha}}\right)^{k} ((\log T)^{\beta} + k^{\beta})$$
$$\ll \begin{cases} T^{\alpha}(\log T)^{\beta} & (\alpha \neq 0), \\ (\log T)^{\beta+1} & (\alpha = 0). \end{cases}$$

§3. Proof of Theorem 1.1

In this section, we regard σ_1, s_2 as constants. We divide the proof into two cases.

Proof of Theorem 1.1 for $\sigma_1 + \sigma_2 > 2$. We set

$$a_m = \frac{1}{m^{\sigma_1}} \left(\zeta(s_2) - \sum_{n=1}^{m} \frac{1}{n^{s_2}}\right)$$

for $m \in \mathbb{N}$. If we assume that $\sigma_2 > 1$, then we have

$$\zeta_2(s_1, s_2) = \sum_{m=1}^{\infty} \frac{1}{m^{\sigma_1 + it_1}} \sum_{n=m+1}^{\infty} \frac{1}{n^{s_2}}$$
$$= \sum_{m=1}^{\infty} a_m m^{-it_1}.$$

The last series converges absolutely in $\sigma_1 + \sigma_2 > 2$. Since

$$\sum_{m=1}^{\infty} m|a_m|^2 = \sum_{m=1}^{\infty} \frac{1}{m^{2\sigma_1 - 1}} \left|\zeta(s_2) - \sum_{n=1}^{m} \frac{1}{n^{s_2}}\right|^2$$

converges by (2.1), we have

$$I^{[1]}(T) = \zeta_2^{[1]}(2\sigma_1, s_2)T + O(1)$$

by Lemma 2.1. ∎

In the case $3/2 \leq \sigma_1 + \sigma_2 \leq 2$, we use the following lemma.

LEMMA 3.1. *Let $s_1 = \sigma_1 + it_1$ with $t_1 \geq 1$, let $s_2 = \sigma_2 + it_2 \in \mathbb{C}$, and let $N \in \mathbb{N}$. Let $C > 1$ be a given constant. Assume that the point $(s_1, s_2) \in \mathbb{C}^2$ does not encounter the singularities of $\zeta_2(s_1, s_2)$. If $1 < |t_1 + t_2| < 2\pi N/C$, then we have*

$$\zeta_2(s_1, s_2) = \sum_{m \leq N} \frac{1}{m^{s_1}} \Big(\zeta(s_2) - \sum_{n=1}^{m} \frac{1}{n^{s_2}} \Big) + O(t_1^{-1} N^{2-\sigma_1-\sigma_2})$$

for $\sigma_1 + \sigma_2 > 1$ and any fixed σ_1, s_2.

Proof. Let $l \in \mathbb{N}$ with $\sigma_2 > -2l$. To obtain the analytic continuation of $\zeta_2(s_1, s_2)$, we regard s_1 and s_2 as complex variables, and we assume that $\sigma_1, \sigma_2 > 1$ temporarily. For any $N \in \mathbb{N}$, we have, say,

$$\zeta_2(s_1, s_2) = \sum_{m=1}^{N} \frac{1}{m^{s_1}} \sum_{n=m+1}^{\infty} \frac{1}{n^{s_2}} + \sum_{m=N+1}^{\infty} \frac{1}{m^{s_1}} \sum_{n=m+1}^{\infty} \frac{1}{n^{s_2}} = V_1 + V_2.$$

Since

$$V_1 = \sum_{m=1}^{N} \frac{1}{m^{s_1}} \Big(\zeta(s_2) - \sum_{n=1}^{m} \frac{1}{n^{s_2}} \Big),$$

V_1 is continued meromorphically to $\mathbb{C}^2$. By setting $M = 2l+1$ in Lemma 2.2, we have, say,

$$\begin{aligned}
V_2 &= \sum_{m=N+1}^{\infty} \frac{1}{m^{s_1}} \Big(\frac{m^{1-s_2}}{s_2 - 1} - \frac{m^{-s_2}}{2} + \sum_{k=1}^{M-1} \frac{B_{k+1}}{(k+1)!} (s_2)_k m^{-s_2-k} + R_{M,m}(s_2) \Big) \\
&= \frac{1}{s_2 - 1} \sum_{m=N+1}^{\infty} \frac{1}{m^{s_1+s_2-1}} - \frac{1}{2} \sum_{m=N+1}^{\infty} \frac{1}{m^{s_1+s_2}} \\
&\quad + \sum_{k=1}^{M-1} \frac{B_{k+1}}{(k+1)!} (s_2)_k \sum_{m=N+1}^{\infty} \frac{1}{m^{s_1+s_2+k}} + \sum_{m=N+1}^{\infty} \frac{1}{m^{s_1}} R_{M,m}(s_2) \\
&= \frac{1}{s_2 - 1} \Big(\zeta(s_1 + s_2 - 1) - \sum_{m=1}^{N} \frac{1}{m^{s_1+s_2-1}} \Big) - \frac{1}{2} \sum_{m=N+1}^{\infty} \frac{1}{m^{s_1+s_2}} \\
&\quad + \sum_{k=1}^{M-1} \frac{B_{k+1}}{(k+1)!} (s_2)_k \sum_{m=N+1}^{\infty} \frac{1}{m^{s_1+s_2+k}} + \sum_{m=N+1}^{\infty} \frac{1}{m^{s_1}} R_{M,m}(s_2) \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Since I_4 converges absolutely for $\sigma_2 > -M+1 = -2l$ and $\sigma_1+\sigma_2 > -1$, V_2 is continued meromorphically to $\sigma_2 > -2l$ and $\sigma_1+\sigma_2 > 1$. Now, we regard σ_1, s_2 as constants. By Lemma 2.3, we have $I_1 \ll t_1^{-1}N^{2-\sigma_1-\sigma_2}$. Also, we can easily obtain $I_2, I_3, I_4 \ll t_1^{-1}N^{2-\sigma_1-\sigma_2}$. This implies the lemma. ∎

Proof of Theorem 1.1 for $3/2 \le \sigma_1+\sigma_2 \le 2$. Let

$$a_m = m^{-\sigma_1}\Big(\zeta(s_2) - \sum_{n=1}^{m} n^{-s_2}\Big),$$

and let

$$m_0 = \max\Big\{m \in \mathbb{N} \,\Big|\, \frac{T}{2^m} > |t_2|+1\Big\}.$$

Note that

$$\sum_{m=1}^{\infty} |a_m|^2 = \zeta_2^{[1]}(2\sigma_1, s_2)$$

(in the case $\sigma_1+\sigma_2 > 3/2$) and that

$$m_0 < \frac{\log T - \log(|t_2|+1)}{\log 2} \le m_0+1$$

hold. We take $T \ge 2$ and $N \in \mathbb{N}$ with $|t_2|+1 < T$ and $3T < 2\pi N/C$, where $C > 1$, and we assume that $T < t_1 < 2T$. Then we have

$$1 < t_1 - |t_2| < |t_1+t_2| < |t_1|+|t_2| < 3T < \frac{2\pi N}{C}.$$

Therefore, we can use Lemma 3.1, and we have, say,

$$\zeta_2(s_1,s_2) = \sum_{m=1}^{N} a_m m^{-it_1} + O(t_1^{-1}N^{2-\sigma_1-\sigma_2}) = I_1 + I_2.$$

Since $a_m \ll m^{-\sigma_1-\sigma_2+1}$ by Corollary 2.1, we obtain

$$\sum_{m=1}^{N} m a_m^2 \ll \sum_{m=1}^{N} m^{3-2\sigma_1-2\sigma_2} \ll \begin{cases} \log N & (\sigma_1+\sigma_2 = 2), \\ N^{4-2\sigma_1-2\sigma_2} & (\sigma_1+\sigma_2 < 2) \end{cases}$$

and

$$I_1 \ll \sum_{m=1}^{N} a_m \ll \sum_{m=1}^{N} m^{1-\sigma_1-\sigma_2} \ll \begin{cases} \log N & (\sigma_1+\sigma_2 = 2), \\ N^{2-\sigma_1-\sigma_2} & (\sigma_1+\sigma_2 < 2). \end{cases}$$

Therefore, we have

$$\int_T^{2T} |I_1|^2 \, dt_1 = T \sum_{m=1}^{N} |a_m|^2 + \begin{cases} O(\log N) & (\sigma_1 + \sigma_2 = 2), \\ O(N^{4-2\sigma_1-2\sigma_2}) & (\sigma_1 + \sigma_2 < 2) \end{cases}$$

by Lemma 2.1 and

$$\begin{aligned} \int_T^{2T} |I_1 I_2| \, dt_1 &\ll N^{2-\sigma_1-\sigma_2} \max_{T<t_1<2T} |I_1| \\ &\ll \begin{cases} \log N & (\sigma_1 + \sigma_2 = 2), \\ N^{4-2\sigma_1-2\sigma_2} & (\sigma_1 + \sigma_2 < 2). \end{cases} \end{aligned}$$

On the other hand, we have

$$\int_T^{2T} |I_2|^2 \, dt_1 \ll N^{4-2\sigma_1-2\sigma_2} \int_T^{2T} \frac{dt_1}{t_1^2} \ll T^{-1} N^{4-2\sigma_1-2\sigma_2}.$$

Therefore, we have

$$\begin{aligned} &\int_T^{2T} \left|\zeta_2(s_1, s_2)\right|^2 dt_1 \\ &\quad = T \sum_{m=1}^{N} |a_m|^2 + \begin{cases} O(\log N) & (\sigma_1 + \sigma_2 = 2), \\ O(N^{4-2\sigma_1-2\sigma_2}) & (\sigma_1 + \sigma_2 < 2). \end{cases} \end{aligned}$$

By setting $N = [T] + 1$, we obtain

$$\begin{aligned} &\int_T^{2T} \left|\zeta_2(s_1, s_2)\right|^2 dt_1 \\ &\quad = T \sum_{m \le T} |a_m|^2 + \begin{cases} O(\log T) & (\sigma_1 + \sigma_2 = 2), \\ O(T^{4-2\sigma_1-2\sigma_2}) & (\sigma_1 + \sigma_2 < 2). \end{cases} \end{aligned} \tag{3.1}$$

Therefore, in the case $\sigma_1 + \sigma_2 > 3/2$, we have

$$\begin{aligned} &\int_T^{2T} \left|\zeta_2(s_1, s_2)\right|^2 dt_1 \\ &\quad = \zeta_2^{[1]}(2\sigma_1, s_2) T + \begin{cases} O(\log T) & (\sigma_1 + \sigma_2 = 2), \\ O(T^{4-2\sigma_1-2\sigma_2}) & (3/2 < \sigma_1 + \sigma_2 < 2). \end{cases} \end{aligned}$$

By this relation and Remark 2.1, we obtain

$$\begin{aligned}
&\int_{|t_2|+1}^{T} \left|\zeta_2(s_1,s_2)\right|^2 dt_1 \\
&\quad = \int_{T/2^{m_0}}^{T} \left|\zeta_2(s_1,s_2)\right|^2 dt_1 + O(1) \\
&\quad = \sum_{1\le k\le m_0} \int_{T/2^k}^{T/2^{k-1}} \left|\zeta_2(s_1,s_2)\right|^2 dt_1 + O(1) \\
&\quad = \zeta_2^{[1]}(2\sigma_1,s_2)T \sum_{1\le k\le m_0} \frac{1}{2^k} \\
&\qquad + \begin{cases} O(\sum_{1\le k\le m_0} \log \frac{T}{2^k}) & (\sigma_1+\sigma_2=2), \\ O(\sum_{1\le k\le m_0} (\frac{T}{2^k})^{4-2\sigma_1-2\sigma_2}) & (3/2<\sigma_1+\sigma_2<2) \end{cases} \\
&\quad = \zeta_2^{[1]}(2\sigma_1,s_2)T + \begin{cases} O((\log T)^2) & (\sigma_1+\sigma_2=2), \\ O(T^{4-2\sigma_1-2\sigma_2}) & (3/2<\sigma_1+\sigma_2<2). \end{cases}
\end{aligned}$$

This implies the theorem for $3/2<\sigma_1+\sigma_2\le 2$.

In the case $\sigma_1+\sigma_2=3/2$, since

$$a_m = m^{-\sigma_1}\Big(\zeta(s_2) - \sum_{n=1}^{m} n^{-s_2}\Big) = \frac{m^{1-\sigma_1-s_2}}{s_2-1} + O(m^{-\sigma_1-\sigma_2})$$

by Lemma 2.2, we have

$$|a_m|^2 = \frac{m^{-1}}{|s_2-1|^2} + O(m^{-2}).$$

Therefore, we have

$$\int_{T}^{2T} \left|\zeta_2(s_1,s_2)\right|^2 dt_1 = \frac{T\log T}{|s_2-1|^2} + O(T)$$

by (3.1). By Remark 2.1 and this relation, we obtain the theorem. ∎

§4. Proof of Theorem 1.2

In this section, we regard σ_2, s_1 as constants. We divide the proof into three cases.

Proof of Theorem 1.2 for $\sigma_2 > 1$ and $\sigma_1+\sigma_2 > 2$. We set

$$a_n = \frac{1}{n^{\sigma_2}} \sum_{m=1}^{n-1} \frac{1}{m^{s_1}}$$

for $n \in \mathbb{N}$. We have

$$\zeta_2(s_1,s_2) = \sum_{n=2}^{\infty} \Big(\sum_{m=1}^{n-1} \frac{1}{m^{s_1}} \Big) \frac{1}{n^{\sigma_2+it_2}} = \sum_{n=2}^{\infty} a_n n^{-it_2}.$$

Since

$$\sum_{n=2}^{\infty} n|a_n|^2 = \sum_{n=2}^{\infty} \Big(\sum_{m=1}^{n-1} \frac{1}{m^{s_1}} \Big) \frac{1}{n^{2\sigma_2-1}}$$

converges by (2.2), we have

$$I^{[2]}(T) = \zeta_2^{[2]}(s_1, 2\sigma_2)T + O(1)$$

by Lemma 2.1. ∎

We use the following lemma in the cases $\sigma_1 > 1$, $1/2 < \sigma_2 \le 1$ and $\sigma_1 \le 1$, $3/2 \le \sigma_1+\sigma_2 \le 2$.

LEMMA 4.1. *Let $s_1 = \sigma_1 + it_1$, let $s_2 = \sigma_2 + it_2 \in \mathbb{C}$ with $t_2 \ge 1$, and let $N \in \mathbb{N}$ with $N > e^2$. Let $C > 1$ be a given constant. Assume that the point $(s_1,s_2) \in \mathbb{C}^2$ does not encounter the singularities of $\zeta_2(s_1,s_2)$. If $1 < t_2 < 2\pi N/C$ and $1 < |t_1+t_2| < 2\pi N/C$, then we have*

$$\zeta_2(s_1,s_2) = \sum_{2\le n\le N} \Big(\sum_{m=1}^{n-1} \frac{1}{m^{s_1}} \Big) \frac{1}{n^{s_2}} + \begin{cases} O(t_2^{-1}N^{1-\sigma_2} + t_2^{-1}N^{2-\sigma_1-\sigma_2}) & (s_1 \ne 1), \\ O(t_2^{-1}N^{1-\sigma_2}\log N) & (s_1 = 1) \end{cases}$$

for $\sigma_2 \ge 1/2$, $\sigma_1+\sigma_2 > 1$ and any fixed σ_2, s_1.

Proof. Let $l \in \mathbb{N}$ with $\sigma_1 > -2l$. To obtain the analytic continuation of $\zeta_2(s_1,s_2)$, we regard s_1 and s_2 as complex variables and assume that $\sigma_1, \sigma_2 > 1$ temporarily. For any $N \in \mathbb{N}$, we have, say,

$$\zeta_2(s_1,s_2) = \sum_{2\le n\le N} \Big(\sum_{m=1}^{n-1} \frac{1}{m^{s_1}} \Big) \frac{1}{n^{s_2}} + \sum_{n>N} \Big(\sum_{m=1}^{n-1} \frac{1}{m^{s_1}} \Big) = U_1 + U_2. \tag{4.1}$$

The term U_1 is obviously holomorphic in $\mathbb{C}^2$. By setting $M = 2l + 1$ in Lemma 2.2, we have, say,

$$
\begin{aligned}
U_2 &= \sum_{n>N} \Big(\sum_{m=1}^{n} \frac{1}{m^{s_1}} - \frac{1}{n^{s_1}} \Big) \frac{1}{n^{s_2}} \\
&= \sum_{n>N} \Big(\zeta(s_1) - \frac{n^{1-s_1}}{s_1 - 1} - \frac{n^{-s_1}}{2} \\
&\quad - \sum_{k=1}^{M-1} \frac{B_{k+1}}{(k+1)!} (s_1)_k n^{-s_1-k} - R_{M,n}(s_1) \Big) \frac{1}{n^{s_2}} \\
&= \zeta(s_1) \sum_{n>N} \frac{1}{n^{s_2}} + \frac{1}{1-s_1} \sum_{n>N} \frac{1}{n^{s_1+s_2-1}} - \frac{1}{2} \sum_{n>N} \frac{1}{n^{s_1+s_2}} \\
&\quad - \sum_{k=1}^{M-1} \frac{B_{k+1}}{(k+1)!} (s_1)_k \sum_{n>N} \frac{1}{n^{s_1+s_2+k}} - \sum_{n>N} \frac{1}{n^{s_2}} R_{M,n}(s_1) \\
&= \zeta(s_1) \Big(\zeta(s_2) - \sum_{n=1}^{N} \frac{1}{n^{s_2}} \Big) \\
&\quad + \frac{1}{1-s_1} \Big(\zeta(s_1 + s_2 - 1) - \sum_{n=1}^{N} \frac{1}{n^{s_1+s_2-1}} \Big) \\
&\quad - \frac{1}{2} \sum_{n>N} \frac{1}{n^{s_1+s_2}} - \sum_{k=1}^{M-1} \frac{B_{k+1}}{(k+1)!} (s_1)_k \sum_{n>N} \frac{1}{n^{s_1+s_2+k}} \\
&\quad - \sum_{n>N} \frac{1}{n^{s_2}} R_{M,n}(s_1) \\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
\tag{4.2}
$$

Since I_5 converges absolutely for $\sigma_1 > -M + 1 = -2l$ and $\sigma_1 + \sigma_2 > -1$, U_2 is continued meromorphically to $\sigma_2 > 0$, $\sigma_1 > -2l$, and $\sigma_1 + \sigma_2 > 1$. Now, we regard σ_2, s_1 as constants.

In the case $s_1 \neq 1$, by Lemma 2.3 we have $I_1 \ll |s_1 - 1|^{-1} t_2^{-1} N^{1-\sigma_2}$ and $I_2 \ll |s_1 - 1|^{-1} t_2^{-1} N^{2-\sigma_1-\sigma_2}$. Also, we can easily obtain $I_3, I_4, I_5 \ll t_2^{-1} N^{2-\sigma_1-\sigma_2}$. This implies the lemma for $s_1 \neq 1$. In the case $s_1 = 1$, we obtain the lemma by using the maximum modulus principle. □

We prove Theorem 1.2 for $\sigma_1 > 1$, $1/2 \leq \sigma_2 \leq 1$ or $\sigma_1 \leq 1$, $3/2 \leq \sigma_1 + \sigma_2 \leq 2$. We divide the proof into the case $s_1 \neq 1$ and the case $s_1 = 1$.

Proof of Theorem 1.2 for $s_1 \neq 1$. We prove the theorem by the same argument as in the proof of Theorem 1.1.

Let

$$a_n = n^{-\sigma_2} \sum_{m=1}^{n-1} m^{-s_1}.$$

Note that

$$\sum_{n=2}^{\infty} |a_n|^2 = \zeta_2^{[2]}(s_1, 2\sigma_2)$$

in the case $\sigma_1 + \sigma_2 > 3/2$ and $\sigma_2 > 1/2$. We take $T \geq 2$ and $N \in \mathbb{N}$ with $N > e^2$, $|t_1| + 1 < T$, and $3T < 2\pi N/C$, where $C > 1$, and we assume that $T < t_2 < 2T$. Then we can use Lemma 4.1, and we have, say,

$$\zeta_2(s_1, s_2) = \sum_{n=2}^{N} a_n n^{-it_2} + O(t_2^{-1} N^{1-\sigma_2} + t_2^{-1} N^{2-\sigma_1-\sigma_2}) = I_1 + I_2.$$

Since

$$a_n \ll \begin{cases} n^{-\sigma_2} & (\sigma_1 \geq 1), \\ n^{-\sigma_1-\sigma_2+1} & (\sigma_1 < 1) \end{cases}$$

by Corollary 2.1, we obtain

$$\sum_{n=2}^{N} n a_n^2 \ll \begin{cases} \log N & (\sigma_2 = 1, \sigma_1 \geq 1), \\ N^{2-2\sigma_2} & (\sigma_2 < 1, \sigma_1 \geq 1), \\ \log N & (\sigma_1 + \sigma_2 = 2, \sigma_1 < 1), \\ N^{4-2\sigma_1-2\sigma_2} & (\sigma_1 + \sigma_2 < 2, \sigma_1 < 1) \end{cases}$$

and

$$I_1 \ll \sum_{n=2}^{N} a_n \ll \begin{cases} \log N & (\sigma_2 = 1, \sigma_1 \geq 1), \\ N^{1-\sigma_2} & (\sigma_2 < 1, \sigma_1 \geq 1), \\ \log N & (\sigma_1 + \sigma_2 = 2, \sigma_1 < 1), \\ N^{2-\sigma_1-\sigma_2} & (\sigma_1 + \sigma_2 < 2, \sigma_1 < 1). \end{cases}$$

Therefore, we have

$$\int_T^{2T} |I_1|^2 \, dt_2 = T \sum_{n=2}^{N} |a_n|^2 + \begin{cases} O(\log N) & (\sigma_2 = 1, \sigma_1 \geq 1), \\ O(N^{2-2\sigma_2}) & (\sigma_2 < 1, \sigma_1 \geq 1), \\ O(\log N) & (\sigma_1 + \sigma_2 = 2, \sigma_1 < 1), \\ O(N^{4-2\sigma_1-2\sigma_2}) & (\sigma_1 + \sigma_2 < 2, \sigma_1 < 1) \end{cases}$$

by Lemma 2.1 and

$$\int_T^{2T} |I_1 I_2| \, dt_2 \ll \begin{cases} (N^{1-\sigma_2} + N^{2-\sigma_1-\sigma_2}) \log N & (\sigma_2 = 1, \sigma_1 \geq 1), \\ (N^{1-\sigma_2} + N^{2-\sigma_1-\sigma_2}) N^{1-\sigma_2} & (\sigma_2 < 1, \sigma_1 \geq 1), \\ (N^{1-\sigma_2} + N^{2-\sigma_1-\sigma_2}) \log N & (\sigma_1 + \sigma_2 = 2, \sigma_1 < 1), \\ (N^{1-\sigma_2} + N^{2-\sigma_1-\sigma_2}) N^{2-\sigma_1-\sigma_2} & (\sigma_1 + \sigma_2 < 2, \sigma_1 < 1) \end{cases}$$

$$\ll \begin{cases} \log N & (\sigma_2 = 1, \sigma_1 \geq 1), \\ N^{2-2\sigma_2} & (\sigma_2 < 1, \sigma_1 \geq 1), \\ \log N & (\sigma_1 + \sigma_2 = 2, \sigma_1 < 1), \\ N^{4-2\sigma_1-2\sigma_2} & (\sigma_1 + \sigma_2 < 2, \sigma_1 < 1). \end{cases}$$

On the other hand, we have

$$\int_T^{2T} |I_2|^2 \, dt_2 \ll T^{-1}(N^{2-2\sigma_2} + N^{4-2\sigma_1-2\sigma_2}).$$

Therefore, we have

$$\int_T^{2T} \left|\zeta_2(s_1, s_2)\right|^2 dt_2 = T \sum_{n=2}^{N} |a_n|^2 + \begin{cases} O(\log N) & (\sigma_2 = 1, \sigma_1 \geq 1), \\ O(N^{2-2\sigma_2}) & (\sigma_2 < 1, \sigma_1 \geq 1), \\ O(\log N) & (\sigma_1 + \sigma_2 = 2, \sigma_1 < 1), \\ O(N^{4-2\sigma_1-2\sigma_2}) & (\sigma_1 + \sigma_2 < 2, \sigma_1 < 1). \end{cases}$$

By setting $N = [T] + 1$, we obtain

$$\begin{aligned} &\int_T^{2T} \left|\zeta_2(s_1, s_2)\right|^2 dt_2 \\ &= T \sum_{n \leq T} |a_n|^2 + \begin{cases} O(\log T) & (\sigma_2 = 1, \sigma_1 \geq 1), \\ O(T^{2-2\sigma_2}) & (\sigma_2 < 1, \sigma_1 \geq 1), \\ O(\log T) & (\sigma_1 + \sigma_2 = 2, \sigma_1 < 1), \\ O(T^{4-2\sigma_1-2\sigma_2}) & (\sigma_1 + \sigma_2 < 2, \sigma_1 < 1). \end{cases} \end{aligned} \tag{4.3}$$

Therefore, in the case $\sigma_1+\sigma_2>3/2$ and $\sigma_2>1/2$, we have

$$\int_T^{2T}\left|\zeta_2(s_1,s_2)\right|^2 dt_2 \\ = \zeta_2^{[2]}(s_1,2\sigma_2)T+\begin{cases} O(\log T) & (\sigma_2=1,\sigma_1\geq 1), \\ O(T^{2-2\sigma_2}) & (\sigma_2<1,\sigma_1\geq 1), \\ O(\log T) & (\sigma_1+\sigma_2=2,\sigma_1<1), \\ O(T^{4-2\sigma_1-2\sigma_2}) & (\sigma_1+\sigma_2<2,\sigma_1<1). \end{cases} \tag{4.4}$$

In the case $\sigma_1>1, \sigma_2=1/2$, since

$$a_n=n^{-\sigma_2}\sum_{m=1}^{n-1}m^{-s_1}=n^{-\sigma_2}\big(\zeta(s_1)+O(n^{-\sigma_1+1})\big)$$

by Lemma 2.2, we have

$$|a_n|^2=n^{-1}\left|\zeta(s_1)+O(n^{-\sigma_1+1})\right|^2=n^{-1}\left|\zeta(s_1)\right|^2+O(n^{-\sigma_1}).$$

Therefore, we obtain

$$\int_T^{2T}\left|\zeta_2(s_1,s_2)\right|^2 dt_2=\left|\zeta(s_1)\right|^2 T\log T+O(T) \tag{4.5}$$

by (4.3). In the case $\sigma_1<1$ and $\sigma_1+\sigma_2=3/2$, since

$$a_n=n^{-\sigma_2}\Big(\frac{n^{-s_1+1}}{s_1-1}+O(n^{-\sigma_1})+O(1)\Big)$$

by Lemma 2.2, we have

$$\begin{aligned}|a_n|^2&=n^{-2\sigma_2}\Big(\Big|\frac{n^{-s_1+1}}{s_1-1}\Big|^2+O(n^{-2\sigma_1+1})+O(n^{-\sigma_1+1})\Big)\\&=\frac{n^{-1}}{|s_1-1|^2}+O(n^{-2})+O(n^{-2+\sigma_1}).\end{aligned}$$

Therefore, we obtain

$$\int_T^{2T}\left|\zeta_2(s_1,s_2)\right|^2 dt_2=\frac{T\log T}{|s_1-1|^2}+O(T) \tag{4.6}$$

by (4.3). In the case $\sigma_1 = 1$ and $\sigma_2 = 1/2$, since

$$a_n = n^{-\sigma_2}\Big(\zeta(s_1) - \frac{n^{-s_1+1}}{s_1 - 1} + O(n^{-\sigma_1})\Big)$$

by Lemma 2.2, we have

$$\sum_{n\le T} |a_n|^2 = \sum_{n\le T}\Big(n^{-1}\Big|\zeta(s_1) - \frac{n^{-s_1+1}}{s_1 - 1}\Big|^2 + O(n^{-2})\Big)$$

by Corollary 2.1. Since we have

$$\begin{aligned}&\sum_{n\le T} n^{-1}\Big|\zeta(s_1) - \frac{n^{-s_1+1}}{s_1 - 1}\Big|^2 \\ &\quad= \big(\big|\zeta(s_1)\big|^2 + |s_1-1|^{-2}\big)\log T - 2\sum_{n\le T}\Re\Big(\overline{\zeta(s_1)}\frac{n^{-s_1}}{s_1-1}\Big) + O(1) \\ &\quad= \big(\big|\zeta(s_1)\big|^2 + |s_1-1|^{-2}\big)\log T + O(1)\end{aligned}$$

by Corollary 2.1, we have

$$\sum_{n\le T} |a_n|^2 = \big(\big|\zeta(s_1)\big|^2 + |s_1-1|^{-2}\big)\log T + O(1).$$

Therefore, we obtain

$$\int_T^{2T} \big|\zeta_2(s_1, s_2)\big|^2 dt_2 = \big(\big|\zeta(s_1)\big|^2 + |s_1-1|^{-2}\big)T\log T + O(T) \tag{4.7}$$

by (4.3). By (4.4), (4.5), (4.6), and (4.7), we can obtain the theorem by the same argument as in the proof of Theorem 1.1. □

Proof of Theorem 1.2 for $s_1 = 1$. We prove the theorem by the same argument as in the proof of Theorem 1.1.

Hereafter we use the same notation as in the preceding proof. Note that, in this case, we have $I_2 = O(t_2^{-1}N^{1-\sigma_2}\log N)$ by using Lemma 4.1. Since $a_n \ll n^{-\sigma_2}\log n$, we obtain

$$\sum_{n=2}^{N} n|a_n|^2 \ll \sum_{n=2}^{N} n^{1-2\sigma_2}(\log n)^2 \ll \begin{cases} O((\log N)^3) & (\sigma_2 = 1), \\ O(N^{2-2\sigma_2}(\log N)^2) & (\sigma_2 < 1) \end{cases}$$

and

$$I_1 \ll \sum_{n=2}^{N} |a_n| \ll \begin{cases} (\log N)^2 & (\sigma_2 = 1), \\ N^{1-\sigma_2}\log N & (\sigma_2 < 1). \end{cases}$$

Therefore, we have

$$\int_T^{2T} |I_1|^2\, dt_2 = T\sum_{n=2}^{N} |a_n|^2 + \begin{cases} O((\log N)^3) & (\sigma_2 = 1), \\ O(N^{2-2\sigma_2}(\log N)^2) & (\sigma_2 < 1) \end{cases}$$

by Lemma 2.1 and

$$\int_T^{2T} |I_1 I_2|\, dt_2 \ll \begin{cases} O((\log N)^3) & (\sigma_2 = 1), \\ O(N^{2-2\sigma_2}(\log N)^2) & (\sigma_2 < 1). \end{cases}$$

On the other hand, we have

$$\int_T^{2T} |I_2|^2\, dt_2 \ll T^{-1}N^{2-2\sigma_2}(\log N)^2.$$

Therefore, by setting $N = [T]+1$, we obtain

$$\int_T^{2T} \bigl|\zeta_2(s_1,s_2)\bigr|^2\, dt_2 = T\sum_{n\le T} |a_n|^2 + \begin{cases} O((\log T)^3) & (\sigma_2 = 1), \\ O(T^{2-2\sigma_2}(\log T)^2) & (\sigma_2 < 1). \end{cases} \tag{4.8}$$

In the case $\sigma_2 > 1/2$, we have

$$\begin{aligned} &\int_T^{2T} \bigl|\zeta_2(s_1,s_2)\bigr|^2\, dt_2 \\ &= \zeta_2^{[2]}(s_1, 2\sigma_2)T + \begin{cases} O((\log T)^3) & (\sigma_2 = 1), \\ O(T^{2-2\sigma_2}(\log T)^2) & (\sigma_2 < 1). \end{cases} \end{aligned} \tag{4.9}$$

In the case $\sigma_2 = 1/2$, since

$$|a_n|^2 = n^{-1}\Bigl(\sum_{m=1}^{n-1} m^{-1}\Bigr)^2 = \frac{(\log n)^2}{n} + O\Bigl(\frac{\log n}{n}\Bigr)$$

and

$$\sum_{n=2}^{N} \frac{(\log n)^2}{n} = \int_1^N x^{-1}(\log x)^2\, dx + O(1) = \frac{(\log N)^3}{3} + O(1)$$

hold, we have

$$\int_T^{2T} \left|\zeta_2(s_1,s_2)\right|^2 dt_2 = \frac{T(\log T)^3}{3} + O\big(T(\log T)^2\big) \tag{4.10}$$

by (4.8). By (4.9) and (4.10), we can obtain the theorem by the same argument as in the proof of Theorem 1.1. ∎

§5. Proof of Theorem 1.3

We divide the proof into four cases.

Proof of Theorem 1.3 for $\sigma_2 > 1$ and $\sigma_1 + \sigma_2 > 2$. We set

$$a_k = \Big(\sum_{\substack{m|k\\ m<\sqrt{k}}} \frac{1}{m^{\sigma_1-\sigma_2}}\Big)\frac{1}{k^{\sigma_2}}$$

for $k \in \mathbb{N}$. We have

$$\begin{aligned}\zeta_2(s_1,s_2) &= \sum_{1\le m<n} \frac{1}{m^{\sigma_1} n^{\sigma_2} (mn)^{it}} \\ &= \sum_{k\ge 2}\Big(\sum_{\substack{mn=k\\ m<n}} \frac{1}{m^{\sigma_1} n^{\sigma_2}}\Big)\frac{1}{k^{it}} \\ &= \sum_{k\ge 2}\Big(\sum_{\substack{m|k\\ m<\sqrt{k}}} \frac{1}{m^{\sigma_1-\sigma_2}}\Big)\frac{1}{k^{\sigma_2+it}} \\ &= \sum_{k\ge 2} a_k k^{-it}.\end{aligned}$$

Since

$$\sum_{k\ge 2} k|a_k|^2 = \sum_{k\ge 2}^{\infty}\Big(\sum_{\substack{m|k\\ m<\sqrt{k}}} \frac{1}{m^{\sigma_1-\sigma_2}}\Big)^2 \frac{1}{k^{2\sigma_2-1}}$$

converges by (2.3), we have

$$I^{\square}(T) = \zeta_2^{\square}(\sigma_1,\sigma_2)T + O(1)$$

by Lemma 2.1. ∎

We use the following lemma in the cases $\sigma_1 > 1$, $1/2 < \sigma_2 \le 1$ and $\sigma_1 \le 1$, $3/2 < \sigma_1 + \sigma_2 \le 2$.

LEMMA 5.1. *Let $\sigma_1 + \sigma_2 > 1$, let $\sigma_2 > 0$, let $s_1 = \sigma_1 + it$, and let $s_2 = \sigma_2 + it$. Then*

$$\zeta_2(s_1, s_2) = \sum_{n \le t} n^{-s_2} \sum_{m=1}^{n-1} m^{-s_1} + \begin{cases} O(t^{-\sigma_2}) & (\sigma_1 > 1), \\ O(t^{-\sigma_2+\epsilon}) & (\sigma_1 = 1), \\ O(t^{1-\sigma_1-\sigma_2}) & (\sigma_1 < 1) \end{cases}$$

holds for $t \ge 2$, where the implied constants depend on σ_1, σ_2.

To prove Lemma 5.1, we use the following lemma and corollary.

LEMMA 5.2 ([6, Lemma 2.2]). *Let $s = \sigma + it$, $|t| > 1$. For $N > (1/4)|t|$, $m \ge 1$, and $\sigma > -2m - 1$, we have*

$$\zeta(s) = \sum_{n \le N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{N^{-s}}{2} + \sum_{k=1}^{2m} \frac{B_{k+1}}{(k+1)!} (s)_k N^{-(s+k)} + O\big(|t|^{2m+1} N^{-\sigma-2m-1}\big),$$

where the implied constant does not depend on t.

COROLLARY 5.1 ([6, Corollary 2.3]). *Let $s = \sigma + it$, $|t| > 1$. For $N > (1/4)|t|$ and $\sigma > -3$, we have*

$$\zeta(s) = \sum_{n \le N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{N^{-s}}{2} + \frac{s}{12} N^{-s-1} + O\big(|t|^3 N^{-\sigma-3}\big),$$

where the implied constant does not depend on t.

The following proof is similar to that in [6, Section 4.1].

Proof of Lemma 5.1. Let $l \in \mathbb{N}$ with $\sigma_1 > -2l$. We use (4.1) and (4.2). Hence, we obtain the analytic continuation of $\zeta_2(s_1, s_2)$ for $\sigma_2 > 0$, $\sigma_1 > -2l$, and $\sigma_1 + \sigma_2 > 1$. Now, we set $s_1 = \sigma_1 + it$, $s_2 = \sigma_2 + it$ with $t \ge 1$ and $N = [t]$.

Then we have

$$\begin{aligned}
I_1 &= \zeta(s_1)\Big(\frac{N^{1-s_2}}{s_2-1} - \frac{N^{-s_2}}{2} + \frac{s_2}{12}N^{-s_2-1} + O\big(|t|^3 N^{-\sigma_2-3}\big)\Big) \\
&\ll \big|\zeta(s_1)\big| t^{-\sigma_2} \\
&\ll \begin{cases} t^{-\sigma_2} & (\sigma_1 > 1), \\ t^{-\sigma_2+\epsilon} & (\sigma_1 = 1), \\ t^{1-\sigma_1-\sigma_2} & (\sigma_1 < 1) \end{cases}
\end{aligned}$$

for $\sigma_2 > -3$ by Corollary 5.1. Similarly, we have

$$\begin{aligned}
I_2 = \frac{1}{1-s_1}\Big(&\frac{N^{2-s_1-s_2}}{s_1+s_2-2} - \frac{1}{2}N^{1-s_1-s_2} + \frac{s_1+s_2-1}{12}N^{-s_1-s_2} \\
&+ O\big(|t|^3 N^{1-\sigma_1-\sigma_2-3}\big)\Big) \\
\ll\, & t^{1-\sigma_1-\sigma_2}
\end{aligned}$$

for $\sigma_1 + \sigma_2 > -2$. Since $\sigma_1 + \sigma_2 > 1$, we have

$$I_j \ll t^{1-\sigma_1-\sigma_2} \quad (j = 3, 4).$$

On the other hand, $R_{M,n}(s_1) = O(t^M n^{-\sigma_1-M})$ for $\sigma_1 > -M$ by Lemma 5.2. Hence, we have

$$I_5 \ll t^M \sum_{n>N} \frac{1}{n^{\sigma_1+\sigma_2+M}} \ll t^{1-\sigma_1-\sigma_2}.$$

This implies the lemma. ∎

We prove Theorem 1.3 for $\sigma_1 > 1$, $1/2 < \sigma_2 \le 1$ or $\sigma_1 \le 1$, $3/2 < \sigma_1 + \sigma_2 \le 2$. First we consider the case $\sigma_1 > \sigma_2$. In particular, this condition is satisfied when $\sigma_1 > 1$ and $1/2 < \sigma_2 \le 1$.

Proof of Theorem 1.3 for $\sigma_1 > \sigma_2$. If we set

$$A(s_1, s_2) = \sum_{n \le t} n^{-s_2} \sum_{m=1}^{n-1} m^{-s_1},$$

then we have, say,

$$\begin{aligned}
&\int_2^T \left|A(s_1,s_2)\right|^2 dt \\
&= \int_2^T \Big(\sum_{n_1\le t} n_1^{-s_2} \sum_{m_1=1}^{n_1-1} m_1^{-s_1} \sum_{n_2\le t} n_2^{-\overline{s_2}} \sum_{m_2=1}^{n_2-1} m_2^{-\overline{s_1}}\Big)\, dt \\
&= \sum_{2\le n_1\le T} \sum_{m_1=1}^{n_1-1} \sum_{2\le n_2\le T} \sum_{m_2=1}^{n_2-1} n_1^{-\sigma_2} m_1^{-\sigma_1} n_2^{-\sigma_2} m_2^{-\sigma_1} \int_{M(n_1,n_2)}^T \Big(\frac{m_2 n_2}{m_1 n_1}\Big)^{it} dt \\
&= \sum_{m_1n_1=m_2n_2} \sum_{\substack{1\le m_1\le n_1-1\\ 2\le n_1\le T}} \sum_{\substack{1\le m_2\le n_2-1\\ 2\le n_2\le T}} n_1^{-\sigma_2} m_1^{-\sigma_1} n_2^{-\sigma_2} m_2^{-\sigma_1} \big(T - M(n_1,n_2)\big) \\
&\quad + \sum_{m_1n_1\ne m_2n_2} \sum_{\substack{1\le m_1\le n_1-1\\ 2\le n_1\le T}} \sum_{\substack{1\le m_2\le n_2-1\\ 2\le n_2\le T}} n_1^{-\sigma_2} m_1^{-\sigma_1} n_2^{-\sigma_2} m_2^{-\sigma_1} \\
&\quad \times \frac{\exp(iT\log(\frac{m_2n_2}{m_1n_1})) - \exp(iM(n_1,n_2)\log(\frac{m_2n_2}{m_1n_1}))}{i\log(\frac{m_2n_2}{m_1n_1})} \\
&= S_1 T - S_2 + S_3,
\end{aligned}$$

where $M(n_1,n_2) = \max(n_1,n_2)$. First, we rewrite

$$\begin{aligned}
S_1 &= \sum_{2\le k\le T} \Big(\sum_{\substack{mn=k\\ m<n\le T}} m^{-\sigma_1} n^{-\sigma_2}\Big)^2 + \sum_{T<k<T^2} \Big(\sum_{\substack{mn=k\\ m<n\le T}} m^{-\sigma_1} n^{-\sigma_2}\Big)^2 \\
&= \sum_{k=2}^{\infty} \Big(\sum_{\substack{mn=k\\ m<n}} m^{-\sigma_1} n^{-\sigma_2}\Big)^2 - \sum_{k>T} \Big(\sum_{\substack{mn=k\\ m<n}} m^{-\sigma_1} n^{-\sigma_2}\Big)^2 \\
&\quad + \sum_{T<k<T^2} \Big(\sum_{\substack{mn=k\\ m<n\le T}} m^{-\sigma_1} n^{-\sigma_2}\Big)^2.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{k>T} \Big(\sum_{mn=k} m^{-\sigma_1} n^{-\sigma_2}\Big)^2 &= \sum_{k>T} \Big(\sum_{m|k} m^{-\sigma_1} m^{\sigma_2} k^{-\sigma_2}\Big)^2 \\
&\ll \sum_{k>T} k^{-2\sigma_2+\epsilon} \ll T^{1-2\sigma_2+\epsilon},
\end{aligned}$$

we have

$$S_1 = \zeta_2^{\square}(\sigma_1, \sigma_2) + O(T^{1-2\sigma_2+\epsilon}).$$

Next, we rewrite, say,

$$\begin{aligned}
S_2 &\ll \sum_{m_1 n_1 = m_2 n_2} \sum_{\substack{1\le m_1\le n_1-1 \\ 2\le n_1\le T}} \sum_{\substack{1\le m_2\le n_2-1 \\ 2\le n_2\le T}} n_1^{-\sigma_2} m_1^{-\sigma_1} n_2^{-\sigma_2} m_2^{-\sigma_1} O(n_1+n_2) \\
&\ll \sum_{2\le k<T^2} \Big(\sum_{\substack{mn=k \\ 1\le m<n\le T}} m^{-\sigma_1} n^{1-\sigma_2}\Big)\Big(\sum_{\substack{mn=k \\ 1\le m<n\le T}} m^{-\sigma_1} n^{-\sigma_2}\Big) \\
&\ll \sum_{2\le k\le T} \Big(\sum_{\substack{mn=k \\ 1\le m<n\le T}} m^{-\sigma_1} n^{1-\sigma_2}\Big)\Big(\sum_{\substack{mn=k \\ 1\le m<n\le T}} m^{-\sigma_1} n^{-\sigma_2}\Big) \\
&\quad + \sum_{T<k<T^2} \Big(\sum_{\substack{mn=k \\ 1\le m<n\le T}} m^{-\sigma_1} n^{1-\sigma_2}\Big)\Big(\sum_{\substack{mn=k \\ 1\le m<n\le T}} m^{-\sigma_1} n^{-\sigma_2}\Big) \\
&= A_1 + A_2.
\end{aligned}$$

Since we have

$$\begin{aligned}
A_1 &= \sum_{2\le k\le T} k^{1-2\sigma_2} \Big(\sum_{\substack{m|k \\ m<\sqrt{k}}} m^{\sigma_2-\sigma_1-1}\Big)\Big(\sum_{\substack{m|k \\ m<\sqrt{k}}} m^{\sigma_2-\sigma_1}\Big) \\
&\ll \sum_{2\le k\le T} k^{1-2\sigma_2+\epsilon} \ll T^{2-2\sigma_2+\epsilon}
\end{aligned}$$

and

$$\begin{aligned}
A_2 &\ll \sum_{T<k<T^2} k^{-2\sigma_1} \Big(\sum_{\substack{n|k \\ n\le T}} n^{1+\sigma_1-\sigma_2}\Big)\Big(\sum_{\substack{n|k \\ n\le T}} n^{\sigma_1-\sigma_2}\Big) \\
&\ll T^{1+2\sigma_1-2\sigma_2+\epsilon} \sum_{T<k<T^2} k^{-2\sigma_1} \ll T^{2-2\sigma_2+\epsilon},
\end{aligned}$$

we have $S_2 \ll T^{2-2\sigma_2+\epsilon}$. Next, we have, say,

$$\begin{aligned}
S_3 = &\sum_{m_1 n_1 \ne m_2 n_2} \sum_{\substack{1\le m_1\le n_1-1 \\ 2\le n_1\le T}} \sum_{\substack{1\le m_2\le n_2-1 \\ 2\le n_2\le T}} n_1^{-\sigma_2} m_1^{-\sigma_1} n_2^{-\sigma_2} m_2^{-\sigma_1} \\
&\times \frac{\exp(iT\log(\frac{m_2 n_2}{m_1 n_1})) - \exp(iM(n_1,n_2)\log(\frac{m_2 n_2}{m_1 n_1}))}{i\log(\frac{m_2 n_2}{m_1 n_1})}
\end{aligned}$$

$$\ll \sum_{m_1n_1<m_2n_2} \sum_{\substack{1\le m_1\le n_1-1\\ 2\le n_1\le T}} \sum_{\substack{1\le m_2\le n_2-1\\ 2\le n_2\le T}} n_1^{-\sigma_2}m_1^{-\sigma_1}n_2^{-\sigma_2}m_2^{-\sigma_1}\frac{1}{\log(\frac{m_2n_2}{m_1n_1})}$$

$$= \sum_{m_1n_1<m_2n_2<2m_1n_1} \sum_{\substack{1\le m_1\le n_1-1\\ 2\le n_1\le T}} \sum_{\substack{1\le m_2\le n_2-1\\ 2\le n_2\le T}} n_1^{-\sigma_2}m_1^{-\sigma_1}n_2^{-\sigma_2}m_2^{-\sigma_1}\frac{1}{\log(\frac{m_2n_2}{m_1n_1})}$$

$$+ \sum_{m_2n_2\ge 2m_1n_1} \sum_{\substack{1\le m_1\le n_1-1\\ 2\le n_1\le T}} \sum_{\substack{1\le m_2\le n_2-1\\ 2\le n_2\le T}} n_1^{-\sigma_2}m_1^{-\sigma_1}n_2^{-\sigma_2}m_2^{-\sigma_1}\frac{1}{\log(\frac{m_2n_2}{m_1n_1})}$$

$$= B_1 + B_2.$$

We have $B_2 \ll T^{2-2\sigma_2}$ in the case $\sigma_1 > 1$. In the case $\sigma_1 \le 1$ we have

$$B_2 \ll \sum_{\substack{1\le m_1\le n_1-1\\ 2\le n_1\le T}} \sum_{\substack{1\le m_2\le n_2-1\\ 2\le n_2\le T}} n_1^{-\sigma_2}m_1^{-\sigma_1}n_2^{-\sigma_2}m_2^{-\sigma_1} \ll T^{4-2\sigma_1-2\sigma_2+\epsilon}. \tag{5.1}$$

Hence, we have

$$B_2 \ll \begin{cases} T^{2-2\sigma_2+\epsilon} & (\sigma_1 > 1),\\ T^{4-2\sigma_1-2\sigma_2+\epsilon} & (\sigma_1 \le 1).\end{cases}$$

Next we evaluate B_1. In the case $\sigma_1 > 1$ we have

$$B_1 \ll \sum_{r\le T^2} \sum_{\substack{m_2n_2-r=m_1n_1\\ r<m_2n_2}} \sum_{1\le m_1<n_1\le T} \sum_{1\le m_2<n_2\le T} n_1^{-\sigma_2}m_1^{-\sigma_1}n_2^{-\sigma_2}m_2^{-\sigma_1}\frac{m_1n_1}{r}$$

$$\ll \sum_{r\le T^2} \sum_{2\le n_1\le T} \sum_{1\le m_2<n_2\le T} \sum_{\substack{n_1|(m_2n_2-r)\\ r<m_2n_2}} n_1^{1-\sigma_2}n_2^{-\sigma_2}m_2^{-\sigma_1}\frac{1}{r}$$

$$\ll \sum_{r\le T^2} \sum_{2\le n_1\le T} \sum_{1\le m_2<n_2\le T} T^{1-\sigma_2+\epsilon}n_2^{-\sigma_2}m_2^{-\sigma_1}\frac{1}{r} \ll T^{2-2\sigma_2+\epsilon}.$$

If $\sigma_1 \le 1$ we have

$$B_1 \ll \sum_{r\le T^2} \sum_{1\le m_1<n_1\le T} \sum_{1\le m_2<n_2\le T} \sum_{\substack{m_2n_2=m_1n_1+r\\ r<m_2n_2}} n_1^{-\sigma_2}m_1^{-\sigma_1}n_2^{-\sigma_2}m_2^{-\sigma_1} \times \frac{m_1n_1}{r}$$

$$\ll \sum_{r\le T^2} \sum_{1\le m_1<n_1\le T} \sum_{1\le m_2<n_2\le T} \sum_{m_2n_2=m_1n_1+r} n_1^{-\sigma_2}m_1^{-\sigma_1}n_2^{-\sigma_2}m_2^{-\sigma_1} \tag{5.2}$$

$$\times \frac{m_1 n_1}{r} \frac{m_2 n_2}{m_1 n_1}$$
$$\ll \sum_{r \le T^2} \sum_{1 \le m_1 < n_1 \le T} n_1^{-\sigma_2} m_1^{-\sigma_1} \frac{T^{2-\sigma_1-\sigma_2+\epsilon}}{r} \ll T^{4-2\sigma_1-2\sigma_2+\epsilon},$$

since we are in the case $\sigma_2 \le \sigma_1 \le 1$. Hence, we have

$$B_1 \ll \begin{cases} T^{2-2\sigma_2+\epsilon} & (\sigma_1 > 1), \\ T^{4-2\sigma_1-2\sigma_2+\epsilon} & (\sigma_1 \le 1). \end{cases}$$

This implies that

$$S_3 \ll \begin{cases} T^{2-2\sigma_2+\epsilon} & (\sigma_1 > 1), \\ T^{4-2\sigma_1-2\sigma_2+\epsilon} & (\sigma_1 \le 1). \end{cases}$$

Therefore, we have

$$\int_2^T \left| A(s_1, s_2) \right|^2 dt = \zeta_2^{\square}(\sigma_1, \sigma_2) T + \begin{cases} O(T^{2-2\sigma_2+\epsilon}) & (\sigma_1 > 1), \\ O(T^{4-2\sigma_1-2\sigma_2+\epsilon}) & (\sigma_1 \le 1). \end{cases}$$

Now, if we set

$$\lambda = \begin{cases} -\sigma_2 & (\sigma_1 > 1), \\ -\sigma_2 + \epsilon & (\sigma_1 = 1), \\ 1 - \sigma_1 - \sigma_2 & (\sigma_1 < 1), \end{cases}$$

then we have

$$\begin{aligned} \int_2^T \left| \zeta(s_1, s_2) \right|^2 dt &= \int_2^T \left| A(s_1, s_2) + O(t^\lambda) \right|^2 dt \\ &= \int_2^T \left| A(s_1, s_2) \right|^2 dt + O\Big(\int_2^T \left| A(s_1, s_2) t^\lambda \right| dt \Big) + O(1). \end{aligned}$$

By the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \int_2^T \left| A(s_1, s_2) t^\lambda \right| dt &\ll \Big(\int_2^T \left| A(s_1, s_2) \right|^2 dt \Big)^{1/2} \Big(\int_2^T t^{2\lambda} \, dt \Big)^{1/2} \\ &\ll T^{1/2}. \end{aligned}$$

This implies the theorems. □

Next, we consider the case $\sigma_1 \le \sigma_2$.

Proof of Theorem 1.3 for $\sigma_1 \le \sigma_2$. Hereafter we use the same notation as in the preceding proof. First we evaluate S_1. Since

$$\begin{aligned}
&\sum_{k>T}\Big(\sum_{\substack{mn=k\\ m<n}} m^{-\sigma_1}n^{-\sigma_2}\Big)^2 \\
&\quad = \sum_{k>T}\Big(\sum_{\substack{m|k\\ m<\sqrt{k}}} m^{-\sigma_1}m^{\sigma_2}k^{-\sigma_2}\Big)^2 \\
&\quad \ll \sum_{k>T} k^{-2\sigma_2}(k^{\frac{1}{2}(\sigma_2-\sigma_1)+\epsilon})^2 \\
&\quad \ll \sum_{k>T} k^{-\sigma_1-\sigma_2+\epsilon} \ll T^{1-\sigma_1-\sigma_2+\epsilon},
\end{aligned}$$

we have

$$S_1 = \zeta_2^{\square}(\sigma_1,\sigma_2) + O(T^{1-\sigma_1-\sigma_2+\epsilon}).$$

Next we evaluate S_2. Since

$$\begin{aligned}
&\sum_{\substack{mn=k\\ m<n}} m^{-\sigma_1}n^{-\sigma_2} = \sum_{\substack{m|k\\ m<\sqrt{k}}} m^{\sigma_2-\sigma_1}k^{-\sigma_2} \ll k^{-\frac{1}{2}(\sigma_1+\sigma_2)+\epsilon}, \\
&\sum_{\substack{mn=k\\ m<n}} m^{-\sigma_1}n^{1-\sigma_2} = \sum_{\substack{m|k\\ m<\sqrt{k}}} k^{1-\sigma_2}m^{\sigma_2-\sigma_1-1} \\
&\qquad\qquad \ll \begin{cases} k^{1-\sigma_2+\epsilon} & (\sigma_2-\sigma_1-1\le 0), \\ k^{\frac{1}{2}(1-\sigma_1-\sigma_2)+\epsilon} & (\sigma_2-\sigma_1-1>0) \end{cases}
\end{aligned} \tag{5.3}$$

hold, we have

$$\begin{aligned}
A_1 &\ll \begin{cases} \sum_{2\le k\le T} k^{-\frac{1}{2}(\sigma_1+\sigma_2)+\epsilon}k^{1-\sigma_2+\epsilon} & (\sigma_2-\sigma_1-1\le 0), \\ \sum_{2\le k\le T} k^{-\frac{1}{2}(\sigma_1+\sigma_2)+\epsilon}k^{\frac{1}{2}(1-\sigma_1-\sigma_2)+\epsilon} & (\sigma_2-\sigma_1-1>0), \end{cases} \\
&= \begin{cases} \sum_{2\le k\le T} k^{1-\frac{1}{2}\sigma_1-\frac{3}{2}\sigma_2+\epsilon} & (\sigma_2-\sigma_1-1\le 0), \\ \sum_{2\le k\le T} k^{\frac{1}{2}-\sigma_1-\sigma_2+\epsilon} & (\sigma_2-\sigma_1-1>0). \end{cases}
\end{aligned}$$

We note that $1-(1/2)\sigma_1-(3/2)\sigma_2<-1$ is equivalent to $\sigma_2>-(1/3)\sigma_1+(4/3)$. Hence, we have

$$A_1 \ll \begin{cases} T^{2-\frac{1}{2}\sigma_1-\frac{3}{2}\sigma_2+\epsilon} & (\sigma_2-\sigma_1-1\le 0 \text{ and } \sigma_2\le -\frac{1}{3}\sigma_1+\frac{4}{3}), \\ 1 & (\text{otherwise}) \end{cases}$$

because $\sigma_1+\sigma_2>3/2$. Similarly, we have

$$\begin{aligned} A_2 &\ll \begin{cases} \sum_{T<k<T^2} k^{1-\frac{1}{2}\sigma_1-\frac{3}{2}\sigma_2+\epsilon} & (\sigma_2-\sigma_1-1\le 0), \\ \sum_{T<k<T^2} k^{\frac{1}{2}-\sigma_1-\sigma_2+\epsilon} & (\sigma_2-\sigma_1-1>0), \end{cases} \\ &\ll \begin{cases} T^{4-\sigma_1-3\sigma_2+\epsilon} & (\sigma_2-\sigma_1-1\le 0 \text{ and } \sigma_2\le -\frac{1}{3}\sigma_1+\frac{4}{3}), \\ 1 & (\text{otherwise}). \end{cases} \end{aligned}$$

Therefore, we have

$$S_2 \ll \begin{cases} T^{4-\sigma_1-3\sigma_2+\epsilon} & (\sigma_2-\sigma_1-1\le 0 \text{ and } \sigma_2\le -\frac{1}{3}\sigma_1+\frac{4}{3}), \\ 1 & (\text{otherwise}). \end{cases}$$

Next we evaluate S_3. Since estimation (5.1) remains also valid in this case, we have $B_2 \ll T^{4-2\sigma_1-2\sigma_2+\epsilon}$. Since we have

$$\begin{aligned} \sum_{1\le m_2<n_2\le T} \sum_{m_2n_2=m_1n_1+r} n_2^{1-\sigma_2} m_2^{1-\sigma_1} &\ll \left(\frac{m_1n_1+r}{T}\right)^{1-\sigma_2} T^{1-\sigma_1+\epsilon} \\ &\ll T^{2-\sigma_1-\sigma_2+\epsilon} \end{aligned}$$

for $\sigma_1\le 1$, $\sigma_2>1$, $0\ne m_1n_1+r\ll T^2$, estimation (5.2) also remains valid in this case. Therefore, we have $S_3\ll T^{4-2\sigma_1-2\sigma_2+\epsilon}$. Since $4-2\sigma_1-2\sigma_2-(4-\sigma_1-3\sigma_2)=\sigma_2-\sigma_1\ge 0$, we have

$$\int_2^T \left|A(s_1,s_2)\right|^2 dt = \zeta_2^{\square}(\sigma_1,\sigma_2)T + O(T^{4-2\sigma_1-2\sigma_2+\epsilon}).$$

By the same argument as in the case $\sigma_1>\sigma_2$, we obtain the theorem. ∎

Proof of Theorem 1.3 for $\sigma_1>1$ and $\sigma_2=1/2$. By [11, Theorem 2.2] we have

$$\int_2^T \left|\zeta(1/2+it)\right|^2 \left|\zeta(\sigma_1+it)\right|^2 dt \sim \frac{\zeta(2\sigma_1)\zeta(\sigma_1+1/2)^2}{\zeta(2\sigma_1+1)} T\log T.$$

By (1.3) and the Cauchy–Schwarz inequality, we have

$$I^{\square}(T) \sim \int_2^T \left|\zeta(1/2+it)\right|^2 \left|\zeta(\sigma_1+it)\right|^2 dt.$$

This completes the proof. ∎

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