Nagoya Math. J. **217** (2015), 133–159
DOI 10.1215/00277630-2888085

THE M-SET OF $\lambda\exp(z)/z$ HAS INFINITE AREA

GUOPING ZHAN AND LIANGWEN LIAO

Abstract. It is known that the Fatou set of the map $\exp(z)/z$ defined on the punctured plane $\mathbb{C}^*$ is empty. We consider the M-set of $\lambda\exp(z)/z$ consisting of all parameters λ for which the Fatou set of $\lambda\exp(z)/z$ is empty. We prove that the M-set of $\lambda\exp(z)/z$ has infinite area. In particular, the Hausdorff dimension of the M-set is 2. We also discuss the area of complement of the M-set.

§1. Introduction and main results

The exponential family $E_\lambda(z)=\lambda\exp(z)$ is the simplest family of transcendental entire functions which is topologically complete. For $\lambda=1$, the Julia set of e^z is the whole complex plane $\mathbb{C}$ (see [11, Main Theorem]). Moreover, it is proved in [8] that for almost all $z\in\mathbb{C}$, the ω-limit set consists of the infinity and the postsingular orbit; in particular, e^z is not recurrent. (A map f is recurrent if, for every set K of positive area, there is an integer $n\geq 1$ such that $f^n(K)\cap K$ has positive area, where f^n is the nth iterate of f.) For the exponential family $E_\lambda(z)=\lambda\exp(z)$, the M-set of all λ-values for which E_λ has no Fatou set was first studied in [7], where some topological structure of the M-set was described. From [12], one knows that the M-set has Hausdorff dimension 2. But it is still unknown whether the M-set has positive area. (For more information about the dynamics of the exponential family, see, e.g., [4]–[6], [15], [13], [14], [16]–[18].)

The family of functions F_λ with parameter $\lambda\in\mathbb{C}^*$ mapping the punctured plane $\mathbb{C}^*$ to itself, defined by $F_\lambda: z\mapsto\lambda\exp(z)/z$, may be regarded as the simplest family of transcendental meromorphic functions on $\mathbb{C}$ with exactly one pole which is a Picard exceptional value. For $\lambda=1$, it is proved in [19, Theorem 1.6] that the set Λ of all points in $\mathbb{C}^*$ whose ω-limit set of $\exp(z)/z$ does not equal $\{0,\infty\}$ has zero Lebesgue measure. In particular, the map

Received May 2, 2013. Revised March 17, 2014. Accepted April 3, 2014.
2010 Mathematics Subject Classification. Primary 37F10, 37F45; Secondary 32H50, 30D05.

$\exp(z)/z$ is not recurrent. Moreover, the set Λ has Hausdorff dimension 2 (see [20, Theorem 1.1]). In this paper, we consider the M-set of F_λ consisting of all parameters λ for which the Fatou set of F_λ is empty. Before stating the main result, let us introduce some notation and definitions.

NOTATION. Let $\mathbb{C}$ denote the complex plane, let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ denote the punctured plane, and let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let F_λ^n be the nth iterate of F_λ for all $n \in \mathbb{N}$, and let $\mathcal{J}(F_\lambda)$ be the Julia set of F_λ for each $\lambda \in \mathbb{C}^*$. For $\rho > 0$ and $z \in \mathbb{C}$, let $D(z, \rho)$ denote an open disk centered at z with radius ρ. For a bounded set $X \subset \mathbb{C}$, let $\operatorname{area}(X)$ denote the Euclidean area of X.

DEFINITIONS.

(1) The ω-limit set of F_λ at $z \in \mathbb{C}^*$, denoted by $\omega_{F_\lambda}(z)$, consists of all accumulation points of $\{F_\lambda^n(z)\}_{n=0}^\infty$ on $\widehat{\mathbb{C}}$.

(2) As in [19], we denote

$$F(z) = \exp(z)/z$$

for all $z \in \mathbb{C}^*$. Recall that 1 is the only critical point of $F_\lambda : z \mapsto \lambda \exp(z)/z$. Furthermore, let

$$G_n(\lambda) = F_\lambda^{n+1}(1)$$

for $n \geq 1$ and $\lambda \in \mathbb{C}^*$. Moreover, define

$$W = \{\lambda \in \mathbb{C}^* \mid \omega_{F_\lambda}(1) \subset \{0, \infty\}\},$$
$$M = \{\lambda \in \mathbb{C}^* \mid \mathcal{J}(F_\lambda) = \mathbb{C}\},$$

and

$$M^c = \mathbb{C}^* \setminus M.$$

We prove the following.

MAIN THEOREM. *The M-set of the family F_λ has infinite area. In particular, the Hausdorff dimension of the M-set is* 2.

REMARK. The Main Theorem leads us to pose the following question: Does the complement of the M-set have infinite area? In the end, we show that the area of complement of the M-set is positive. In this paper, we are not able to prove that the complement of the M-set has finite or infinite area.

§2. Preliminary lemmas

LEMMA 2.1. *The set W is contained in the set M.*

Proof. Note that for each $\lambda \in \mathbb{C}^*$, F_λ has exactly one critical point 1 and one asymptotic value 0, which implies that F_λ has exactly two finite singular values, 0 and λe. It follows that F_λ has neither Baker domains nor wandering domains (see [1], [2], [9]).

If $\lambda \in W$, then any accumulation point of forward iterations of the critical point can be only either 0 or ∞; this implies that the closure of the forward orbit of the critical point contains no line segment. Using the facts that the boundary of a Siegel disk or a Herman ring is contained in the closure of the forward orbits of the singular values and that any periodic attracting or parabolic component of F_λ will attract the forward orbit of a singular value (see [2]), we can conclude that F has no Siegel disks, Herman rings, or attracting or parabolic periodic components. Therefore, the Fatou set of F_λ is empty, and the Julia set of F_λ is the whole complex plane $\mathbb{C}$. Hence, the set W is contained in the set M. ∎

The following is well known (refer to [3, Chapter I, Theorem 1.4]).

LEMMA 2.2. *If f is univalent on a domain D, and if $z_0 \in D$, then*

$$\frac{1}{4}\left|f'(z_0)\right| \operatorname{dist}(z_0, \partial D) \le \operatorname{dist}\big(f(z_0), \partial\big(f(D)\big)\big) \le 4\left|f'(z_0)\right| \operatorname{dist}(z_0, \partial D).$$

For two Lebesgue measurable subsets A and B of $\mathbb{C}$, we call

$$\operatorname{dens}(A, B) := \frac{\operatorname{area}(A \cap B)}{\operatorname{area}(B)}$$

the *density* of A in B.

Let us introduce a criterion due to McMullen [10], which provides a tool for constructing a nested intersection of dynamically defined sets with positive area.

LEMMA 2.3 (Nesting conditions). *For all $k \ge 0$, let $\mathcal{E}_k$ denote a finite collection of subsets in $\mathbb{C}$ such that every two elements in $\mathcal{E}_k$ have an intersection of measure zero, and let E_k denote the union of all elements in $\mathcal{E}_k$. Suppose that the sequence $(\mathcal{E}_k)_{k\ge 0}$ satisfies the following nesting conditions:*

(C_1) *every $U \in \mathcal{E}_{k+1}$ is contained in a unique $U' \in \mathcal{E}_k$;*

(C_2) *every $U' \in \mathcal{E}_k$ contains at least one element of $\mathcal{E}_{k+1}$;*

(C_3) *for all k and all U in $\mathcal{E}_k$,*

$$\mathrm{dens}(E_{k+1}, U) \geq \Delta_k.$$

Then for the set $E := \bigcap_{k=0}^{\infty} E_k$, we have

$$\mathrm{dens}(E, E_0) \geq \prod_{k=0}^{\infty} \Delta_k.$$

§3. Proof of Main Theorem

For $m, l \in \mathbb{Z}$, define

$$S_{m,l} := \{\lambda \in \mathbb{C} \mid m \leq \mathrm{Re}(\lambda) \leq m+1, l \leq \mathrm{Im}(\lambda) \leq l+1\}$$

and

$$\mathcal{B} := \{S_{m,l} \mid m, l \in \mathbb{Z}\}.$$

For all $t > 0$, let

$$V_t^+ = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) \geq t\},$$
$$V_t^- = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) \leq -t\}.$$

Let $K \subseteq \mathbb{C}$ be a bounded subset. Suppose that f is a univalent function in a neighborhood of K. We call

$$T(f_{|K}) := \frac{\sup_{z \in K} |f'(z)|}{\inf_{z \in K} |f'(z)|}$$

the *distortion of f on K*. It is easy to see that

$$T(f_{|K}) = T(f^{-1}_{|f(K)}) \tag{3.1}$$

and that, for any two Lebesgue measurable subsets A and B of K,

$$\mathrm{dens}\big(f(A), f(B)\big) \leq T(f_{|K})^2 \,\mathrm{dens}(A, B). \tag{3.2}$$

In the following, all squares are closed squares whose sides are parallel to the coordinate axes. Applying the argument of [21, Lemma 2.5], we have the following.

LEMMA 3.1. *Let $t>0$ and $Q\subset\mathbb{C}$ be a square with side length 1. Suppose that f is a univalent map defined in a neighborhood of Q and that there is a constant $C>0$ such that $T(f_{|Q})<C$. Then for any $z_0\in Q$,*

$$\operatorname{dens}\Big(\bigcup S_{m,l}, f(Q)\Big)\geq 1-C^3\Big(\frac{2\sqrt{2}t+21}{|f'(z_0)|}+\frac{12}{C|f'(z_0)|^2}\Big),$$

where the union set takes over all $S_{m,l}\in\mathcal{B}$ and $S_{m,l}\subset f(Q)\cap(V_t^+\cup V_t^-)$.

REMARK. Lemma 3.1 is for the case of vertical strip $\{\lambda\in\mathbb{C}\mid|\mathrm{Re}(\lambda)|\geq t\}$, which is a version of [21, Lemma 2.5] for the case of horizontal strip $\{\lambda\in\mathbb{C}\mid|\mathrm{Im}(\lambda)|\geq t\}$; [21, Lemma 2.5] is crucial for estimating the area of escaping parameters of the sine family $\lambda\sin z$ in squares lying away from the imaginary axis, since the forward orbits of escaping parameters go away from the imaginary axis. However, for the family $\lambda\exp(z)/z$, the forward orbits of parameters in the set W go away from the real axis or approach 0 (the pole), so Lemma 3.1 is crucial for estimating the area of the set W in squares lying away from the real axis. By the way, the constant C is not essential for application, since the distortion of forward orbits of parameters in the set W is controlled by a power of e, while the derivative of their forward orbits is larger than an iterate of the exponential.

To prove the Main Theorem, it suffices to prove the following.

LEMMA 3.2. *For each square $S_{m,m}\in\mathcal{B}$ with $m\geq 10^2$, there is a constant $\alpha>0$ such that*

$$\operatorname{area}(S_{m,m}\cap M)\geq\alpha.$$

3.1. Proof of Lemma 3.2

Take a fixed square $S_{m,m}\in\mathcal{B}$ with $m\geq 10^2$. For simplicity, denote

$$Q_0=S_{m,m},\qquad \mathcal{Q}_0=\{Q_0\}$$

and

$$x_0=2m,\qquad x_n=2\cdot\exp^n(m)$$

for all integers $n\geq 1$, where $\exp^n(m)$ is the nth iterate of m under exp. Let $\widetilde{Q}_0$ be an open square with the same center of Q_0 and side length 2, which is a neighborhood of Q_0.

PROPOSITION 3.3. *The map G_1 is univalent in $\widetilde{Q}_0$ with*

$$\inf_{\lambda\in Q_0}\big|G_1'(\lambda)\big|\geq\exp(me)$$

and

$$T(G_{1|Q_0}) \le \exp(e).$$

Proof. Recall that

$$G_1(\lambda) = F_\lambda^2(1) = \exp(\lambda e)/e.$$

Since $\widetilde{Q}_0$ is contained in a horizontal strip of width less than $2\pi/e$ and parallel to the real axis, G_1 is univalent in $\widetilde{Q}_0$.

For all $\lambda \in Q_0$, we have $|G_1'(\lambda)| = \exp(\lambda e)| \ge \exp(me)$. Hence,

$$\inf_{\lambda \in Q_0} \left| G_1'(\lambda) \right| \ge \exp(me)$$

and

$$T(G_{1|Q_0}) = \frac{\sup_{\lambda \in Q_0} |G_1'(\lambda)|}{\inf_{\lambda \in Q_0} |G_1'(\lambda)|} \le \frac{\exp(me + e)}{\exp(me)} = \exp(e). \qquad \square$$

Since Q_0 is mapped away by G_1, we consider the set $G_1(Q_0)$. It follows from Lemma 2.2 and Proposition 3.3 that $G_1(Q_0) \cap (V_{x_1}^+ \cup V_{x_1}^-)$ contains many squares in $\mathcal{B}$. So we can define for $\mu \in \{+,-\}$

$$\begin{aligned}
\mathcal{P}_{1,1}^\mu &:= \left\{ S \in \mathcal{B} \,\middle|\, S \subset G_1(Q_0) \cap V_{x_1}^\mu \right\}, \\
P_1 &:= \bigcup_{S \in \mathcal{P}_{1,1}^\mu | \mu \in \{+,-\}} S, \\
\mathcal{Q}_{1,1}^\mu &:= \left\{ K \subset Q_0 \,\middle|\, G_1(K) \in \mathcal{P}_{1,1}^\mu \right\}, \\
\mathcal{Q}_1 &:= \left\{ K \in \mathcal{Q}_{1,1}^\mu \,\middle|\, \mu \in \{+,-\} \right\}, \\
Q_1 &:= \bigcup_{K \in \mathcal{Q}_1} K.
\end{aligned}$$

From the definitions, we can see that $\mathcal{Q}_{1,1}^+ \cap \mathcal{Q}_{1,1}^- = \emptyset$ and that every two elements in $\mathcal{Q}_{1,1}^\mu$ with $\mu \in \{+,-\}$ have an intersection of measure zero. So $\mathcal{Q}_1$ is a finite collection of subsets in $\mathbb{C}$ satisfying that every two elements in $\mathcal{Q}_1$ have an intersection of measure zero.

Proposition 3.4. *For Q_0, we have*

$$\operatorname{dens}(Q_1, Q_0) \ge 1 - \exp\left(-\frac{x_0}{8}\right).$$

Proof. By Proposition 3.3, G_1 is univalent in a neighborhood of Q_0. We can take an inverse branch of G_1 which maps $G_1(\widetilde{Q}_0)$ to $\widetilde{Q}_0$, denoted by φ_1. Using (3.1) and Proposition 3.3, we have

$$T(\varphi_1) := T(\varphi_{1|G_1(Q_0)}) = T(G_{1|Q_0}) \leq \exp(e). \tag{3.3}$$

Since $m \geq 10^2$, applying Lemma 3.1 and Proposition 3.3 we have

$$\begin{aligned}
&\operatorname{dens}\big(P_1, G_1(Q_0)\big)\\
&\quad \geq 1 - \exp(3e)\Big(\frac{2\sqrt{2}x_1 + 21}{\inf_{\lambda\in Q_0}|G_1'(\lambda)|} + \frac{12}{\exp(e)\cdot(\inf_{\lambda\in Q_0}|G_1'(\lambda)|)^2}\Big)\\
&\quad \geq 1 - \exp(3e)\Big(\frac{2\sqrt{2}x_1 + 21}{\exp(me)} + \frac{12}{\exp(2me+e)}\Big)\\
&\quad \geq 1 - \exp\Big(-\frac{x_0}{4}\Big).
\end{aligned} \tag{3.4}$$

Since $\varphi_1 \circ G_1 = \mathrm{id}$ on Q_0 and $G_1(Q_0 \setminus Q_1) \subset G_1(Q_0) \setminus P_1$, applying (3.2)–(3.4) we obtain

$$\begin{aligned}
\operatorname{dens}(Q_1, Q_0) &= 1 - \operatorname{dens}(Q_0 \setminus Q_1, Q_0)\\
&= 1 - \operatorname{dens}\big(\varphi_1 \circ G_1(Q_0 \setminus Q_1), \varphi_1 \circ G_1(Q_0)\big)\\
&\geq 1 - T(\varphi_1)^2 \operatorname{dens}\big(G_1(Q_0 \setminus Q_1), G_1(Q_0)\big)\\
&\geq 1 - T(\varphi_1)^2 \operatorname{dens}\big(G_1(Q_0) \setminus P_1, G_1(Q_0)\big)\\
&\geq 1 - \exp(2e)\big(1 - \operatorname{dens}\big(P_1, G_1(Q_0)\big)\big)\\
&\geq 1 - \exp\Big(-\frac{x_0}{8}\Big).
\end{aligned} \tag{3.5}$$

∎

PROPOSITION 3.5. *For each $K \in \mathcal{Q}_{1,1}^+$, the map G_2 is univalent in a neighborhood $\widetilde{K}$ of K with*

$$\inf_{\lambda\in K}\Big|\frac{G_2'(\lambda)}{G_1'(\lambda)}\Big| \geq \exp\Big(\frac{3x_1}{4}\Big)$$

and

$$T(G_{2|K}) \leq \exp(2e).$$

For each $K \in \mathcal{Q}^-_{1,1}$, G_3 *is univalent in a neighborhood* $\widetilde{K}$ *of* K *with*

$$\inf_{\lambda\in K}\left|\frac{G_3'(\lambda)}{G_1'(\lambda)}\right| \geq \exp\Big(\frac{3x_1}{4}\Big)$$

and

$$T(G_{3|K}) \leq \exp(3e).$$

Proof. If $K \in \mathcal{Q}^+_{1,1}$, then there is a unique $S \in \mathcal{P}^+_{1,1}$ such that $G_1(K) = S$. Note that $G_1(\widetilde{Q}_0)$ is a simply connected domain and contains S. Denote

$$r = \frac{1}{2}\min\big\{1, \operatorname{dist}\big(S, \partial\big(G_1(\widetilde{Q}_0)\big)\big)\big\} > 0.$$

Let $\widetilde{S}$ be an open square with the same center of S and side length $1+2r$. Then $\widetilde{S}$ is a neighborhood of S and contained in $V^+_{x_1-1}$. Recall that φ_1 is the inverse branch of G_1 which maps $G_1(\widetilde{Q}_0)$ to $\widetilde{Q}_0$. Define $\widetilde{K} := \varphi_1(\widetilde{S})$; then $\widetilde{K}$ is a neighborhood of K with $\widetilde{K} \subset \widetilde{Q}_0$.

Recall that $F(z) = \exp(z)/z$. Suppose that $G_2(a) = G_2(b)$ with $a, b \in \widetilde{K}$. That is,

$$aF\big(G_1(a)\big) = bF\big(G_1(b)\big).$$

Then

$$|a-b|\big|F\big(G_1(a)\big)\big| = |b|\big|F\big(G_1(b)\big) - F\big(G_1(a)\big)\big|. \tag{3.6}$$

Since $G_1(\widetilde{K}) = \widetilde{S} \subset V^+_{x_1-1}$ and $\widetilde{K} \subset \widetilde{Q}_0$, applying Proposition 3.3 we have

$$\begin{aligned}
\big|F\big(G_1(b)\big) - F\big(G_1(a)\big)\big| &\geq |a-b| \inf_{\lambda\in\widetilde{K}}\left|\frac{\mathrm{d}F(G_1(\lambda))}{\mathrm{d}\lambda}\right| \\
&\geq |a-b|\Big(\inf_{\lambda\in\widetilde{K}}\left|F\big(G_1(\lambda)\big)G_1'(\lambda)\frac{G_1(\lambda)-1}{G_1(\lambda)}\right|\Big) \\
&\geq |a-b|\big(\inf_{\nu\in\widetilde{Q}_0}\big|F(\nu)\big|\big)\big(\inf_{\lambda\in\widetilde{Q}_0}\big|G_1'(\lambda)\big|\big)\Big(\inf_{\nu\in\widetilde{S}}\left|\frac{\nu-1}{\nu}\right|\Big) \\
&\geq |a-b|\big(\inf_{\nu\in\widetilde{Q}_0}\big|F(\nu)\big|\big)\cdot\frac{1}{2}\exp(me-e).
\end{aligned}$$

Moreover,

$$\big|F\big(G_1(a)\big)\big| \leq \sup_{\nu\in\widetilde{Q}_0}\big|F(\nu)\big|$$

and

$$\frac{\sup_{\nu\in\widetilde{Q}_0}|F(\nu)|}{\inf_{\nu\in\widetilde{Q}_0}|F(\nu)|}\le 2e^2.$$

Since $|b|\ge m-1\ge 99$, it follows from (3.6) that

$$4e^2|a-b|\ge 99|a-b|\exp(99e).$$

This implies that $a=b$. Therefore, G_2 is univalent in $\widetilde{K}$.

By calculation,

$$G_2'(\lambda)=F\big(G_1(\lambda)\big)\Big(1+\lambda G_1'(\lambda)\frac{G_1(\lambda)-1}{G_1(\lambda)}\Big), \tag{3.7}$$

$$\frac{G_2'(\lambda)}{G_1'(\lambda)}=F\big(G_1(\lambda)\big)\Big(\frac{1}{G_1'(\lambda)}+\lambda\frac{G_1(\lambda)-1}{G_1(\lambda)}\Big). \tag{3.8}$$

Note that $G_1(K)=S\subset G_1(Q_0)\cap V_{x_1}^+$ and $K\subset Q_0$. Then $|\lambda|\ge m\ge 10^2$ for all $\lambda\in K$ and $|\nu|\le\exp(2x_0)$ for all $\nu\in S$. Using Proposition 3.3 with (3.7) and (3.8), we have

$$\inf_{\lambda\in K}\Big|\frac{G_2'(\lambda)}{G_1'(\lambda)}\Big|\ge\inf_{\nu\in S}\big|F(\nu)\big|\cdot\frac{1}{2}\inf_{\lambda\in K}|\lambda|\ge\frac{m}{2}\cdot\frac{\exp(x_1)}{\exp(2x_0)}\ge\exp\Big(\frac{3x_1}{4}\Big)$$

and

$$\begin{aligned}T(G_{2|K})&\le\frac{\sup_{\nu\in S}|F(\nu)|}{\inf_{\nu\in S}|F(\nu)|}\cdot\frac{\sup_{\lambda\in K}|1+\lambda G_1'(\lambda)\frac{G_1(\lambda)-1}{G_1(\lambda)}|}{\inf_{\lambda\in K}|1+\lambda G_1'(\lambda)\frac{G_1(\lambda)-1}{G_1(\lambda)}|}\\&\le\sqrt{2}e\cdot\sqrt{2}T(G_{1|Q_0})\le\sqrt{2}e\cdot\sqrt{2}\exp(e)\le\exp(2e).\end{aligned}$$

If $K\in\mathcal{Q}_{1,1}^-$, then there also exists a corresponding neighborhood $\widetilde{K}$ of K such that $\widetilde{K}\subset\widetilde{Q}_0$ and $G_1(\widetilde{K})$ is an open square contained in $V_{x_1-1}^-$ with side length no more than 2.

Suppose that $G_3(a)=G_3(b)$ with $a,b\in\widetilde{K}$. Similar to (3.6), we have

$$|a-b|\big|F\big(G_2(a)\big)\big|=|b|\big|F\big(G_2(b)\big)-F\big(G_2(a)\big)\big|. \tag{3.9}$$

Let $A=G_2(\widetilde{K})$; then A is contained in a small annulus with outer radius no more than $\exp(-x_1)$ and $\mathrm{mod}(A)\le 2e^2$. In particular, $|G_2(\lambda)|\le\exp(-x_1)$

for each $\lambda \in \widetilde{K}$. Also note that $\widetilde{K} \subset \widetilde{Q}_0$; combining $|G_1(\lambda)| \geq x_1$ for each $\lambda \in \widetilde{K}$ with Proposition 3.3 and (3.7), we have

$$\begin{aligned} \inf_{\lambda \in \widetilde{K}} \Big| \frac{G_2(\lambda) - 1}{G_2(\lambda)} G_2'(\lambda) \Big| &\geq \frac{1}{\sqrt{2}} \inf_{\lambda \in \widetilde{K}} \Big| \frac{G_2'(\lambda)}{G_2(\lambda)} \Big| \\ &\geq \frac{1}{2} \inf_{\lambda \in \widetilde{Q}_0} \big| G_1'(\lambda) \big| \geq \frac{1}{2} \exp(me - e). \end{aligned} \tag{3.10}$$

Hence,

$$\begin{aligned} &\big| F\big(G_2(b)\big) - F\big(G_2(a)\big) \big| \\ &\quad \geq |a - b| \inf_{\lambda \in \widetilde{K}} \Big| \frac{\mathrm{d}F(G_2(\lambda))}{\mathrm{d}\lambda} \Big| \\ &\quad \geq |a - b| \Big(\inf_{\lambda \in \widetilde{K}} \big| F\big(G_2(\lambda)\big) \big| \Big) \Big(\inf_{\lambda \in \widetilde{K}} \Big| \frac{G_2(\lambda) - 1}{G_2(\lambda)} G_2'(\lambda) \Big| \Big) \\ &\quad \geq |a - b| \Big(\inf_{\nu \in A} \big| F(\nu) \big| \Big) \cdot \frac{1}{2} \exp(me - e). \end{aligned}$$

Moreover,

$$\big| F\big(G_2(a)\big) \big| \leq \sup_{\nu \in A} \big| F(\nu) \big|$$

and

$$\frac{\sup_{\nu \in A} |F(\nu)|}{\inf_{\nu \in A} |F(\nu)|} \leq \operatorname{mod}(A) \cdot \exp\big(2 \cdot \exp(-x_1)\big) \leq 4e^2. \tag{3.11}$$

Since $|b| \geq m - 1 \geq 99$, it follows from (3.9) that

$$8e^2 |a - b| \geq 99 |a - b| \exp(99e).$$

This implies that $a = b$. Therefore, G_3 is univalent in $\widetilde{K}$.

By calculation,

$$G_3'(\lambda) = F\big(G_2(\lambda)\big) \Big(1 + \lambda G_2'(\lambda) \frac{G_2(\lambda) - 1}{G_2(\lambda)} \Big), \tag{3.12}$$

$$\frac{G_3'(\lambda)}{G_1'(\lambda)} = F\big(G_2(\lambda)\big) \Big(\frac{G_2(\lambda)}{G_1'(\lambda)} + \lambda \frac{(G_1(\lambda) - 1)^2}{G_1(\lambda)} \Big). \tag{3.13}$$

Note that $G_2(K)\subset A$, $G_1(K)\subset V^-_{x_1}$, $K\subset Q_0$, and $|\lambda|\geq m\geq 10^2$ for all $\lambda\in K$. Then $|\nu|\leq\exp(-x_1)$ for all $\nu\in G_2(K)$ and $|G_1(\lambda)|\geq x_1$ for all $\lambda\in K$. Using Proposition 3.3 and (3.13), we have

$$\begin{aligned}\inf_{\lambda\in K}\left|\frac{G_3'(\lambda)}{G_1'(\lambda)}\right| &\geq \inf_{\nu\in A}|F(\nu)|\cdot\frac{1}{2}\inf_{\lambda\in K}|\lambda| \\ &\geq \frac{m}{2}\cdot\frac{\exp(x_1)}{2} \\ &\geq \exp\Big(\frac{3x_1}{4}\Big).\end{aligned}$$

By Proposition 3.3 with (3.11) and (3.12),

$$\begin{aligned}T(G_{3|K}) &\leq \frac{\sup_{\nu\in A}|F(\nu)|}{\inf_{\nu\in A}|F(\nu)|}\cdot\frac{\sup_{\lambda\in K}|1+\lambda G_2'(\lambda)\frac{G_2(\lambda)-1}{G_2(\lambda)}|}{\inf_{\lambda\in K}|1+\lambda G_2'(\lambda)\frac{G_2(\lambda)-1}{G_2(\lambda)}|} \\ &\leq 4e^2\cdot 2\frac{\sup_{\lambda\in Q_0}|\frac{G_2'(\lambda)}{G_2(\lambda)}|}{\inf_{\lambda\in Q_0}|\frac{G_2'(\lambda)}{G_2(\lambda)}|} \\ &\leq 4e^2\cdot 4T(G_{1|Q_0})\leq 4e^2\cdot 4\exp(e)\leq\exp(3e).\end{aligned}$$ □

Note that if $K'\in\mathcal{Q}^+_{1,1}$, then K' is mapped away by G_2; if $K'\in\mathcal{Q}^-_{1,1}$, then K' is mapped into a neighborhood of 0 (the pole) by G_2, before being mapped away by G_3. So we consider the set $G_2(K')$ for each $K'\in\mathcal{Q}^+_{1,1}$ and the set $G_3(K')$ for each $K'\in\mathcal{Q}^-_{1,1}$. By Lemma 2.2 with Propositions 3.3 and 3.5, $G_2(K')\cap(V^+_{x_2}\cup V^-_{x_2})$ contains many squares in $\mathcal{B}$ for each $K'\in\mathcal{Q}^+_{1,1}$, and $G_3(K')\cap(V^+_{x_2}\cup V^-_{x_2})$ contains many squares in $\mathcal{B}$ for each $K'\in\mathcal{Q}^-_{1,1}$. Define for $\mu\in\{+,-\}$

$$\mathcal{P}^\mu_{2,1}:=\bigcup_{K'\in\mathcal{Q}^+_{1,1}}\{S\in\mathcal{B}\,|\,S\subset G_2(K')\cap V^\mu_{x_2}\},$$

$$\mathcal{P}^\mu_{2,2}:=\bigcup_{K'\in\mathcal{Q}^-_{1,1}}\{S\in\mathcal{B}\,|\,S\subset G_3(K')\cap V^\mu_{x_2}\},$$

$$P_2:=\bigcup_{S\in\mathcal{P}^\mu_{2,j}|\mu\in\{+,-\},1\leq j\leq 2}S,$$

$$\mathcal{Q}^\mu_{2,1}:=\{K\subset Q_1\,|\,G_2(K)\in\mathcal{P}^\mu_{2,1}\},$$

$$\mathcal{Q}^{\mu}_{2,2} := \big\{K \subset Q_1 \bigm| G_3(K) \in \mathcal{P}^{\mu}_{2,2}\big\},$$
$$\mathcal{Q}_2 := \big\{K \in \mathcal{Q}^{\mu}_{2,j} \bigm| \mu \in \{+,-\}, 1 \le j \le 2\big\},$$
$$Q_2 := \bigcup_{K \in \mathcal{Q}_2} K.$$

From the definitions, for $\mu \in \{+,-\}$ and $1 \le j \le 2$, we have $\mathcal{Q}^{+}_{2,j} \cap \mathcal{Q}^{-}_{2,j} = \emptyset$, and every two elements in $\mathcal{Q}^{\mu}_{2,j}$ have an intersection of measure zero. Using $\mathcal{Q}^{+}_{1,1} \cap \mathcal{Q}^{-}_{1,1} = \emptyset$ and the facts that, for $\mu \in \{+,-\}$, every $K \in \mathcal{Q}^{\mu}_{2,1}$ (resp., $\mathcal{Q}^{\mu}_{2,2}$) is contained in a unique $K' \in \mathcal{Q}^{+}_{1,1}$ (resp., $\mathcal{Q}^{-}_{1,1}$) and that every $K' \in \mathcal{Q}^{+}_{1,1}$ (resp., $\mathcal{Q}^{-}_{1,1}$) contains at least one element of $\mathcal{Q}^{+}_{2,1} \cup \mathcal{Q}^{-}_{2,1}$ (resp., $\mathcal{Q}^{+}_{2,2} \cup \mathcal{Q}^{-}_{2,2}$), we have $\mathcal{Q}^{\mu_1}_{2,j_1} \cap \mathcal{Q}^{\mu_2}_{2,j_2} = \emptyset$ for any two distinct pairs (j_1, μ_1) and (j_2, μ_2).

Therefore, $\mathcal{Q}_2$ is a finite collection of subsets in $\mathbb{C}$ satisfying that every two elements in $\mathcal{Q}_2$ have an intersection of measure zero and that every $K \in \mathcal{Q}_2$ is contained in a unique $K' \in \mathcal{Q}_1$, with each $K' \in \mathcal{Q}_1$ containing at least one element in $\mathcal{Q}_2$.

PROPOSITION 3.6. *For each $K \in \mathcal{Q}_1$, we have*

$$\operatorname{dens}(Q_2, K) \ge 1 - \exp\Big(-\frac{x_1}{8}\Big).$$

Proof. If $K \in \mathcal{Q}^{+}_{1,1}$, by Proposition 3.5 G_2 is univalent in a neighborhood $\widetilde{K}$ of K. We can take an inverse branch of G_2 which maps $G_2(\widetilde{K})$ to $\widetilde{K}$, denoted by φ_2. Using (3.1) and Proposition 3.5, we have

$$T(\varphi_2) := T(\varphi_{2|G_2(K)}) = T(G_{2|K}) \le \exp(2e). \tag{3.14}$$

Recall that φ_1 is the inverse branch of G_1 which maps $G_1(\widetilde{Q}_0)$ to $\widetilde{Q}_0$. By construction of $\mathcal{Q}^{+}_{1,1}$, there is a unique square $S \in \mathcal{P}^{+}_{1,1}$ such that $K = \varphi_1(S)$ for each $K \in \mathcal{Q}^{+}_{1,1}$, so Proposition 3.5 implies that $G_2 \circ \varphi_1$ is univalent in a neighborhood $\widetilde{S}$ of S. By (3.1) with (3.3) and (3.14), we have

$$T(G_2 \circ \varphi_{1|S}) \le T(G_{2|K}) \cdot T(G_{1|K}) \le T(\varphi_2) \cdot T(\varphi_1) \le \exp(3e)$$

and

$$\inf_{\nu \in S} \big|(G_2 \circ \varphi_1)'(\nu)\big| = \inf_{\lambda \in K} \Big|\frac{G_2'(\lambda)}{G_1'(\lambda)}\Big| \ge \exp\Big(\frac{3x_1}{4}\Big).$$

This, together with Lemma 3.1 and Proposition 3.5, implies that

$$\begin{aligned}\operatorname{dens}\big(P_2, G_2(K)\big) &= \operatorname{dens}\big(P_2, G_2\circ\varphi_1(S)\big)\\ &\geq 1-\exp(9e)\Big(\frac{2\sqrt{2}x_2+21}{\exp(\frac{3x_1}{4})}+\frac{12}{\exp(3e+\frac{3x_1}{2})}\Big) \qquad (3.15)\\ &\geq 1-\frac{\exp(10e)}{\exp(\frac{x_1}{4})}.\end{aligned}$$

Since $\varphi_2\circ G_2 = \mathrm{id}$ on K and $G_2(K\setminus Q_2)\subset G_2(K)\setminus P_2$, we can repeat the argument of (3.5) with (3.14) and (3.15) to obtain

$$\begin{aligned}\operatorname{dens}(Q_2, K) &\geq 1-\exp(4e)\big(1-\operatorname{dens}\big(P_2, G_2(K)\big)\big)\\ &\geq 1-\frac{\exp(14e)}{\exp(\frac{x_1}{4})}\geq 1-\exp\Big(-\frac{x_1}{8}\Big).\end{aligned}$$

If $K\in\mathcal{Q}_{1,1}^-$, then by (3.1) and Proposition 3.5, G_3 is univalent in a neighborhood $\widetilde{K}$ of K and there is an inverse branch φ_3 of G_3 which maps $G_3(\widetilde{K})$ to $\widetilde{K}$ with

$$T(\varphi_3) := T(\varphi_{3|G_3(K)}) = T(G_{3|K}) \leq \exp(3e). \qquad (3.16)$$

By construction of $\mathcal{Q}_{1,1}^-$, there is a unique square $S\in\mathcal{P}_{1,1}^-$ such that $K=\varphi_1(S)$, so Proposition 3.5 implies that $G_3\circ\varphi_1$ is univalent in a neighborhood $\widetilde{S}$ of S. By (3.1) with (3.3) and (3.16),

$$T(G_3\circ\varphi_{1|S}) \leq T(G_{3|K})\cdot T(G_{1|K}) \leq T(\varphi_3)\cdot T(\varphi_1)\leq \exp(4e)$$

and

$$\inf_{\nu\in S}\big|(G_3\circ\varphi_1)'(\nu)\big| = \inf_{\lambda\in K}\Big|\frac{G_3'(\lambda)}{G_1'(\lambda)}\Big| \geq l\cdot\exp\Big(\frac{3x_1}{4}\Big).$$

This, together with Lemma 3.1 and Proposition 3.5, implies that

$$\begin{aligned}\operatorname{dens}\big(P_2, G_3(K)\big) &= \operatorname{dens}\big(P_2, G_3\circ\varphi_1(S)\big)\\ &\geq 1-\exp(12e)\Big(\frac{2\sqrt{2}x_2+21}{\exp(\frac{3x_1}{4})}+\frac{12}{\exp(4e+\frac{3x_1}{2})}\Big) \qquad (3.17)\\ &\geq 1-\frac{\exp(13e)}{\exp(\frac{x_1}{4})}.\end{aligned}$$

Since $\varphi_3 \circ G_3 = \mathrm{id}$ on K and $G_3(K \setminus Q_2) \subset G_3(K) \setminus P_2$, we can also repeat the argument of (3.5) with (3.16) and (3.17) to obtain

$$\begin{aligned}\mathrm{dens}(Q_2, K) &\geq 1 - \exp(6e)\big(1 - \mathrm{dens}\big(P_2, G_2(K)\big)\big) \\ &\geq 1 - \frac{\exp(19e)}{\exp(\frac{x_1}{4})} \geq 1 - \exp\Big(-\frac{x_1}{8}\Big).\end{aligned}$$

∎

Let $\mu \in \{+, -\}$. From the above construction, $\mathcal{Q}_1$ consists of two members $\mathcal{Q}_{1,1}^+$ and $\mathcal{Q}_{1,1}^-$, where $\mathcal{Q}_{1,1}^\mu$ is generated by pullback of $G_1(Q_0) \cap V_{x_1}^\mu$ with an inverse branch of G_1, so $\mathcal{Q}_{1,1}^+$ and $\mathcal{Q}_{1,1}^-$ can be viewed as the "twins" of generation 1 of $\mathcal{Q}_0$; $\mathcal{Q}_2$ consists of four members $\mathcal{Q}_{2,1}^+$, $\mathcal{Q}_{2,1}^-$, $\mathcal{Q}_{2,2}^+$, and $\mathcal{Q}_{2,2}^-$, where $\mathcal{Q}_{2,1}^\mu$ is generated by pullback of $G_2(K') \cap V_{x_2}^\mu$ with an inverse branch of G_2 for each $K' \in \mathcal{Q}_{1,1}^+$, and $\mathcal{Q}_{2,2}^\mu$ is generated by pullback of $G_3(K') \cap V_{x_2}^\mu$ with an inverse branch of G_3 for each $K' \in \mathcal{Q}_{1,1}^-$, so $\mathcal{Q}_{2,1}^+$ and $\mathcal{Q}_{2,1}^-$ can be viewed as the "twins" of generation 2 of $\mathcal{Q}_{1,1}^+$, and $\mathcal{Q}_{2,2}^+$ and $\mathcal{Q}_{2,2}^-$ can be viewed as the "twins" of generation 3 of $\mathcal{Q}_{1,1}^-$.

For integers $1 \leq n \leq 2$ and $1 \leq i \leq 2^{n-1}$, let $t_{n,i}$ denote the number of generations of $\mathcal{Q}_{n,i}^+$ and $\mathcal{Q}_{n,i}^-$. Then

$$t_{1,1} = 1, \qquad t_{2,1} = 2 = 1 + t_{2-1,(1+1)/2}, \qquad t_{2,2} = 3 = 2 + t_{2-1,2/2}.$$

So we can use induction to define, for all integers $n \geq 3$ and $1 \leq i \leq 2^{n-1}$,

$$t_{n,i} = 1 + t_{n-1,(i+1)/2} \quad (\text{if } i \in I_{1,n}), \qquad t_{n,i} = 2 + t_{n-1,i/2} \quad (\text{if } i \in I_{2,n}),$$

where

$$\begin{aligned}I_{1,n} &:= \{i \in \mathbb{N} : i \text{ is odd and } 1 \leq i \leq 2^{n-1}\}, \\ I_{2,n} &:= \{i \in \mathbb{N} : i \text{ is even and } 1 \leq i \leq 2^{n-1}\}.\end{aligned}$$

We check that, for all integers $n \geq 1$ and $1 \leq i \leq 2^{n-1}$,

$$n \leq t_{n,i} \leq 2n - 1.$$

Let $\mu \in \{+, -\}$. For each integer $n \geq 3$, we shall use induction to construct $\mathcal{Q}_n$ consisting of 2^n members $\mathcal{Q}_{n,i}^+$ and $\mathcal{Q}_{n,i}^-$ $(1 \leq i \leq 2^{n-1})$ such that $\mathcal{Q}_{n,i}^\mu$ is generated by pullback of $G_{t_{n,i}}(K') \cap V_{x_n}^\mu$ with an inverse branch of $G_{t_{n,i}}$ for each $K' \in \mathcal{Q}_{n-1,(i+1)/2}^+$ (resp., $K' \in \mathcal{Q}_{n-1,i/2}^-$) if $i \in I_{1,n}$ (resp., $i \in I_{2,n}$), and then $t_{n,i}$ is the number of generation of $\mathcal{Q}_{n,i}^+$ and $\mathcal{Q}_{n,i}^-$, so $\mathcal{Q}_{n,i}^+$ and $\mathcal{Q}_{n,i}^-$ can be viewed as the "twins" of generation $t_{n,i}$ of $\mathcal{Q}^+_{n-1,(i+1)/2}$ (resp., $\mathcal{Q}^-_{n-1,i/2}$) if $i \in I_{1,n}$ (resp., $i \in I_{2,n}$). This can be seen in the following structure of sequence $(\mathcal{Q}_n)_{n\in\mathbb{N}}$:

$$\mathcal{Q}_0$$

$$\mathcal{Q}_1: \quad \underbrace{\mathcal{Q}^+_{1,1} \quad \mathcal{Q}^-_{1,1}}_{\text{generation 1 of } \mathcal{Q}_0}$$

$$\mathcal{Q}_2: \quad \underbrace{\mathcal{Q}^+_{2,1} \quad \mathcal{Q}^-_{2,1}}_{\text{generation 2 of } \mathcal{Q}^+_{1,1},} \qquad \underbrace{\mathcal{Q}^+_{2,2} \quad \mathcal{Q}^-_{2,2}}_{\text{generation 3 of } \mathcal{Q}^-_{1,1}}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathcal{Q}_n: \quad \cdots \quad \underbrace{\mathcal{Q}^+_{n,i} \quad \mathcal{Q}^-_{n,i}}_{\text{generation } t_{n,i} \text{ of } \mathcal{Q}^+_{n-1,(i+1)/2} \text{ (resp., } \mathcal{Q}^-_{n-1,i/2}) \text{ if } i\in I_{1,n} \text{ (resp., } i\in I_{2,n})} \quad \cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\big(\text{structure of sequence } (\mathcal{Q}_n)_{n\in\mathbb{N}}\big)$$

Let $n \geq 2$ be an integer. Assume that, for all integers $2 \leq s \leq n$, $1 \leq i \leq 2^{s-1}$, and $\mu \in \{+,-\}$, all such $\mathcal{P}^\mu_{s,i}, P_s, \mathcal{Q}^\mu_{s,i}$, $\mathcal{Q}_s$, and Q_s are well defined, satisfying the following properties.

(i) For each $K \in \mathcal{Q}^+_{s-1,i}$, the map $G_{1+t_{s-1,i}}$ is univalent in a neighborhood $\widetilde{K}$ of K with

$$\inf_{\lambda\in K}\left|\frac{G'_{1+t_{s-1,i}}(\lambda)}{G'_{t_{s-1,i}}(\lambda)}\right| \geq \exp\Big(\frac{3x_{s-1}}{4}\Big)$$

and

$$T(G_{1+t_{s-1,i}|K}) \leq \exp\big((1+t_{s-1,i})e\big).$$

For each $K \in \mathcal{Q}^-_{s-1,i}$, the map $G_{2+t_{s-1,i}}$ is univalent in a neighborhood $\widetilde{K}$ of K with

$$\inf_{\lambda\in K}\left|\frac{G'_{2+t_{s-1,i}}(\lambda)}{G'_{t_{s-1,i}}(\lambda)}\right| \geq \exp\Big(\frac{3x_{s-1}}{4}\Big)$$

and

$$T(G_{2+t_{s-1,i}|K}) \leq \exp\big((2+t_{s-1,i})e\big).$$

(ii) Each $\mathcal{Q}_s$ is a finite collection of subsets in $\mathbb{C}$ satisfying that every two elements in $\mathcal{Q}_s$ have an intersection of measure zero and that every $K' \in \mathcal{Q}_s$

is contained in a unique $K \in \mathcal{Q}_{s-1}$, with each $K \in \mathcal{Q}_{s-1}$ containing at least one element in $\mathcal{Q}_s$.

(iii) For each $K \in \mathcal{Q}_{s-1}$, we have

$$\mathrm{dens}(Q_s, K) \geq 1 - \exp\Big(-\frac{x_{s-1}}{8}\Big).$$

For $s = n+1$, we can inductively prove the following.

PROPOSITION 3.7. *Let $1 \leq i \leq 2^{n-1}$. For each $K' \in \mathcal{Q}^+_{n,i}$, the map $G_{1+t_{n,i}}$ is univalent in a neighborhood $\widetilde{K}'$ of K' with*

$$\inf_{\lambda \in K'} \Big| \frac{G'_{1+t_{n,i}}(\lambda)}{G'_{t_{n,i}}(\lambda)} \Big| \geq \exp\Big(\frac{3x_n}{4}\Big)$$

and

$$T(G_{1+t_{n,i}|K'}) \leq \exp\big((1+t_{n,i})e\big).$$

For each $K' \in \mathcal{Q}^-_{n,i}$, the map $G_{2+t_{n,i}}$ is univalent in a neighborhood $\widetilde{K}'$ of K' with

$$\inf_{\lambda \in K'} \Big| \frac{G'_{2+t_{n,i}}(\lambda)}{G'_{t_{n,i}}(\lambda)} \Big| \geq \exp\Big(\frac{3x_n}{4}\Big)$$

and

$$T(G_{2+t_{n,i}|K'}) \leq \exp\big((2+t_{n,i})e\big).$$

Proof. If $K' \in \mathcal{Q}^+_{n,i}$ with $i \in I_{1,n}$ (resp., $i \in I_{2,n}$), then there exist a unique $K \in \mathcal{Q}_{n-1,(i+1)/2}$ (resp., $K \in \mathcal{Q}_{n-1,i/2}$) and a unique $S' \in \mathcal{P}^+_{n,i}$ such that $K' \subset K$ and $G_{t_{n,i}}(K') = S'$. Note that $t_{n,1} = 1 + t_{n-1,(i+1)/2}$ if $i \in I_{1,n}$, and $t_{n,i} = 2 + t_{n-1,i/2}$ if $i \in I_{2,n}$. By the hypothesis (i) for $s = n$, $G_{t_{n,i}}$ is univalent in a neighborhood $\widetilde{K}$ of K, so $G_{t_{n,i}}(\widetilde{K})$ is a simply connected domain and contains S'. Denote

$$r = \frac{1}{2}\min\big\{1, \mathrm{dist}\big(S', \partial\big(G_{t_{n,i}}(\widetilde{K})\big)\big)\big\} > 0.$$

Let $\widetilde{S}'$ be an open square with the same center of S' and side length $1+2r$. Then $\widetilde{S}'$ is a neighborhood of S' and contained in $V^+_{x_n-1}$. Define $\widetilde{K}' := \varphi_{t_{n,i}}(\widetilde{S}')$, where $\varphi_{t_{n,i}}$ is the inverse branch of $G_{t_{n,i}}$ which maps $G_{t_{n,i}}(\widetilde{K})$ to $\widetilde{K}$. Then $\widetilde{K}'$ is a neighborhood of K' with $\widetilde{K}' \subset \widetilde{K}$.

Recall that $F(z) = \exp(z)/z$. Suppose that $G_{1+t_{n,i}}(a) = G_{1+t_{n,i}}(b)$ with $a, b \in \widetilde{K}'$. Similar to (3.6), we have

$$|a-b|\big|F\big(G_{t_{n,i}}(a)\big)\big| = |b|\big|F\big(G_{t_{n,i}}(b)\big) - F\big(G_{t_{n,i}}(a)\big)\big|. \tag{3.18}$$

By $\widetilde{K}' \subset \widetilde{K} \subset \widetilde{Q}_0$ and hypothesis (i) for $2 \le s \le n$,

$$\inf_{\lambda\in\widetilde{K}'}\left|G'_{t_{n,i}}(\lambda)\right| \ge \inf_{\lambda\in\widetilde{Q}_0}\left|G'_1(\lambda)\right| \ge \exp(me-e).$$

Combining this and $G_{t_{n,i}}(\widetilde{K}') = \widetilde{S}' \subset V^+_{x_n-1}$, we have

$$\begin{aligned}
&\left|F\big(G_{t_{n,i}}(b)\big) - F\big(G_{t_{n,i}}(a)\big)\right| \\
&\quad\ge |a-b|\cdot \inf_{\lambda\in\widetilde{K}'}\left|\frac{\mathrm{d}F(G_{t_{n,i}}(\lambda))}{\mathrm{d}\lambda}\right| \\
&\quad\ge |a-b|\Big(\inf_{\lambda\in\widetilde{K}'}\Big|F\big(G_{t_{n,i}}(\lambda)\big)G'_{t_{n,i}}(\lambda)\frac{G_{t_{n,i}}(\lambda)-1}{G_{t_{n,i}}(\lambda)}\Big|\Big) \\
&\quad\ge |a-b|\big(\inf_{\nu\in\widetilde{S}'}\left|F(\nu)\right|\big)\big(\inf_{\lambda\in\widetilde{K}'}\left|G'_{t_{n,i}}(\lambda)\right|\big)\Big(\inf_{\nu\in\widetilde{S}'}\Big|\frac{\nu-1}{\nu}\Big|\Big) \\
&\quad\ge |a-b|\big(\inf_{\nu\in\widetilde{S}'}\left|F(\nu)\right|\big)\cdot\frac{1}{2}\exp(me-e).
\end{aligned}$$

Moreover,

$$\left|F\big(G_{t_{n,i}}(a)\big)\right| \le \sup_{\nu\in\widetilde{S}'}\left|F(\nu)\right|$$

and

$$\frac{\sup_{\nu\in\widetilde{S}'}|F(\nu)|}{\inf_{\nu\in\widetilde{S}'}|F(\nu)|} \le 2e^2.$$

This, together with (3.18) and $|b| \ge m-1 \ge 99$, implies that

$$4e^2|a-b| \ge 99|a-b|\exp(99e).$$

So $a=b$. Therefore, $G_{1+t_{n,i}}$ is univalent in $\widetilde{K}'$.

By calculation,

$$G'_{1+t_{n,i}}(\lambda) = F\big(G_{t_{n,i}}(\lambda)\big)\Big(1+\lambda G'_{t_{n,i}}(\lambda)\frac{G_{t_{n,i}}(\lambda)-1}{G_{t_{n,i}}(\lambda)}\Big), \tag{3.19}$$

$$\frac{G'_{1+t_{n,i}}(\lambda)}{G'_{t_{n,i}}(\lambda)} = F\big(G_{t_{n,i}}(\lambda)\big)\Big(\frac{1}{G'_{t_{n,i}}(\lambda)}+\lambda\frac{G_{t_{n,i}}(\lambda)-1}{G_{t_{n,i}}(\lambda)}\Big). \tag{3.20}$$

Recall that $x_0 = 2m, x_j = 2\cdot\exp^j(m)$ for all integers $j \ge 1$, and note that $G_{t_{n,i}}(K') = S' \subset V^+_{x_n}$ and that $K' \subset K \subset S_{m,l}$. Then $|\lambda| \ge m \ge 10^2$ for all

$\lambda \in K'$ and $|\nu| \le \exp(2\sum_{j=0}^{n-1} x_j)$ for all $\nu \in S'$. By hypothesis (i) for $s=n$ with (3.19) and (3.20), we have

$$\inf_{\lambda\in K'}\Big|\frac{G'_{1+t_{n,i}}(\lambda)}{G'_{t_{n,i}}(\lambda)}\Big| \ge \inf_{\nu\in S'}\big|F(\nu)\big|\cdot\frac{1}{2}\inf_{\lambda\in K'}|\lambda|$$
$$\ge \frac{m}{2}\cdot\frac{\exp(x_n)}{\exp(2\sum_{j=0}^{n-1}x_j)} \ge \exp\Big(\frac{3x_n}{4}\Big)$$

and

$$T(G_{1+t_{n,i}|K'}) \le \frac{\sup_{\nu\in S'}|F(\nu)|}{\inf_{\nu\in S'}|F(\nu)|}\cdot\frac{\sup_{\lambda\in K'}|1+\lambda G'_{t_{n,i}}(\lambda)\frac{G_{t_{n,i}}(\lambda)-1}{G_{t_{n,i}}(\lambda)}|}{\inf_{\lambda\in K'}|1+\lambda G'_{t_{n,i}}(\lambda)\frac{G_{t_{n,i}}(\lambda)-1}{G_{t_{n,i}}(\lambda)}|}$$
$$\le \sqrt{2}e\cdot\sqrt{2}T(G_{t_{n,i}|K}) \le \sqrt{2}e\cdot\sqrt{2}\exp(t_{n,i}\cdot e)$$
$$\le \exp\big((1+t_{n,i})e\big).$$

If $K' \in \mathcal{Q}^-_{n,i}$ with $i \in I_{1,n}$ (resp., $i \in I_{2,n}$), then there also exist a unique $K \in \mathcal{Q}_{n-1,(i+1)/2}$ (resp., $K \in \mathcal{Q}_{n-1,i/2}$) and a unique $S' \in \mathcal{P}^-_{n,i}$ such that $K' \subset K$ and $G_{t_{n,i}}(K') = S'$. By hypothesis (i) for $s=n$, there is also a corresponding neighborhood $\widetilde{K}'$ of K' such that $\widetilde{K}' \subset \widetilde{K} \subset \widetilde{Q}_0$ and $G_{t_{n,i}}(\widetilde{K}')$ is an open square contained in $V^-_{x_n-1}$ with side length no more than 2.

Suppose that $G_{2+t_{n,i}}(a) = G_{2+t_{n,i}}(b)$ with $a, b \in \widetilde{K}'$. Similar to (3.18), we have

$$|a-b|\big|F(G_{1+t_{n,i}})\big| = |b|\big|F\big(G_{1+t_{n,i}}(b)\big) - F\big(G_{1+t_{n,i}}(a)\big)\big|. \tag{3.21}$$

Let $A' = G_{1+t_{n,i}}(\widetilde{K}')$; then A' is contained in a small annulus with outer radius no more than $\exp(-x_n)$ and $\mathrm{mod}(A') \le 2e^2$, so $|G_{1+t_{n,i}}(\lambda)| \le \exp(-x_n)$ for each $\lambda \in \widetilde{K}'$. By $\widetilde{K}' \subset \widetilde{K} \subset \widetilde{Q}_0$ and hypothesis (i) for $2 \le s \le n$,

$$\inf_{\lambda\in\widetilde{K}'}\big|G'_{t_{n,i}}(\lambda)\big| \ge \inf_{\lambda\in\widetilde{Q}_0}\big|G'_1(\lambda)\big| \ge \exp(me-e).$$

This, together with (3.19) and $|G_{t_{n,i}}(\lambda)| \ge x_n$ for each $\lambda \in \widetilde{K}'$, implies that

$$\inf_{\lambda\in\widetilde{K}'}\Big|\frac{G_{1+t_{n,i}}(\lambda)-1}{G_{1+t_{n,i}}(\lambda)}G'_{1+t_{n,i}}(\lambda)\Big| \ge \frac{1}{\sqrt{2}}\cdot\inf_{\lambda\in\widetilde{K}'}\Big|\frac{G'_{1+t_{n,i}}(\lambda)}{G_{1+t_{n,i}}(\lambda)}\Big|$$
$$\ge \frac{1}{2}\cdot\inf_{\lambda\in\widetilde{K}'}\big|G'_{t_{n,i}}(\lambda)\big| \ge \frac{1}{2}\cdot\exp(me-e)$$

and that

$$\begin{aligned}
&\big|F\big(G_{1+t_{n,i}}(b)\big)-F\big(G_{1+t_{n,i}}(a)\big)\big|\\
&\quad\geq |a-b|\inf_{\lambda\in\widetilde{K}'}\Big|\frac{\mathrm{d}F(G_{1+t_{n,i}}(\lambda))}{\mathrm{d}\lambda}\Big|\\
&\quad\geq |a-b|\big(\inf_{\lambda\in\widetilde{K}'}\big|F\big(G_{1+t_{n,i}}(\lambda)\big)\big|\big)\Big(\inf_{\lambda\in\widetilde{K}'}\Big|\frac{G_{1+t_{n,i}}(\lambda)-1}{G_{1+t_{n,i}}(\lambda)}G'_{1+t_{n,i}}(\lambda)\Big|\Big)\\
&\quad\geq |a-b|\big(\inf_{\nu\in A'}\big|F(\nu)\big|\big)\cdot\frac{1}{2}\cdot\exp(me-e).
\end{aligned}$$

Moreover,

$$\big|F\big(G_{1+t_{n,i}}(a)\big)\big|\leq\sup_{\nu\in A'}\big|F(\nu)\big|$$

and

$$\frac{\sup_{\nu\in A'}|F(\nu)|}{\inf_{\nu\in A'}|F(\nu)|}\leq \mathrm{mod}(A')\cdot\exp\big(2\cdot\exp(-x_n)\big)\leq 4e^2.$$

This, together with (3.21) and $|b|\geq m-1\geq 99$, implies that

$$8e^2|a-b|\geq 99|a-b|\exp(99e).$$

So $a=b$. Therefore, $G_{2+t_{n,i}}$ is univalent in $\widetilde{K}$.

By calculation,

$$G'_{2+t_{n,i}}(\lambda)=F\big(G_{1+t_{n,i}}(\lambda)\big)\Big(1+\lambda G'_{1+t_{n,i}}(\lambda)\frac{G_{1+t_{n,i}}(\lambda)-1}{G_{1+t_{n,i}}(\lambda)}\Big),\tag{3.22}$$

$$\frac{G'_{2+t_{n,i}}(\lambda)}{G'_{t_{n,i}}(\lambda)}=F\big(G_{1+t_{n,i}}(\lambda)\big)\Big(\frac{G_{1+t_{n,i}}(\lambda)}{G'_{t_{n,i}}(\lambda)}+\lambda\frac{(G_{t_{n,i}}(\lambda)-1)^2}{G_{t_{n,i}}(\lambda)}\Big).\tag{3.23}$$

Note that $G_{1+t_{n,i}}(K')\subset A'$, $G_{t_{n,i}}(K')\subset V^-_{x_n}$, $K'\subset Q_0$, and $|\lambda|\geq m\geq 10^2$ for all $\lambda\in K'$. Then $|\nu|\leq\exp(-x_n)$ for all $\nu\in G_{1+t_{n,i}}(K')\subset A'$ and $|G_{t_{n,i}}(\lambda)|\geq x_n$ for all $\lambda\in K'$. By hypothesis (i) for $s=n$ with (3.22) and (3.23), we have

$$\inf_{\lambda\in K'}\Big|\frac{G'_{2+t_{n,i}}(\lambda)}{G'_{t_{n,i}}(\lambda)}\Big|\geq\inf_{\nu\in A'}\big|F(\nu)\big|\cdot\frac{1}{2}\inf_{\lambda\in K'}|\lambda|\geq\frac{m}{2}\cdot\frac{\exp(x_n)}{2}\geq\exp\Big(\frac{3x_n}{4}\Big)$$

and

$$T(G_{2+t_{n,i}|K'}) \leq \frac{\sup_{\nu\in A'}|F(\nu)|}{\inf_{\nu\in A'}|F(\nu)|}\cdot\frac{\sup_{\lambda\in K'}|1+\lambda G'_{1+t_{n,i}}(\lambda)\frac{G_{1+t_{n,i}}(\lambda)-1}{G_{1+t_{n,i}}(\lambda)}|}{\inf_{\lambda\in K'}|1+\lambda G'_{1+t_{n,i}}(\lambda)\frac{G_{1+t_{n,i}}(\lambda)-1}{G_{1+t_{n,i}}(\lambda)}|}$$

$$\leq 4e^2\cdot 2\frac{\sup_{\lambda\in K'}|\frac{G'_{1+t_{n,i}}(\lambda)}{G_{1+t_{n,i}}(\lambda)}|}{\inf_{\lambda\in K'}|\frac{G'_{1+t_{n,i}}(\lambda)}{G_{1+t_{n,i}}(\lambda)}|}\leq 4e^2\cdot 4T(G_{t_{n,i}|K})$$

$$\leq 4e^2\cdot 4\exp(t_{n,i}\cdot e)\leq \exp\big((2+t_{n,i})e\big). \qquad \square$$

Let $1\leq i\leq 2^{n-1}$. Note that if $K'\in\mathcal{Q}^+_{n,i}$, then K' is mapped away by $G_{1+t_{n,i}}$; if $K'\in\mathcal{Q}^-_{n,i}$, then K' is mapped into a neighborhood of 0 (the pole) by $G_{1+t_{n,i}}$, before being mapped away by $G_{2+t_{n,i}}$. So we consider the set $G_{1+t_{n,i}}(K')$ for each $K'\in\mathcal{Q}^+_{n,i}$, and the set $G_{2+t_{n,i}}(K')$ for each $K'\in\mathcal{Q}^-_{n,i}$. By hypothesis (i) for $2\leq s\leq n$ with Lemma 2.2 and Proposition 3.7, $G_{1+t_{n,i}}(K')\cap(V^+_{x_{n+1}}\cup V^-_{x_{n+1}})$ contains many squares in $\mathcal{B}$ for each $K'\in\mathcal{Q}^+_{n,i}$, and $G_{2+t_{n,i}}(K')\cap(V^+_{x_{n+1}}\cup V^-_{x_{n+1}})$ contains many squares in $\mathcal{B}$ for each $K'\in\mathcal{Q}^-_{n,i}$. For each integer $1\leq j\leq 2^n$, we have two cases.

If $j\in I_{1,n+1}$, then $t_{n+1,j}=1+t_{n,(j+1)/2}$. Define for $\mu\in\{+,-\}$

$$\mathcal{P}^\mu_{n+1,j} := \bigcup_{K'\in\mathcal{Q}^+_{n,(j+1)/2}}\big\{S\in\mathcal{B}\,\big|\,S\subset G_{t_{n+1,j}}(K')\cap V^\mu_{x_{n+1}}\big\},$$

$$\mathcal{Q}^\mu_{n+1,j} := \big\{K\subset Q_n\,\big|\,G_{t_{n+1,j}}(K)\in\mathcal{P}^\mu_{n+1,j}\big\}.$$

If $j\in I_{2,n+1}$, then $t_{n+1,j}=2+t_{n,j/2}$. Define for $\mu\in\{+,-\}$

$$\mathcal{P}^\mu_{n+1,j} := \bigcup_{K'\in\mathcal{Q}^-_{n,j/2}}\big\{S\in\mathcal{B}\,\big|\,S\subset G_{t_{n+1,j}}(K')\cap V^\mu_{x_{n+1}}\big\},$$

$$\mathcal{Q}^\mu_{n+1,j} := \big\{K\subset Q_n\,\big|\,G_{t_{n+1,j}}(K)\in\mathcal{P}^\mu_{n+1,j}\big\}.$$

Furthermore, we define

$$P_{n+1} := \bigcup_{S\in\mathcal{P}^\mu_{n+1,j}|\mu\in\{+,-\},1\leq j\leq 2^n} S,$$

$$\mathcal{Q}_{n+1} := \big\{K\in\mathcal{Q}^\mu_{n+1,j}\,\big|\,\mu\in\{+,-\},1\leq j\leq 2^n\big\},$$

$$Q_{n+1} := \bigcup_{K\in\mathcal{Q}_{n+1}} K.$$

By the definitions, for $1 \le j \le 2^n$ and $\mu \in \{+,-\}$, $\mathcal{Q}^+_{n+1,j} \cap \mathcal{Q}^-_{n+1,j} = \emptyset$, and every two elements of $\mathcal{Q}^\mu_{n+1,j}$ have an intersection of measure zero. By hypothesis (ii) for $s = n$, we have $\mathcal{Q}^{\mu_1}_{n,j_1} \cap \mathcal{Q}^{\mu_2}_{n,j_2} = \emptyset$ for any two distinct pairs (j_1,μ_1) and (j_2,μ_2); for $\mu \in \{+,-\}$, every $K \in \mathcal{Q}^\mu_{n+1,j}$ with $j \in I_{1,n+1}$ (resp., $\mathcal{Q}^\mu_{n+1,j}$ with $j \in I_{2,n+1}$) is contained in a unique $K' \in \mathcal{Q}^+_{n,(j+1)/2}$ (resp., $\mathcal{Q}^-_{n,j/2}$); and every $K' \in \mathcal{Q}^+_{n,j}$ (resp., $\mathcal{Q}^-_{n,j}$) contains at least one element in $\mathcal{Q}^+_{n+1,2j-1} \cup \mathcal{Q}^-_{n+1,2j-1}$ (resp., $\mathcal{Q}^+_{n+1,2j} \cup \mathcal{Q}^-_{n+1,2j}$). This implies that $\mathcal{Q}^{\mu_1}_{n+1,j_1} \cap \mathcal{Q}^{\mu_2}_{n+1,j_2} = \emptyset$ for any two distinct pairs (j_1,μ_1) and (j_2,μ_2).

Therefore, $\mathcal{Q}_{n+1}$ is a finite collection of subsets in $\mathbb{C}$ satisfying that every two elements in $\mathcal{Q}_{n+1}$ have an intersection of measure zero and that every $K \in \mathcal{Q}_{n+1}$ is contained in a unique $K' \in \mathcal{Q}_n$, with each $K' \in \mathcal{Q}_n$ containing at least one element in $\mathcal{Q}_{n+1}$.

PROPOSITION 3.8. *For each $K' \in \mathcal{Q}_n$, we have*

$$\operatorname{dens}(Q_{n+1}, K') \ge 1 - \exp\Big(-\frac{x_n}{8}\Big).$$

Proof. Let $K' \in \mathcal{Q}^+_{n,i}$. By Proposition 3.7, $G_{1+t_{n,i}}$ is univalent in a neighborhood $\widetilde{K}'$ of K'. We can take an inverse branch of $G_{1+t_{n,i}}$ which maps $G_{1+t_{n,i}}(\widetilde{K}')$ to $\widetilde{K}'$, denoted by $\varphi_{1+t_{n,i}}$. Using (3.1) and Proposition 3.7, we have

$$\begin{aligned} T(\varphi_{1+t_{n,i}}) &:= T(\varphi_{1+t_{n,i}|G_{1+t_{n,i}}(K')}) \\ &= T(G_{1+t_{n,i}|K'}) \le \exp\big((1+t_{n,i})e\big). \end{aligned} \tag{3.24}$$

Recall that $\varphi_{t_{n,i}}$ is the inverse branch of $G_{t_{n,i}}$ which maps $G_{t_{n,i}}(\widetilde{K})$ to $\widetilde{K}$, where K is the unique element of $\mathcal{Q}_{n-1,(i+1)/2}$ (resp., $\mathcal{Q}_{n-1,i/2}$) such that $K' \subset K$ for $K' \in \mathcal{Q}_{n,i}$ with $i \in I_{1,n}$ (resp., $I_{2,n}$). By construction of $\mathcal{Q}^+_{n,i}$, there is a unique square $S' \in \mathcal{P}^+_{n,i}$ such that $K' = \varphi_{t_{n,i}}(S')$, so Proposition 3.7 implies that $G_{1+t_{n,i}} \circ \varphi_{t_{n,i}}$ is univalent in a neighborhood $\widetilde{S}'$ of S'. Since $K' \subset K$, by hypothesis (i) for $s = n$ with (3.1) and (3.24), we have

$$\begin{aligned} T(G_{1+t_{n,i}} \circ \varphi_{t_{n,i}|S'}) &\le T(G_{1+t_{n,i}|K'}) \cdot T(G_{t_{n,i}|K}) \\ &\le T(\varphi_{1+t_{n,i}}) \cdot \exp(t_{n,i} \cdot e) \le \exp\big((1+2t_{n,i})e\big) \end{aligned}$$

and

$$\inf_{\nu \in S'} \big|(G_{1+t_{n,i}} \circ \varphi_{t_{n,i}})'(\nu)\big| = \inf_{\lambda \in K'} \Big|\frac{G'_{1+t_{n,i}}(\lambda)}{G'_{t_{n,i}}(\lambda)}\Big| \ge \exp\Big(\frac{3x_n}{4}\Big).$$

This, together with Lemma 3.1 and Proposition 3.7, implies that

$$\begin{aligned} &\operatorname{dens}\big(P_{n+1}, G_{1+t_{n,i}}(K')\big) \\ &\quad = \operatorname{dens}\big(P_{n+1}, G_{1+t_{n,i}} \circ \varphi_{t_{n,i}}(S')\big) \\ &\quad \geq 1 - \exp\big((3+6t_{n,i})e\big) \\ &\qquad \times \Big(\frac{2\sqrt{2}x_{n+1}+21}{\exp(\frac{3x_n}{4})} + \frac{12}{\exp(\frac{3x_n}{2}+(1+2t_{n,i})e)}\Big) \\ &\quad \geq 1 - \frac{\exp((4+6t_{n,i})e)}{\exp(\frac{x_n}{4})}. \end{aligned} \tag{3.25}$$

Note that $x_n = 2\exp^n(m)$ and that $n \leq t_{n,i} \leq 2n-1$ for all integers $1 \leq i \leq 2^{n-1}$. Since $\varphi_{1+t_{n,i}} \circ G_{1+t_{n,i}} = \mathrm{id}$ on K' and $G_{1+t_{n,i}}(K' \setminus Q_{n+1}) \subset G_{1+t_{n,i}}(K') \setminus P_{n+1}$, we can repeat the argument of (3.5) with (3.24) and (3.25) to obtain

$$\begin{aligned} \operatorname{dens}(Q_{n+1}, K') &\geq 1 - \exp\big((2+2t_{n,i})e\big)\big(1 - \operatorname{dens}\big(P_{n+1}, G_{1+t_{n,i}}(K')\big)\big) \\ &\geq 1 - \frac{\exp((6+8t_{n,i})e)}{\exp(\frac{x_n}{4})} \\ &\geq 1 - \frac{\exp(16ne-2e)}{\exp(\frac{x_n}{4})} \geq 1 - \exp\Big(-\frac{x_n}{8}\Big). \end{aligned}$$

If $K' \in \mathcal{Q}_{n,i}^-$, then by (3.1) and Proposition 3.7, $G_{2+t_{n,i}}$ is univalent in a neighborhood $\widetilde{K}'$ of K', and there is an inverse branch $\varphi_{2+t_{n,i}}$ of $G_{2+t_{n,i}}$ which maps $G_{2+t_{n,i}}(\widetilde{K}')$ to $\widetilde{K}'$ with

$$\begin{aligned} T(\varphi_{2+t_{n,i}}) &:= T(\varphi_{2+t_{n,i}|G_{2+t_{n,i}}(K')}) \\ &= T(G_{2+t_{n,i}|K'}) \leq \exp\big((2+t_{n,i})e\big). \end{aligned} \tag{3.26}$$

By construction of $\mathcal{Q}_{n,i}^-$, there is a unique square $S' \in \mathcal{P}_{n,i}^-$ such that $K' = \varphi_{t_{n,i}}(S')$, so Proposition 3.7 implies that $G_{2+t_{n,i}} \circ \varphi_{t_{n,i}}$ is univalent in a neighborhood $\widetilde{S}'$ of S', where $\varphi_{t_{n,i}}$ is the same as in the proof of the case of $K' \in K' \in \mathcal{Q}_{n,i}^+$. By hypothesis (i) for $s=n$ with (3.1) and (3.26), we have

$$\begin{aligned} T(G_{2+t_{n,i}} \circ \varphi_{t_{n,i}|S'}) &\leq T(G_{2+t_{n,i}|K'}) \cdot T(G_{t_{n,i}|K}) \\ &\leq T(\varphi_{2+t_{n,i}}) \cdot \exp(t_{n,i} \cdot e) \leq \exp\big((2+2t_{n,i})e\big) \end{aligned}$$

and

$$\inf_{\nu\in S'}\big|(G_{2+t_{n,i}}\circ\varphi_{t_{n,i}})'(\nu)\big| = \inf_{\lambda\in K'}\Big|\frac{G'_{2+t_{n,i}}(\lambda)}{G'_{t_{n,i}}(\lambda)}\Big| \geq \exp\Big(\frac{3x_n}{4}\Big).$$

This, together with Lemma 3.1 and Proposition 3.7, implies that

$$\begin{aligned}
&\operatorname{dens}\big(P_{n+1}, G_{2+t_{n,i}}(K')\big)\\
&\quad= \operatorname{dens}\big(P_{n+1}, G_{2+t_{n,i}}\circ\varphi_{t_{n,i}}(S')\big)\\
&\quad\geq 1-\exp\big((6+6t_{n,i})e\big)\\
&\qquad\times\Big(\frac{2\sqrt{2}x_{n+1}+21}{\exp(\frac{3x_n}{4})}+\frac{12}{\exp(\frac{3x_n}{2}+(2+2t_{n,i})e)}\Big)\\
&\quad\geq 1-\frac{\exp((7+6t_{n,i})e)}{\exp(\frac{x_n}{4})}.
\end{aligned}\tag{3.27}$$

Also note that $x_n = 2\exp^n(m)$ and $n \leq t_{n,i} \leq 2n-1$ for all integers $1 \leq i \leq 2^{n-1}$. Since $\varphi_{2+t_{n,i}}\circ G_{2+t_{n,i}} = \mathrm{id}$ on K' and $G_{2+t_{n,i}}(K'\setminus Q_{n+1}) \subset G_{2+t_{n,i}}(K')\setminus P_{n+1}$, we can repeat the argument of (3.5) with (3.26) and (3.27) to obtain

$$\begin{aligned}
\operatorname{dens}(Q_{n+1}, K') &\geq 1-\exp\big((4+2t_{n,i})e\big)\big(1-\operatorname{dens}\big(P_{n+1}, G_{1+t_{n,i}}(K')\big)\big)\\
&\geq 1-\frac{\exp((11+8t_{n,i})e)}{\exp(\frac{x_n}{4})}\\
&\geq 1-\frac{\exp(16ne+3e)}{\exp(\frac{x_n}{4})} \geq 1-\exp\Big(-\frac{x_n}{8}\Big).
\end{aligned}$$

□

By the above construction, the sequence $(\mathcal{Q}_n)_{n\geq 0}$ satisfies the nesting conditions of Lemma 2.3. Denote $Q = \bigcap_{n=0}^{\infty} Q_n$ and

$$\delta_n = 1-\exp\Big(-\frac{x_n}{8}\Big)$$

for all integers $n \geq 0$. Applying Lemma 2.3, we have

$$\operatorname{dens}(Q, Q_0) \geq \prod_{n=0}^{\infty}\delta_n.$$

Note that $x_0 = 2m$ and $x_n = 2\cdot\exp^n(m)$ for all integers $n \geq 1$; then $x_n \geq (n+1)m$ for all integers $n \geq 0$. This, together with $m \geq 10^2$, implies that

$$\exp\Big(-\frac{x_n}{8}\Big) \leq \exp\Big(-\frac{(n+1)m}{8}\Big) \leq \exp\Big(-\frac{m}{8}\Big) < \frac{1}{2}\tag{3.28}$$

for all integers $n \geq 0$. Using (3.28) and $\log(1-t) > -2t$ for all $t \in (0, 1/2)$, we have

$$\log\Big(\prod_{n=0}^{\infty} \delta_n\Big) = \sum_{n=0}^{\infty} \log\Big(1 - \exp\Big(-\frac{x_n}{8}\Big)\Big)$$
$$\geq -2\sum_{n=0}^{\infty} \exp\Big(-\frac{x_n}{8}\Big) \geq -2\sum_{n=0}^{\infty} \exp\Big(-\frac{(n+1)m}{8}\Big)$$
$$\geq -4\exp\Big(-\frac{m}{8}\Big).$$

Using $\exp(t) \geq 1 + t$ for all $t \in \mathbb{R}$, we obtain

$$\text{dens}(Q, Q_0) \geq 1 - 4\exp\Big(-\frac{m}{8}\Big). \tag{3.29}$$

Let $\lambda \in Q$, and let $n \geq 1$. From the construction of Q, we have three cases for each $G_n(\lambda)$.

(i) If $G_n(\lambda) \in S$ for $S \in \mathcal{P}^+_{k,j}$ with $k \geq 1$ and $1 \leq j \leq 2^{k-1}$, then $\operatorname{Re} G_n(\lambda) \geq x_k$ and

$$G_{n+1}(\lambda) \in S' \quad \text{for } S' \in \mathcal{P}^+_{k+1,2j-1} \cup \mathcal{P}^-_{k+1,2j-1} \text{ with } \big|\operatorname{Re} G_{n+1}(\lambda)\big| \geq x_{k+1}.$$

(ii) If $G_n(\lambda) \in S$ for $S \in \mathcal{P}^-_{k,j}$ with $k \geq 1$ and $1 \leq j \leq 2^{k-1}$, then $\operatorname{Re} G_n(\lambda) \leq -x_k$ and

$$G_{n+1}(\lambda) \in G_{1+t_{k,j}}(K) \quad \text{for } K \in \mathcal{Q}^-_{k,j} \text{ with } \big|G_{n+1}(\lambda)\big| \leq \exp(-x_k).$$

(iii) If $G_n(\lambda) \in G_{1+t_{k,j}}(K)$ for $K \in \mathcal{Q}^-_{k,j}$ with $k \geq 1$ and $1 \leq j \leq 2^{k-1}$, then $|G_n(\lambda)| \leq \exp(-x_k)$ and

$$G_{n+1}(\lambda) \in S \quad \text{for } S \in \mathcal{P}^+_{k+1,2j} \cup \mathcal{P}^-_{k+1,2j} \text{ with } \big|\operatorname{Re} G_{n+1}(\lambda)\big| \geq x_{k+1}.$$

This, together with $x_k = 2\exp^k(m)$, implies that each accumulation point of $\{G_n(\lambda)\}_{n\geq 1}$ on $\widehat{\mathbb{C}}$ is either 0 or ∞. So $\omega_{F_\lambda}(1) \subset \{0, \infty\}$ and $Q \subset W$. By (3.29),

$$\text{dens}(W, Q_0) \geq \text{dens}(Q, Q_0) \geq 1 - 4\exp\Big(-\frac{m}{8}\Big).$$

Recall that $Q_0 = S_{m,m}$. Using $\text{meas}(S_{m,m}) = 1$ and $m \geq 10^2$, we have

$$\text{area}(S_{m,m} \cap W) \geq 1 - 4\exp\Big(-\frac{m}{8}\Big) \geq 1 - 4\exp\Big(-\frac{10^2}{8}\Big) := \alpha > 0.$$

By Lemma 2.1,

$$\operatorname{area}(S_{m,m} \cap M) \geq \operatorname{area}(S_{m,m} \cap W) \geq \alpha > 0.$$

Hence, Lemma 3.2 follows.

The Main Theorem then follows from Lemma 3.2.

REMARK. The family F_λ has the property that the forward orbit of a singular point (the only critical point 1) goes far away after visiting a small neighborhood of 0 (the pole), while this property does not hold for the exponential family. This leads to the fact that an approach analogous to the one used in the proof of the Main Theorem cannot be applied for the exponential family.

Finally, we prove the following.

PROPOSITION 3.9. *The area of the complement M^c of the M-set is positive.*

Proof. If F_λ has an attracting fixed point, say, t, then $\lambda \exp(t)/t = t$ and $|F'_\lambda(t)| < 1$. So $\lambda = t^2 e^{-t}$ with $|t-1| < 1$. Let

$$\Omega = \{\lambda \in \mathbb{C}^* \mid F_\lambda \text{ has an attracting fixed point}\}.$$

Then

$$\Omega = \left\{\lambda = t^2 e^{-t} : t \in D(1,1)\right\} \subset M^c.$$

Denote $\lambda(t) = t^2 e^{-t}$ for all $t \in D(1,1)$. Note that $\lambda(t)$ is analytic at the point $t = 1$, which is not a critical point of $\lambda(t)$. There exists an $r \in (0,1)$ such that $\lambda(t)$ is univalent in $D(1,r)$. For all $t \in D(1,r)$, we have

$$\left|\lambda'(t)\right| = \left|t(2-t)e^{-t}\right| \geq (1-r)^2 e^{-(r+1)}.$$

This implies that

$$\operatorname{area}(M^c) \geq \operatorname{area}(\Omega) \geq \int\int_{D(1,r)} \left|\lambda'(t)\right|^2 d\sigma \geq \pi r^2 (1-r)^4 e^{-2(r+1)} > 0. \qquad \square$$

Acknowledgments. The authors would like to express their deep thanks to the anonymous referees for their very detailed suggestions, which helped to improve this article. Zhan's work was supported by the Research Starting Fund of Zhejiang University of Technology (grant 109006929). Liao's work was supported by National Natural Science Foundation of China grants 10871089 and 11271179.

References

[1] I. N. Baker, J. Kotus, and Y. Lü, *Iterates of meromorphic functions, IV: Critically finite functions*, Results Math. **22** (1992), 651–656. MR 1189754. DOI 10.1007/BF03323112.

[2] W. Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. (N.S.) **29** (1993), 151–188. MR 1216719. DOI 10.1090/S0273-0979-1993-00432-4.

[3] L. Carleson and T. W. Gamelin, *Complex Dynamics*, Universitext, Springer, New York, 1993. MR 1230383. DOI 10.1007/978-1-4612-4364-9.

[4] R. Devaney, *Julia sets and bifurcation diagrams for exponential maps*, Bull. Amer. Math. Soc. (N.S.) **11** (1984), 167–171. MR 0741732. DOI 10.1090/S0273-0979-1984-15253-4.

[5] R. Devaney, N. Fagella, and X. Jarque, *Hyperbolic components of the complex exponential family*, Fund. Math. **174** (2002), 193–215. MR 1924998. DOI 10.4064/fm174-3-1.

[6] R. Devaney and L. R. Goldberg, *Uniformization of attracting basins for exponential maps*, Duke Math. J. **55** (1987), 253–266. MR 0894579. DOI 10.1215/S0012-7094-87-05513-X.

[7] R. Devaney, L. Goldberg, and J. H. Hubbard, *A dynamical approximation to the exponential map by polynomials*, preprint, 1985.

[8] M. Y. Lyubich, *Measurable dynamics of the exponential* (in Russian), Sibirsk. Mat. Zh. **28**, no. 5 (1987), 111–127; English translation in Siberian Math. J. **28**, no. 5 (1987), 780–797. MR 0924986.

[9] P. M. Makienko, *Iterations of analytic functions in* $\mathbb{C}^*$ (in Russian), Dokl. Akad. Nauk **297**, no. 1 (1987), 35–37; English translation in Dokl. Math. **36**, no. 3 (1988), 418–420. MR 0916928.

[10] C. McMullen, *Area and Hausdorff dimension of Julia sets of entire functions*, Trans. Amer. Math. Soc. **300**, no. 1 (1987), 329–342. MR 0871679. DOI 10.2307/2000602.

[11] M. Misiurewicz, *On iterates of* e^z, Ergodic Theory Dynam. Systems **1** (1981), 103–106. MR 0627790.

[12] W. Qiu, *Hausdorff dimension of the* M*-set of* $\lambda \exp(z)$, Acta Math. Sinica (N.S.) **10** (1994), 362–368. MR 1416147. DOI 10.1007/BF02582032.

[13] L. Rempe, *Topological dynamics of exponential maps on their escaping sets*, Ergodic Theory Dynam. Systems **26** (2006), 1939–1975. MR 2279273. DOI 10.1017/S0143385706000435.

[14] ———, *The escaping set of the exponential*, Ergodic Theory Dynam. Systems **30** (2010), 595–599. MR 2599894. DOI 10.1017/S014338570900008X.

[15] ———, *Dynamics of exponential maps*, Ph.D. dissertation, Christian-Albrechts-Universität zu Kiel, Kiel, Germany, 2003.

[16] L. Rempe and D. Schleicher, "Combinatorics of bifurcations in exponential parameter space" in *Transcendental Dynamics and Complex Analysis*, London Math. Soc. Lecture Note Ser. **348**, Cambridge University Press, Cambridge, 2008, 317–370. MR 2458808. DOI 10.1017/CBO9780511735233.014.

[17] ———, *Bifurcations in the space of exponential maps*, Invent. Math. **175** (2009), 103–135. MR 2461427. DOI 10.1007/s00222-008-0147-5.

[18] X. Wang and G. Zhang, *Most of the maps near the exponential are hyperbolic*, Nagoya Math. J. **191** (2008), 135–148. MR 2451223.

[19] G. P. Zhan, *Non-recurrence of* $\exp(z)/z$, Acta Math. Sin. (Engl. Ser.) **29** (2013), 703–716. MR 3029285. DOI 10.1007/s10114-012-1451-y.

[20] ———, *Hausdorff dimension of a fractal set of* $\exp(z)/z$, preprint, 2012.
[21] G. P. Zhan and L.-W. Liao, *Area of non-escaping parameters of the sine family*, Houston J. Math. **38** (2012), 493–524. MR 2954649.

Guoping Zhan
Department of Mathematics
Zhejiang University of Technology
Hangzhou 310023
People's Republic of China
zhangp@zjut.edu.cn

Liangwen Liao
Department of Mathematics
Nanjing University
Nanjing 210093
People's Republic of China
maliao@nju.edu.cn