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THE MONODROMY REPRESENTATION AND TWISTED PERIOD RELATIONS FOR APPELL'S HYPERGEOMETRIC FUNCTION F_4

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To Professor Kyoichi Takano on his seventieth birthday

Abstract. We consider the system $\mathcal{F}_4(a,b,c)$ of differential equations annihilating Appell's hypergeometric series $F_4(a,b,c;x)$. We find the integral representations for four linearly independent solutions expressed by the hypergeometric series F_4. By using the intersection forms of twisted (co)homology groups associated with them, we provide the monodromy representation of $\mathcal{F}_4(a,b,c)$ and the twisted period relations for the fundamental systems of solutions of $\mathcal{F}_4$.

§1. Introduction

Appell's hypergeometric series $F_4(a,b,c;x)$ of variables $x=(x_1,x_2)$ with complex parameters $a,b,c=(c_1,c_2)$ is defined by

$$F_4(a,b,c;x)=\sum_{(n_1,n_2)\in\mathbb{N}^2}\frac{(a,n_1+n_2)(b,n_1+n_2)}{(c_1,n_1)(c_2,n_2)(1,n_1)(1,n_2)}x_1^{n_1}x_2^{n_2},$$

where $c_1,c_2\notin -\mathbb{N}=\{0,-1,-2,\dots\}$ and $(c_1,n_1)=c_1(c_1+1)\cdots(c_1+n-1)=\Gamma(c_1+n_1)/\Gamma(c_1)$. This series converges in the set

$$\mathbb{D}=\{x\in\mathbb{C}^2 \mid \sqrt{|x_1|}+\sqrt{|x_2|}<1\},$$

satisfies

$$F_4(a,b,c;x)=F_4(b,a,c;x),$$

and admits the integral representations (2.3), (2.4), and (2.5) (see Section 2). The system $\mathcal{F}_4(a,b,c)$ of differential equations annihilating Appell's hypergeometric series $F_4(a,b,c;x)$ is a holonomic system of rank 4 with the singular locus S given in (2.1). A fundamental system of solutions of

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$\mathcal{F}_4(a,b,c)$ in a simply connected domain U in $\mathbb{D}-S$ is expressed in terms of Appell's hypergeometric series F_4 with different parameters (see (2.2) for their explicit forms).

In this paper, we find the twisted cycles associated with the integrand in (2.3) which correspond to the solutions (2.2). We evaluate the intersection numbers of several twisted cycles. By using the intersection numbers, as in [15] and [17], we provide the monodromy representation of $\mathcal{F}_4(a,b,c)$ (see Theorem 4.1). We provide a basis for the twisted cohomology group associated with the integrand in (2.3), and we evaluate the intersection matrix for this basis (see Theorem 5.1). By the compatibility of the pairings of twisted (co)homology groups, we have the identity (6.1) for the intersection matrices and the period matrices for our bases of twisted (co)homology groups (for details, refer to Theorem 6.1). This identity implies twisted period relations, which are quadratic relations between a fundamental system of solutions of $\mathcal{F}_4$ and those of $\mathcal{F}_4$ with different parameters. We present some examples in Corollary 6.1.

There have been several studies of monodromy representations of the system $\mathcal{F}_4(a,b,c)$ under the condition

$$c_1, c_2, a, a-c_1, a-c_2, a-c_1-c_2, b, b-c_1, b-c_2, b-c_1-c_2 \notin \mathbb{Z}$$

(see [9], [10], [19]). It is determined in [11] that representation matrices are valid even when c_1, c_2 are positive integers and that the system $\mathcal{F}_4(a,b,c)$ is irreducible if and only if $c_1, c_2 \notin \mathbb{Z}$ are removed from the above. Our expression of the monodromy representation is independent of the choice of fundamental systems of solutions of $\mathcal{F}_4(a,b,c)$, and it is valid even in the case $c_1, c_2 \in \mathbb{Z}$. We represent circuit transforms as matrices by assigning fundamental systems of solutions of $\mathcal{F}_4(a,b,c)$ (see Corollary 4.1 and Remark 4.4).

Twisted period relations for Lauricella's system $\mathcal{F}_D$ and Appell's system $\mathcal{F}_2$, $\mathcal{F}_3$ are studied in [5] and [14]. We can obtain an explicit form of that for $\mathcal{F}_4$ by evaluating the intersection matrix for the basis of the twisted cohomology group. We show that the intersection matrix H of twisted cycles corresponding to the fundamental system of solutions of $\mathcal{F}_4(a,b,c)$ in U is diagonal. This fact is key to obtaining several simple formulas for $F_4(a,b,c;x)$ that arise from the identity (6.1). There is another application of the intersection form of twisted cohomology groups. We have a Pfaffian system of $\mathcal{F}_4(a,b,c)$ using it as in [16]. For this, we refer the reader to the forthcoming paper [8].

Appell's system $\mathcal{F}_4(a,b,c)$ is generalized to Lauricella's system $\mathcal{F}_C(a,b,c)$ of rank 2^m with m-variables. A fundamental system of solutions of $\mathcal{F}_C(a,b,c)$ near the origin is expressed in terms of Lauricella's hypergeometric series $F_C(a,b,c;x)$. Their integral representations have been given in [6]; here, 2^m twisted cycles corresponding to them are constructed, and the intersection numbers of these twisted cycles are evaluated. These results together with some intersection numbers of twisted closed m-forms imply that there are twisted period relations for the fundamental systems of $\mathcal{F}_C$. Similar results for Lauricella's system $\mathcal{F}_A(a,b,c)$ have been obtained in [7].

§2. Appell's system $\mathcal{F}_4(a,b,c)$

In this section, we collect some facts about Appell's system $\mathcal{F}_4(a,b,c)$ of hypergeometric differential equations annihilating $F_4(a,b,c;x)$. (For more details, see [2].)

Let ∂_i $(i=1,2)$ be the partial differential operator with respect to x_i. The function $F_4(a,b,c;x)$ satisfies differential equations

$$\begin{aligned}&\big[x_1(1-x_1)\partial_1^2-x_2^2\partial_2^2-2x_1x_2\partial_1\partial_2\\&\quad+\big\{c_1-(a+b+1)x_1\big\}\partial_1-(a+b+1)x_2\partial_2-ab\big]f(x)=0,\\&\big[x_2(1-x_2)\partial_2^2-x_1^2\partial_1^2-2x_1x_2\partial_1\partial_2\\&\quad+\big\{c_2-(a+b+1)x_2\big\}\partial_2-(a+b+1)x_1\partial_1-ab\big]f(x)=0.\end{aligned}$$

The system generated by them is called *Appell's hypergeometric system $\mathcal{F}_4(a,b,c)$ of differential equations*. Though the function $F_4(a,b,c;x)$ is not defined for the case $c_1,c_2\in-\mathbb{N}$, the system $\mathcal{F}_4(a,b,c)$ is defined in this case, and it is a holonomic system of rank 4 with the singular locus

$$\begin{aligned}S&=\big\{(x_1,x_2)\in\mathbb{C}^2\;\big|\;x_1x_2R(x)=0\big\}\cup L_\infty,\\R(x)&=x_1^2+x_2^2-2x_1x_2-2x_1-2x_2+1,\end{aligned}\tag{2.1}$$

where L_∞ is the line at infinity in the projective plane $\mathbb{P}^2$. We set $X=\mathbb{P}^2-S$. We denote by $\mathcal{F}_4(a,b,c;U)$ the vector space of solutions of $\mathcal{F}_4(a,b,c)$ in a simply connected domain $U\subset X\cap\mathbb{D}$.

If $c_1,c_2\notin\mathbb{Z}$, then $\mathcal{F}_4(a,b,c;U)$ is spanned by

$$\begin{aligned}&F_4(a,b,c;x),\\&x_1^{1-c_1}F_4(a+1-c_1,b+1-c_1,2-c_1,c_2;x),\end{aligned}\tag{2.2}$$

$$x_2^{1-c_2}F_4(a+1-c_2,b+1-c_2,c_1,2-c_2;x),$$
$$x_1^{1-c_1}x_2^{1-c_2}F_4(a+2-c_1-c_2,b+2-c_1-c_2,2-c_1,2-c_2;x).$$

Note that $x_1^{1-c_1}$ and $x_2^{1-c_2}$ are single-valued holomorphic functions in U.

For sufficiently small positive real numbers x_1 and x_2, $F_4(a,b,c;x)$ admits the following integral representations:

$$\begin{aligned}&G_1\int_{\varDelta_1} t_1^{-c_1}t_2^{-c_2}(1-t_1-t_2)^{c_1+c_2-a-2}\left(1-\frac{x_1}{t_1}-\frac{x_2}{t_2}\right)^{-b}dt_1\wedge dt_2,\\ &c_1,c_2,a-c_1-c_2\notin\mathbb{Z},\end{aligned}\tag{2.3}$$

$$\begin{aligned}&G_2\int_{\sqrt{-1}\mathbb{R}_x^2} t_1^{-c_1}t_2^{-c_2}(1-t_1-t_2)^{c_1+c_2-a-2}\left(1-\frac{x_1}{t_1}-\frac{x_2}{t_2}\right)^{-b}dt_1\wedge dt_2,\\ &\mathrm{Re}(c_1-a)<1,\mathrm{Re}(c_2-a)<1,\end{aligned}\tag{2.4}$$

$$\begin{aligned}&G_3\int_{D} t_1^{a-1}t_2^{b-1}(1-t_1+t_1t_2x_2)^{c_1-a-1}(1-t_2+t_1t_2x_1)^{c_2-b-1}\,dt_1\wedge dt_2,\\ &\mathrm{Re}(c_1)>\mathrm{Re}(a)>0,\mathrm{Re}(c_2)>\mathrm{Re}(b)>0.\end{aligned}\tag{2.5}$$

Here

$$G_1=\frac{\varGamma(1-a)}{\varGamma(1-c_1)\varGamma(1-c_2)\varGamma(c_1+c_2-a-1)},$$
$$G_2=\frac{\varGamma(c_1)\varGamma(c_2)\varGamma(a-c_1-c_2+2)}{(2\pi\sqrt{-1})^2\varGamma(a)},$$
$$G_3=\frac{\varGamma(c_1)\varGamma(c_2)}{\varGamma(a)\varGamma(b)\varGamma(c_1-a)\varGamma(c_2-b)}.$$

In addition, $\varDelta_1$ is the formal sum

$$\begin{aligned}\varDelta_1=\triangle&+\frac{(\circlearrowleft_1\times I_1)}{1-\gamma_1^{-1}}+\frac{(\circlearrowleft_2\times I_2)}{1-\gamma_2^{-1}}+\frac{(\circlearrowleft_3\times I_3)}{1-\gamma_1\gamma_2\alpha^{-1}}\\ &+\frac{(\circlearrowleft_1\times\circlearrowleft_2)}{(1-\gamma_1^{-1})(1-\gamma_2^{-1})}+\frac{(\circlearrowleft_2\times\circlearrowleft_3)}{(1-\gamma_2^{-1})(1-\gamma_1\gamma_2\alpha^{-1})}\\ &+\frac{(\circlearrowleft_3\times\circlearrowleft_1)}{(1-\gamma_1\gamma_2\alpha^{-1})(1-\gamma_1^{-1})}\end{aligned}$$

of 2-dimensional real surfaces $\triangle$, with boundary components I_i $(i=1,2,3)$ given in Figure 1, where $\circlearrowleft_i$ $(i=1,2)$ is a positively oriented circle in the

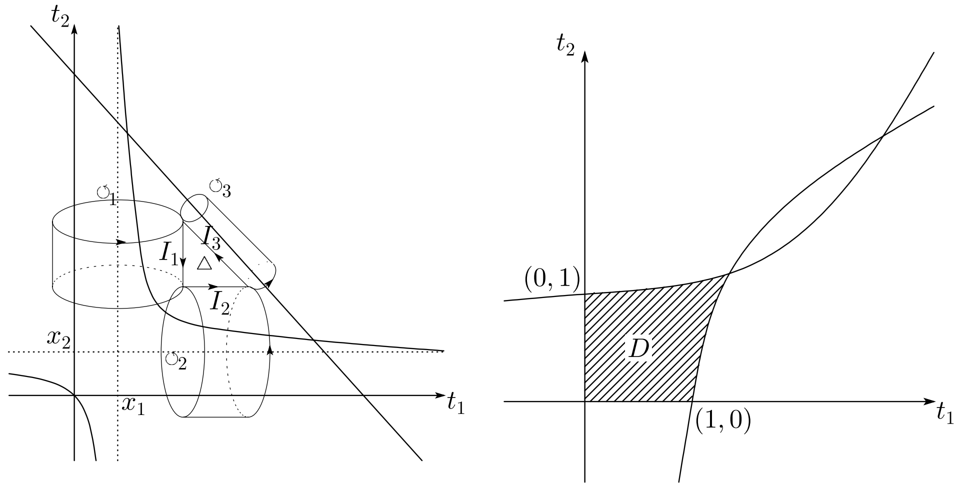


Figure 1: Domains of integrals.

t_i-space starting from the projection of I_i to this space and surrounding the divisors $t_i = 0$ and $Q(t,x) = t_1t_2 - t_1x_2 - t_2x_1 = 0$ for $t \in I_i$, $\circlearrowleft_3$ is a positively oriented circle with a small radius in the orthogonal complement of the divisor $L(t) = 1 - t_1 - t_2 = 0$ starting from the projection of I_3 to this space and surrounding the divisor, $\alpha = e^{2\pi\sqrt{-1}a}$, $\beta = e^{2\pi\sqrt{-1}b}$, $\gamma_i = e^{2\pi\sqrt{-1}c_i}$ $(i = 1, 2)$,

$$\sqrt{-1}\mathbb{R}^2_x = \left\{(\sqrt{x_1}, \sqrt{x_2}) + (s_1, s_2)\sqrt{-1} \;\middle|\; s_1, s_2 \in \mathbb{R}\right\} \subset \mathbb{C}^2, \quad (\sqrt{x_1}, \sqrt{x_2}) \in \triangle,$$

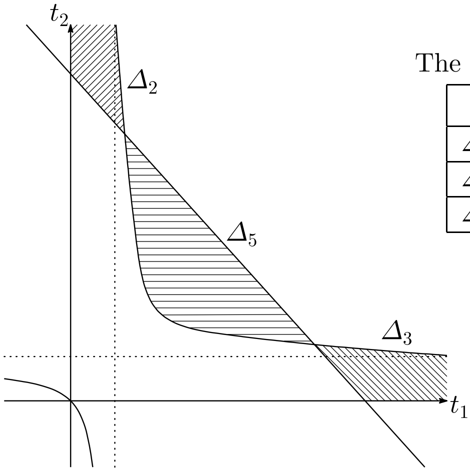
and D is the bounded connected component of

$$\left\{(t_1, t_2) \in \mathbb{R}^2 \;\middle|\; t_1, t_2, 1 - t_1 + t_1t_2x_2, 1 - t_2 + t_1t_2x_1 > 0\right\}$$

(see Figure 1). The argument of each factor of the integrand of (2.3) at any point $t = (t_1, t_2) \in \triangle$ is 0, that of (2.3) at the starting point of the circle $\circlearrowleft_i$ $(i = 1, 2, 3)$ is 0, that of (2.4) at $(t_1, t_2) = (\sqrt{x_1}, \sqrt{x_2})$ is 0, and that of (2.5) at any point $t = (t_1, t_2) \in D$ is 0. For these integral representations of $F_4(a, b, c; x)$, we refer the reader to [1], [18], and [3].

For $x \in U$, we set

$$f_i(x) = \int_{\Delta_i} t_1^{-c_1} t_2^{-c_2} (1 - t_1 - t_2)^{c_1 + c_2 - a - 2} \left(1 - \frac{x_1}{t_1} - \frac{x_2}{t_2}\right)^{-b} dt_1 \wedge dt_2, \tag{2.6}$$
$$(i = 1, \ldots, 5),$$



The arguments of the factors of the integrand

	t_1	t_2	$1-t_1-t_2$	$1-\frac{x_1}{t_1}-\frac{x_2}{t_2}$
Δ_2	0	0	$-\pi$	$-\pi$
Δ_3	0	0	$-\pi$	$-\pi$
Δ_5	0	0	0	0

Figure 2: Domains of integrals.

Table 1: Convergence conditions.

f_1	$c_1,\ c_2,\ a-c_1-c_2 \notin \mathbb{Z}$
f_2	$\mathrm{Re}(b-c_1+1),\ \mathrm{Re}(c_1+c_2-a-1),\ \mathrm{Re}(1-b),\ \mathrm{Re}(a-c_1+1)>0$
f_3	$\mathrm{Re}(b-c_2+1),\ \mathrm{Re}(c_1+c_2-a-1),\ \mathrm{Re}(1-b),\ \mathrm{Re}(a-c_2+1)>0$
f_4	$c_1,\ c_2,\ b-c_1-c_2 \notin \mathbb{Z}$
f_5	$\mathrm{Re}(c_1+c_2-a-1),\ \mathrm{Re}(1-b)>0$

where Δ_2, Δ_3, and Δ_5 are given in Figure 2, and Δ_4 is the image of Δ_1 under the involution

$$\imath : (t_1, t_2) \mapsto \left(\frac{x_1}{t_1}, \frac{x_2}{t_2}\right),$$

on

$$\mathbb{C}_x^2 = \left\{(t_1,t_2) \in \mathbb{C}^2 \;\middle|\; t_1 t_2 (1-t_1-t_2)(t_1 t_2 - t_1 x_2 - t_2 x_1) \neq 0\right\}.$$

The conditions for their convergence are as follows in Table 1.

LEMMA 2.1. *We have*

$$f_1(x) = \frac{\Gamma(1-c_1)\Gamma(1-c_2)\Gamma(c_1+c_2-a-1)}{\Gamma(1-a)} F_4(a,b,c_1,c_2;x),$$

$$
\begin{aligned}
f_2(x) &= \frac{\Gamma(a+1-c_1)\Gamma(b+1-c_1)\Gamma(1-b)\Gamma(c_1+c_2-a-1)}{\Gamma(2-c_1)\Gamma(c_2)} \\
&\quad \times e^{-\pi\sqrt{-1}(c_1+c_2-a-b)} x_1^{1-c_1} F_4(a+1-c_1, b+1-c_1, 2-c_1, c_2; x), \\
f_3(x) &= \frac{\Gamma(a+1-c_2)\Gamma(b+1-c_2)\Gamma(1-b)\Gamma(c_1+c_2-a-1)}{\Gamma(c_1)\Gamma(2-c_2)} \\
&\quad \times e^{-\pi\sqrt{-1}(c_1+c_2-a-b)} x_2^{1-c_2} F_4(a+1-c_2, b+1-c_2, c_1, 2-c_2; x), \\
f_4(x) &= \frac{\Gamma(c_1-1)\Gamma(c_2-1)\Gamma(1-b)}{\Gamma(c_1+c_2-b-1)} \\
&\quad \times x_1^{1-c_1} x_2^{1-c_2} F_4(a+2-c_1-c_2, b+2-c_1-c_2, 2-c_1, 2-c_2; x).
\end{aligned}
$$

Proof. Note that the first equality is nothing but the integral representation (2.3). We will show the last equality. The transformation $\imath$ satisfies $\imath = \imath^{-1}$, and it implies that

$$
\begin{aligned}
f_4 &= x_1^{1-c_1} x_2^{1-c_2} \\
&\quad \times \int_{\Delta_1} t_1^{c_1-2} t_2^{c_2-2} \Big(1 - \frac{x_1}{t_1} - \frac{x_2}{t_2}\Big)^{c_1+c_2-a-2} (1-t_1-t_2)^{-b}\, dt_1 \wedge dt_2 \\
&= x_1^{1-c_1} x_2^{1-c_2} \frac{\Gamma(c_1-1)\Gamma(c_2-1)\Gamma(1-b)}{\Gamma(c_1+c_2-b-1)} \\
&\quad \times F_4(b+2-c_1-c_2, a+2-c_1-c_2, 2-c_1, 2-c_2; x).
\end{aligned}
$$

To obtain the second equality, we use an orientation-reversing transformation

$$
(s_1, s_2) \mapsto (t_1, t_2) = \Big(x_1 s_1, \frac{1}{s_2}\Big),
$$

which sends the domain D to Δ_2. This transformation leads to

$$
\begin{aligned}
f_2 &= -x_1^{1-c_1} \int_{-D} s_1^{-c_1} s_2^{c_2-2} \Big(1 - x_1 s_1 - \frac{1}{s_2}\Big)^{c_1+c_2-a-2} \\
&\quad \times \Big(1 - \frac{1}{s_1} - s_2 x_2\Big)^{-b} ds_1 \wedge ds_2 \\
&= x_1^{1-c_1} \int_D s_1^{b-c_1} s_2^{a-c_1} (s_2 - x_1 s_1 s_2 - 1)^{c_1+c_2-a-2} \\
&\quad \times (s_1 - 1 - x_2 s_1 s_2)^{-b} ds_1 \wedge ds_2 \\
&= e^{-\pi\sqrt{-1}(c_1+c_2-a-b)} x_1^{1-c_1}
\end{aligned}
$$

$$\times \frac{\Gamma(b+1-c_1)\Gamma(a+1-c_1)\Gamma(1-b)\Gamma(c_1+c_2-a-1)}{\Gamma(2-c_1)\Gamma(c_2)}$$
$$\times F_4(b+1-c_1, a+1-c_1, 2-c_1, c_2; x)$$

by (2.5). We can obtain the third equality in a similar way. □

§3. Twisted homology group

Below, we will regard the parameters a, b, c_1, and c_2 as indeterminants, and we will assume that

$$(3.1) \qquad a, a-c_1, a-c_2, a-c_1-c_2, b, b-c_1, b-c_2, b-c_1-c_2, c_1, c_2 \notin \mathbb{Z}$$

when we assign them to complex numbers. Set

$$\lambda_1 = b-c_1+1, \qquad \lambda_2 = b-c_2+1, \qquad \lambda_3 = c_1+c_2-a-1, \qquad \lambda_4 = -b,$$

and let $\mathbb{C}(\mu)$ be the rational function field of $\mu_1 = e^{2\pi\sqrt{-1}\lambda_1}, \dots, \mu_4 = e^{2\pi\sqrt{-1}\lambda_4}$ over $\mathbb{C}$.

We define a subset $\mathfrak{X}$ in $(\mathbb{P}^1 \times \mathbb{P}^1) \times \mathbb{P}^2$ by

$$\mathfrak{X} = \{(t,x) \in \mathbb{C}^2 \times X \mid t_1 t_2 L(t) Q(t,x) \neq 0\},$$
$$L(t) = 1 - t_1 - t_2, \qquad Q(t,x) = t_1 t_2 - t_2 x_1 - t_1 x_2.$$

There is a natural projection

$$\mathrm{pr} : \mathfrak{X} \ni (t,x) \mapsto x \in X;$$

note that $\mathbb{C}^2_x = \mathrm{pr}^{-1}(x)$ for a fixed $x \in X$. Let

$$u = u(t,x) = t_1^{\lambda_1} t_2^{\lambda_2} L(t)^{\lambda_3} Q(t,x)^{\lambda_4} = t_1^{b+1-c_1} t_2^{b+1-c_2} L(t)^{c_1+c_2-a-1} Q(t,x)^{-b}$$

be a function of (t,x) in a simply connected neighborhood of $(\dot{t},\dot{x}) = (\sqrt{2}, \sqrt{2}, 1, 1)/8 \in \mathfrak{X}$. Along any path in $\mathfrak{X}$ starting with $(\dot{t},\dot{x})$, we can make the analytic continuation of u. Though this continuation depends on the path, it is single valued and holomorphic around the endpoint of the path.

Let σ be a k-chain in $\mathbb{C}^2_x$ for a fixed $x \in X$. We define a twisted k-chain σ^u by σ loading a branch of u on it. We denote the $\mathbb{C}(\mu)$-vector space of finite sums of twisted k-chains by $\mathcal{C}_k(\mathbb{C}^2_x, u)$. We define the boundary operator $\partial^u : \mathcal{C}_k(\mathbb{C}^2_x, u) \to \mathcal{C}_{k-1}(\mathbb{C}^2_x, u)$ by

$$\sigma^u \mapsto \partial(\sigma)^{u|_{\partial(\sigma)}},$$

where ∂ is the usual boundary operator and $u|_{\partial(\sigma)}$ is the restriction of u to $\partial(\sigma)$. We have a complex

$$\mathcal{C}_\bullet(\mathbb{C}_x^2, u) : \cdots \xrightarrow{\partial^u} \mathcal{C}_k(\mathbb{C}_x^2, u) \xrightarrow{\partial^u} \mathcal{C}_{k-1}(\mathbb{C}_x^2, u) \xrightarrow{\partial^u} \cdots$$

and its kth homology group $H_k(\mathcal{C}_\bullet(\mathbb{C}_x^2, u))$. Similarly we have a complex $\mathcal{C}_\bullet^{lf}(\mathbb{C}_x^2, u)$ of locally finite sums of twisted chains and its kth homology group $H_k(\mathcal{C}_\bullet^{lf}(\mathbb{C}_x^2, u))$. It is shown in [1] that

$$H_k\big(\mathcal{C}_\bullet(\mathbb{C}_x^2, u)\big) \simeq H_k\big(\mathcal{C}_\bullet^{lf}(\mathbb{C}_x^2, u)\big),$$

$$\dim_{\mathbb{C}(\mu)} H_k\big(\mathcal{C}_\bullet(\mathbb{C}_x^2, u)\big) = \begin{cases} 4 & \text{if } k = 2, \\ 0 & \text{otherwise}, \end{cases}$$

for any fixed $x \in X$. Thus, we have a map

$$\mathrm{reg} : H_2\big(\mathcal{C}_\bullet^{lf}(\mathbb{C}_x^2, u)\big) \to H_2\big(\mathcal{C}_\bullet(\mathbb{C}_x^2, u)\big),$$

which is the inverse of the natural map $H_2(\mathcal{C}_\bullet(\mathbb{C}_x^2, u)) \to H_2(\mathcal{C}_\bullet^{lf}(\mathbb{C}_x^2, u))$.

We regard the integral (2.6) as the pairing between the form

$$\varphi_1 = d\log\Big(\frac{t_1}{L(t)}\Big) \wedge d\log\Big(\frac{t_2}{L(t)}\Big) = \frac{dt_1 \wedge dt_2}{t_1 t_2 L(t)}$$

and $\varDelta_i$ loaded with a branch of u, which represents an element of $H_2(\mathcal{C}_\bullet^{lf}(\mathbb{C}_x^2, u))$ $(i = 1, \ldots, 5)$. The images of the element above under the map reg will be denoted by $\varDelta_i^u \in H_2(\mathcal{C}_\bullet(\mathbb{C}_x^2, u))$ for $i = 1, \ldots, 5$.

By considering $1/u$ instead of u, we have $H_2(\mathcal{C}_\bullet(\mathbb{C}_x^2, 1/u))$ and its elements $\varDelta_1^{1/u}, \ldots, \varDelta_5^{1/u}$. There is the intersection pairing $\mathcal{I}_h$ between $H_2(\mathcal{C}_\bullet(\mathbb{C}_x^2, u))$ and $H_2(\mathcal{C}_\bullet(\mathbb{C}_x^2, 1/u))$. It is defined as follows. Let $\varDelta^u$ and $\acute{\varDelta}^{1/u}$ be elements of $H_2(\mathcal{C}_\bullet(\mathbb{C}_x^2, u))$ and $H_2(\mathcal{C}_\bullet(\mathbb{C}_x^2, 1/u))$ given by

$$\varDelta^u = \sum_{i \in I} d_i D_i^{u_i}, \qquad \acute{\varDelta}^{1/u} = \sum_{j \in J} \acute{d}_j \acute{D}_j^{1/u_j}, \quad d_i, \acute{d}_j \in \mathbb{C}(\mu),$$

where $D_i^{u_i}$ denotes a singular 2-simplex D_i loaded with a branch u_i of u. Then their intersection number is

$$\mathcal{I}_h(\varDelta^u, \acute{\varDelta}^{1/u}) = \sum_{i \in I, j \in J} \sum_{p \in D_i \cap \acute{D}_j} d_i \acute{d}_j (D_i \cdot \acute{D}_j)_p \frac{u_i(p)}{u_j(p)},$$

where $(D_i \cdot \acute{D}_j)_p$ is the topological intersection number of 2-chains D_i and $\acute{D}_j$ at p. The intersection from $\mathcal{I}_h$ is bilinear. Since

$$\varDelta^{1/u} = \sum_{i \in I} d_i^\vee D_i^{1/u_i}, \qquad \acute{\varDelta}^u = \sum_{j \in J} \acute{d}_j^\vee \acute{D}_j^{u_j},$$

for the above $\varDelta^u$ and $\acute{\varDelta}^{1/u}$, we have

$$\mathcal{I}_h(\acute{\varDelta}^u, \varDelta^{1/u}) = \mathcal{I}_h(\varDelta^u, \acute{\varDelta}^{1/u})^\vee, \tag{3.2}$$

where $z(\mu_1, \dots, \mu_4)^\vee = z(1/\mu_1, \dots, 1/\mu_4)$ for $z(\mu_1, \dots, \mu_4) \in \mathbb{C}(\mu)$.

Lemma 3.1. *The intersection numbers $\mathcal{I}_h(\varDelta_i^u, \varDelta_i^{1/u})$ $(i = 1, \dots, 4)$ are*

$$\mathcal{I}_h(\varDelta_1^u, \varDelta_1^{1/u}) = \frac{1 - (\mu_1\mu_4)(\mu_2\mu_4)(\mu_3)}{(1-\mu_1\mu_4)(1-\mu_2\mu_4)(1-\mu_3)} = \frac{-(1-\alpha)\gamma_1\gamma_2}{(\alpha - \gamma_1\gamma_2)(1-\gamma_1)(1-\gamma_2)},$$

$$\begin{aligned}
\mathcal{I}_h(\varDelta_2^u, \varDelta_2^{1/u}) &= \frac{(1-\mu_1\mu_4)(1-\mu_3(\mu_2\mu_3\mu_4)^{-1})}{(1-\mu_1)(1-\mu_4)(1-\mu_3)(1-(\mu_2\mu_3\mu_4)^{-1})} \\
&= \frac{\alpha\beta\gamma_1(1-\gamma_1)(1-\gamma_2)}{(\alpha-\gamma_1)(\alpha-\gamma_1\gamma_2)(\beta-\gamma_1)(1-\beta)},
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_h(\varDelta_3^u, \varDelta_3^{1/u}) &= \frac{(1-\mu_2\mu_4)(1-\mu_3(\mu_1\mu_3\mu_4)^{-1})}{(1-\mu_2)(1-\mu_4)(1-\mu_3)(1-(\mu_1\mu_3\mu_4)^{-1})} \\
&= \frac{\alpha\beta\gamma_2(1-\gamma_1)(1-\gamma_2)}{(\alpha-\gamma_2)(\alpha-\gamma_1\gamma_2)(1-\beta)(\beta-\gamma_2)},
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_h(\varDelta_4^u, \varDelta_4^{1/u}) &= \frac{1-(\mu_1\mu_4)^{-1}(\mu_2\mu_4)^{-1}(\mu_4)}{(1-(\mu_1\mu_4)^{-1})(1-(\mu_2\mu_4)^{-1})(1-\mu_4)} \\
&= \frac{-(\beta-\gamma_1\gamma_2)}{(1-\beta)(1-\gamma_1)(1-\gamma_2)}.
\end{aligned}$$

Proof. To compute $\mathcal{I}_h(\varDelta_1^u, \varDelta_1^{1/u})$, we have only to follow [20, Chapter VIII, Section 3, Example 3.1], by considering the contribution of the divisor $Q(t,x) = 0$. By using the involution $\imath$, we can evaluate $\mathcal{I}_h(\varDelta_4^u, \varDelta_4^{1/u})$. For the rest, transform $\varDelta_i$ $(i = 2, 3)$ to the domain D in expression (2.5) as in the proof of Lemma 2.1; regard it as a quadrilateral, and apply [20, Chapter VIII, Section 3, Example 3.2]. □

For a small simply connected neighborhood U of $\dot{x}$, we have a family

$$\bigcup_{x \in U} H_2(\mathcal{C}_\bullet(\mathbb{C}_x^2, u)),$$

which can be naturally identified with $\mathcal{F}_4(a,b,c;U)$ by (2.6). Since a path ρ_x in X connecting $\dot{x}$ and x defines the isomorphism

$$(\rho_x)_* : H_2\big(\mathcal{C}_\bullet(\mathbb{C}^2_{\dot{x}}, u)\big) \to H_2\big(\mathcal{C}_\bullet(\mathbb{C}^2_x, u)\big),$$

we have a local system

$$\mathcal{H}_2(X) = \bigcup_{x \in X} H_2\big(\mathcal{C}_\bullet(\mathbb{C}^2_x, u)\big)$$

over X. Its stalk over x is denoted by $H_2(\mathcal{C}_\bullet(\mathbb{C}^2_x, u))$.

Similarly, we have a local system

$$\mathcal{H}_2^\vee(X) = \bigcup_{x \in X} H_2\big(\mathcal{C}_\bullet(\mathbb{C}^2_x, 1/u)\big)$$

over X with respect to $1/u$. The local triviality of these local systems $\mathcal{H}_2(X)$ and $\mathcal{H}_2^\vee(X)$ implies the following.

Proposition 3.1. *The intersection number is invariant under the deformation; that is,*

$$\mathcal{I}_h\big((\rho_x)_*(\Delta^u), (\rho_x)_*(\acute{\Delta}^{1/u})\big) = \mathcal{I}_h(\Delta^u, \acute{\Delta}^{1/u})$$

for any $\Delta^u \in H_2(\mathcal{C}_\bullet(\mathbb{C}^2_{\dot{x}}, u))$, $\acute{\Delta}^{1/u} \in H_2(\mathcal{C}_\bullet(\mathbb{C}^2_{\dot{x}}, 1/u))$, and any path ρ_x in X connecting $\dot{x}$ and x.

§4. Monodromy representation

A loop ρ in X with base point $\dot{x}$ induces a linear transformation of the stalk $H_2(\mathcal{C}_\bullet(\mathbb{C}^2_{\dot{x}}, u))$ of $\mathcal{H}_2(X)$ over $\dot{x}$. By this correspondence, we have a homomorphism

$$\mathcal{M} : \pi_1(X, \dot{x}) \to GL\big(H_2\big(\mathcal{C}_\bullet(\mathbb{C}^2_{\dot{x}}, u)\big)\big),$$

which is called the *monodromy representation* of the local system $\mathcal{H}_2(X)$. Note that we can regard it as the monodromy representation of the system $\mathcal{F}_4(a,b,c)$ by the identification of $\mathcal{F}_4(a,b,c;U)$ for a small neighborhood U of $\dot{x}$ with $\bigcup_{x \in U} H_2(\mathcal{C}_\bullet(\mathbb{C}^2_x, u))$. It is shown in [10, Appendix] that the fundamental group $\pi_1(X, \dot{x})$ is generated by three loops $\rho_i : [0,1] \to X$ ($i = 1,2,3$):

$$\rho_1 : \theta \mapsto \Big(\frac{\exp(2\pi\sqrt{-1}\theta)}{8}, \frac{1}{8}\Big),$$

$$\rho_2 : \theta \mapsto \Big(\frac{1}{8}, \frac{\exp(2\pi\sqrt{-1}\theta)}{8}\Big),$$
$$\rho_3 : \theta \mapsto \Big(\frac{2-\exp(2\pi\sqrt{-1}\theta)}{8}, \frac{2-\exp(2\pi\sqrt{-1}\theta)}{8}\Big).$$

Note that the loop ρ_i $(i=1,2)$ turns the divisor $x_i=0$ positively, and ρ_3 turns the divisor $R(x)=0$ positively. We put $\mathcal{M}_i = \mathcal{M}(\rho_i)$ $(i=1,2,3)$.

PROPOSITION 4.1. *The elements $\Delta_1^u, \dots, \Delta_4^u$ span $H_2(\mathcal{C}_\bullet(\mathbb{C}_{\dot{x}}^2, u))$. With respect to the basis ${}^t(\Delta_1^u, \dots, \Delta_4^u)$, $\mathcal{M}_1$ and $\mathcal{M}_2$ are represented by matrices*

$$\mathrm{diag}(1, \gamma_1^{-1}, 1, \gamma_1^{-1}) \quad \textit{and} \quad \mathrm{diag}(1, 1, \gamma_2^{-1}, \gamma_2^{-1}),$$

respectively, where $\mathrm{diag}(z_1, \dots, z_n)$ *denotes the diagonal matrix with diagonal entries $z_1, \dots, z_n$.*

Proof. Recall that the solutions f_i are defined by the integrals over Δ_i in (2.6) and that they admit local expressions as in Lemma 2.1. We have

$$ {}^t\big((\rho_1)_*(\Delta_1^u), \dots, (\rho_1)_*(\Delta_4^u)\big) = \mathrm{diag}(1, \gamma_1^{-1}, 1, \gamma_1^{-1})\, {}^t(\Delta_1^u, \dots, \Delta_4^u),$$
$$ {}^t\big((\rho_2)_*(\Delta_1^u), \dots, (\rho_2)_*(\Delta_4^u)\big) = \mathrm{diag}(1, 1, \gamma_2^{-1}, \gamma_2^{-1})\, {}^t(\Delta_1^u, \dots, \Delta_4^u),$$

since the local behavior of f_i is the same as that of Δ_i. ∎

LEMMA 4.1. *If $i \neq j$ $(1 \le i, j \le 4)$, then*

$$\mathcal{I}_h(\Delta_i^u, \Delta_j^{1/u}) = 0.$$

The intersection matrix $H = (\mathcal{I}_h(\Delta_i^u, \Delta_j^{1/u}))_{1 \le i,j \le 4}$ is a diagonal matrix with entries as given in Lemma 3.1.

Proof. By Propositions 3.1 and 4.1, we have

$$\mathcal{I}_h(\Delta_i^u, \Delta_j^{1/u}) = \mathcal{I}_h\big((\rho_1)_*(\Delta_i^u), (\rho_1)_*(\Delta_j^{1/u})\big) = \mathcal{I}_h(\gamma_1^{-1}\Delta_i^u, \Delta_j^{1/u})$$
$$= \gamma_1^{-1}\mathcal{I}_h(\Delta_i^u, \Delta_j^{1/u})$$

for $i = 2, 4$ and $j = 1, 3$. Since $\gamma_1 \neq 1$, $\mathcal{I}_h(\Delta_i^u, \Delta_j^{1/u}) = 0$ for $i = 2, 4$ and $j = 1, 3$. By (3.2), we have $\mathcal{I}_h(\Delta_i^u, \Delta_j^{1/u}) = 0$ for $i = 1, 3$ and $j = 2, 4$. To show that $\mathcal{I}_h(\Delta_i^u, \Delta_j^{1/u}) = 0$ for $(i, j) = (1, 3), (2, 4), (3, 1), (4, 2)$, use the map $(\rho_2)_*$. ∎

REMARK 4.1. The eigenspace V_1^u of $\mathcal{M}_1$ with eigenvalue 1 is spanned by $\varDelta_1^u$ and $\varDelta_3^u$. The eigenspace of $\mathcal{M}_1$ with eigenvalue $1/\gamma_1$ is characterized by

$$\big\{\varDelta^u \in H_2\big(\mathcal{C}_\bullet(\mathbb{C}^2_{\dot{x}},u)\big) \,\big|\, \mathcal{I}_h(\varDelta^u,\varDelta_1^{1/u}) = \mathcal{I}_h(\varDelta^u,\varDelta_3^{1/u}) = 0\big\}.$$

The eigenspace V_2^u of $\mathcal{M}_2$ with eigenvalue 1 is spanned by $\varDelta_1^u$ and $\varDelta_2^u$. The eigenspace of $\mathcal{M}_2$ with eigenvalue $1/\gamma_2$ is characterized by

$$\big\{\varDelta^u \in H_2\big(\mathcal{C}_\bullet(\mathbb{C}^2_{\dot{x}},u)\big) \,\big|\, \mathcal{I}_h(\varDelta^u,\varDelta_1^{1/u}) = \mathcal{I}_h(\varDelta^u,\varDelta_2^{1/u}) = 0\big\}.$$

Note that the linear transformation $\mathcal{M}_i$ $(i=1,2)$ is determined by the subspace V_i^u, the eigenvalue $1/\gamma_i$, and the intersection form $\mathcal{I}_h$, under the condition $c_i \notin \mathbb{Z}$ when we assign complex values to the parameters.

We characterize the linear transformation $\mathcal{M}_3$ by determining its eigenvalues and eigenspaces. The following is the key lemma of this section.

LEMMA 4.2. *We have*

$$\mathcal{M}_3(\varDelta_5^u) = -\mu_3\mu_4\varDelta_5^u = -\frac{\gamma_1\gamma_2}{\alpha\beta}\varDelta_5^u, \quad \mathcal{M}_3(\varDelta^u) = \varDelta^u$$

for any $\varDelta^u \in (\varDelta_5^{1/u})^\perp = \{\varDelta^u \in H_2(\mathcal{C}_\bullet(\mathbb{C}^2_{\dot{x}},u)) \,|\, \mathcal{I}_h(\varDelta^u,\varDelta_5^{1/u}) = 0\}$.

Proof. We express $\varDelta_5$ in terms of the coordinates $s=(s_1,s_2)=(t_1/x_1, t_2/x_2)$. Since $L(t)$ and $Q(t,x)$ are expressed as

$$1 - s_1x_1 - s_2x_2, \qquad x_1x_2(s_1s_2 - s_1 - s_2),$$

in terms of these coordinates, we set

$$L(s,x) = 1 - s_1x_1 - s_2x_2, \qquad Q(s) = s_1s_2 - s_1 - s_2.$$

The intersection points P_1 and P_2 of the curves defined by $L(s,x)=0$ and $Q(s)=0$ are

$$\Big(\frac{1+x_1-x_2+\sqrt{R(x)}}{2x_1}, \frac{1-x_1+x_2-\sqrt{R(x)}}{2x_2}\Big),$$
$$\Big(\frac{1+x_1-x_2-\sqrt{R(x)}}{2x_1}, \frac{1-x_1+x_2+\sqrt{R(x)}}{2x_2}\Big).$$

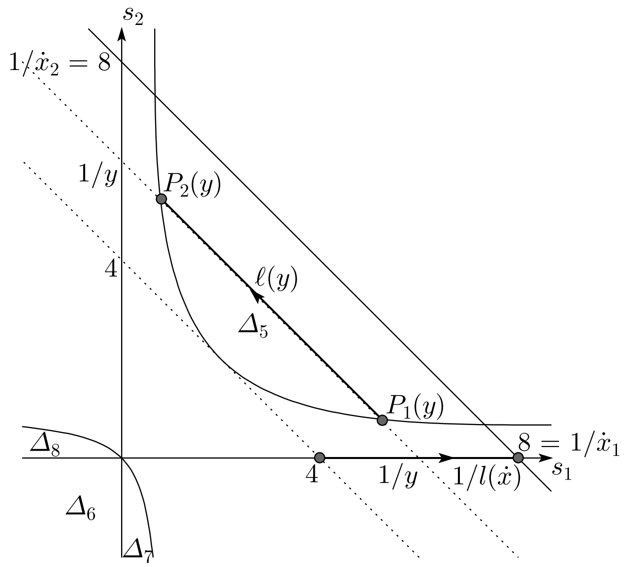


Figure 3: Cycles $\Delta_5, \ldots, \Delta_8$.

Note that $R(x) = 1 - 4x_1$ for $x = (x_1, x_1) \in \rho_3$. When $x_1 = x_2 = 1/4$, $R(x)$ vanishes and $Q(s) = 0$ is tangent to $L(s, x) = 0$. For $\dot{x} = (1/8, 1/8)$, we regard Δ_5 as

$$\bigcup_{y \in l(\dot{x}_1)} \ell(y),$$

where $l(\dot{x}_1)$ is the segment connecting $1/4$ and $\dot{x}_1 = 1/8$, and $\ell(y)$ is the segment connecting the intersection points of $L(s, x) = 0$ and $Q(s) = 0$ for $x = (y, y)$ with $y \in l(\dot{x}_1)$ (see Figure 3). For a fixed $x = (x_1, x_1)$ in the loop ρ_3, the segment $l(x_1)$ is expressed as

$$\frac{1}{4} + \left(x_1 - \frac{1}{4}\right) q_1$$

by a parameter $q_1 \in [0, 1]$. For an element $y = 1/4 + (x_1 - 1/4)q_1 \in l(x_1)$, the segment $\ell(y)$ is expressed as

$$P_1(y) + \big(P_2(y) - P_1(y)\big) q_2,$$

by a parameter $q_2 \in [0, 1]$, where $P_1(y)$ and $P_2(y)$ are the intersection points P_1 and P_2 for $x = (y, y)$. Hence, Δ_5 is expressed by $(q_1, q_2) \in [0, 1] \times [0, 1]$ as

$$(s_1,s_2)=\Big(\frac{2(1+(1-2q_2)\sqrt{(1-4x_1)q_1})}{1-(1-4x_1)q_1},\frac{2(1-(1-2q_2)\sqrt{(1-4x_1)q_1})}{1-(1-4x_1)q_1}\Big)$$
(4.1)

for a fixed $x=(x_1,x_1)$ in the loop ρ_3.

By the continuation of $\sqrt{1-4x_1}$ along the loop ρ_3, its sign changes. We regard this sign change in the deformation of Δ_5 along ρ_3 as a bijection of Δ_5 with the reversing orientation given by

$$r:[0,1]\times[0,1]\ni(q_1,q_2)\mapsto(q_1,1-q_2)\in[0,1]\times[0,1].$$

We deform the pullbacks of s_1, s_2, $L(s,x)$, and $Q(s)$ to $[0,1]\times[0,1]$ by (4) along ρ_3 and apply r to them. It is easy to see that those of s_1 and s_2 are invariant under the deformation and the action. Since those of $L(s,x)$ and $Q(s)$ are expressed as

$$\frac{(1-q_1)(1-4x_1)}{1-(1-4x_1)q_1},\qquad \frac{16q_1q_2(1-q_2)(1-4x_1)}{(1-q_1(1-4x_1))^2},$$

their arguments increase by 2π under the deformation, and they are invariant under r. Thus, the pullback of $s_1^{\lambda_1}s_2^{\lambda_2}L(s,x)^{\lambda_3}Q(s)^{\lambda_4}$ to $[0,1]\times[0,1]$ by (4) is multiplied by $\mu_3\mu_4$ under the deformation along ρ_3 and the action r. By considering the orientation of Δ_5, we have

$$\mathcal{M}_3(\Delta_5^u)=-\mu_3\mu_4\Delta_5^u.$$

It is easy to see by Figure 3 that three chambers

$$\begin{aligned}\Delta_6&=\{(s_1,s_2)\in\mathbb{R}^2\mid s_1,s_2<0\},\\ \Delta_7&=\{(s_1,s_2)\in\mathbb{R}^2\mid s_1,Q(s)>0,s_2<0\},\\ \Delta_8&=\{(s_1,s_2)\in\mathbb{R}^2\mid s_2,Q(s)>0,s_1<0\}\end{aligned}$$

are invariant under the deformation along ρ_3. Thus, the elements Δ_i^u ($i=6,7,8$) of $H_2(\mathcal{C}_\bullet(\mathbb{C}^2_{\dot x},u))$ corresponding to Δ_i are eigenvectors of $\mathcal{M}_3$ with eigenvalue 1. Since they do not intersect Δ_5 topologically, they belong to $(\Delta_5^{1/u})^\perp$. To show that they are linearly independent, we compute the intersection numbers $H_{ij}=\mathcal{I}_h(\Delta_i^u,\Delta_j^{1/u})$ ($6\le i,j\le 8$).

By using results of [13, Sections 3, 4], we have the following.

$$
\begin{aligned}
H_{66} &= 1 + \frac{1}{\mu_0 - 1} + \frac{1}{\mu_1 - 1} + \frac{1}{\mu_2 - 1} + \frac{\mu_{12} - 1}{(\mu_{124} - 1)(\mu_1 - 1)(\mu_2 - 1)} \\
&\qquad + \frac{\mu_{01} - 1}{(\mu_{014} - 1)(\mu_0 - 1)(\mu_1 - 1)} + \frac{\mu_{02} - 1}{(\mu_{024} - 1)(\mu_0 - 1)(\mu_2 - 1)}, \\
H_{67} &= -\frac{1}{\mu_1 - 1}\Big(1 + \frac{1}{\mu_{124} - 1} + \frac{1}{\mu_{014} - 1}\Big), \\
H_{68} &= -\frac{1}{\mu_2 - 1}\Big(1 + \frac{1}{\mu_{124} - 1} + \frac{1}{\mu_{024} - 1}\Big), \\
H_{77} &= 1 + \frac{1}{\mu_1 - 1} + \frac{1}{\mu_4 - 1} + \frac{\mu_{14} - 1}{(\mu_{124} - 1)(\mu_1 - 1)(\mu_4 - 1)} \\
&\qquad + \frac{\mu_{14} - 1}{(\mu_{014} - 1)(\mu_1 - 1)(\mu_4 - 1)}, \\
H_{78} &= -\frac{\mu_1 \mu_4}{(\mu_4 - 1)(\mu_{124} - 1)}, \\
H_{88} &= 1 + \frac{1}{\mu_2 - 1} + \frac{1}{\mu_4 - 1} + \frac{\mu_{24} - 1}{(\mu_{124} - 1)(\mu_2 - 1)(\mu_4 - 1)} \\
&\qquad + \frac{\mu_{24} - 1}{(\mu_{024} - 1)(\mu_2 - 1)(\mu_4 - 1)},
\end{aligned}
$$

and $H_{ji} = H_{ij}^{\vee}$ for $6 \le i < j \le 8$, where

$$
\mu_0 = \frac{1}{\mu_1 \mu_2 \mu_3 \mu_4^2} = \alpha, \qquad \mu_{ij} = \mu_i \mu_j, \qquad \mu_{ijk} = \mu_i \mu_j \mu_k.
$$

Since

$$
\begin{aligned}
&\det(H_{ij})_{6 \le i,j \le 8} \\
&\quad = \frac{\beta^2(\alpha - \gamma_1\gamma_2)^2(\alpha\beta + \gamma_1\gamma_2)}{(\alpha - 1)(\alpha - \gamma_1)(\alpha - \gamma_2)(\beta - 1)^2(\beta - \gamma_1)(\beta - \gamma_2)(\beta - \gamma_1\gamma_2)},
\end{aligned}
$$

if $\alpha\beta + \gamma_1\gamma_2 \neq 0$ when we assign complex values to the parameters, then they span the eigenspace of $\mathcal{M}_3$ with eigenvalue 1 and the space $(\varDelta_5^{1/u})^{\perp}$. $\square$

To represent $\mathcal{M}_3$ by a matrix, we express $\varDelta_5^u$ by a linear combination of $\varDelta_1^u, \dots, \varDelta_4^u$.

LEMMA 4.3. *We have*

$$\mathcal{I}_h(\Delta_5^u, \Delta_1^{1/u}) = \frac{1-(\mu_1\mu_4)(\mu_2\mu_4)(\mu_3)}{(1-\mu_1\mu_4)(1-\mu_2\mu_4)(1-\mu_3)} = \frac{-(1-\alpha)\gamma_1\gamma_2}{(\alpha-\gamma_1\gamma_2)(1-\gamma_1)(1-\gamma_2)},$$

$$\mathcal{I}_h(\Delta_5^u, \Delta_2^{1/u}) = \mathcal{I}_h(\Delta_5^u, \Delta_3^{1/u}) = \frac{\mu_3\mu_4}{(1-\mu_3)(1-\mu_4)} = \frac{-\gamma_1\gamma_2}{(\alpha-\gamma_1\gamma_2)(1-\beta)},$$

$$\begin{aligned}\mathcal{I}_h(\Delta_5^u, \Delta_4^{1/u}) &= \frac{1-(\mu_1\mu_4)^{-1}(\mu_2\mu_4)^{-1}(\mu_4)}{(1-(\mu_1\mu_4)^{-1})(1-(\mu_2\mu_4)^{-1})(1-\mu_4)} \\ &= \frac{-(\beta-\gamma_1\gamma_2)}{(1-\beta)(1-\gamma_1)(1-\gamma_2)}.\end{aligned}$$

The twisted cycle Δ_5^u is expressed as

$$\Delta_1^u - \frac{\gamma_2(\alpha-\gamma_1)(\beta-\gamma_1)}{\alpha\beta(1-\gamma_1)(1-\gamma_2)}\Delta_2^u - \frac{\gamma_1(\alpha-\gamma_2)(\beta-\gamma_2)}{\alpha\beta(1-\gamma_1)(1-\gamma_2)}\Delta_3^u + \Delta_4^u,$$

which leads to

$$\mathcal{I}_h(\Delta_5^u, \Delta_5^{1/u}) = \frac{1+\mu_3\mu_4}{(1-\mu_3)(1-\mu_4)} = \frac{-(\alpha\beta+\gamma_1\gamma_2)}{(\alpha-\gamma_1\gamma_2)(1-\beta)}.$$

Proof. By the results in [20, Chapter VIII, Section 3.4], we can compute the intersection numbers $\mathcal{I}_h(\Delta_5^u, \Delta_i^{1/u})$ for $i = 2, 3$. Among the components of Δ_1, only $\triangle$ intersects with $\sqrt{-1}\mathbb{R}_x^2$ at $(\sqrt{x_1}, \sqrt{x_2})$. Since their topological intersection number at this point is -1, we have

$$(\sqrt{-1}\mathbb{R}_x^2)^{1/u} = \frac{(1-\gamma_1)(1-\gamma_2)(\alpha-\gamma_1\gamma_2)}{(1-\alpha)\gamma_1\gamma_2}\Delta_1^{1/u}$$

by (2.4). This implies that

$$\mathcal{I}_h(\Delta_5^u, \Delta_1^{1/u}) = \frac{-(1-\alpha)\gamma_1\gamma_2}{(\alpha-\gamma_1\gamma_2)(1-\gamma_1)(1-\gamma_2)}.$$

We can evaluate the intersection number $\mathcal{I}_h(\Delta_5^u, \Delta_4^{1/u})$ in a similar way. Lemma 4.1 together with Lemma 3.1 implies the expression of Δ_5^u as a linear combination of Δ_i^u $(i = 1, \dots, 4)$. ∎

REMARK 4.2.

(i) The eigenspace of $\mathcal{M}_3$ with eigenvalue 1 is characterized by Δ_5^u and the intersection form $\mathcal{I}_h$.

(ii) If $\alpha\beta+\gamma_1\gamma_2=0$, then $\mathcal{I}_h(\Delta_5^u,\Delta_5^{1/u})=0$. In this case, the 3-dimensional space $(\Delta_5^{1/u})^{\perp}$ contains the cycle Δ_5^u and coincides with the eigenspace of $\mathcal{M}_3$ with eigenvalue 1. Since $H_2(\mathcal{C}_\bullet(\mathbb{C}^2_{\dot{x}},u))$ is not spanned by eigenvectors of $\mathcal{M}_3$, its representation is not diagonalizable.

PROPOSITION 4.2. *With respect to the basis* ${}^t(\Delta_1^u,\Delta_2^u,\Delta_3^u,\Delta_4^u)$, $\mathcal{M}_3$ *is represented by the matrix*

$$\mathrm{id}_4-(1+\gamma_1\gamma_2\alpha^{-1}\beta^{-1})\frac{H^te_5^\vee e_5}{e_5H^te_5^\vee}=\mathrm{id}_4-\frac{(\beta-1)(\alpha-\gamma_1\gamma_2)}{\alpha\beta}H^te_5^\vee e_5,$$

where id_4 *is the unit matrix of size* 4, *and*

$$e_5=\Big(1,-\frac{\gamma_2(\alpha-\gamma_1)(\beta-\gamma_1)}{\alpha\beta(\gamma_2-1)(\gamma_1-1)},-\frac{\gamma_1(\alpha-\gamma_2)(\beta-\gamma_2)}{\alpha\beta(\gamma_2-1)(\gamma_1-1)},1\Big),$$
$$e_5^\vee=\Big(1,-\frac{(\alpha-\gamma_1)(\beta-\gamma_1)}{\gamma_1(\gamma_1-1)(\gamma_2-1)},-\frac{(\alpha-\gamma_2)(\beta-\gamma_2)}{\gamma_2(\gamma_1-1)(\gamma_2-1)},1\Big),$$

corresponding to Δ_5^u *and* $\Delta_5^{1/u}$ *by the expression in Lemma 4.3.*

Proof. We set $M=\mathrm{id}_4-(1+\gamma_1\gamma_2\alpha^{-1}\beta^{-1})H^te_5^\vee(e_5H^te_5^\vee)^{-1}e_5$. Since

$$\mathcal{I}_h(\Delta^u,\Delta_5^{1/u})=(d_1,\dots,d_4)H^te_5^\vee,$$

for $\Delta^u=(d_1,\dots,d_4){}^t(\Delta_1^u,\Delta_2^u,\Delta_3^u,\Delta_4^u)$, we have

$$e_5M=e_5-(1+\gamma_1\gamma_2\alpha^{-1}\beta^{-1})e_5H^te_5^\vee(e_5H^te_5^\vee)^{-1}e_5=-\frac{\gamma_1\gamma_2}{\alpha\beta}e_5,$$
$$(d_1,\dots,d_4)M=(d_1,\dots,d_4),$$

for $(d_1,\dots,d_4)$ satisfying $(d_1,\dots,d_4)H^te_5^\vee=0$. Thus, the eigenvalues of M are $-\gamma_1\gamma_2/(\alpha\beta)$ and 1, e_5 is an eigenvector with eigenvalue $-\gamma_1\gamma_2/(\alpha\beta)$, and the eigenspace with eigenvalue 1 is characterized by the equality $(d_1,\dots,d_4)H^te_5^\vee=0$. Since e_5 corresponds to Δ_5 and $(d_1,\dots,d_4)H^te_5^\vee=\mathcal{I}_h(\Delta^u,\Delta_5^{1/u})$ for $\Delta^u=d_1\Delta_1^u+\cdots+d_4\Delta_4^u$, the linear transformation represented by M coincides with $\mathcal{M}_3$ by Lemma 4.2. Note that

$$\frac{1+\gamma_1\gamma_2\alpha^{-1}\beta^{-1}}{e_5H^te_5^\vee}=\frac{(\beta-1)(\alpha-\gamma_1\gamma_2)}{\alpha\beta}$$

by Lemma 4.3. The representation matrix of $\mathcal{M}_3$ on the right-hand side is valid even in the case $\alpha\beta+\gamma_1\gamma_2=0$. ∎

Note that $\mathcal{M}_1, \mathcal{M}_2$, and $\mathcal{M}_3$ are represented by the matrices in Propositions 4.1 and 4.2 with respect to the basis ${}^t(\Delta_1^u, \Delta_2^u, \Delta_3^u, \Delta_4^u)$. However, this basis degenerates when we assign an integer to c_i $(i=1,2)$. For example, if $c_1=1$, then $\gamma_1=1$ and $\mathcal{M}_1$ is represented by the unit matrix; we see that this expression is not valid in this case. Hence, we give expressions of $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ in terms of the intersection form $\mathcal{I}_h$, which are independent of the choice of a basis of $H_2(\mathcal{C}_\bullet(\mathbb{C}^2_{\dot{x}}, u))$ and are valid even for integer values of c_1, c_2. As we have mentioned in Remarks 4.1 and 4.2, $\mathcal{M}_i$ are determined by the eigenspaces V_1^u, V_2^u, the eigenvector Δ_5^u, and the intersection form $\mathcal{I}_h$. We take a basis of $H_2(\mathcal{C}_\bullet(\mathbb{C}^2_{\dot{x}}, u))$ consisting of bases of these subspaces. We set

$$\hat{\Delta}^u_{1234} = {}^t(\hat{\Delta}^u_1, \hat{\Delta}^u_2, \hat{\Delta}^u_3, \hat{\Delta}^u_4) = P\,{}^t(\Delta^u_1, \Delta^u_2, \Delta^u_3, \Delta^u_5),$$

where

$$P = \begin{pmatrix} \frac{\alpha\beta(1-\gamma_1)(1-\gamma_2)}{(1-\alpha)(1-\beta)\gamma_1\gamma_2} & 0 & 0 & 0 \\ \frac{-\alpha\beta(1-\gamma_2)}{(1-\alpha)(1-\beta)\gamma_2} & \frac{\gamma_1}{1-\gamma_1} & 0 & 0 \\ \frac{-\alpha\beta(1-\gamma_1)}{(1-\alpha)(1-\beta)\gamma_1} & 0 & \frac{\gamma_2}{1-\gamma_2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

LEMMA 4.4. *The integrals*

$$\hat{f}(x) = \int_{\hat{\Delta}_i} u(t,x)\varphi_1 \quad (i=1,2,3)$$

are well defined even in the case $c_1, c_2 \in \mathbb{Z}$ when we assign complex values to the parameters.

Proof. By Lemma 2.1, we have

$$\hat{f}_1(x) = G_4 \sum_{n\in\mathbb{N}^2} \frac{\Gamma(a+n_1+n_2)\Gamma(b+n_1+n_2)}{\Gamma(c_1+n_1)\Gamma(c_2+n_2)\Gamma(1+n_1)\Gamma(1+n_2)} x_1^{n_1} x_2^{n_2},$$

$$f_2(x) = G_4 \sum_{n\in\mathbb{N}^2} \frac{\Gamma(a+1-c_1+n_1+n_2)\Gamma(b+1-c_1+n_1+n_2)}{\Gamma(2-c_1+n_1)\Gamma(c_2+n_2)\Gamma(1+n_1)\Gamma(1+n_2)} x_1^{n_1+1-c_1} x_2^{n_2},$$

$$f_3(x) = G_4 \sum_{n\in\mathbb{N}^2} \frac{\Gamma(a+1-c_2+n_1+n_2)\Gamma(b+1-c_2+n_1+n_2)}{\Gamma(c_1+n_1)\Gamma(2-c_2+n_2)\Gamma(1+n_1)\Gamma(1+n_2)} x_1^{n_1} x_2^{n_2+1-c_2},$$

$$\hat{f}_2(x) = G_4 \frac{\gamma_1}{1-\gamma_1}\big(f_2(x) - \hat{f}_1(x)\big),$$

$$\hat{f}_3(x) = G_4 \frac{\gamma_2}{1-\gamma_2}\big(f_3(x) - \hat{f}_1(x)\big),$$

where $G_4 = \Gamma(1-b)\Gamma(c_1+c_2-a-1)e^{\pi\sqrt{-1}(a+b-c_1-c_2)}$. It is clear that $\hat{f}_1(x)$ is well defined for $c_1, c_2 \in \mathbb{Z}$. We claim that

$$\lim_{c_1 \to m} \frac{f_2(x) - \hat{f}_1(x)}{c_1 - m}$$

converges to a nonzero function for any $m \in \mathbb{Z}$. Let m be a fixed integer, and put $c_1 = m - \varepsilon$. Then $f_2(x)/G_4$ is

$$\sum_{\substack{n_1' \geq 1-m \\ n_2 \geq 0}} \frac{\Gamma(a+n_1'+n_2+\varepsilon)\Gamma(b+n_1'+n_2+\varepsilon)}{\Gamma(1+n_1'+\varepsilon)\Gamma(c_2+n_2)\Gamma(n_1'+m)\Gamma(1+n_2)} x_1^{n_1'+\varepsilon} x_2^{n_2},$$

where $n_1' = n_1 + 1 - m$. If $m \geq 2$, then we have

$$\lim_{\varepsilon \to 0} \frac{1}{\Gamma(1+n_1'+\varepsilon)} = 0$$

for $1-m \leq n_1' < 0$. If $m \leq 0$, then the terms $1/\Gamma(c_1+n_1)$ $(0 \leq n_1 \leq -m)$ in the series expressing $\hat{f}_1(x)$ converge to 0 as $c_1 \to m$. Thus, $f_2(x)$ converges to $\hat{f}_1(x)$ with $c_1 = m$ as $\varepsilon \to 0$. Since the poles of the Γ-function are simple, we have this claim. Similarly, we can show that $\hat{f}_3(x)$ is well defined for $c_1, c_2 \in \mathbb{Z}$. ∎

The intersection matrix $\hat{H} = (\mathcal{I}_h(\hat{\Delta}_i^u, \hat{\Delta}_j^{1/u}))_{1 \leq i,j \leq 4}$ is given by

$$\begin{pmatrix}
\frac{-\alpha\beta(1-\gamma_1)(1-\gamma_2)}{(1-\alpha)(\alpha-\gamma_1\gamma_2)(1-\beta)^2} & \frac{-\alpha\beta(1-\gamma_2)}{(1-\alpha)(\alpha-\gamma_1\gamma_2)(1-\beta)^2} & \frac{-\alpha\beta(1-\gamma_1)}{(1-\alpha)(\alpha-\gamma_1\gamma_2)(1-\beta)^2} & \frac{-\alpha\beta}{(\alpha-\gamma_1\gamma_2)(1-\beta)} \\
\frac{\alpha\beta\gamma_1(1-\gamma_2)}{(1-\alpha)(\alpha-\gamma_1\gamma_2)(1-\beta)^2} & \frac{\alpha\beta(\alpha\beta-\gamma_1)\gamma_1(1-\gamma_2)}{(1-\alpha)(\alpha-\gamma_1)(\alpha-\gamma_1\gamma_2)(1-\beta)^2(\beta-\gamma_1)} & \frac{\alpha\beta\gamma_1}{(1-\alpha)(\alpha-\gamma_1\gamma_2)(1-\beta)^2} & 0 \\
\frac{\alpha\beta(1-\gamma_1)\gamma_2}{(1-\alpha)(\alpha-\gamma_1\gamma_2)(1-\beta)^2} & \frac{\alpha\beta\gamma_2}{(1-\alpha)(\alpha-\gamma_1\gamma_2)(\beta-1)^2} & \frac{\alpha\beta(\alpha\beta-\gamma_2)(1-\gamma_1)\gamma_2}{(1-\alpha)(\alpha-\gamma_2)(\alpha-\gamma_1\gamma_2)(\beta-1)^2(\beta-\gamma_2)} & 0 \\
\frac{-\gamma_1\gamma_2}{(\alpha-\gamma_1\gamma_2)(1-\beta)} & 0 & 0 & \frac{-(\alpha\beta+\gamma_1\gamma_2)}{(\alpha-\gamma_1\gamma_2)(1-\beta)}
\end{pmatrix},$$

and its determinant is

$$\frac{\alpha^3\beta^3(\beta-\gamma_1\gamma_2)\gamma_1^2\gamma_2^2}{(1-\alpha)(\alpha-\gamma_1)(\alpha-\gamma_2)(\alpha-\gamma_1\gamma_2)^3(1-\beta)^5(\beta-\gamma_1)(\beta-\gamma_2)}.$$

Let $\hat{H}_{12}$ (resp., $\hat{H}_{13}$) be the submatrix of $\hat{H}$ made by entries $(1,1)$, $(1,2)$, $(2,1)$, and $(2,2)$ (resp., $(1,1)$, $(1,3)$, $(3,1)$, and $(3,3)$).

THEOREM 4.1. *The linear transformations $\mathcal{M}_i = \mathcal{M}(\rho_i)$ $(i=1,2,3)$ of $H_2(\mathcal{C}_\bullet(\mathbb{C}^2_x, u))$ are expressed as*

$$\mathcal{M}_1(\Delta^u) = \frac{1}{\gamma_1}\Delta^u + \Big(1-\frac{1}{\gamma_1}\Big)\big(\mathcal{I}_h(\Delta^u,\hat{\Delta}_1^{1/u}),\mathcal{I}_h(\Delta^u,\hat{\Delta}_3^{1/u})\big)(\hat{H}_{13})^{-1}\begin{pmatrix}\hat{\Delta}_1^u\\ \hat{\Delta}_3^u\end{pmatrix},$$

$$\mathcal{M}_2(\Delta^u) = \frac{1}{\gamma_2}\Delta^u + \Big(1-\frac{1}{\gamma_2}\Big)\big(\mathcal{I}_h(\Delta^u,\hat{\Delta}_1^{1/u}),\mathcal{I}_h(\Delta^u,\hat{\Delta}_2^{1/u})\big)(\hat{H}_{12})^{-1}\begin{pmatrix}\hat{\Delta}_1^u\\ \hat{\Delta}_2^u\end{pmatrix},$$

$$\begin{aligned}\mathcal{M}_3(\Delta^u) &= \Delta^u - \Big(1+\frac{\gamma_1\gamma_2}{\alpha\beta}\Big)\frac{\mathcal{I}_h(\Delta^u,\Delta_5^{1/u})}{\mathcal{I}_h(\Delta_5^u,\Delta_5^{1/u})}\Delta_5^u\\ &= \Delta^u - \frac{(\beta-1)(\alpha-\gamma_1\gamma_2)}{\alpha\beta}\mathcal{I}_h(\Delta^u,\Delta_5^{1/u})\Delta_5^u.\end{aligned}$$

Proof. By Proposition 4.1 and Lemma 4.1, the eigenspace of $\mathcal{M}_1$ with eigenvalue 1 is spanned by Δ_1^u and Δ_3^u, and that with eigenvalue γ_1^{-1} is its orthogonal complement

$$\big\{\Delta^u \in H_2\big(\mathcal{C}_\bullet(\mathbb{C}^2_x,u)\big) \,\big|\, \mathcal{I}_h(\Delta^u,\Delta_1^{1/u}) = \mathcal{I}_h(\Delta^u,\Delta_3^{1/u}) = 0\big\}.$$

The elements $\hat{\Delta}_1^u$ and $\hat{\Delta}_3^u$ belong to the eigenspace of $\mathcal{M}_1$ with eigenvalue 1, and they are linearly independent. Set

$$\mathcal{M}_1'(\Delta^u) = \frac{1}{\gamma_1}\Delta^u + \Big(1-\frac{1}{\gamma_1}\Big)\big(\mathcal{I}_h(\Delta^u,\hat{\Delta}_1^{1/u}),\mathcal{I}_h(\Delta^u,\hat{\Delta}_3^{1/u})\big)(\hat{H}_{13})^{-1}\begin{pmatrix}\hat{\Delta}_1^u\\ \hat{\Delta}_3^u\end{pmatrix}.$$

We can easily check that

$$\mathcal{M}_1'(\Delta^u) = \begin{cases}\hat{\Delta}_i^u & \text{if } \Delta^u = \hat{\Delta}_i^u \ (i=1,3),\\ \frac{1}{\gamma_1}\Delta^u & \text{if } \mathcal{I}_h(\Delta^u,\hat{\Delta}_1^{1/u}) = \mathcal{I}_h(\Delta^u,\hat{\Delta}_3^{1/u}) = 0,\end{cases}$$

by the property

$$\begin{aligned}&\big(\mathcal{I}_h(\Delta^u,\hat{\Delta}_1^{1/u}),\mathcal{I}_h(\Delta^u,\hat{\Delta}_3^{1/u})\big)(\hat{H}_{13})^{-1}\\&\quad=\begin{cases}(1,0) & \text{if } \Delta^u=\hat{\Delta}_1^u,\\(0,1) & \text{if } \Delta^u=\hat{\Delta}_3^u,\\(0,0) & \text{if } \mathcal{I}_h(\Delta^u,\hat{\Delta}_1^{1/u})=\mathcal{I}_h(\Delta^u,\hat{\Delta}_3^{1/u})=0.\end{cases}\end{aligned}$$

Since the eigenvalues and eigenspaces of $\mathcal{M}_1$ coincide with those of $\mathcal{M}_1'$, we have $\mathcal{M}_1=\mathcal{M}_1'$. We obtain the expression of $\mathcal{M}_2$ in a similar way. Set

$$\mathcal{M}_3'(\Delta^u)=\Delta^u-\Big(1+\frac{\gamma_1\gamma_2}{\alpha\beta}\Big)\frac{\mathcal{I}_h(\Delta^u,\Delta_5^{1/u})}{\mathcal{I}_h(\Delta_5^u,\Delta_5^{1/u})}\Delta_5^u.$$

By the property

$$\frac{\mathcal{I}_h(\Delta^u,\Delta_5^{1/u})}{\mathcal{I}_h(\Delta_5^u,\Delta_5^{1/u})}=\begin{cases}1 & \text{if } \Delta^u=\Delta_5^u,\\0 & \text{if } \Delta^u\in(\Delta_5^{1/u})^{\perp},\end{cases}$$

we see that

$$\mathcal{M}_3'(\Delta^u)=\begin{cases}-\frac{\gamma_1\gamma_2}{\alpha\beta}\Delta_5^u & \text{if } \Delta^u=\Delta_5^u,\\\Delta^u & \text{if } \Delta^u\in(\Delta_5^{1/u})^{\perp},\end{cases}$$

which shows that $\mathcal{M}_3=\mathcal{M}_3'$ by Lemma 4.2. The second expression of $\mathcal{M}_3$ is obtained by the equality

$$\mathcal{I}_h(\Delta_5^u,\Delta_5^{1/u})=\frac{-(\alpha\beta+\gamma_1\gamma_2)}{(\alpha-\gamma_1\gamma_2)(1-\beta)}$$

in Lemma 4.3. ∎

REMARK 4.3.

(i) We note that when we assign integers to c_1 and c_2, although Δ_1^u, Δ_2^u, and Δ_3^u are linearly dependent, $\hat{\Delta}_1^u$, $\hat{\Delta}_2^u$, and $\hat{\Delta}_3^u$ remain linearly independent.

(ii) Since we have

$$\begin{aligned}(\hat{H}_{12})^{-1}&=\frac{(\alpha-\gamma_1\gamma_2)(1-\beta)}{\alpha\beta\gamma_1^2(1-\gamma_2)}\\&\quad\times\begin{pmatrix}(\alpha\beta-\gamma_1)\gamma_1 & (\alpha-\gamma_1)(\beta-\gamma_1)\\-(\alpha-\gamma_1)(\beta-\gamma_1)\gamma_1 & -(\alpha-\gamma_1)(\beta-\gamma_1)(1-\gamma_1)\end{pmatrix},\end{aligned}$$

$$(\hat{H}_{13})^{-1} = \frac{(\alpha-\gamma_1\gamma_2)(1-\beta)}{\alpha\beta(1-\gamma_1)\gamma_2^2} \times \begin{pmatrix} (\alpha\beta-\gamma_2)\gamma_2 & (\alpha-\gamma_2)(\beta-\gamma_2) \\ -(\alpha-\gamma_2)(\beta-\gamma_2)\gamma_2 & -(\alpha-\gamma_2)(\beta-\gamma_2)(1-\gamma_2) \end{pmatrix},$$

the factors $1-\gamma_1$ and $1-\gamma_2$ are canceled in the expression of $\mathcal{M}_1$ and $\mathcal{M}_2$. Theorem 4.1 is valid even in the case $c_1, c_2, a+b-c_1-c_2-(1/2)\in\mathbb{Z}$ when we assign complex values to the parameters.

COROLLARY 4.1. *The linear transformations $\mathcal{M}_i$ $(i=1,2,3)$ are represented by matrices M_i with respect to the basis $\hat{\Delta}^u_{1234} = {}^t(\hat{\Delta}^u_1,\dots,\hat{\Delta}^u_4)$ as $\mathcal{M}_i(\hat{\Delta}^u_{1234}) = M_i\hat{\Delta}^u_{1234}$, where*

$$\begin{aligned} M_1 &= \frac{1}{\gamma_1}\mathrm{id}_4 + \Big(1-\frac{1}{\gamma_1}\Big)\hat{H}({}^te_1,{}^te_3)(\hat{H}_{13})^{-1}\begin{pmatrix} e_1 \\ e_3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{\gamma_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\alpha\beta-\gamma_2}{\alpha\beta} & 0 & \frac{(\alpha-\gamma_2)(\beta-\gamma_2)}{\alpha\beta\gamma_2} & \frac{1}{\gamma_1} \end{pmatrix}, \\ M_2 &= \frac{1}{\gamma_2}\mathrm{id}_4 + \Big(1-\frac{1}{\gamma_2}\Big)\hat{H}({}^te_1,{}^te_2)(\hat{H}_{12})^{-1}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & \frac{1}{\gamma_2} & 0 \\ \frac{\alpha\beta-\gamma_1}{\alpha\beta} & \frac{(\alpha-\gamma_1)(\beta-\gamma_1)}{\alpha\beta\gamma_1} & 0 & \frac{1}{\gamma_2} \end{pmatrix}, \\ M_3 &= \mathrm{id}_4 - \Big(1+\frac{\gamma_1\gamma_2}{\alpha\beta}\Big)\frac{\hat{H}{}^te_4e_4}{e_4\hat{H}{}^te_4} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{-\gamma_1\gamma_2}{\alpha\beta} \end{pmatrix}, \end{aligned}$$

and e_i is the ith unit row vector of $\mathbb{Z}^4$.

Proof. The matrix M_3 is obtained in the same way as in the proof of Proposition 4.2. By the expression of $\mathcal{M}_1$ in Theorem 4.1, we give its representation matrix with respect to the basis $\hat{\Delta}^u_{1234}$. Set $\Delta^u = (d_1,\dots,d_4)\hat{\Delta}^u_{1234}$. Since

$$\mathcal{I}_h(\Delta^u, \hat{\Delta}_i^{1/u}) = (d_1,\dots,d_4)\hat{H}{}^te_i \quad (i=1,\dots,4),$$

we have

$$\big(\mathcal{I}_h(\varDelta^u,\hat{\varDelta}_1^{1/u}),\mathcal{I}_h(\varDelta^u,\hat{\varDelta}_3^{1/u})\big)=(d_1,\dots,d_4)\hat{H}({}^te_1,{}^te_3).$$

Note that

$$\begin{pmatrix}\hat{\varDelta}_1^u\\ \hat{\varDelta}_3^u\end{pmatrix}=\begin{pmatrix}e_1\\ e_3\end{pmatrix}\hat{\varDelta}_{1234}^u.$$

Thus, we have

$$\begin{aligned}&\big(\mathcal{I}_h(\varDelta^u,\hat{\varDelta}_1^{1/u}),\mathcal{I}_h(\varDelta^u,\hat{\varDelta}_3^{1/u})\big)(\hat{H}_{13})^{-1}\begin{pmatrix}\hat{\varDelta}_1^u\\ \hat{\varDelta}_3^u\end{pmatrix}\\ &\quad=(d_1,\dots,d_4)\hat{H}({}^te_1,{}^te_3)(\hat{H}_{13})^{-1}\begin{pmatrix}e_1\\ e_3\end{pmatrix}\hat{\varDelta}_{1234}^u,\end{aligned}$$

which implies that M_1 is the representation matrix of $\mathcal{M}_1$. We obtain the matrix M_2 in a similar way. ∎

REMARK 4.4. With respect to the basis $P'^t(\varDelta_1^u,\varDelta_2^u,\varDelta_3^u,\varDelta_4^u)$ of $H_2(\mathcal{C}_\bullet(\mathbb{C}^2_{\dot{x}},u))$ for

$$P'=\begin{pmatrix}\frac{\alpha\beta(1-\gamma_1)(1-\gamma_2)}{(1-\alpha)(1-\beta)\gamma_1\gamma_2} & 0 & 0 & 0\\ \frac{-\alpha\beta(1-\gamma_2)}{(1-\alpha)(1-\beta)\gamma_2} & \frac{\gamma_1}{1-\gamma_1} & 0 & 0\\ \frac{-\alpha\beta(1-\gamma_1)}{(1-\alpha)(1-\beta)\gamma_1} & 0 & \frac{\gamma_2}{1-\gamma_2} & 0\\ \frac{\alpha\beta}{(1-\alpha)(1-\beta)} & \frac{-\gamma_1\gamma_2}{(1-\gamma_1)(1-\gamma_2)} & \frac{-\gamma_1\gamma_2}{(1-\gamma_1)(1-\gamma_2)} & \frac{\alpha\beta\gamma_1\gamma_2}{(\alpha-\gamma_1\gamma_2)(\beta-\gamma_1\gamma_2)}\end{pmatrix},$$

$\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ are represented by matrices

$$\begin{pmatrix}1&0&0&0\\ 1&\gamma_1^{-1}&0&0\\ 0&0&1&0\\ 0&0&1&\gamma_1^{-1}\end{pmatrix},\qquad \begin{pmatrix}1&0&0&0\\ 0&1&0&0\\ 1&0&\gamma_2^{-1}&0\\ 0&1&0&\gamma_2^{-1}\end{pmatrix},$$

$$\begin{pmatrix}-\frac{\gamma_1\gamma_2}{\alpha\beta} & \frac{\gamma_1\gamma_2}{\alpha\beta}-\frac{1}{\gamma_1} & \frac{\gamma_1\gamma_2}{\alpha\beta}-\frac{1}{\gamma_2} & -\frac{(\alpha-\gamma_1\gamma_2)(\beta-\gamma_1\gamma_2)}{\alpha\beta\gamma_1\gamma_2}\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\end{pmatrix},$$

respectively. These representations of $\mathcal{M}_i$ are also valid even in the case $c_1,c_2,a+b-c_1-c_2-(1/2)\in\mathbb{Z}$ when we assign complex values to the parameters.

§5. Twisted cohomology group

Recall that

$$\lambda_1 = b+1-c_1, \qquad \lambda_2 = b+1-c_2,$$
$$\lambda_3 = -a+c_1+c_2-1, \qquad \lambda_4 = -b,$$
$$\mathfrak{X} = \left\{(t,x) \in \mathbb{C}^2 \times X \;\middle|\; t_1 t_2 L(t) Q(t,x) \neq 0\right\} \subset (\mathbb{P}^1 \times \mathbb{P}^1) \times \mathbb{P}^2,$$
$$\mathbb{C}_x^2 = \mathrm{pr}^{-1}(x), \quad \mathrm{pr} : \mathfrak{X} \ni (t,x) \mapsto x \in X.$$

In this section, we regard vector spaces as defined over the rational function field $\mathbb{C}(\lambda) = \mathbb{C}(\lambda_1, \ldots, \lambda_4) = \mathbb{C}(a, b, c_1, c_2)$. We denote the vector space of rational functions on $\mathbb{P}^2$ with poles only along S by $\mathcal{O}_X(*S)$. Note that $\mathcal{O}_X(*S)$ admits the structure of an algebra over $\mathbb{C}(\lambda)$. We set

$$\mathfrak{S} = (\mathbb{P}^1 \times \mathbb{P}^1) \times \mathbb{P}^2 - \mathfrak{X}.$$

Let $\Omega_{\mathfrak{X}}^k(*\mathfrak{S})$ be the vector space of rational k-forms on $\mathfrak{X}$ with poles only along $\mathfrak{S}$, and let $\Omega_{\mathfrak{X}}^{p,q}(*\mathfrak{S})$ be the subspace of $\Omega_{\mathfrak{X}}^{p+q}(*\mathfrak{S})$ consisting of elements that are p-forms with respect to the variables t_1, t_2. We set

$$\omega = d_t \log\bigl(u(t,x)\bigr) = \lambda_1 \frac{dt_1}{t_1} + \lambda_2 \frac{dt_2}{t_2} + \lambda_3 \frac{d_t L(t)}{L(t)} + \lambda_4 \frac{d_t Q(t,x)}{Q(t,x)} \in \Omega_{\mathfrak{X}}^{1,0}(*\mathfrak{S}),$$

where d_t is the exterior derivative with respect to the variables t_1, t_2. Note that

$$d_t L(t) = -dt_1 - dt_2, \qquad d_t Q(t,x) = (t_2 - x_2)\, dt_1 + (t_1 - x_1)\, dt_2.$$

By a twisted exterior derivative $\nabla = d_t + \omega\wedge$ on $\mathfrak{X}$, we define quotient spaces

$$\mathcal{H}^k(\nabla) = \ker\bigl(\nabla : \Omega_{\mathfrak{X}}^{k,0}(*\mathfrak{S}) \to \Omega_{\mathfrak{X}}^{k+1,0}(*\mathfrak{S})\bigr) / \nabla\bigl(\Omega_{\mathfrak{X}}^{k-1,0}(*\mathfrak{S})\bigr) \quad (k = 0, 1, 2),$$

where we regard $\Omega_{\mathfrak{X}}^{-1,0}(*\mathfrak{S})$ as the zero vector space. Each of them admits the structure of a vector bundle over X.

We consider the structure of the fiber of $\mathcal{H}^k(\nabla)$ at x. Let $\Omega_{\mathbb{C}_x^2}^p(*x)$ be the vector space of rational p-forms on $\mathbb{C}_x^2$ with poles only along the pole divisor of the pullback $\omega_x = \imath_x^*(\omega)$ of ω by the map $\imath_x : \mathbb{C}_x^2 \to \mathfrak{X}$. There is a natural map from each fiber of $\mathcal{H}^k(\nabla)$ at x to the rational twisted cohomology group

$$H^k\bigl(\Omega_{\mathbb{C}_x^2}^\bullet(*x), \nabla_x\bigr) = \ker\bigl(\nabla_x : \Omega_{\mathbb{C}_x^2}^k(*x) \to \Omega_{\mathbb{C}_x^2}^{k+1}(*x)\bigr) / \nabla_x\bigl(\Omega_{\mathbb{C}_x^2}^{k-1}(*x)\bigr)$$

on $\mathbb{C}_x^2$ with respect to the twisted exterior derivative $\nabla_x = d_t + \omega_x\wedge$.

FACTS 5.1 ([1], [4]).

(i) We have
$$\dim H^k\big(\Omega^\bullet_{\mathbb{C}^2_x}(*x),\nabla_x\big)=\begin{cases}4 & \text{if } k=2,\\ 0 & \text{if } k=0,1.\end{cases}$$

(ii) There is a canonical isomorphism
$$\begin{aligned}\jmath_x : H^2\big(\Omega^\bullet_{\mathbb{C}^2_x}(*x),\nabla_x\big)&\to H^2\big(\mathcal{E}^\bullet_c(x),\nabla_x\big)\\ &=\ker\big(\nabla_x:\mathcal{E}^2_c(x)\to\mathcal{E}^3_c(x)\big)/\nabla_x\big(\mathcal{E}^1_c(x)\big),\end{aligned}$$
where $\mathcal{E}^k_c(x)$ is the vector space of smooth k-forms with compact support in $\mathbb{C}^2_x$.

We have a twisted exterior derivative $\nabla^\vee = d_t-\omega\wedge$ for $-\omega$ and
$$\mathcal{H}^2(\nabla^\vee)=\Omega^{2,0}_{\mathfrak{X}}(*\mathfrak{S})/\nabla^\vee\big(\Omega^{1,0}_{\mathfrak{X}}(*\mathfrak{S})\big),$$
$$H^2\big(\Omega^\bullet_{\mathbb{C}^2_x}(*x),\nabla^\vee_x\big)=\Omega^2_{\mathbb{C}^2_x}(*x)/\nabla^\vee_x\big(\Omega^1_{\mathbb{C}^2_x}(*x)\big).$$
The $\mathcal{O}_X(*S)$-module $\mathcal{H}^2(\nabla^\vee)$ can be regarded as a vector bundle over X.

For any fixed $x\in X$, we define the intersection form between $H^2(\Omega^\bullet_{\mathbb{C}^2_x}(*x),\nabla)$ and $H^2(\Omega^\bullet_{\mathbb{C}^2_x}(*x),\nabla^\vee)$ by
$$\mathcal{I}_c(\varphi_x,\varphi'_x)=\int_{\mathbb{C}^2_x}\jmath_x(\varphi_x)\wedge\varphi'_x\in\mathbb{C}(\alpha),$$
where $\varphi_x,\varphi'_x\in\Omega^2_{\mathbb{C}^2_x}(*x)$, $\jmath_x$ is given in Fact 5.1. This integral converges since $\jmath_x(\varphi_x)$ is a smooth 2-form on $\mathbb{C}^2_x$ with compact support. It is bilinear over $\mathbb{C}(\alpha)$.

We take four elements
$$\varphi_1=d_t\log\left(\frac{t_1}{L(t)}\right)\wedge d_t\log\left(\frac{t_2}{L(t)}\right)=\frac{dt_1\wedge dt_2}{t_1t_2L(t)},$$
$$\varphi_2=d_t\log(t_2)\wedge d_t\log\big(L(t)\big)=\frac{dt_1\wedge dt_2}{t_2L(t)},$$
$$\varphi_3=-d_t\log(t_1)\wedge d_t\log\big(L(t)\big)=\frac{dt_1\wedge dt_2}{t_1L(t)},$$
$$\varphi_4=\frac{t_1\wedge t_2}{L(t)Q(t,x)}$$
of $\mathcal{H}^2(\nabla)$, and we denote $\imath_x^*(\varphi_i)\in H^2(\Omega^\bullet_{\mathbb{C}^2_x}(*x),\nabla_x)$ by $\varphi_{x,i}$. Since $\nabla^\vee(\varphi_i)=0$, $\nabla^\vee_x(\varphi_{x,i})=0$, we can regard φ_i and $\varphi_{x,i}$ as elements of $\mathcal{H}^2(\nabla^\vee)$ and $H^2(\Omega^\bullet_{\mathbb{C}^2_x}(*x),\nabla^\vee_x)$, respectively. The intersection numbers $\mathcal{I}_c(\varphi_{x,i},\varphi_{x,j})$

$(\varphi_{x,i} \in H^2(\Omega^{\bullet}_{\mathbb{C}^2_x}(*x), \nabla_x),\ \varphi_{x,j} \in H^2(\Omega^{\bullet}_{\mathbb{C}^2_x}(*x), \nabla^{\vee}_x)\ 1 \le i,j \le 4)$ are evaluated as follows.

THEOREM 5.1. *The intersection matrix* $(\mathcal{I}_c(\varphi_{x,i}, \varphi_{x,j}))_{1\le i,j\le 4}$ *is* $(2\pi\sqrt{-1})^2 C$, *where* C *is a symmetric matrix with entries*

$$\begin{aligned}
C_{11} &= \Big(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\Big)\Big(\frac{1}{\lambda_3}+\frac{1}{\lambda_{124}}\Big)\\
&= \frac{(-a+1+b)(2b-c_1-c_2+2)}{(-a+c_1+c_2-1)(b-c_1+1)(b-c_2+1)(b-c_1-c_2+2)},\\
C_{12} &= \frac{1}{\lambda_2\lambda_3} = \frac{1}{(b-c_2+1)(-a+c_1+c_2-1)},\\
C_{13} &= \frac{1}{\lambda_1\lambda_3} = \frac{1}{(b-c_1+1)(-a+c_1+c_2-1)},\\
C_{14} &= 0,\\
C_{22} &= \Big(\frac{1}{\lambda_0}+\frac{1}{\lambda_2}\Big)\Big(\frac{1}{\lambda_3}+\frac{1}{\lambda_{\bar{1}34}}\Big)\\
&= \frac{(c_1-1)(a+b-c_2)}{(a-1)(a-c_2)(b-c_2+1)(-a+c_1+c_2-1)},\\
C_{23} &= \frac{-1}{\lambda_0\lambda_3} = \frac{-1}{(a-1)(-a+c_1+c_2-1)},\\
C_{24} &= 0,\\
C_{33} &= \Big(\frac{1}{\lambda_0}+\frac{1}{\lambda_1}\Big)\Big(\frac{1}{\lambda_3}+\frac{1}{\lambda_{\bar{2}34}}\Big)\\
&= \frac{(c_2-1)(a+b-c_1)}{(a-1)(a-c_1)(b-c_1+1)(-a+c_1+c_2-1)},\\
C_{34} &= 0,\\
C_{44} &= \frac{2}{\lambda_3\lambda_4 R(x)} = \frac{2}{(-a+c_1+c_2-1)(-b)R(x)},
\end{aligned}$$

where

$$\begin{aligned}
\lambda_0 &= -\lambda_1-\lambda_2-\lambda_3-2\lambda_4 = a-1,\\
\lambda_{124} &= \lambda_1+\lambda_2+\lambda_4 = b-c_1-c_2+2,\\
\lambda_{\bar{1}34} &= -\lambda_1-\lambda_3-\lambda_4 = a-c_2,\\
\lambda_{\bar{2}34} &= -\lambda_2-\lambda_3-\lambda_4 = a-c_1.
\end{aligned}$$

Table 2: Residues of ω_x.

Component	Residue
E_∞	$\lambda_0 = a-1$
$t_1 = 0$	$\lambda_1 = b - c_1 + 1$
$t_2 = 0$	$\lambda_2 = b - c_2 + 1$
$L(t) = 0$	$\lambda_3 = -a + c_1 + c_2 - 1$
$Q(t,x) = 0$	$\lambda_4 = -b$
E_0	$\lambda_{124} = b - c_1 - c_2 + 2$
$t_1 = \infty$	$\lambda_{\overline{1}34} = a - c_2$
$t_2 = \infty$	$\lambda_{\overline{2}34} = a - c_1$

The determinant of C is

$$-4b/\big((a-1)(a-c_1)(a-c_2)(-a+c_1+c_2-1)^3 \\ \times (b-c_1+1)(b-c_2+1)(b-c_1-c_2+2)R(x)\big).$$

Proof. We blow up $\mathbb{P}^1 \times \mathbb{P}^1 (\supset \mathbb{C}_x^2)$ at the two points $(0,0)$ and (∞,∞) so that the pole divisor of ω_x is normally crossing. We tabulate the residue of ω_x at each component of the pole divisor in Table 2, where E_0 and E_∞ are the exceptional divisors corresponding to the points $(0,0)$ and (∞,∞), respectively. To evaluate C_{11}, we find the intersection points of components of the pole divisor of $\varphi_{x,1}$. There are four points

$$\{t_1 = 0\} \cap E_0, \qquad \{t_2 = 0\} \cap E_0,$$
$$\{t_1 = 0\} \cap \big\{L(t) = 0\big\}, \qquad \{t_2 = 0\} \cap \big\{L(t) = 0\big\}$$

(see Figure 4). For every intersection point, we compute the reciprocal of the product of the residues of ω_x along the components passing it. The results in [14, Section 5] imply that C_{11} is given by their sum:

$$\frac{1}{\lambda_1\lambda_{124}} + \frac{1}{\lambda_2\lambda_{124}} + \frac{1}{\lambda_1\lambda_3} + \frac{1}{\lambda_2\lambda_3}.$$

Similarly, we can evaluate C_{22} and C_{33}.

Let us evaluate C_{12}. The intersection points of the components of the pole divisor of $\varphi_{x,2}$ are

$$\{t_2 = 0\} \cap \big\{L(t) = 0\big\}, \qquad \{t_2 = 0\} \cap \{t_1 = \infty\},$$

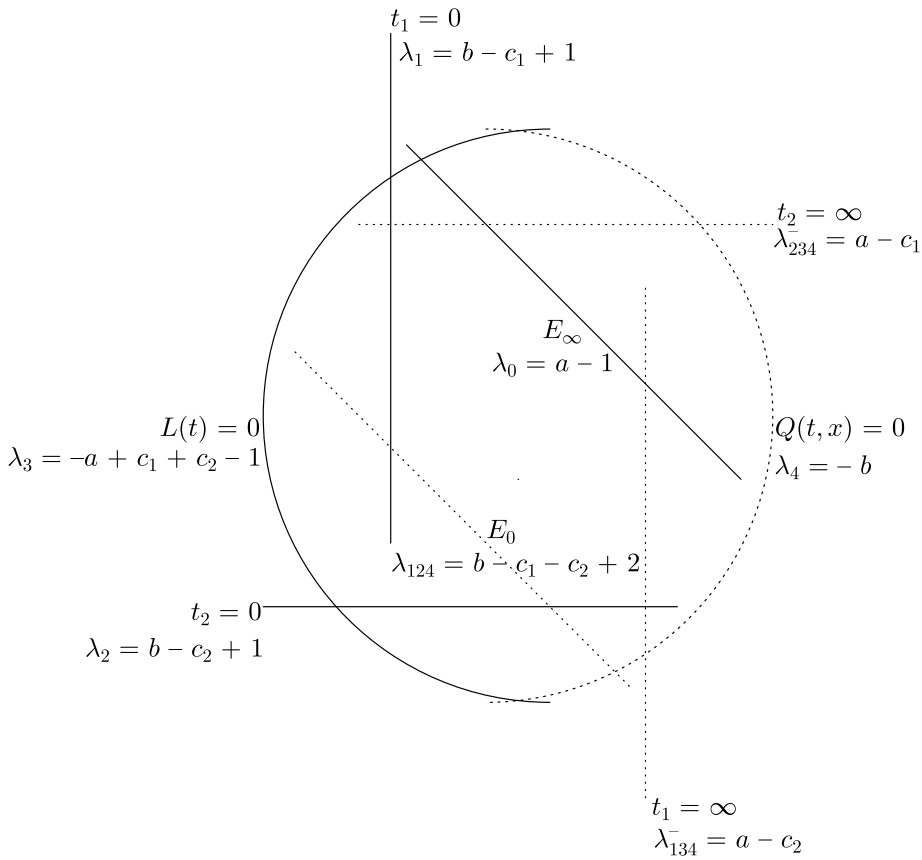


Figure 4: Pole divisor of ω_x.

$$\{L(t) = 0\} \cap E_\infty, \qquad \{t_1 = \infty\} \cap E_\infty;$$

$\{t_2 = 0\} \cap \{L(t) = 0\}$ is the common intersection point of the pole divisors of $\varphi_{x,1}$ and $\varphi_{x,2}$. By regarding $L(t)$ and t_2 as local coordinates around this point, we express $\varphi_{x,1}$ and $\varphi_{x,2}$ in terms of them:

$$\varphi_{x,1} = -\frac{dL(t) \wedge dt_2}{(1 - L(t) - t_2)t_2 L(t)}, \qquad \varphi_{x,2} = -\frac{dL(t) \wedge dt_2}{t_2 L(t)}.$$

Since $1/(1 - L(t) - t_2) = 1$ for $(L(t), t_2) = (0, 0)$, the intersection number C_{12} is given by the reciprocal of the product of the residues of ω_x along the components passing the point $(L(t), t_2) = (0, 0)$, that, is $1/(\lambda_2\lambda_3)$. Similarly, we can evaluate C_{13}. To evaluate C_{23}, we express $\varphi_{x,2}$ and $\varphi_{x,3}$ in terms of coordinates $s_1 = 1/t_1$, $s_2 = t_2/t_1$ around $\{L(t) = 0\} \cap E_\infty$ represented by $(s_1, s_2) = (0, -1)$. Since

$$\varphi_{x,2} = \frac{-ds_1 \wedge ds_2}{s_1(s_1 - 1 - s_2)}, \qquad \varphi_{x,3} = \frac{-ds_1 \wedge ds_2}{s_1 s_2(s_1 - 1 - s_2)},$$
$$[s_2]_{(s_1,s_2)=(0,-1)} = -1,$$

and the residue of ω_x along $\{L(t) = 0\}$ and that along E_∞ are λ_3 and λ_0, respectively, we have $C_{23} = -1/(\lambda_0\lambda_3)$.

The pole divisor of φ_4 consists of $L(t) = 0$ and $Q(t,x) = 0$. They intersect at the two points P_1 and P_2. Since the pole divisor of $\varphi_{x,i}$ $(i = 1,2,3)$ does not contain $Q(t,x) = 0$, we have $C_{i4} = 0$ for $i = 1,2,3$. To compute C_{44}, we express φ_4 around the intersection points P_1 and P_2 in terms of the local coordinates $L(t)$ and $Q(t,x)$. A straightforward calculation implies that

$$\varphi_4 = \frac{(-1)^i \, dL(t) \wedge dQ(t,x)}{L(t)Q(t,x)\sqrt{R(x) + L(t)^2 - 2(1 - x_1 - x_2)L(t) - 4Q(t,x)}}$$

around P_i $(i = 1,2)$, where the function $(-1)^i/(R(x) + L(t)^2 - 2(1 - x_1 - x_2)L(t) - 4Q(t,x))^{1/2}$ is a single-valued holomorphic function around P_i with value $(-1)^i/\sqrt{R(x)}$ at this point. We have

$$C_{44} = \frac{1}{\lambda_3\lambda_4}\frac{-1}{\sqrt{R(x)}}\frac{-1}{\sqrt{R(x)}} + \frac{1}{\lambda_3\lambda_4}\frac{1}{\sqrt{R(x)}}\frac{1}{\sqrt{R(x)}} = \frac{2}{\lambda_3\lambda_4 R(x)}.$$

The determinant of C is obtained by a straightforward calculation. $\square$

Note that the matrix C is well defined and that $\det(C) \neq 0$ for any $x \in X$ under our assumption. The natural map from each fiber of $\mathcal{H}^2(\nabla)$ at x to $H^2(\Omega^\bullet_{\mathbb{C}^2_x}(*x), \nabla_x)$ is surjective. The $\mathbb{C}(\lambda)$-span of the classes of $\varphi_1, \dots, \varphi_4 \in \mathcal{H}^2(\nabla)$ (resp., $\in \mathcal{H}^2(\nabla^\vee)$) is denoted by $\mathcal{H}^2_{\mathbb{C}(\lambda)}(\nabla)$ (resp., $\mathcal{H}^2_{\mathbb{C}(\lambda)}(\nabla^\vee)$). The intersection form $\mathcal{I}_c$ is regarded as a map from $\mathcal{H}^2_{\mathbb{C}(\lambda)}(\nabla) \times \mathcal{H}^2_{\mathbb{C}(\lambda)}(\nabla^\vee)$ to $\mathcal{O}(*S)$.

§6. Twisted period relations

Note that in this case, among $H_2(\mathcal{C}_\bullet(\mathbb{C}^2_x, u))$, $H_2(\mathcal{C}_\bullet(\mathbb{C}^2_x, 1/u))$, $H^2(\Omega^\bullet_{\mathbb{C}^2_x}(*x), \nabla_x)$, and $H^2(\Omega^\bullet_{\mathbb{C}^2_x}(*x), \nabla^\vee_x)$, there are the intersection pairings $\mathcal{I}_h$ and $\mathcal{I}_c$, and the pairings which yield solutions of $\mathcal{F}_4$ with various parameters. We have two isomorphisms from $H_2(\mathcal{C}_\bullet(\mathbb{C}^2_x, u))$ to $H^2(\Omega^\bullet_{\mathbb{C}^2_x}(*x), \nabla^\vee_x)$ by regarding them as the dual spaces of $H_2(\mathcal{C}_\bullet(\mathbb{C}^2_x, 1/u))$ and those of $H^2(\Omega^\bullet_{\mathbb{C}^2_x}(*x), \nabla_x)$. As shown in [12, Section 1.5], these isomorphisms coincide. This compatibility implies the following.

THEOREM 6.1. *The intersection matrices H and $(2\pi\sqrt{-1})^2 C$ and the period matrices*

$$\Pi_+(x) = \Big(\int_{\Delta_j} u\varphi_{x,i}\Big)_{0\le i,j\le 4}, \qquad \Pi_-(x) = \Big(\int_{\Delta_j} (1/u)\varphi_{x,i}\Big)_{0\le i,j\le 4}$$

satisfy

$$\Pi_+(x)^t H^{-1t}\Pi_-(x) = (2\pi\sqrt{-1})^2 C. \tag{6.1}$$

COROLLARY 6.1. *The identity (6.1) implies twisted period relations*

$$\begin{aligned}
&\frac{1-a}{1-a_{12}} F_4(a,b,c_1,c_2;x)F_4(2-a,-b,2-c_1,2-c_2;x)\\
&\quad - \frac{b(1-a_1)}{b_1(1-a_{12})} F_4(a_1,b_1,2-c_1,c_2;x)F_4(2-a_1,-b_1,c_1,2-c_2;x)\\
&\quad - \frac{b(1-a_2)}{b_2(1-a_{12})} F_4(a_2,b_2,c_1,2-c_2;x)F_4(2-a_2,-b_2,2-c_1,c_2;x)\\
&\quad + \frac{b}{b_{12}} F_4(a_{12},b_{12},2-c_1,2-c_2;x)F_4(2-a_{12},-b_{12},c_1,c_2;x)\\
&= \frac{(1-a+b)(b_1+b_2)(1-c_1)(1-c_2)}{(1-a_{12})b_1b_2b_{12}},
\end{aligned}$$

$$\begin{aligned}
&\frac{1-a}{1-a_{12}} F_4(a,b+1,c_1,c_2;x)F_4(2-a,1-b,2-c_1,2-c_2;x)\\
&\quad - \frac{b_1(1-a_1)}{b(1-a_{12})} F_4(a_1,b_1+1,2-c_1,c_2;x)F_4(2-a_1,1-b_1,c_1,2-c_2;x)\\
&\quad - \frac{b_2(1-a_2)}{b(1-a_{12})} F_4(a_2,b_2+1,c_1,2-c_2;x)F_4(2-a_2,1-b_2,2-c_1,c_2;x)\\
&\quad + \frac{b_{12}}{b} F_4(a_{12},b_{12}+1,2-c_1,2-c_2;x)F_4(2-a_{12},1-b_{12},c_1,c_2;x)\\
&= \frac{2(1-c_1)(1-c_2)}{(1-a_{12})(-b)R(x)},
\end{aligned}$$

$$
\begin{aligned}
&\frac{1-a}{1-a_{12}}F_4(a,b,c_1,c_2;x)F_4(2-a,1-b,2-c_1,2-c_2;x)\\
&\qquad-\frac{1-a_1}{1-a_{12}}F_4(a_1,b_1,2-c_1,c_2;x)F_4(2-a_1,1-b_1,c_1,2-c_2;x)\\
&\qquad-\frac{1-a_2}{1-a_{12}}F_4(a_2,b_2,c_1,2-c_2;x)F_4(2-a_2,1-b_2,2-c_1,c_2;x)\\
&\qquad+F_4(a_{12},b_{12},2-c_1,2-c_2;x)F_4(2-a_{12},1-b_{12},c_1,c_2;x)\\
&\quad=0,
\end{aligned}
$$

where

$$
\begin{aligned}
a_1&=a-c_1+1, & a_2&=a-c_2+1, & a_{12}&=a-c_1-c_2+2,\\
b_1&=b-c_1+1, & b_2&=b-c_2+1, & b_{12}&=b-c_1-c_2+2.
\end{aligned}
$$

Proof. Compare the $(1,1)$-entries of the both sides of (6.1). Then we have

$$
\big(f_1(x),\dots,f_4(x)\big)\,{}^tH^{-1}\,{}^t\big(f_1^\vee(x),\dots,f_4^\vee(x)\big)=\mathcal{I}_c(\varphi_{x,1},\varphi_{x,1}), \tag{6.2}
$$

where

$$
\begin{aligned}
f_1^\vee(x)&=\frac{\Gamma(c_1-1)\Gamma(c_2-1)\Gamma(1-c_1-c_2+a)}{\Gamma(-1+a)}F_4(2-a,-b,2-c_1,2-c_2;x),\\
f_2^\vee(x)&=\frac{\Gamma(c_1-b-1)\Gamma(c_1-a+1)\Gamma(1+b)\Gamma(1-c_1-c_2+a)}{\Gamma(c_1)\Gamma(2-c_2)}\\
&\qquad\times e^{-\pi\sqrt{-1}(a+b-c_1-c_2)}x_1^{c_1-1}F_4(c_1-a+1,c_1-b-1,c_1,2-c_2;x),\\
f_3^\vee(x)&=\frac{\Gamma(c_2-a+1)\Gamma(c_2-b-1)\Gamma(1+b)\Gamma(1-c_1-c_2+a)}{\Gamma(2-c_1)\Gamma(c_2)}\\
&\qquad\times e^{-\pi\sqrt{-1}(a+b-c_1-c_2)}x_2^{c_2-1}F_4(c_2-a+1,c_2-b-1,2-c_1,c_2;x),\\
f_4^\vee(x)&=\frac{x_1^{c_1-1}x_2^{c_2-1}\Gamma(1-c_1)\Gamma(1-c_2)\Gamma(1+b)}{\Gamma(3-c_1-c_2+b)}\\
&\qquad\times F_4(c_1+c_2-a,c_1+c_2-b-2,c_1,c_2;x).
\end{aligned}
$$

Since H is diagonal, we can easily evaluate $H^{-1}={}^tH^{-1}$. By multiplying both sides of (6.2) by $(1-c_1)(1-c_2)/(2\pi\sqrt{-1})^2$ and using the formula $\Gamma(a)\Gamma(1-a)=\pi/\sin(\pi a)$, we reduce this relation to the first identity. By multiplying the identities arising from the $(4,4)$ and $(1,4)$-entries of (6.1) by $(1-c_1)(1-c_2)/(2\pi\sqrt{-1})^2$, we have the second and third equalities in this corollary. ∎

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