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MODULAR FORMS OF HALF-INTEGRAL WEIGHTS ON $\mathrm{SL}(2,\mathbb{Z})$

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Abstract. In this paper, we prove that, for an integer r with $(r,6)=1$ and $0<r<24$ and a nonnegative even integer s, the set

$$\left\{\eta(24\tau)^r f(24\tau) : f(\tau)\in M_s(1)\right\}$$

is isomorphic to

$$S^{\mathrm{new}}_{r+2s-1}\left(6,-\left(\frac{8}{r}\right),-\left(\frac{12}{r}\right)\right)\otimes\left(\frac{12}{\cdot}\right)$$

as Hecke modules under the Shimura correspondence. Here $M_s(1)$ denotes the space of modular forms of weight s on $\Gamma_0(1)=\mathrm{SL}(2,\mathbb{Z})$, $S^{\mathrm{new}}_{2k}(6,\epsilon_2,\epsilon_3)$ is the space of newforms of weight $2k$ on $\Gamma_0(6)$ that are eigenfunctions with eigenvalues ϵ_2 and ϵ_3 for Atkin–Lehner involutions W_2 and W_3, respectively, and the notation $\otimes(12/\cdot)$ means the twist by the quadratic character $(12/\cdot)$. There is also an analogous result for the cases $(r,6)=3$.

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§1. Introduction

Let

$$\theta(\tau)=\sum_{n\in\mathbb{Z}} q^{n^2},\quad q=e^{2\pi i\tau},$$

be the Jacobi theta function. Then Shimura's theory of modular forms of half-integral weights can be described as follows. Let k be a positive integer, let N be a positive integer, and let χ be a Dirichlet character modulo $4N$.

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We say that a holomorphic function $f:\mathbb{H}=\{\tau:\operatorname{Im}\tau>0\}\to\mathbb{C}$ is a *modular form of half-integral weight* $k+1/2$ on $\Gamma_0(4N)$ with character χ if it is holomorphic at each cusp and satisfies

$$\frac{f(\gamma\tau)}{f(\tau)}=\chi(d)\frac{\theta(\gamma\tau)^{2k+1}}{\theta(\tau)^{2k+1}}$$

for all $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(4N)$. Let $M_{k+1/2}(4N,\chi)$ denote the space of these functions. Shimura [19] showed how the Hecke theory can be extended to these spaces. More importantly, he showed that if $f\in M_{k+1/2}(4N,\chi)$ is a Hecke eigenform, then there is a corresponding Hecke eigenform of integral weight $2k$ with character χ^2 that shares the same eigenvalues. Moreover, he conjectured that the level of this modular form of integral weight can be taken to be $2N$. This conjecture was later proved by Niwa [16] (see also [21]). In the literature, this correspondence between modular forms of half-integral weights and modular forms of integral weights is called the *Shimura correspondence*.

In [19], the correspondence was proved by using Weil's characterization of Hecke eigenforms in terms of L-functions. From the representation-theoretical point of view, this correspondence amounts to a correspondence from certain automorphic representations of the metaplectic double cover of $\mathrm{GL}(2,\mathbb{A}_\mathbb{Q})$ to automorphic representations of $\mathrm{GL}(2,\mathbb{A}_\mathbb{Q})$, where $\mathbb{A}_\mathbb{Q}$ denotes the adèle ring of $\mathbb{Q}$ (see [6], [10], and [24] for more details).

In general, the Shimura correspondence is not one-to-one. In order to get a multiplicity-one result, Kohnen, in the fundamental works [13] and [14], introduced a subspace of $M_{k+1/2}(4N,\chi)$ and developed a newform theory for this subspace that is parallel to the Atkin–Lehner–Li theory of newforms for modular forms of integral weights. To state Kohnen's result, let χ be a Dirichlet character modulo $4N$, and set $\epsilon=\chi(-1)$. Let $S^+_{k+1/2}(4N,(4\epsilon/\cdot)\chi)$ be the subspace consisting of cusp forms of half-integral weight $k+1/2$ and character $(4\epsilon/\cdot)\chi$ on $\Gamma_0(4N)$ whose Fourier expansions are of the form

$$\sum_{\epsilon(-1)^k n\equiv 0,1 \bmod 4} a_n e^{2\pi i n\tau}.$$

Then Kohnen proved that, under the assumptions that N is odd and square-free and that χ is a quadratic character, the image of $S^+_{k+1/2}(4N,(4\epsilon/\cdot)\chi)$ under the Shimura correspondence is $S_{2k}(N)$. Moreover, there is a canonically defined subspace $S^{\mathrm{new}}_{k+1/2}(4N,(4\epsilon/\cdot)\chi)\subset S^+_{k+1/2}(4N,(4\epsilon/\cdot)\chi)$ such that

$S^{\mathrm{new}}_{k+1/2}(4N,(4\epsilon/\cdot)\chi) \simeq S^{\mathrm{new}}_{k+1/2}(N)$ as Hecke modules. In particular, the strong multiplicity-one theorem holds for $S^{\mathrm{new}}_{k+1/2}(4N,(4\epsilon/\cdot)\chi)$. Kohnen's work was later extended by several authors in various directions (see [7], [8], [23]).

Modular forms of half-integral weights are closely related to many problems in number theory. For example, let $p(n)$ denote the number of ways to write a positive integer n as unordered sums of positive integers. Then the generating function of the partition function $p(n)$ is equal to

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^m}.$$

If we set $q = e^{2\pi i\tau}$, then the infinite product above is essentially the reciprocal of the Dedekind eta function, which is well known to be a modular form of weight $1/2$ on $\Gamma_0(576)$ with character $(12/\cdot)$. Using this fact, along with the Shimura correspondence and properties of Galois representations attached to cusp forms, Ono [18, Theorem 1] proved that for every prime m greater than 3, there is a positive proportion of primes ℓ such that the congruence

$$p\Big(\frac{m\ell^3 n+1}{24}\Big) \equiv 0 \bmod m$$

holds for all integers n relatively prime to ℓ. This result was later extended by several authors (see [1], [2], [28]). For example, in [28], the current author showed that, for every prime m greater than 3 and every prime different from 2, 3, and m, there is an explicitly computable integer k such that

$$p\Big(\frac{m\ell^k n+1}{24}\Big) \equiv 0 \bmod m$$

for all integers n relatively prime to ℓ. A key ingredient in [28] is the Hecke invariance of the space

$$\mathcal{S}_{r,s} = \big\{\eta(24\tau)^r f(24\tau) : f(\tau) \in M_s(1)\big\}, \tag{1}$$

where r is an odd integer between 0 and 24, s is a nonnegative even integer, and $M_s(1)$ is the space of modular forms of weight s on $\Gamma_0(1) = \mathrm{SL}(2,\mathbb{Z})$. That is, even though the space $M_{k+1/2}(576,(12/\cdot))$ itself has a huge dimension, it contains several subspaces of small dimensions that are invariant under the action of Hecke algebra. The invariance of these spaces was first proved by Garvan [9, Proposition 3.1] and later rediscovered by the current author independently (see Section 4 of the arXiv version of [28] (arXiv:0904.2530 [math.NT]) for a proof of the invariance).

Now recall that a well-known result of Waldspurger [25, Theorem 1] states that if f is a Hecke eigenform of half-integral weight $k+1/2$ and if F is the corresponding Hecke eigenform of integral weight $2k$, then for square-free n, the square of the nth Fourier coefficient of f is essentially proportional to the special value at $s=k$ of $L(F\otimes\chi_{(-1)^k n},s)$, where $\chi_{(-1)^k n}$ is the Kronecker character of the quadratic field $\mathbb{Q}(\sqrt{(-1)^k n})$ (see also [15]). Using this result of Waldspurger, Guo and Ono [11] related the arithmetic of the partition function $p(n)$ to the arithmetic of certain motives. Specifically, let $13\le\ell\le 31$ be a prime. Let r be the unique integer between 0 and 24 such that $r\equiv-\ell \bmod 24$, and let $s=(\ell-r-2)/2$. Then the space $\mathcal{S}_{r,s}$ defined in (1) is 1-dimensional and spanned by $g_\ell(\tau)=\eta(24\tau)^r E_s(24\tau)$, where $E_s(\tau)$ denotes the Eisenstein series. It is known that

$$g_\ell(\tau)\equiv\sum_{n=0}^\infty p\left(\frac{\ell n+1}{24}\right)q^n \bmod \ell. \tag{2}$$

Then Guo and Ono showed that if we let $G_\ell(\tau)$ be the unique Hecke eigenform in $S_{\ell-3}(6)$ with Fourier expansion

$$G_\ell(\tau)=q+\left(\frac{8}{r}\right)2^{(\ell-5)/2}q^2+\left(\frac{12}{r}\right)3^{(\ell-5)/2}q^3+\cdots,$$

then the image of $g_\ell(\tau)$ under the Shimura correspondence is $G_\ell\otimes(12/\cdot)$, which is a newform of level 144. (Note that $g_\ell(\tau)$ is contained in Kohnen's $+$-space.) In view of Waldspurger's result and (2), this means that the values of the partition function modulo ℓ are related to the values at the center point of the L-function of G_ℓ twisted by quadratic Dirichlet characters. Thus, assuming the truth of the Bloch–Kato conjecture, the arithmetic properties of the partition function are related to those of certain motives associated to G_ℓ.

Now observe that, by [3, Theorem 3], the function $G_\ell(\tau)$ is contained in the Atkin–Lehner eigensubspace of $S_{\ell-3}^{\mathrm{new}}(6)$ with eigenvalues $-(8/r)$ and $-(12/r)$ for the Atkin–Lehner involutions W_2 and W_3, respectively. In other words, for the few cases considered in [11], the Shimura correspondence yields an isomorphism

$$\mathcal{S}_{r,s}\simeq S_{r+2s-1}^{\mathrm{new}}\left(6,-\left(\frac{8}{r}\right),-\left(\frac{12}{r}\right)\right)\otimes\left(\frac{12}{\cdot}\right)$$

as Hecke modules, where $S_{2k}^{\mathrm{new}}(6,\epsilon_2,\epsilon_3)$ denotes the space of newforms of weight $2k$ on $\Gamma_0(6)$ that are eigenfunctions with eigenvalues ϵ_2 and ϵ_3 for

W_2 and W_3. (Note that the Hecke algebras on the two sides are isomorphic. Thus, we may talk about isomorphisms as Hecke modules.) It is natural to ask whether this isomorphism holds in general. The purpose of this paper is to prove that this is indeed the case.

THEOREM 1. *Let r be an integer satisfying $(r,6)=1$ and $0<r<24$, and let s be a nonnegative even integer. Let*

$$\mathcal{S}_{r,s}=\left\{\eta(24\tau)^r f(24\tau): f(\tau)\in M_s(1)\right\}\subset S_{r/2+s}\Big(576,\Big(\frac{12}{\cdot}\Big)\Big),$$

where $M_s(1)$ denotes the space of modular forms of weight s on $\Gamma_0(1)=$ SL$(2,\mathbb{Z})$. Then the Shimura correspondence yields an isomorphism

$$\mathcal{S}_{r,s}\simeq S^{\text{new}}_{r+2s-1}\Big(6,-\Big(\frac{8}{r}\Big),-\Big(\frac{12}{r}\Big)\Big)\otimes\Big(\frac{12}{\cdot}\Big)$$

as Hecke modules.

For odd integers r that are divisible by 3, we have also an analogous result.

THEOREM 2. *Let r be an odd integer satisfying $0<r<8$, and let s be a nonnegative even integer. Let*

$$\mathcal{S}_{3r,s}=\left\{\eta(8\tau)^{3r} f(8\tau): f(\tau)\in M_s(1)\right\}\subset S_{3r/2+s}\Big(64,\Big(\frac{-4}{\cdot}\Big)\Big).$$

Then the Shimura correspondence yields an isomorphism

$$\mathcal{S}_{3r,s}\simeq S^{\text{new}}_{3r+2s-1}\Big(2,-\Big(\frac{8}{r}\Big)\Big)\otimes\Big(\frac{-4}{\cdot}\Big)$$

as Hecke modules.

COROLLARY 1. *The multiplicity-one property holds for the spaces $\mathcal{S}_{r,s}$ defined in Theorems 1 and 2.*

REMARK 2. Note that the space $S^{\text{new}}_{2k}(6,\epsilon_2,\epsilon_3)\otimes(12/\cdot)$ is contained in $S^{\text{new}}_{2k}(144,-,-)$, regardless of whether ϵ_2,ϵ_3 are 1 or -1. Also, $S^{\text{new}}_{2k}(2,\epsilon_2)\otimes(-4/\cdot)$ is a subspace of $S^{\text{new}}_{2k}(16,-)$ for both $\epsilon_2=1$ and $\epsilon_2=-1$.

It turns out that the Hecke invariance of $\mathcal{S}_{r,s}$ and the explicit Shimura correspondence in Theorems 1 and 2 are best explained in terms of modular forms of half-integral weight of η-type. Namely, in Shimura's setting, a function is called a *modular form of half-integral weight* if its transformation is comparable with the Jacobi theta function. In a similar way, we say that a function $f(\tau)$ is a *modular form of η-type* if its transformation is comparable with the Dedekind eta function, that is, if $f(\tau)$ satisfies

$$\frac{f(\gamma\tau)}{f(\tau)} = (c\tau+d)^s \frac{\eta(\gamma\tau)^r}{\eta(\tau)^r}$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in a subgroup Γ of $\mathrm{SL}(2,\mathbb{Z})$, where s is assumed to be a nonnegative even integer and r is an odd integer between 0 and 24. Then it is easy to show that modular forms of η-type on $\mathrm{SL}(2,\mathbb{Z})$ are essentially just the functions in $\mathcal{S}_{r,s}$ defined in Theorems 1 and 2 (see Proposition 6 below). This explains the Hecke invariance of the spaces $\mathcal{S}_{r,s}$.

At first sight, the introduction of the notion of modular forms of η-type is superficial since if $f(\tau)$ is such a function, then $f(24\tau)$ is just a modular form of half-integral weight with character $(12/\cdot)$ in the sense of Shimura, and we do not get any new modular forms in this way. However, if a modular form of half-integral weight on a congruence subgroup $\Gamma_0(4N)$ in the sense of Shimura happens to be a modular form of η-type on a larger group, then the extra symmetries from this larger group will give us additional information about the function. This is the reason that we can determine such a precise image of $S_{r,s}$ under the Shimura correspondence. (In the proof of Theorems 1 and 2, we will work with modular forms of η-type on $\mathrm{SL}(2,\mathbb{Z})$ instead of modular forms on the much smaller group $\Gamma_0(576)$ in the sense of Shimura.)

Our proof of Theorems 1 and 2 is classical. That is, since the Hecke modules involved are all semisimple, to prove the theorems, it suffices to show that the traces coincide for all Hecke operators. It will be interesting to have a representation-theoretical proof of the results. Note that here we prove only Theorem 1; the proof of Theorem 2 is similar, but much simpler, and will be omitted.

The rest of this article has the following organization. In Section 2, we first define modular forms of η-type in more detail. We then define Hecke operators and introduce several basic properties of them. We also describe Shimura's abstract trace formula (see [20, Theorem 4.5]). In Section 3, which constitutes the principal part of our paper, we compute the traces of Hecke operators on the space of modular forms of η-type. In Section 4, we determine the traces of Hecke operators on $S_{2k}^{\mathrm{new}}(6,\epsilon_2,\epsilon_3)$. In Section 5, we show that the traces coincide and thereby establish Theorem 1.

§2. Preliminaries

In this section, we give a more detailed definition of modular forms of η-type. We then define Hecke operators on these modular forms and review Shimura's trace formula for Hecke operators.

2.1. Modular forms of (η^r, s)-type

Notation 3. Throughout the rest of this article, we let r and s be fixed integers with $(r,6)=1$, $0<r<24$ and s even. Set also $k=(r-1)/2+s$. Let $\mathfrak{G}_{k+1/2}$ be the group of pairs $(A,\phi(\tau))$, where $A=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in \mathrm{GL}^+(2,\mathbb{Q})$, $\phi(\tau)$ is a holomorphic function $\mathbb{H}\mapsto\mathbb{C}$ satisfying

$$\big|\phi(\tau)\big| = (\det A)^{-k/2-1/4}|c\tau+d|^{k+1/2},$$

and the group law is defined by

$$\big(A,\phi(\tau)\big)\big(B,\psi(\tau)\big)=\big(AB,\phi(B\tau)\psi(\tau)\big).$$

Consider the subgroup Γ^* of $\mathfrak{G}_{k+1/2}$ defined by

$$\Gamma^*=\Gamma^*_{r,s}=\left\{\left(\gamma,\frac{\eta(\gamma\tau)^r}{\eta(\tau)^r}(c\tau+d)^s\right):\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{SL}(2,\mathbb{Z})\right\}.$$

For an element γ in $\mathrm{SL}(2,\mathbb{Z})$, we let γ^* denote the element in Γ^* whose first component is γ. Naturally, if G is a subgroup of $\mathrm{SL}(2,\mathbb{Z})$, then we let G^* be the subgroup $\{\gamma^*:\gamma\in G\}$.

Here let us recall a well-known formula for $\eta(\gamma\tau)/\eta(\tau)$.

Lemma 4 ([26, pp. 125–127]). *Let $\gamma=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in\mathrm{SL}(2,\mathbb{Z})$ with $c\ge 0$. Then we have*

$$\frac{\eta(\gamma\tau)}{\eta(\tau)}=\epsilon(a,b,c,d)(c\tau+d)^{1/2},$$

where

$$\epsilon(a,b,c,d)=\begin{cases}\left(\frac{d}{c}\right)i^{(1-c)/2}e^{2\pi i(bd(1-c^2)+c(a+d)-3)/24} & \text{if } c \text{ is odd},\\ \left(\frac{c}{d}\right)e^{2\pi i(ac(1-d^2)+d(b-c+3)-3)/24} & \text{if } c \text{ is even}.\end{cases}\tag{3}$$

We now define modular forms of η-type.

Definition 5. Let $f:\mathbb{H}\to\mathbb{C}$ be a holomorphic function on the upper half-plane. We define the action of $\gamma^*=(\gamma,\phi_\gamma(\tau))$ on f by

$$(f\mid\gamma^*)(\tau)=\phi_\gamma(\tau)^{-1}f(\gamma\tau).$$

Let G be a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ of finite index. If the function f satisfies

$$(f \mid \gamma^*)(\tau) = f(\tau)$$

for all $\gamma^* \in G^*$ and is holomorphic at each cusp of G, then we say that f is a *modular form of* (η^r, s)*-type* on G. If, in addition, f vanishes at each cusp of G, we say that f is a *cusp form* of (η^r, s)-type. The space of cusp forms of (η^r, s)-type on G will be denoted by $\mathcal{S}_{r,s}(G)$. If $G = \Gamma_0(N)$, we simply write it as $\mathcal{S}_{r,s}(N)$.

In the case $N = 1$, the space $\mathcal{S}_{r,s}(1)$ has a very simple description.

Proposition 6. *For $N = 1$, we have*

$$\mathcal{S}_{r,s}(1) = \left\{\eta(\tau)^r f(\tau) : f \in M_s(1)\right\},$$

where $M_s(1)$ is the space of modular forms of weight s on $\mathrm{SL}(2,\mathbb{Z})$.

Proof. Assume that $g(\tau) \in \mathcal{S}_{r,s}(1)$. We have

$$g(\tau + 1) = e^{2\pi i r/24} g(\tau).$$

Thus, the Fourier expansion of $g(\tau)$ takes the form $q^{r/24}(a_0 + a_1 q + \cdots)$. Since $\eta(\tau)$ is nonvanishing throughout $\mathbb{H}$, the function $g(\tau)/\eta(\tau)^r$ is holomorphic on $\mathbb{H}$. Moreover, it is easy to see that for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, one has

$$\frac{g(\gamma\tau)}{\eta(\gamma\tau)^r} = (c\tau + d)^s \frac{g(\tau)}{\eta(\tau)^r}.$$

Therefore, $g(\tau)/\eta(\tau)^r$ is a modular form of weight s on $\mathrm{SL}(2,\mathbb{Z})$. This proves the proposition. ∎

2.2. Hecke operators on $\mathcal{S}_{r,s}(N)$

Notation 7. Let N be a positive integer. For a positive integer n, let

$$\begin{aligned}
&\mathscr{M}_n(N) \\
&\quad = \Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \Gamma_0(N) \\
&\quad = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}, ad - bc = n, N \,\middle|\, c, (a,N) = 1, (a,b,c,d) = 1 \right\},
\end{aligned}$$

and let $\mathscr{M}_n(N)^*$ denote the subset

$$\mathscr{M}_n(N)^* = \Gamma_0(N)^* \left(\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}, n^{k/2+1/4} \right) \Gamma_0(N)^*$$

of $\mathfrak{G}_{k+1/2}$.

LEMMA 8. *If n is a positive integer relatively prime to 6, then for each $\gamma \in \mathscr{M}_{n^2}(N)$, there exists a unique element γ^* in $\mathscr{M}_{n^2}(N)^*$ such that the first component of γ^* is γ.*

Proof. It suffices to prove the case $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & n^2 \end{smallmatrix}\right)$. We are required to show that if $A, B \in \Gamma_0(N)$ are matrices such that $A\left(\begin{smallmatrix} 1 & 0 \\ 0 & n^2 \end{smallmatrix}\right)B^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & n^2 \end{smallmatrix}\right)$, then

$$A^* \left(\begin{pmatrix} 1 & 0 \\ 0 & n^2 \end{pmatrix}, n^{k+1/2} \right) = \left(\begin{pmatrix} 1 & 0 \\ 0 & n^2 \end{pmatrix}, n^{k+1/2} \right) B^*. \tag{4}$$

Assume that $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. By Lemma 4, we have

$$\begin{aligned} A^* & \left(\begin{pmatrix} 1 & 0 \\ 0 & n^2 \end{pmatrix}, n^{k+1/2} \right) \\ & = \left(\begin{pmatrix} a & bn^2 \\ c & dn^2 \end{pmatrix}, \epsilon(a,b,c,d)^r (c\tau/n^2 + d)^{s+r/2} n^{k+1/2} \right), \end{aligned}$$

where $\epsilon(a,b,c,d)$ is defined by (3). Now the assumption that $A\left(\begin{smallmatrix} 1 & 0 \\ 0 & n^2 \end{smallmatrix}\right)B^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & n^2 \end{smallmatrix}\right)$ implies that if $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $B = \left(\begin{smallmatrix} a & bn^2 \\ c/n^2 & d \end{smallmatrix}\right)$. In particular, we have $n^2 \mid c$. Thus,

$$\begin{aligned} & \left(\begin{pmatrix} 1 & 0 \\ 0 & n^2 \end{pmatrix}, n^{k+1/2} \right) B^* \\ & = \left(\begin{pmatrix} a & bn^2 \\ c & dn^2 \end{pmatrix}, n^{k+1/2} \epsilon(a, bn^2, c/n^2, d)^r (c\tau/n^2 + d)^{s+r/2} \right). \end{aligned}$$

Since n is assumed to be relatively prime to 6, we have $n^2 \equiv 1 \bmod 24$, and hence

$$\epsilon(a,b,c,d) = \epsilon(a, bn^2, c/n^2, d).$$

This establishes (4) and the lemma. $\square$

The group $\Gamma_0(N)$ acts on $\mathscr{M}_{n^2}(N)$ by matrix multiplication on the left. It is clear that if $f \in \mathcal{S}_{r,s}(N)$ and α and β are two elements of $\mathscr{M}_{n^2}(N)$ that are equivalent under the left action of $\Gamma_0(N)$, then

$$f \mid \alpha^* = f \mid \beta^*.$$

Moreover, since $\left(\begin{smallmatrix} 1 & 0 \\ 0 & n^2 \end{smallmatrix}\right)^{-1}\Gamma_0(N)\left(\begin{smallmatrix} 1 & 0 \\ 0 & n^2 \end{smallmatrix}\right)$ and $\Gamma_0(N)$ are commensurable, there are finitely many right cosets in $\Gamma_0(N)\backslash\mathscr{M}_{n^2}(N)$. Thus, for each positive integer n with $(n,6) = 1$, we can define a linear operator on $\mathcal{S}_{r,s}(N)$.

LEMMA 9. *The mapping*

$$\left[\mathscr{M}_{n^2}(N)^*\right] : f \longmapsto f \mid \left[\mathscr{M}_{n^2}(N)^*\right] = \sum_{\gamma \in \Gamma_0(N) \backslash \mathscr{M}_{n^2}(N)} f \mid \gamma^*$$

is a linear operator on $\mathcal{S}_{r,s}(N)$*.*

We now define Hecke operators on $\mathcal{S}_{r,s}(N)$.

DEFINITION 10. For a positive integer n with $(n,6)=1$, the *Hecke operator* T_{n^2} on $\mathcal{S}_{r,s}(N)$ is defined by

$$T_{n^2} : f \longmapsto n^{k-3/2} \sum_{ad=n, a|d} af \mid \left[\mathscr{M}_{(d/a)^2}(N)\right].$$

PROPOSITION 11. *Let p be a prime such that $p \nmid 6N$. Then for* $f(\tau) = \sum_{n=1}^{\infty} a_f(n) q^{n/24} \in \mathcal{S}_{r,s}(N)$*, we have*

$$T_{p^2} : f(\tau)$$
$$\mapsto \sum_{n=1}^{\infty} \left(a_f(p^2 n) + \left(\frac{12}{p}\right)\left(\frac{(-1)^k n}{p}\right) p^{k-1} a_f(n) + p^{2k-1} a_f(n/p^2) \right) q^{n/24}.$$

Proof. One way to prove the proposition would be to utilize the standard coset representatives of $\Gamma_0(N) \backslash \mathscr{M}_{p^2}(N)$ given by

$$\begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} p & a \\ 0 & p \end{pmatrix}, \qquad \begin{pmatrix} 1 & b \\ 0 & p^2 \end{pmatrix}, \quad a = 1, \dots, p-1, b = 0, \dots, p^2 - 1,$$

and then apply formulas similar to those in (5) in the proof of Lemma 17 below to get the conclusion. Here, however, because it is well known that $f(24\tau)$ is a modular form of half-integral weight on $\Gamma_0(576)$ with character $(12/\cdot)$ in the sense of Shimura, we can actually skip those tedious computations. Indeed, from the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{S}_{r,s} \arrow[r, "T_{p^2}"] \arrow[d, "\left(\begin{smallmatrix} 24 & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] & \mathcal{S}_{r,s} \arrow[d, "\left(\begin{smallmatrix} 24 & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \\
S_{k+1/2}\left(576, \left(\frac{12}{\cdot}\right)\right) \arrow[r, "T_{p^2}"] & S_{k+1/2}\left(576, \left(\frac{12}{\cdot}\right)\right)
\end{tikzcd}$$

and the formula given in [19, p. 450] for the Hecke operator T_{p^2} on $S_{k+1/2}(576, (12/\cdot))$ in terms of Fourier coefficients, we immediately get the conclusion. $\square$

REMARK 12. The linear operators $[\mathscr{M}_n(N)]$ can also be defined for non-square integers n and integers that are not relatively prime to 6, but it turns out that they are actually the zero operator. The reason is that for such integers n, there is more than one element in $\mathscr{M}_n(N)$ with the same first component, and the actions of these elements cancel out each other.

2.3. Shimura's trace formula

Here we state a trace formula of Shimura [20, Theorem 4.5], adapted to our setting. Our description of the trace formula mostly follows that of Kohnen [14, Section 4].

DEFINITION 13. Let N and n be positive integers. For $\gamma \in \mathscr{M}_n(N)$, we say that γ is

(1) a *scalar* element if $\gamma = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ for some integer a (this happens only when $n = 1$),
(2) a *parabolic* element if the fixed point of γ is a single cusp in $\mathbb{P}^1(\mathbb{Q})$,
(3) a *hyperbolic* element if the fixed points of γ are two distinct real numbers, and
(4) an *elliptic* element if the fixed points of γ are a pair of conjugate complex numbers.

Two elements γ_1 and γ_2 in $\mathscr{M}_n(N)$ are *equivalent* if

(1) γ_1 and γ_2 are scalars and $\gamma_1 = \gamma_2$;
(2) γ_1 and γ_2 are hyperbolic or elliptic, and there exists an element $\sigma \in \Gamma_0(N)$ such that $\sigma\gamma_1\sigma^{-1} = \gamma_2$; or
(3) γ_1 and γ_2 are parabolic, and there exist $\sigma \in \Gamma_0(N)$ and α in the stabilizer subgroup inside $\Gamma_0(N)$ of the cusp fixed by γ_2 such that $\sigma\gamma_1\sigma^{-1} = \alpha\gamma_2$.

Now for $\gamma \in \mathscr{M}_{n^2}(N)$, we define a number $J(\gamma)$ as follows.

(1) If γ is a scalar, then we set

$$J(\gamma) = \frac{1}{24}\left(k - \frac{1}{2}\right)\left[\mathrm{SL}(2,\mathbb{Z}) : \Gamma_0(N)\right].$$

(2) Assume that γ is parabolic with fixed point $a/c \in \mathbb{P}^1(\mathbb{Q})$. Let $\sigma \in \mathrm{SL}(2,\mathbb{Z})$ be a matrix such that $\sigma\infty = a/c$. Then the stabilizer subgroup of a/c inside $\Gamma_0(N)$ is generated by $\sigma\left(\begin{smallmatrix} 1 & w \\ 0 & 1 \end{smallmatrix}\right)\sigma^{-1}$ and $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, where $w = N/(c^2, N)$ is the width of the cusp a/c. Now we write

$$\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}^* = \left(\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}, e^{-2\pi i\mu}\right)$$

with $0 \le \mu < 1$. If

$$\sigma^{*-1}\gamma^*\sigma^* = \left(\pm\begin{pmatrix} n & nuw \\ 0 & n \end{pmatrix}, \eta\right),$$

then let

$$J(\gamma) = \begin{cases} -\frac{1}{2\eta}e^{-2\pi i u\mu}(1-2\mu) & \text{if } u \in \mathbb{Z}, \\ -\frac{1}{2\eta}e^{-2\pi i u\mu}(1-i\cot\pi u) & \text{if } u \notin \mathbb{Z}. \end{cases}$$

(3) If γ is hyperbolic and the fixed points are not cusps, then set $J(\gamma) = 0$.

(4) Assume that γ is hyperbolic fixing (two distinct) cusps. Then the eigenvalues of γ are two integers λ and λ'. We assume that $|\lambda| > |\lambda'|$. Let $\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)$ be an eigenvector associated to λ' with $a, c \in \mathbb{Z}$ and $(a,c) = 1$. Find an element $\sigma \in \mathrm{SL}(2,\mathbb{Z})$ such that $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then $\sigma^{-1}\gamma\sigma = \left(\begin{smallmatrix} \lambda' & x \\ 0 & \lambda \end{smallmatrix}\right)$ for some integer x. If

$$\sigma^*\gamma^*\sigma^{-1*} = \left(\begin{pmatrix} \lambda' & x \\ 0 & \lambda \end{pmatrix}, \eta\right),$$

then set

$$J(\gamma) = \frac{1}{2}\Big(\eta\Big(\frac{\lambda'}{\lambda} - 1\Big)\Big)^{-1}.$$

(5) Assume that $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is elliptic. Then the eigenvalues of γ are a pair of conjugate complex numbers ρ and $\overline{\rho}$. We assume that $\operatorname{sgn}\operatorname{Im}\rho = \operatorname{sgn} c$. If

$$\gamma^* = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, u(c\tau+d)^{k+1/2}\right),$$

then we set

$$J(\gamma) = \big(wu\rho^{k-1/2}(\rho - \overline{\rho})\big)^{-1},$$

where w denotes the number of elements in $\Gamma_0(N)$ that commute with γ.

Then according to Shimura's formula [20, p. 273], the trace of the operator $[\mathscr{M}_{n^2}(N)]$ on $\mathcal{S}_{r,s}(N)$ is as follows (compare [14, p. 49]).

Proposition 14. *For positive integers N and n with $(n, 6N) = 1$, the trace of the linear operator $[\mathscr{M}_{n^2}(N)^*]$ on $\mathcal{S}_{r,s}(N)$ is given by*

$$\operatorname{tr}\big[\mathscr{M}_{n^2}(N)^*\big] = \sum_\gamma J(\gamma),$$

where the sum runs over representatives of equivalence classes as per Definition 13 and $J(\gamma)$ are defined as in the paragraph preceding the proposition.

§3. Traces of Hecke operators on $\mathcal{S}_{r,s}(1)$

In this section, we will compute the trace of Hecke operators on $\mathcal{S}_{r,s}(1)$. The contributions of scalar, parabolic, hyperbolic, and elliptic classes will be determined separately in individual subsections.

Throughout this section, we write $\mathscr{M}_{n^2}(1)$ and $\mathscr{M}_{n^2}(1)^*$ simply as $\mathscr{M}_{n^2}$ and $\mathscr{M}_{n^2}^*$, respectively. All equivalences mentioned here refer to the equivalence relation described in Definition 13.

3.1. Scalar cases

Since any element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathscr{M}_{n^2}$ satisfies $(a,b,c,d)=1$, scalar elements exist only in $\mathscr{M}_1^*$, and they are $(\pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right),1)$.

Proposition 15. *The contribution of scalar elements in $\mathscr{M}_{n^2}$ to the trace of $[\mathscr{M}_{n^2}^*]$ is*

$$\begin{cases} \frac{1}{12}(k-\frac{1}{2}) & \text{if } n=1, \\ 0 & \text{otherwise.} \end{cases}$$

3.2. Parabolic cases

The contribution of the parabolic classes to the trace of $\mathscr{M}_{n^2}$ is summarized in Proposition 21. The proof is divided into several steps.

Lemma 16. *The inequivalent parabolic elements in $\mathscr{M}_{n^2}$ are*

$$\begin{pmatrix} n & a \\ 0 & n \end{pmatrix}, \qquad a=1,\dots,n, (a,n)=1.$$

Proof. Since $\mathrm{SL}(2,\mathbb{Z})$ has only one inequivalent cusp ∞, a parabolic element is conjugate to

$$\pm\begin{pmatrix} n & b \\ 0 & n \end{pmatrix}$$

for some b with $(b,n)=1$. Since the stabilizer subgroup of ∞ inside $\mathrm{SL}(2,\mathbb{Z})$ is generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, we see that the inequivalent parabolic elements are given as in the statement. ∎

Lemma 17. *Let a be an integer relatively prime to n. The contribution of the class of $\left(\begin{smallmatrix} n & a \\ 0 & n \end{smallmatrix}\right)$ to the trace of $[\mathscr{M}_{n^2}^*]$ on $\mathcal{S}_{r,s}(1)$ is*

$$\begin{cases} -(r-12)/24 & \text{if } n=a=1, \\ -(-i)^{r(n-1)/2}\left(\frac{-a}{n}\right)\frac{e^{-2\pi i(r\ell+1)a/n}}{1-e^{-2\pi ia/n}} & \text{if } n>1, \end{cases}$$

where $\ell=(n^2-1)/24$.

Proof. Assume first that $n=a=1$. We have $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)^*=\left(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right),e^{2\pi ir/24}\right)$. Then the numbers μ, η, and u in the definition of $J(\gamma)$ in Proposition 14 are

$$\mu=\frac{24-r}{24},\qquad \eta=e^{2\pi ir/24},\qquad u=1,$$

respectively. Thus, the contribution to the trace is $-(r-12)/24$.

Now assume that $n>1$. We have $(n,a)=1$. Let α and β be integers such that $\alpha n+\beta a=1$ and $\beta>0$. We have

$$\begin{pmatrix}n&a\\0&n\end{pmatrix}=\begin{pmatrix}a&-\alpha\\n&\beta\end{pmatrix}\begin{pmatrix}-1&0\\0&-n^2\end{pmatrix}\begin{pmatrix}-n\beta&-1\\1&0\end{pmatrix}.$$

By Lemma 4, we have

$$\begin{pmatrix}a&-\alpha\\n&\beta\end{pmatrix}^*=\left(\begin{pmatrix}a&-\alpha\\n&\beta\end{pmatrix},\left(\frac{a}{n}\right)i^{r(1-n)/2}e^{2\pi ir(n(a+\beta)-3)/24}(n\tau+\beta)^{k+1/2}\right)$$

and

$$\begin{pmatrix}-n\beta&-1\\1&0\end{pmatrix}^*=\left(\begin{pmatrix}-n\beta&-1\\1&0\end{pmatrix},e^{2\pi ir(-n\beta-3)/24}\tau^{k+1/2}\right).$$

Then

$$\begin{pmatrix}-1&0\\0&-n^2\end{pmatrix}^*\begin{pmatrix}-n\beta&-1\\1&0\end{pmatrix}^*=\left(\begin{pmatrix}n\beta&1\\-n^2&0\end{pmatrix},e^{2\pi ir(-n\beta-3)/24}(n\tau)^{k+1/2}\right)$$

and

$$\begin{aligned}&\begin{pmatrix}n&a\\0&n\end{pmatrix}^*\\ &=\left(\begin{pmatrix}n&a\\0&n\end{pmatrix},\left(\frac{a}{n}\right)i^{r(1-n)/2}e^{2\pi ir(na-6)/24}\left(-\frac{1}{n\tau}\right)^{k+1/2}(n\tau)^{k+1/2}\right)\\ &=\left(\begin{pmatrix}n&a\\0&n\end{pmatrix},\left(\frac{a}{n}\right)e^{2\pi ir(n(a-3)+3)/24}\right).\end{aligned}\tag{5}$$

Now the stabilizer of ∞ is generated by $\left(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right),e^{2\pi ir/24}\right)$ and $\left(\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right),1\right)$. Thus, according to Proposition 14 and (5), the contribution of the class of $\left(\begin{smallmatrix}n&a\\0&n\end{smallmatrix}\right)$ to the trace is

$$-\frac{1}{2}\left(\frac{a}{n}\right)e^{-2\pi ir(n(a-3)+3)/24}e^{-2\pi ia(24-r)/(24n)}\big(1-i\cot(\pi a/n)\big).$$

Simplifying the expression, we get the lemma. $\square$

The proof of Proposition 21 involves sums of the form $\sum_{u\leq N/j}\chi(u)$ for some Dirichlet character χ. Here we recall a formula from [22, (6), p. 276] for such sums. Note that for a Dirichlet character χ modulo M, the generalized Bernoulli numbers $B_{m,\chi}$ are defined by the power series

$$\sum_{a=1}^{M}\frac{\chi(a)te^{at}}{e^{Mt}-1}=\sum_{m=0}^{\infty}B_{m,\chi}\frac{t^m}{m!}.$$

If χ is the trivial character modulo 1, then $B_{m,\chi}$ is just the Bernoulli numbers B_m. We have

$$B_{0,\chi}=\frac{1}{M}\sum_{a=1}^{M}\chi(a)=\begin{cases}\phi(M)/M & \text{if } \chi \text{ is principal,}\\ 0 & \text{otherwise.}\end{cases} \tag{6}$$

Also, $B_{1,\chi}=0$ if χ is an even character. Moreover, if χ is an imprimitive Dirichlet character induced from χ_1, then we have the relation

$$B_{1,\chi}=B_{1,\chi_1}\sum_{d|M}\mu(d)\chi_1(d)=B_{1,\chi_1}\prod_{p|M}\big(1-\chi_1(p)\big). \tag{7}$$

Also, if $d<0$ is a fundamental discriminant, then we have

$$B_{1,(d/\cdot)}=-H(d), \tag{8}$$

where $H(d)$ is the Hurwitz class number.

Lemma 18 ([22, (6), p. 276]). *Let M be a positive integer, and let χ be a Dirichlet character modulo M. Let $N>0$ be a multiple of M, and let j be a positive integer relatively prime to N. Then we have*

$$\sum_{0\leq u<N/j}\chi(u)=-B_{1,\chi}+\frac{\overline{\chi(j)}}{\phi(j)}\sum_{\psi \bmod j}\overline{\psi(-N)}B_{1,\chi\psi}(N),$$

where the sum runs over all Dirichlet characters ψ modulo j and

$$\begin{aligned}B_{1,\chi\psi}&(N)\\ &=B_{0,\chi\psi}N+B_{1,\chi\psi}\\ &=\begin{cases}B_{1,\chi\psi} & \text{if } \chi\psi \text{ is a nonprincipal character modulo } jM,\\ N\phi(jM)/jM & \text{if } \chi\psi \text{ is the principal character modulo } jM.\end{cases}\end{aligned}$$

Proof. If $M>1$, then the formula is just [22, (6), p. 276] with $m=1$. If $M=1$, that is, if $\chi(u)=1$ for all $u\in\mathbb{Z}$, then we should modify the definition of $\mathcal{L}_\chi(t)$ of [22, p. 274] to $\mathcal{L}_\chi(t)=\sum_{n=0}^\infty \chi(n)e^{nt}$. Then following the argument of [22, p. 274, up to (6)], we see that our formula holds. The details of the proof are omitted. ∎

LEMMA 19. *Let n be a positive odd integer greater than 1. If $n\equiv 1 \bmod 4$, then*

$$\sum_{a \bmod n}\left(\frac{a}{n}\right)\frac{1}{1-e^{2\pi ia/n}}=\begin{cases}\phi(n)/2 & \text{if } n \text{ is a square},\\ 0 & \text{otherwise.}\end{cases}$$

If $n\equiv 3 \bmod 4$, then

$$\sum_{a \bmod n}\left(\frac{a}{n}\right)\frac{1}{1-e^{2\pi ia/n}}=i\sqrt{n}H(-n).$$

Here $H(-n)$ is the Hurwitz class number, that is, $H(-3)=1/3$, $H(-4)=1/2$, and $H(-n)$ is the ideal class number of the quadratic order of the discriminant $-n$ if $n\neq 3,4$.

Proof. Assume that $n\equiv 1 \bmod 4$. Then $(\cdot/n)$ is an even function. Let S denote the sum. We have

$$2S=\sum_{a \bmod n}\left(\frac{a}{n}\right)\left(\frac{1}{1-e^{2\pi ia/n}}+\frac{1}{1-e^{-2\pi ia/n}}\right)=\sum_{a \bmod n}\left(\frac{a}{n}\right).$$

If n is a square, then $(\cdot/n)$ is the principal Dirichlet character modulo n and we have $S=\phi(n)/2$. If n is not a square, then $(\cdot/n)$ is a nonprincipal Dirichlet character modulo n and the sum S vanishes.

Now assume that $n\equiv 3 \bmod 4$. Set $z=e^{2\pi i/n}$. We first consider the case n is square-free. For an integer t, set

$$S(t)=\sum_{a \bmod n}\left(\frac{a}{n}\right)\frac{z^{ta}}{1-z^a},\qquad T(t)=\sum_{a \bmod n}\left(\frac{a}{n}\right)z^{ta}.$$

The two sums are related by

$$S(t)-S(t+1)=\sum_{a \bmod n}\left(\frac{a}{n}\right)\frac{z^{ta}(1-z^a)}{1-z^a}=\sum_{a \bmod n}\left(\frac{a}{n}\right)z^{ta}=T(t).$$

Since n is assumed to be square-free, we have $T(t)=(t/n)i\sqrt{n}$. It follows that

$$\begin{aligned}\sum_{t=1}^{n}\big(S(0)-S(t)\big)&=\sum_{t=1}^{n}\sum_{a=0}^{t-1}T(a)\\&=\sum_{a=0}^{t-1}T(a)(n-a)\\&=i\sqrt{n}\sum_{a=0}^{n-1}\Big(\frac{a}{n}\Big)(n-a)\\&=in\sqrt{n}H(-n).\end{aligned}$$

Since

$$\sum_{t=1}^{n}S(t)=\sum_{a=1}^{n}\Big(\frac{a}{n}\Big)\frac{1+z^a+\cdots+z^{(n-1)a}}{1-z^a}=0,$$

we conclude that $S=S(0)=i\sqrt{n}H(-n)$ for the case n is square-free.

Now if n is not square-free, we write n as m^2n_0 with square-free n_0. Then using the partial fraction decomposition

$$\frac{\ell}{1-x^\ell}=\sum_{j=0}^{\ell-1}\frac{1}{1-xe^{2\pi ij/\ell}},$$

we get

$$\begin{aligned}S&=\sum_{d|m}\mu(d)\sum_{a=1}^{n/d}\Big(\frac{ad}{n_0}\Big)\frac{1}{1-z^{ad}}\\&=\sum_{d|m}\mu(d)\Big(\frac{d}{n_0}\Big)\sum_{a=1}^{n_0-1}\Big(\frac{a}{n_0}\Big)\sum_{j=0}^{n/dn_0-1}\frac{1}{1-e^{2\pi id(a+jn_0)/n}}\\&=\sum_{d|m}\mu(d)\Big(\frac{d}{n_0}\Big)\sum_{a=1}^{n_0}\Big(\frac{a}{n_0}\Big)\frac{n/dn_0}{1-e^{2\pi ia/n_0}}\\&=im^2\sqrt{n_0}H(-n_0)\sum_{d|m}\frac{\mu(d)}{d}\Big(\frac{d}{n_0}\Big),\end{aligned}$$

where in the last step we use the result for the square-free case computed earlier. Now recall that, for $d>0$ with $d\equiv 0,3 \bmod 4$, the Hurwitz class numbers $H(-m^2D)$ and $H(-D)$ are related by the formula

$$\begin{aligned} H(-m^2D) &= H(-D)m\prod_{p|m}\left(1-\frac{1}{p}\left(\frac{-D}{p}\right)\right) \\ &= H(-D)m\sum_{d|m}\frac{\mu(d)}{d}\left(\frac{-D}{d}\right). \end{aligned} \tag{9}$$

Therefore,

$$S = im\sqrt{n_0}H(-n) = i\sqrt{n}H(-n).$$

This completes the proof of the lemma. ∎

Lemma 20. *For a positive integer $n>1$ with $(n,6)=1$, let $\ell=(n^2-1)/24$. If $n\equiv 1 \bmod 4$, then*

$$\begin{aligned} &\frac{1}{\sqrt{n}}\sum_{a \bmod n}\left(\frac{a}{n}\right)\frac{e^{2\pi i(r\ell+1)a/n}}{1-e^{2\pi ia/n}} \\ &\quad= -\frac{1}{8}\left(\frac{24}{n}\right)\sum_{u=-3,-4,-8,-24}\left(\frac{u}{r}\right)\left(1-\left(\frac{un}{2}\right)\right)\left(1-\left(\frac{un}{3}\right)\right)H(un). \end{aligned} \tag{10}$$

If $n\equiv 3 \bmod 4$, then

$$\begin{aligned} &\frac{1}{i\sqrt{n}}\sum_{a \bmod n}\left(\frac{a}{n}\right)\frac{e^{2\pi i(r\ell+1)a/n}}{1-e^{2\pi ia/n}} \\ &\quad= \frac{1}{8}\left(\frac{24}{n}\right)\sum_{u=1,8,12,24}\left(\frac{u}{r}\right)\left(1-\left(\frac{-un}{2}\right)\right)\left(1-\left(\frac{-un}{3}\right)\right)H(-un). \end{aligned} \tag{11}$$

Proof. Let S denote the sum in question, and for a positive integer ℓ, let z_ℓ denote the ℓth primitive root of unity $e^{2\pi i/\ell}$. Also, write n as $n=m^2n_0$ with square-free n_0. We have, say,

$$\begin{aligned} S &= -\sum_{a \bmod n}\left(\frac{a}{n}\right)\frac{1-z_n^{(r\ell+1)a}}{1-z_n^a}+\sum_{a \bmod n}\left(\frac{a}{n}\right)\frac{1}{1-z_n^a} \\ &= -\sum_{t=0}^{r\ell}\sum_{a \bmod n}\left(\frac{a}{n}\right)z_n^{ta}+\sum_{a \bmod n}\left(\frac{a}{n}\right)\frac{1}{1-z_n^a} = -\sum_{t=0}^{r\ell}T(t)+S'. \end{aligned} \tag{12}$$

The sum S' has been evaluated in Lemma 19. We now consider $T(t)$. We have

$$\begin{aligned} T(t) &= \sum_{d|m} \mu(d) \sum_{a \bmod n/d} \left(\frac{ad}{n_0}\right) z_n^{tad} \\ &= \sum_{d|m} \mu(d) \left(\frac{d}{n_0}\right) \sum_{a \bmod n_0} \left(\frac{a}{n_0}\right) z_n^{tad} \sum_{b=0}^{m^2/d-1} z_{m^2/d}^{bt}. \end{aligned}$$

The inner sum is 0 if $(m^2/d) \nmid t$. Then

$$\begin{aligned} T(t) &= m^2 \sum_{d|m,(m^2/d)|t} \frac{\mu(d)}{d} \left(\frac{d}{n_0}\right) \sum_{a \bmod n_0} \left(\frac{a}{n_0}\right) z_{n_0}^{at/(m^2/d)} \\ &= \epsilon m^2 \sqrt{n_0} \sum_{d|m,(m^2/d)|t} \frac{\mu(d)}{d} \left(\frac{d}{n_0}\right) \left(\frac{t/(m^2/d)}{n_0}\right), \end{aligned} \tag{13}$$

where

$$\epsilon = \begin{cases} 1 & \text{if } n \equiv 1 \bmod 4, \\ i & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

From (13), we obtain

$$\sum_{t=0}^{r\ell} T(t) = \epsilon m^2 \sqrt{n_0} \sum_{d|m} \frac{\mu(d)}{d} \left(\frac{d}{n_0}\right) \sum_{0 \le u \le r\ell/(m^2/d)} \left(\frac{u}{n_0}\right).$$

Now $\ell = (n^2-1)/24$, and r is assumed to be in the range $0 < r < 24$. Therefore, the sum above can also be written as

$$\sum_{t=0}^{r\ell} T(t) = \epsilon m^2 \sqrt{n_0} \sum_{d|m} \frac{\mu(d)}{d} \left(\frac{d}{n_0}\right) \sum_{0 \le u < (rn^2d/m^2)/24} \left(\frac{u}{n_0}\right). \tag{14}$$

Now we apply Lemma 18 to (14) with $\chi = \chi_{n_0} = (\cdot/n_0)$, $M = n_0$, $N = rn^2d/m^2$, and $j = 24$. We consider the three cases

(1) $n_0 = 1$,
(2) $n \equiv 1 \bmod 4$ and $n_0 \neq 1$,
(3) $n \equiv 3 \bmod 4$,

separately.

When $n_0 = 1$, an application of Lemma 18 yields

$$\begin{aligned} \sum_{t=0}^{r\ell} T(t) &= m^2 \sum_{d|m} \frac{\mu(d)}{d} \Big(-B_1 + \frac{1}{8} \sum_{\psi \bmod 24} \overline{\psi(-rm^2 d)} B_{1,\psi}(rm^2 d) \Big) \\ &= \frac{1}{2} m\phi(m) + \frac{1}{8} m^2 \sum_{d|m} \frac{\mu(d)}{d} \sum_{\psi \bmod 24} \overline{\psi(-rm^2 d)} B_{1,\psi}(rm^2 d). \end{aligned} \tag{15}$$

If we let χ_0 denote the principal Dirichlet character modulo 24, then the Dirichlet characters ψ modulo 24 are given by

$$\psi_u = \chi_0(\cdot) \Big(\frac{u}{\cdot} \Big), \quad u = 1, 8, 12, 24, -3, -4, -8, -24. \tag{16}$$

By (6), (7), and (8), we have

$$\begin{aligned} &B_{1,\psi_u}(rm^2 d) \\ &= \begin{cases} rm^2 d\phi(24)/24 = rm^2 d/3 & \text{if } u = 1, \\ 0 & \text{if } u = 8, 12, 24, \\ -(1 - (\frac{u}{2}))(1 - (\frac{u}{3}))H(u) & \text{if } u = -3, -4, -8, -24. \end{cases} \end{aligned} \tag{17}$$

The contribution from the character ψ_1 to (15) is

$$\frac{rm^4}{24} \sum_{d|m} \mu(d) = 0, \tag{18}$$

since $m = \sqrt{n}$ is assumed to be greater than 1. The contributions from ψ_u, for $u = -3, -4, -8, -24$, are

$$\begin{aligned} &\frac{m^2}{8} \Big(\frac{u}{r} \Big) \Big(1 - \Big(\frac{u}{2} \Big) \Big) \Big(1 - \Big(\frac{u}{3} \Big) \Big) H(u) \sum_{d|m} \frac{\mu(d)}{d} \Big(\frac{u}{d} \Big) \\ &= \frac{m}{8} \Big(\frac{u}{r} \Big) \Big(1 - \Big(\frac{u}{2} \Big) \Big) \Big(1 - \Big(\frac{u}{3} \Big) \Big) H(un), \end{aligned} \tag{19}$$

where we have utilized (9). Combining (12), (15), (17), (18), and (19) and using the formula for S' given in Lemma 19, we get formula (10) for the case $n = m^2$.

Now let us consider the case $n \equiv 1 \bmod 4$ but not a perfect square. That is, $n = m^2 n_0$ with square-free $n_0 \neq 1$ and $n_0 \equiv 1 \bmod 4$. In this case, we have $B_{1,(\cdot/n_0)} = B_{1,(\cdot/n_0)\psi_u} = 0$ for $u = 1, 8, 12, 24$, where ψ_u are defined by (16). Then an application of Lemma 18 to (14) yields

$$\begin{aligned}\sum_{t=0}^{r\ell} T(t) =& \frac{m^2\sqrt{n_0}}{8}\sum_{d|m}\frac{\mu(d)}{d}\Big(\frac{d}{n_0}\Big)\Big(\frac{24}{n_0}\Big) \\ & \times \sum_{u=-3,-4,-8,-24} \psi_u(-rn^2d/m^2)B_{1,(\cdot/n_0)\psi_u} \\ =& -\frac{m\sqrt{n}}{8}\Big(\frac{24}{n}\Big)\sum_{d|m}\frac{\mu(d)}{d}\Big(\frac{n_0}{d}\Big) \\ & \times \sum_{u=-3,-4,-8,-24}\Big(\frac{u}{rd}\Big)B_{1,(n_0u/\cdot)}\Big(1-\Big(\frac{un_0}{2}\Big)\Big)\Big(1-\Big(\frac{un_0}{3}\Big)\Big),\end{aligned}$$

where we have used (7). Then by (8) and (9), we get

$$\sum_{t=0}^{r\ell} T(t) = \frac{\sqrt{n}}{8}\Big(\frac{24}{n}\Big)\sum_{u=-3,-4,-8,-24}\Big(\frac{u}{r}\Big)\Big(1-\Big(\frac{un}{2}\Big)\Big)\Big(1-\Big(\frac{un}{3}\Big)\Big)H(un).$$

This gives the evaluation of the sum of $T(t)$ in (12). The term S' in (12) is shown to be 0 in Lemma 19. This establishes (10) for the case n is not a square.

Now assume that $n \equiv 3 \bmod 4$. Then $B_{1,(\cdot/n)\psi_u} = 0$ for $u = -3, -4, -8, -24$, and an application of Lemma 18 to (14) gives us

$$\begin{aligned}\sum_{t=0}^{r\ell} T(t) = im\sqrt{n}\sum_{d|m}\frac{\mu(d)}{d}\Big(\frac{d}{n_0}\Big)\Big(&-B_{1,(\cdot/n_0)} \\ &+\frac{1}{8}\Big(\frac{24}{n_0}\Big)\sum_{u=1,8,12,24}\psi_u(-rn^2d/m^2)B_{1,(\cdot/n_0)\psi_u}\Big).\end{aligned}$$

Using (7), (8), and (9) again, we get

$$\begin{aligned}\sum_{t=0}^{r\ell} T(t) = i\sqrt{n}\Big(&H(-n) \\ &-\frac{1}{8}\Big(\frac{24}{n}\Big)\sum_{u=1,8,12,24}\Big(\frac{u}{r}\Big)\Big(1-\Big(\frac{-un}{2}\Big)\Big)\Big(1-\Big(\frac{-un}{3}\Big)\Big)H(-un)\Big).\end{aligned}$$

Combining this, (12), and Lemma 19, we arrive at the claimed formula. This completes the proof of the lemma. ∎

Proposition 21. *The total contribution of the parabolic classes of $\mathscr{M}_{n^2}$ to the trace of the linear operator $[\mathscr{M}_{n^2}^*]$ on $\mathcal{S}_{r,s}(1)$ is*

$$\frac{\sqrt{n}}{8}\left(\frac{12}{n}\right)\sum_{e=1,2,3,6}\left(\frac{-4e}{r}\right)\left(1-\left(\frac{-en}{3}\right)\right)\big(H(-4en)-H(-en)\big),$$

where, for a negative integer $-d$, we let $H(-d)$ denote the Hurwitz class number of the imaginary quadratic order of the discriminant $-d$. (If $-d$ is not a discriminant, then set $H(-d)=0$.)

Proof. We first consider the case $n=1$. By Lemmas 16 and 17, we find that the total contribution is $-(r-12)/24$. We then verify case by case that

$$-\frac{r-12}{24}=\frac{1}{8}\sum_{e=1,2,3,6}\left(\frac{-4e}{r}\right)\left(1-\left(\frac{-e}{3}\right)\right)\big(H(-4e)-H(-e)\big).$$

We next consider the cases $n>1$. Again, using Lemmas 16 and 17, we find that the total contribution is

$$-(-i)^{r(n-1)/2}\sum_{a=1}^{n}\left(\frac{-a}{n}\right)\frac{e^{-2\pi i(r\ell+1)a/n}}{1-e^{-2\pi ia/n}}.$$

Now for $n\equiv 1 \bmod 4$, we have

$$(-i)^{r(n-1)/2}=\left(\frac{8}{n}\right).$$

Thus, by (10), the contribution to the trace is

$$\frac{\sqrt{n}}{8}\left(\frac{12}{n}\right)\sum_{u=-3,-4,-8,-24}\left(\frac{u}{r}\right)\left(1-\left(\frac{un}{2}\right)\right)\left(1-\left(\frac{un}{3}\right)\right)H(un).$$

Since $H(-12n)=H(-3n)(2-(-3n/2))$, $H(-n)=H(-2n)=H(-6n)=0$, the formula above is equal to that in the statement of the proposition.

Now assume that $n\equiv 3 \bmod 4$. We have

$$i(-i)^{r(n-1)/2}=-\left(\frac{-4}{r}\right)\left(\frac{8}{n}\right).$$

Then from Lemma 17, (11), and $H(-4n)=H(-n)(2-(-n/2))$, we get the claimed formula. This proves the proposition. ∎

3.3. Hyperbolic cases

LEMMA 22. *A complete set of representatives of inequivalent hyperbolic elements in $\mathscr{M}_{n^2}$ whose fixed points are cusps is*

$$\left\{ \pm \begin{pmatrix} a & b+ma \\ 0 & d \end{pmatrix} : \right.$$

$$\left. b = 1, \dots, h, (b,h) = 1, m = 1, \dots, (d-a)/h \text{ with } h = (a,d) \right\},$$

where in the set we let a and d run over all integers satisfying $ad = n^2$ and $0 < a < d$.

Proof. Let M be a hyperbolic element in $\mathscr{M}_{n^2}$ whose fixed points are cusps. Then the eigenvalues of M are two integers a and d satisfying $ad = n^2$. Without loss of generality, we assume that $|d| > |a| > 0$. Let α be the cusp corresponding to the eigenvalue a, and choose an element $\gamma \in \mathrm{SL}(2, \mathbb{Z})$ such that $\sigma\infty = \alpha$. Then

$$\sigma M \sigma^{-1} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

for some integer b. The integer b must satisfy $(a, b, d) = 1$. In other words, a hyperbolic element in $\mathscr{M}_{n^2}$ whose fixed points are cusps is conjugate to a matrix of the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with $ad = n^2$, $|d| > |a| > 0$, and $(b, (a, d)) = 1$. It is clear that if two such matrices $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right)$ are conjugate, then we must have $a = a'$ and $d = d'$. Furthermore, by considering the eigenvector $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, it is easy to see that two matrices $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a & b' \\ 0 & d \end{smallmatrix}\right)$ are conjugate if and only if they are conjugate by $\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$ for some $m \in \mathbb{Z}$, that is, if and only if $(d-a) \mid (b'-b)$. With this information, it is straightforward to verify that the set in the lemma is a complete set of representatives. This proves the lemma. □

PROPOSITION 23. *The total contribution of hyperbolic elements in $\mathscr{M}_{n^2}$ to the trace is* 0.

Proof. By Shimura's formula, the contribution from hyperbolic elements whose fixed points are not cusps is 0. Furthermore, by Lemma 22, a complete set of inequivalent hyperbolic elements whose fixed points are cusps is

$$\left\{ \pm \begin{pmatrix} a & b+ma \\ 0 & d \end{pmatrix} : \right.$$

$$\left. b = 1, \dots, h, (b,h) = 1, m = 1, \dots, (d-a)/h \text{ with } h = (a,d) \right\},$$

where in the set we let a and d run over all integers satisfying $ad=n^2$ and $0<a<d$. Now we have

$$\begin{pmatrix} a & b+ma \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}.$$

Thus, if the contribution from the conjugacy class $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ is A, then the contribution from the class $\left(\begin{smallmatrix} a & b+ma \\ 0 & d \end{smallmatrix}\right)$ is $e^{2\pi irm/24}A$. Since $24 \mid (d-a)/h$, as m runs from 1 to $(d-a)/h$, the contributions from the classes $\left(\begin{smallmatrix} a & b+ma \\ 0 & d \end{smallmatrix}\right)$ cancel out each other. We therefore conclude that the total contribution from hyperbolic elements is 0. $\square$

3.4. Elliptic cases

The contribution of elliptic classes to the trace of $[\mathscr{M}_{n^2}^*]$ will be calculated according to the greatest common divisor of 24 and the trace $t=a+d$ of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathscr{M}_{n^2}$. After relating conjugacy classes of $\mathscr{M}_{n^2}^*$ with equivalence classes of quadratic forms in Section 3.4.1 and some preliminary computation in Section 3.4.2, we evaluate the contribution of elliptic classes case by case. The computation is tedious. Here we will give details only for the case $(t,24)=1$ and a subcase of the case $(t,24)=2$. For the calculation in other cases, we refer the reader to the arXiv version of this paper (see `arXiv:1110.1810v1` [math.NT]).

3.4.1. Quadratic forms. Assume that $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathscr{M}_{n^2}$ is elliptic. Then $t=a+d$, $f=(b,c,d-a)$, and $\operatorname{sgn} c$ are invariants under conjugation by elements of $\mathrm{SL}(2,\mathbb{Z})$. This can be seen from the one-to-one correspondence between the set

$$\Gamma_{n,t,f} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathscr{M}_{n^2} : a+d=t, f=(b,c,d-a), c>0 \right\}$$

and the set $\mathcal{Q}_{(t^2-4n^2)/f^2}$ of all primitive positive definite quadratic forms of the discriminant $(t^2-4n^2)/f^2$, where

$$\mathcal{Q}_D = \{Ax^2+Bxy+Cy^2 : B^2-4AC=D, A>0\},$$

and the correspondence is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longleftrightarrow \frac{1}{f}\left(cx^2+(d-a)xy-by^2\right). \tag{20}$$

Elements γ of $\mathrm{SL}(2,\mathbb{Z})$ act on $\Gamma_{n,t,f}$ by conjugation and on $\mathcal{Q}_{(t^2-4n^2)/f^2}$ by change of variable $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \gamma\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. Moreover, the two group actions are compatible with respect to the correspondence above.

Lemma 24. *The total contribution of elliptic elements in* $\mathscr{M}^*_{n^2}$ *to the trace of the linear operator* $[\mathscr{M}^*_{n^2}]$ *on* $\mathcal{S}_{r,s}(1)$ *is*

$$2\sum_{t,f}\sum_{\gamma\in\Gamma_{n,t,f}/\operatorname{SL}(2,\mathbb{Z})} J(\gamma),$$

where the outer sum runs over all integers t *with* $t^2<4n^2$ *and all positive integers* f *such that* $(t^2-4n^2)/f^2\equiv 0,1 \bmod 4$*; the inner sum runs over all class representatives of* $\Gamma_{n,t,f}$ *under conjugation by* $\operatorname{SL}(2,\mathbb{Z})$*; and* $J(\gamma)$ *is defined as in Definition 13.*

Proof. From the discussion preceding the lemma, we easily see that every elliptic element in $\mathscr{M}_{n^2}$ falls in precisely one of $\Gamma_{n,t,f}$ and $-\Gamma_{n,t,f}$. Also, it is clear that if $\gamma_1,\dots,\gamma_m$ are class representatives of $\Gamma_{n,t,f}$, then $-\gamma_1,\dots,-\gamma_m$ are class representatives of $-\Gamma_{n,t,f}$. We now show that $J(-\gamma)=J(\gamma)$. Then the lemma follows.

Assume that $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_{n,t,f}$ and that

$$\gamma^*=\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}, u(c\tau+d)^{k+1/2}\right).$$

Then by Proposition 14, the contribution of the class of γ to the trace is

$$J(\gamma)=\frac{\rho^{1/2-k}}{wu(\rho-\overline{\rho})},$$

where $\rho=(t+\sqrt{t^2-4n^2})/2$. Now we have

$$\begin{aligned}(-\gamma)^*&=\left(\begin{pmatrix} -a & -b\\ -c & -d\end{pmatrix}, u(c\tau+d)^{k+1/2}\right)\\ &=\left(\begin{pmatrix} -a & -b\\ -c & -d\end{pmatrix}, ue^{2\pi i(2k+1)/4}(-c\tau-d)^{k+1/2}\right).\end{aligned}$$

Thus,

$$J(-\gamma)=\frac{(-\rho)^{1/2-k}}{wue^{2\pi i(2k+1)/4}(-\rho+\overline{\rho})}=\frac{e^{2\pi i(2k-1)/4}\rho^{1/2-k}}{wue^{2\pi i(2k+1)/4}(-\rho+\overline{\rho})}=J(\gamma).$$

This proves the lemma. ∎

Remark 25. Since for $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathscr{M}_{n^2}$ we have $(a,b,c,d)=1$, the integers n, t, and f need to satisfy the following conditions.

(1) The common divisor (n,t,f) must be 1.
(2) The integer (t,f) can only be 1 or 2. Moreover, if $(t,f)=2$, then because $(t^2-4n^2)/4$ is a discriminant, we must have $2\mid t$, but $4\nmid t$.
(3) The integer $(f,t+2n,t-2n)$ can only be 1, 2, or 4, and if the latter two cases occur, then $2\mid t$, but $4\nmid t$.

We first choose a suitable representative γ in each conjugacy class in $\Gamma_{n,t,f}$ and determine γ^*.

Lemma 26. *Each conjugacy class of* $\Gamma_{n,t,f}$ *contains an element* $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ *such that* $c=fp$ *for some prime* $p>4n$ *and* $(c,d)=1$.

Moreover, in the case $2\mid f$ *(which occurs only when* $2\parallel t$*), we may further require that* d *satisfies the congruences*

$$d\equiv\begin{cases} t' \bmod 8 & \text{if } 2\parallel f,\\ t' \bmod 16 & \text{if } 4\parallel f,\\ t' \bmod 8 & \text{if } 8\parallel f \text{ and } (t^2-4n^2)/64 \text{ is even},\\ t'+4 \bmod 8 & \text{if } 8\parallel f \text{ and } (t^2-4n^2)/64 \text{ is odd},\\ t' \bmod 8 & \text{if } 16\mid f,\end{cases}$$

where $t'=t/2$.

Proof. It is well known that a primitive positive definite quadratic form represents infinitely many primes, which implies that each class of quadratic forms in $\mathcal{Q}_{(t^2-4n^2)/f^2}$ contains elements of the form $px^2+uxy+vy^2$ for infinitely many primes p. The matrix corresponding to this form under (20) is

$$\begin{pmatrix}(t-fu)/2 & -fv\\ fp & (t+fu)/2\end{pmatrix}.$$

Now if $p>4n$, then the relation

$$f^2(u^2-4pv)=t^2-4n^2$$

implies that $p\nmid(t+fu)$, and hence $(fp,(t+fu)/2)=(f,(t+fu)/2)$. By Remark 25, this can only be 1 or 2. However, because the determinant of the matrix is the odd integer n^2, $(fp,(t+fu)/2)$ cannot be even. That is, $(fp,(t+fu)/2)=1$. This proves the first part of the statement.

Now assume that $2\parallel f$. From

$$\begin{pmatrix}1 & -u\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}=\begin{pmatrix}a-cu & -cu^2+u(a-d)+b\\ c & d+cu\end{pmatrix},\tag{21}$$

it is clear that we can further assume that $d\equiv t' \bmod 8$.

Assume that $4 \parallel f$. Since $32 \mid (t^2 - 4n^2)$ and $(t^2 - 4n^2)/16$ are discriminant, we must have $64 \mid (t^2 - 4n^2)$. Then the equality $t^2 - 4n^2 = (d-a)^2 + 4bc$ shows that $64 \mid (d-a)^2$, that is, that $8 \mid (d-a)$. Then

$$d = \frac{1}{2}\big(t + (d-a)\big) \equiv t' \bmod 4.$$

By (21) again, we may assume that $d \equiv t' \bmod 16$.

Assume that $8 \parallel f$. We have $64 \mid (t^2 - 4n^2)$ and $8 \mid b, c$. Then $(d-a)^2 = t^2 - 4n^2 - 4bc \equiv (t^2 - 4n^2) \bmod 256$. If $(t^2 - 4n^2)/64$ is even, then $d \equiv a \bmod 16$ and $d = (t + d - a)/2 \equiv t' \bmod 8$. If $(t^2 - 4n^2)/64$ is odd, then $d \equiv a + 8 \bmod 16$ and $d \equiv t' + 4 \bmod 8$.

Finally, if $16 \mid f$, by a computation similar to the case $8 \parallel f$ we find that $d \equiv t' \bmod 8$. This proves the lemma. ∎

For our purpose, we also need to recall some properties of genus characters of integral binary quadratic forms. Let $D < 0$ be a discriminant; that is, $D \equiv 0, 1 \bmod 4$. For each odd prime divisor p of D, one can associate to $\mathcal{Q}_D$ a character by

$$\chi : Ax^2 + Bxy + Cy^2 \longmapsto \left(\frac{A}{p}\right) = \left(\frac{p^*}{A}\right), \quad p^* = (-1)^{(p-1)/2} p.$$

For $D \equiv 0 \bmod 4$, there may also exist characters of the form

$$Ax^2 + Bxy + Cy^2 \longmapsto \left(\frac{u}{A}\right), \quad u \in \{-4, 8, -8\},$$

depending on the residue of $D/4$ modulo 8. Any product of these characters is called a *genus character*. All genus characters can be written as

$$\chi_{D_1} : Ax^2 + Bxy + Cy^2 \longmapsto \left(\frac{D_1}{A}\right)$$

for some (positive or negative) divisor D_1 of D such that D_1 and D/D_1 are both discriminants and vice versa. General properties of genus characters can be found in [4, Chapter 1]. Here we only quote some properties relevant to our calculation of traces.

LEMMA 27. *Let D and D_1 be as above, and let $\chi_{D_1} : \mathcal{Q}_D \to \{\pm 1\}$ be a genus character. Then we have the following properties.*

(1) *The value of χ_{D_1} for a quadratic form $Ax^2 + Bxy + Cy^2 \in \mathcal{Q}_D$ depends only on the genus in which the quadratic form lies. In particular, every quadratic form in the same class takes the same value of χ_{D_1}.*

(2) *We have that* χ_{D_1} *is a trivial character (i.e., mapping every quadratic form to* 1*) if and only if* D_1 *or* D/D_1 *is a square.*
(3) *The sum of* χ_{D_1} *over a complete set of class representatives of* $\mathcal{Q}_D/\mathrm{SL}(2,\mathbb{Z})$ *is*

$$\sum_{Ax^2+Bxy+Cy^2\in\mathcal{Q}_D/\mathrm{SL}(2,\mathbb{Z})}\Big(\frac{D_1}{A}\Big)=\begin{cases}h(D) & \text{if } D_1 \text{ or } D/D_1 \text{ is a square},\\ 0 & \text{otherwise},\end{cases}$$

where $h(D)$ *is the number of classes in* $\mathcal{Q}_D$*.*

Finally, the following formula frequently occurs in our computation.

LEMMA 28. *Let* D *be the discriminant of an imaginary quadratic order. Let* u *be a positive integer. Then we have*

$$\sum_{f|u}\Big(\frac{D}{f}\Big)H\Big(\frac{u^2}{f^2}D\Big)=uH(D).$$

Proof. We have

$$\begin{aligned}\sum_{f|u}\Big(\frac{D}{f}\Big)H\Big(\frac{u^2}{f^2}D\Big)&=H(D)\sum_{f|u}\Big(\frac{D}{f}\Big)\frac{u}{f}\sum_{m|(u/f)}\frac{\mu(m)}{m}\Big(\frac{D}{m}\Big)\\&=H(D)\sum_{n|u}\Big(\frac{D}{n}\Big)\frac{u}{n}\sum_{m|n}\mu(m).\end{aligned}$$

The sum $\sum_{m|n}\mu(m)$ is nonzero only when $n=1$. From this, we get the claimed formula. ∎

3.4.2. Preliminary calculation and notation.

LEMMA 29. *If* $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathscr{M}_{n^2}$ *satisfies* $(c,d)=1$ *and* $c>0$*, then*

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}^*=\left(\begin{pmatrix}a&b\\c&d\end{pmatrix},n^{-k-1/2}\epsilon(a,b,c,d)^r(c\tau+d)^{k+1/2}\right),$$

where

$$\epsilon(a,b,c,d)=\begin{cases}(\frac{d}{c})e^{2\pi i(bd(1-c^2)+c(a+d-3))/24} & \text{if } c \text{ is odd},\\ (\frac{c}{d})e^{2\pi i(ac(1-d^2)+d(b-c+3)-3)/24} & \text{if } c \text{ is even}.\end{cases}$$

Proof. Let $\alpha, \beta \in \mathbb{Z}$ be integers such that $\alpha d + \beta c = 1$. We have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a\beta + b\alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & -\beta \\ c & d \end{pmatrix}.$$

Now

$$\begin{pmatrix} \alpha & -\beta \\ c & d \end{pmatrix}^* = \left(\begin{pmatrix} \alpha & -\beta \\ c & d \end{pmatrix}, \epsilon(\alpha, -\beta, c, d)^r (c\tau + d)^{k+1/2} \right)$$

and

$$\begin{pmatrix} 1 & a\beta + b\alpha \\ 0 & 1 \end{pmatrix}^* = \left(\begin{pmatrix} 1 & a\beta + b\alpha \\ 0 & 1 \end{pmatrix}, e^{2\pi i r(a\beta + b\alpha)/24} \right).$$

Then from

$$\begin{pmatrix} 1 & a\beta + b\alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & -\beta \\ c & d \end{pmatrix} = \begin{pmatrix} a + (1 - n^2)\alpha & b + (n^2 - 1)\beta \\ c & d \end{pmatrix}$$

we deduce that

$$\begin{aligned} e^{2\pi i r(a\beta + b\alpha)/24} \epsilon(\alpha, -\beta, c, d)^r &= \epsilon\big(a + (1 - n^2)\alpha, b + (n^2 - 1)\beta, c, d\big)^r \\ &= \epsilon(a, b, c, d)^r. \end{aligned}$$

Then the lemma follows. ∎

All the 24th roots of unity can be expressed in terms of Jacobi symbols, which we give in the next lemma for future reference.

Lemma 30. *If* $(t, 24) = 1$, *then*

$$e^{2\pi i t/24} = \frac{\sqrt{2}}{4}\left(\frac{8}{t}\right) + \frac{\sqrt{6}}{4}\left(\frac{24}{t}\right) - \frac{i\sqrt{2}}{4}\left(\frac{-8}{t}\right) + \frac{i\sqrt{6}}{4}\left(\frac{-24}{t}\right).$$

If $(t, 12) = 1$, *then*

$$e^{2\pi i t/12} = \frac{\sqrt{3}}{2}\left(\frac{12}{t}\right) + \frac{i}{2}\left(\frac{-4}{t}\right).$$

If $(t, 8) = 1$, *then*

$$e^{2\pi i t/8} = \frac{\sqrt{2}}{2}\left(\frac{8}{t}\right) + \frac{i\sqrt{2}}{2}\left(\frac{-8}{t}\right).$$

If $(t, 6) = 1$, *then*

$$e^{2\pi i t/6} = \frac{1}{2} + \frac{i\sqrt{3}}{2}\left(\frac{-3}{t}\right).$$

If $(t,4)=1$, *then*

$$e^{2\pi it/4}=i\Big(\frac{-4}{t}\Big).$$

If $(t,3)=1$, *then*

$$e^{2\pi it/3}=-\frac{1}{2}+\frac{i\sqrt{3}}{2}\Big(\frac{-3}{t}\Big).$$

Let class representatives $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $\Gamma_{n,t,f}$ be chosen as in Lemma 26. Then by Proposition 14 and Lemma 29, the contribution of the class of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to the trace is

$$J(\gamma)=\frac{n^{k+1/2}}{w_{n,t,f}}\epsilon(a,b,c,d)^{-r}\frac{\rho^{1/2-k}}{\rho-\overline{\rho}}, \tag{22}$$

where

$$w_{n,t,f}=\begin{cases} 6 & \text{if } (t^2-4n^2)/f^2=-3, \\ 4 & \text{if } (t^2-4n^2)/f^2=-4, \\ 2 & \text{otherwise} \end{cases} \tag{23}$$

is the number of elements in $\mathrm{SL}(2,\mathbb{Z})$ commuting with γ,

$$\epsilon(a,b,c,d)=\Big(\frac{d}{c}\Big)e^{-2\pi ic/8}e^{2\pi i(bd(1-c^2)+ct)/24}$$

and $\rho=(t+\sqrt{t^2-4n^2})/2$. In view of Lemma 30, sums of the form $\sum(d/c)\times(e/c)$, $e=\pm1,\pm2,\pm3,\pm6$, will appear frequently in our computation. So we first compute such sums.

Notation 31. For $e=\pm1,\pm2,\pm3,\pm6$ and nonnegative integers ℓ, we set

$$\begin{aligned} L_{e,\ell}(n,t):=\frac{\rho^{1/2-k}}{\rho-\overline{\rho}}\Big(&\sum_{f:2^\ell\|f,(n,t,f)=1}\frac{1}{w_{n,t,f}}\sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_{n,t,f}/\mathrm{SL}(2,\mathbb{Z})}\Big(\frac{d}{c'}\Big)\Big(\frac{e}{c'}\Big) \\ &-3\sum_{f:2^\ell\|f,3|f,(n,t,f)=1}\frac{1}{w_{n,t,f}}\sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_{n,t,f}/\mathrm{SL}(2,\mathbb{Z})}\Big(\frac{d}{c'}\Big)\Big(\frac{e}{c'}\Big)\Big), \end{aligned} \tag{24}$$

where in the inner sums, the representatives $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ are chosen according to Lemma 26, $c'=c/2^\ell$ (i.e., c' is the odd part of c), and $\rho=(t+\sqrt{t^2-4n^2})/2$.

Moreover, for $e=1,2,3,6$ and integers u, we set

$$\begin{aligned} M_{e,\ell}(n,u) := \frac{(\tau/\sqrt{e})^{1-2k}}{\tau-\overline{\tau}}\Big(&\sum_{g:2^\ell \| g,(n,u,g)=1} H\Big(\frac{e^2u^2-4en}{g^2}\Big) \\ &-3\sum_{g:2^\ell\| g, 3|g,(n,u,g)=1} H\Big(\frac{e^2u^2-4en}{g^2}\Big)\Big), \end{aligned} \tag{25}$$

and

$$\begin{aligned} M'_{e,\ell}(n,u) := \frac{(\overline{\tau}/\sqrt{e})^{1-2k}}{\tau-\overline{\tau}}\Big(&\sum_{g:2^\ell \| g,(n,u,g)=1} H\Big(\frac{e^2u^2-4en}{g^2}\Big) \\ &-3\sum_{g:2^\ell\| g, 3|g,(n,u,g)=1} H\Big(\frac{e^2u^2-4en}{g^2}\Big)\Big), \end{aligned} \tag{26}$$

where $\tau=(eu+\sqrt{e^2u^2-4en})/2$ and where g runs over all positive integers satisfying the given conditions such that $(e^2u^2-4en)/g^2$ is a discriminant.

Lemma 32. *Let n be a positive integer relatively prime to 6, and let t be an integer satisfying $t^2<4n^2$. Let*

$$e \in \begin{cases} \{\pm1,\pm3\} & \text{if } (t,24)=1, \\ \{\pm1,\pm2,\pm3,\pm6\} & \text{if } (t,24)=2, \\ \{\pm1\} & \text{if } (t,24)=3, \\ \{\pm2,\pm6\} & \text{if } (t,24)=4 \text{ or } (t,24)=8, \\ \{\pm1,\pm2\} & \text{if } (t,24)=6, \\ \{\pm2\} & \text{if } (t,24)=12 \text{ or } (t,24)=24. \end{cases}$$

If $e(t+2n)$ is a discriminant, that is, if $e(t+2n)\equiv 0,1 \bmod 4$, then there exists a rational number s such that t^2-4n^2 decomposes into a product of two discriminants $s^2e(t+2n)$ and $(t-2n)/(es^2)$.

Proof. Here we consider only the case $(t,24)=2$. When $e=\pm1$, we have $4\mid(t+2n),(t-2n)$, and the statement holds obviously. When $e=\pm2$, we can decompose t^2-4n^2 as $2(t+2n)\cdot(t-2n)/2$ or $(t+2n)/2\cdot 2(t-2n)$ according to whether $8\mid(t-2n)$ or $8\mid(t+2n)$. When $e=\pm3$, the decomposition is $3(t+2n)\cdot(t-2n)/3$ or $(t+2n)/3\cdot 3(t-2n)$ according to whether $3\mid(t-2n)$ or $3\mid(t+2n)$. When $e=\pm6$, the rational number s can be one of 1, 1/2, 1/3, or 1/6, depending on whether $8\mid(t+2n)$ and whether $3\mid(t+2n)$. ∎

We now evaluate $L_{e,\ell}(n,t)$. The computation mostly follows that in [14, Section 4].

LEMMA 33. *Let n be a positive integer relatively prime to 6. For $e \in \{1,2,3,6\}$ and integers u, let*

$$\mu_e(n,u) = \begin{cases} 1 & \textit{if } 3 \nmid u, \\ 1 + \left(\frac{en}{3}\right) & \textit{if } 3 \mid u. \end{cases}$$

Then we have the following formulas.

(1) *Let t and e be integers satisfying the assumptions in Lemma 32.*
If $e(t+2n) \equiv 0, 1 \bmod 4$, then

$$L_{e,0}(n,t) = \frac{1}{2} \begin{cases} 0 & \textit{if } e(t+2n) \textit{ is not a square}, \\ \mu_e(n,u) M_{e,0}(n,u) & \textit{if } t+2n = eu^2, \end{cases} \tag{27}$$

where u is the positive square root of $(t+2n)/e$.
Also, if $-e(t+2n) \equiv 0, 1 \bmod 4$, then

$$L_{-e,0}(n,t) = \frac{i}{2}\left(\frac{-4}{r}\right) \begin{cases} 0 & \textit{if } e(2n-t) \textit{ is not a square}, \\ \mu_e(n,u) M'_{e,0}(n,u) & \textit{if } 2n-t = eu^2, \end{cases} \tag{28}$$

where u is the positive square root of $(2n-t)/e$.

(2) *Assume that $2 \parallel t$ and that $e \in \{\pm 1, \pm 3\}$. Then*

$$L_{e,1}(n,t) = \frac{1}{2} L_{e,0}(n,t).$$

(3) *Assume that $2 \parallel t$, $e \in \{1,3\}$ and that $\ell \geq 2$. Then*

$$L_{e,\ell}(n,t) = \frac{1}{2} \begin{cases} 0 & \textit{if } e(t+2n) \textit{ is not a square}, \\ \mu_e(n,u) M_{e,\ell-1}(n,u)/2 & \textit{if } t+2n = eu^2 \textit{ and } 2 \parallel u, \\ \mu_e(n,u) M_{e,1}(n,u)/2^{\ell-1} & \textit{if } t+2n = eu^2 \textit{ and } 2^{\ell-1} \parallel u, \\ \mu_e(n,u) M_{e,0}(n,u)/2^{\ell} & \textit{if } t+2n = eu^2 \textit{ and } 2^{\ell} \mid u, \end{cases}$$

and

$$L_{-e,\ell}(n,t) = \frac{i}{2}\left(\frac{-4}{r}\right) \begin{cases} 0 & \textit{if } e(2n-t) \textit{ is not a square}, \\ \mu_e(n,u) M'_{e,\ell-1}(n,u)/2 & \textit{if } 2n-t = eu^2 \textit{ and } 2 \parallel u, \\ \mu_e(n,u) M'_{e,1}(n,u)/2^{\ell-1} & \textit{if } 2n-t = eu^2 \textit{ and } 2^{\ell-1} \parallel u, \\ \mu_e(n,u) M'_{e,0}(n,u)/2^{\ell} & \textit{if } 2n-t = eu^2 \textit{ and } 2^{\ell} \mid u. \end{cases}$$

Proof. We first prove (27). Note that in the sum defining $L_{e,0}(n,t)$, the integers f are always odd, and according to the choice of representatives given in Lemma 26, we have $c' = c$. Then from Remark 25, we know that $(f, t+2n, t-2n) = 1$. Thus, if $f_1 = (f, t+2n)$ and $f_2 = (f, t-2n)$, then $f = f_1 f_2$, $f_1^2 \mid (t+2n)$ and $f_2^2 \mid (t-2n)$. Write (d/c) as

$$\Big(\frac{d}{c}\Big) = \Big(\frac{d}{c/f}\Big)\Big(\frac{d}{f_1}\Big)\Big(\frac{d}{f_2}\Big).$$

Since

$$\Big(\frac{d}{c/f}\Big)\Big(\frac{t+2n}{c/f}\Big) = \Big(\frac{ad+2nd+d^2}{c/f}\Big) = \Big(\frac{n^2+2nd+d^2}{c/f}\Big) = 1,$$

we have

$$\Big(\frac{d}{c/f}\Big) = \Big(\frac{(t+2n)/f_1^2}{c/f}\Big).$$

By the same token, we also have

$$\Big(\frac{d}{f_1}\Big) = \Big(\frac{(t-2n)/f_2^2}{f_1}\Big), \qquad \Big(\frac{d}{f_2}\Big) = \Big(\frac{(t+2n)/f_1^2}{f_2}\Big),$$

and hence

$$\Big(\frac{d}{c}\Big)\Big(\frac{e}{c}\Big) = \Big(\frac{e(t+2n)/f_1^2}{c/f}\Big)\Big(\frac{e(t-2n)/f_2^2}{f_1}\Big)\Big(\frac{e(t+2n)/f_1^2}{f_2}\Big). \tag{29}$$

Now $e(t+2n)$ is assumed to be a discriminant. By Lemma 32, we can decompose t^2-4n^2 into a product $es^2(t+2n)\cdot(t-2n)/es^2$ of two discriminants. So by Lemma 27,

$$\begin{aligned} &\frac{1}{w_{n,t,f}} \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_{n,t,f}/\mathrm{SL}(2,\mathbb{Z})} \Big(\frac{e(t+2n)/f_1^2}{c/f}\Big) \\ &\quad= \begin{cases} H((t^2-4n^2)/f^2)/2 & \text{if } e(t+2n)=(eu)^2 \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{30}$$

In the case $e(t+2n) = (eu)^2$, we have

$$\rho = \frac{t+\sqrt{t^2-4n^2}}{2} = \Big(\frac{\sqrt{e(t+2n)}+\sqrt{e(t-2n)}}{2\sqrt{e}}\Big)^2 = \Big(\frac{\tau}{\sqrt{e}}\Big)^2 \tag{31}$$

and

$$\rho - \overline{\rho} = \sqrt{t^2 - 4n^2} = u\sqrt{e^2u^2 - 4en} = u(\tau - \overline{\tau}), \tag{32}$$

where $\tau = (eu + \sqrt{\Delta})/2$ with $\Delta = e^2u^2 - 4en$. Then for the first sum $\sum_f$ defining $L_{e,0}(n,t)$ in (24), we have

$$\begin{aligned} &\sum_{(f,2)=1,(n,t,f)=1} \frac{1}{w_{n,t,f}} \sum_{\gamma} \Big(\frac{d}{c}\Big)\Big(\frac{e}{c}\Big) \\ &= \frac{1}{2} \sum_{(f,2)=1,(n,t,f)=1} \Big(\frac{e(t+2n)/f_1^2}{f_2}\Big)\Big(\frac{e(t-2n)/f_2^2}{f_1}\Big) H\Big(\frac{t^2-4n^2}{f^2}\Big) \\ &= \frac{1}{2} \sum_{(f_2,2)=1,(n,t,f_2)=1} \ \sum_{f_1|u,(f_1,2)=1,(n,t,f_1)=1} \Big(\frac{\Delta/f_2^2}{f_1}\Big) H\Big(\frac{u^2}{f_1^2}\frac{\Delta}{f_2^2}\Big). \end{aligned} \tag{33}$$

The inner sum running over f_1 is the same as

$$\sum_{f_1|u} \Big(\frac{\Delta/f_2^2}{f_1}\Big) H\Big(\frac{u^2}{f_1^2}\frac{\Delta}{f_2^2}\Big) \tag{34}$$

since if $2 \mid f_1$ or $(n,t,f_1) > 1$, then $(\frac{\Delta/f_2^2}{f_1}) = 0$. Then by Lemma 28, the first sum in (24) is equal to

$$\sum_{(f,2)=1,(n,t,f)=1} \frac{1}{w_{n,t,f}} \sum_{\gamma} \Big(\frac{d}{c}\Big)\Big(\frac{e}{c}\Big) = \frac{u}{2} \sum_{(f_2,2)=1,(n,t,f_2)=1} H\Big(\frac{\Delta}{f_2^2}\Big). \tag{35}$$

For the second sum in (24), we have two cases:

$$\begin{cases} 3 \mid f_1, 3 \nmid f_2 & \text{if } 3 \mid u, \\ 3 \mid f_2, 3 \nmid f_1 & \text{if } 3 \nmid u. \end{cases}$$

If $3 \nmid u$, then a computation similar to (33)–(35) yields

$$\sum_{(f,2)=1,3|f,(n,t,f)=1} \frac{1}{w_{n,t,f}} \sum_{\gamma} \Big(\frac{d}{c}\Big)\Big(\frac{e}{c}\Big) = \frac{u}{2} \sum_{(f_2,2)=1,3|f_2,(n,t,f_2)=1} H\Big(\frac{\Delta}{f_2^2}\Big). \tag{36}$$

If $3 \mid u$, then

$$\begin{aligned}
&\sum_{(f,2)=1,3|f,(n,t,f)=1} \frac{1}{w_{n,t,f}} \sum_{\gamma} \Big(\frac{d}{c}\Big)\Big(\frac{e}{c}\Big) \\
&\quad = \frac{1}{2} \sum_{(f,6)=3,(n,t,f)=1} \Big(\frac{e(t+2n)/f_1^2}{f_2}\Big)\Big(\frac{e(t-2n)/f_2^2}{f_1}\Big) H\Big(\frac{t^2-4n}{f^2}\Big) \\
&\quad = \frac{1}{2} \sum_{(f_2,2)=1,(n,t,f_2)=1} \ \sum_{f_1|u,(f_1,6)=3,(n,t,f)=1} \Big(\frac{\Delta/f_2^2}{f_1}\Big) H\Big(\frac{u^2}{f_1^2}\frac{\Delta}{f_2^2}\Big).
\end{aligned}$$

By Lemma 28, the inner sum is equal to

$$\sum_{h|(u/3),(h,2)=1,(n,t,h)=1} \Big(\frac{\Delta/f_2^2}{3h}\Big) H\Big(\frac{(u/3)^2}{f_1^2}\frac{\Delta}{f_2^2}\Big) = \frac{u}{3}\Big(\frac{\Delta}{3}\Big) H\Big(\frac{\Delta}{f_2^2}\Big),$$

and we have

$$\begin{aligned}
&\sum_{(f,2)=1,3|f,(n,t,f)=1} \frac{1}{w_{n,t,f}} \sum_{\gamma} \Big(\frac{d}{c}\Big)\Big(\frac{e}{c}\Big) \\
&\quad = \frac{u}{6}\Big(\frac{\Delta}{3}\Big) \sum_{(f_2,2)=1,(n,t,f_2)=1} H\Big(\frac{\Delta}{f_2^2}\Big).
\end{aligned} \tag{37}$$

Combining (29)–(32) and (35)–(37), we get (27).

The proof of (28) follows the same line of calculation. Equation (29) continues to hold when we replace e by $-e$. By Lemma 27, (30) also remains valid, provided that the condition that $e(t+2n)$ is a square is changed to the condition that $e(2n-t)$ is a square. The computations in (32), (35), (36), and (37) are almost the same. The only significant difference lies at (31). Instead of (31), we have

$$\frac{t+\sqrt{t^2-4n^2}}{2} = \Big(i\frac{\sqrt{e(2n-t)}-\sqrt{-e(t+2n)}}{2\sqrt{e}}\Big)^2 = \Big(i\frac{\overline{\tau}}{\sqrt{e}}\Big)^2$$

and

$$\Big(\frac{t+\sqrt{t^2-4n^2}}{2}\Big)^{1/2-k} = i^{1-2k}\Big(\frac{\overline{\tau}}{\sqrt{e}}\Big)^{1-2k} = i\Big(\frac{-4}{r}\Big)\Big(\frac{\overline{\tau}}{\sqrt{e}}\Big)^{1-2k}, \tag{38}$$

where $\tau = (eu+\sqrt{\Delta})/2$ with $\Delta = e^2u^2 - 4en$. Here in the last step, we have used the assumption $k = (r-1)/2 + s$. This establishes (28).

We now prove the second part of the lemma. Here we consider only the case $e=1,3$. The integers f in the sum defining $L_{e,1}(n,t)$ satisfy $2\parallel f$. Let $f'=f/2$, and set $f_1=(f',t+2n)$ and $f_2=(f',t-2n)$. Then as before, we have $f'=f_1f_2$, $f_1^2\mid(t+2n)$ and $f_2^2\mid(t-2n)$, and (29) remains valid if we replace c by c' and f by f'. For the first sum in $L_{e,\ell}(n,t)$, the computation in (30)–(32) remains unchanged. Now instead of (33), we have

$$\begin{aligned}&\sum_{2\parallel f,(n,t,f)=1}\frac{1}{w_{n,t,f}}\sum_{\gamma}\Big(\frac{d}{c}\Big)\Big(\frac{e}{c}\Big)\\&\quad=\frac{1}{2}\sum_{2\parallel f,(n,t,f)=1}\Big(\frac{e(t+2n)/f_1^2}{f_2}\Big)\Big(\frac{e(t-2n)/f_2^2}{f_1}\Big)H\Big(\frac{t^2-4n^2}{f^2}\Big)\\&\quad=\frac{1}{2}\sum_{(f_2,2)=1,(n,t,f_2)=1}\ \sum_{f_1\mid u,(f_1,2)=1,(n,t,f_1)=1}\Big(\frac{\Delta/f_2^2}{f_1}\Big)H\Big(\frac{(u/2)^2}{f_1^2}\frac{\Delta}{f_2^2}\Big).\end{aligned}$$

Now since $4\mid(\Delta/f_2^2)=(e^2u^2-4en)/f_2^2$, we have

$$H\Big(\frac{(u/2)^2}{f_1^2}\frac{\Delta}{f_2^2}\Big)=\frac{1}{2}H\Big(\frac{u^2}{f_1^2}\frac{\Delta}{f_2^2}\Big).$$

Following the remaining computation, we easily see that the first sum in $L_{e,1}(n,t)$ is equal to one-half of that of $L_{e,0}(n,t)$. Similarly, the second sum in $L_{e,1}(n,t)$ is equal to one-half of that of $L_{e,0}(n,t)$. This proves the second part of the lemma.

The proof of the third part is also similar. Again, we consider only the case $e>0$. The computation for the first sum in $L_{e,\ell}(n,t)$ differs from that for the first sum in $L_{e,0}(n,t)$ mainly at (33). In the case $2\parallel u$, instead of (33), we have

$$\begin{aligned}&\sum_{2^\ell\parallel f,(n,t,f)=1}\frac{1}{w_{n,t,f}}\sum_{\gamma}\Big(\frac{d}{c}\Big)\Big(\frac{e}{c}\Big)\\&\quad=\frac{1}{2}\sum_{(f_2,2)=1,(n,t,f_2)=1}\ \sum_{f_1\mid u,(f_1,2)=1,(n,t,f_1)=1}\Big(\frac{\Delta/f_2^2}{f_1}\Big)H\Big(\frac{(u/2)^2}{f_1^2}\frac{\Delta}{(2^{\ell-1}f_2)^2}\Big),\end{aligned}$$

where $\Delta=e^2u^2-4en$. Then, by Lemma 28, it is equal to

$$\sum_{2^\ell\parallel f,(n,t,f)=1}\frac{1}{w_{n,t,f}}\sum_{\gamma}\Big(\frac{d}{c}\Big)\Big(\frac{e}{c}\Big)=\frac{u}{4}\sum_{(f_2,2)=1,(n,t,f_2)=1}H\Big(\frac{\Delta}{(2^{\ell-1}f_2)^2}\Big).$$

The computation for the case $2^{\ell-1} \parallel u$ is almost the same. In this case, we have

$$\sum_{2^\ell \parallel f,(n,t,f)=1} \frac{1}{w_{n,t,f}} \sum_\gamma \Big(\frac{d}{c}\Big)\Big(\frac{e}{c}\Big) = \frac{u}{2^\ell} \sum_{(f_2,2)=1,(n,t,f_2)=1} H\Big(\frac{\Delta}{(2f_2)^2}\Big).$$

Now if $2^\ell \mid u$, then instead of (33), we have

$$\begin{aligned}\sum_{2^\ell \parallel f,(n,t,f)=1} &\frac{1}{w_{n,t,f}} \sum_\gamma \Big(\frac{d}{c}\Big)\Big(\frac{e}{c}\Big) \\ &= \frac{1}{2} \sum_{(f_2,2)=1,(n,t,f_2)=1} \sum_{f_1 \mid u,(f_1,2)=1,(n,t,f_1)=1} \Big(\frac{\Delta/f_2^2}{f_1}\Big) H\Big(\frac{(u/2^\ell)^2}{f_1^2}\frac{\Delta}{f_2^2}\Big).\end{aligned}$$

Since $4 \mid (\Delta/f_2^2)$, we have

$$H\Big(\frac{(u/2^\ell)^2}{f_1^2}\frac{\Delta}{f_2^2}\Big) = \frac{1}{2^\ell} H\Big(\frac{u^2}{f_1^2}\frac{\Delta}{f_2^2}\Big),$$

and by Lemma 28,

$$\sum_{2^\ell \parallel f,(n,t,f)=1} \frac{1}{w_{n,t,f}} \sum_\gamma \Big(\frac{d}{c}\Big)\Big(\frac{e}{c}\Big) = \frac{u}{2^{\ell+1}} \sum_{(f_2,2)=1,(n,t,f_2)=1} H\Big(\frac{\Delta}{f_2^2}\Big).$$

Together with (31) and (32), this completes the computation for the first sum in $L_{e,\ell}(n,t)$. The computation for the second sum is analogous and is skipped. □

We now utilize Lemma 33 to compute the contribution of $\Gamma_{n,t,f}$ to the trace of $[\mathscr{M}_{n^2}]$. This will be done according to the greatest common divisor of t and 24. To summarize our computation, we fix the following notation.

Notation 34. Given a positive integer n relatively prime to 6, $e \in \{1,2,3,6\}$, and an integer u with $e^2u^2 < 4en$, we let

$$P_k(e,n,u) = \frac{\tau^{2k-1} - \overline{\tau}^{2k-1}}{\tau - \overline{\tau}}, \quad \tau = \frac{eu + \sqrt{e^2u^2 - 4en}}{2}. \tag{39}$$

For nonnegative integers ℓ and m, we set

$$\begin{aligned}A_{\ell,m}(n) = \Big(\frac{12}{n}\Big) \sum_{3\nmid u} \Big(\sum_g H\Big(\frac{\Delta}{g^2}\Big) - 3\sum_{3\mid g} H\Big(\frac{\Delta}{g^2}\Big)\Big) P_k(1,n,u) \\ + \Big(\frac{12}{n}\Big) \sum_{3\mid u} \Big(1 - \Big(\frac{\Delta}{3}\Big)\Big) \sum_g H\Big(\frac{\Delta}{g^2}\Big) P_k(1,n,u),\end{aligned}$$

where $\Delta = u^2 - 4n$, the outer sums run over all integers u satisfying

$$u^2 < 4n, \qquad 2^\ell \parallel u,$$

and the given conditions, and the inner sums run over all positive integers g such that

$$\Delta/g^2 \equiv 0, 1 \bmod 4, \qquad 2^m \parallel g, \qquad (n, u, g) = 1,$$

and the specified conditions are met. We also set

$$A^*_{\ell,m}(n) = \sum_{\ell \le j < \infty} A_{j,m}(n).$$

Similarly, we define

$$\begin{aligned} B_\ell(n) &= \frac{1}{2^{k-1}} \left(\frac{12}{n}\right)\left(\frac{8}{r}\right) \sum_{3\nmid u} \Big(\sum_g H\left(\frac{\Delta}{g^2}\right) - 3 \sum_{3|g} H\left(\frac{\Delta}{g^2}\right)\Big) P_k(2, n, u) \\ &\quad + \frac{1}{2^{k-1}} \left(\frac{12}{n}\right)\left(\frac{8}{r}\right) \sum_{3|u} \Big(1 - \left(\frac{\Delta}{3}\right)\Big) \sum_g H\left(\frac{\Delta}{g^2}\right) P_k(2, n, u), \end{aligned}$$

where $\Delta = 4u^2 - 8n$, the outer sums run over all integers u such that

$$4u^2 < 8n, \qquad 2^\ell \parallel u,$$

and the inner sums run over all positive integers g such that

$$\Delta/g^2 \equiv 0, 1 \bmod 4, \qquad (n, u, g) = 1.$$

Also, we set

$$B^*_\ell(n) = \sum_{\ell \le j < \infty} B_\ell(n).$$

Likewise, define

$$C_{\ell,m}(n) = \frac{1}{3^{k-1}} \left(\frac{12}{n}\right)\left(\frac{12}{r}\right) \sum_{9u^2 < 12n, 2^\ell \parallel u} \sum_g H\left(\frac{\Delta}{g^2}\right) P_k(3, n, u),$$

where $\Delta = 9u^2 - 12n$, and the inner sum runs over all positive integers g with

$$\Delta/g^2 \equiv 0, 1 \bmod 4, \qquad 2^m \parallel g, \qquad (n, u, g) = 1,$$

and let

$$C^*_{\ell,m}(n) = \sum_{\ell \le j < \infty} C_{j,m}(n).$$

Finally, we let

$$D_\ell(n) = \frac{1}{6^{k-1}} \Big(\frac{12}{n}\Big)\Big(\frac{24}{r}\Big) \sum_{36u^2 < 24n, 2^\ell \| u} \sum_g H\Big(\frac{\Delta}{g^2}\Big) P_k(6, n, u),$$

where $\Delta = 36u^2 - 24n$, and the inner sum runs over all positive integers g such that

$$\Delta/g^2 \equiv 0, 1 \bmod 4, \qquad (n, u, g) = 1.$$

Also, let

$$D^*_\ell(n) = \sum_{\ell \le j < \infty} D_j(n).$$

Here we give alternative expressions for $A_{\ell,m}(n)$ and $B_\ell(n)$, which will be used later.

Lemma 35. *For a discriminant Δ of an imaginary quadratic order, we let Δ_0 denote the discriminant of the field $\mathbb{Q}(\sqrt{\Delta})$. Then we have*

$$A_{\ell,m}(n) = \beta(n) \Big(\frac{12}{n}\Big) \sum_u \Big(1 - \Big(\frac{\Delta_0}{3}\Big)\Big) \sum_{g, 3 \nmid \Delta/(\Delta_0 g^2)} H\Big(\frac{\Delta}{g^2}\Big) P_k(1, n, u),$$

where

$$\beta(n) = \begin{cases} 1 & \text{if } n \equiv 1 \bmod 3, \\ 1/2 & \text{if } n \equiv 2 \bmod 3 \end{cases}$$

and the double sum runs over the same u and g as in the definition of $A_{\ell,m}$ satisfying the additional condition that 3 does not divide $\Delta/(\Delta_0 g^2)$, where $\Delta = u^2 - 4n$. Analogously, we also have

$$B_\ell(n) = \frac{\gamma(n)}{2^{k-1}} \Big(\frac{12}{n}\Big)\Big(\frac{8}{r}\Big) \sum_u \Big(1 - \Big(\frac{\Delta_0}{3}\Big)\Big) \sum_{g, 3 \nmid \Delta/(\Delta_0 g^2)} H\Big(\frac{\Delta}{g^2}\Big) P_k(2, n, u),$$

where

$$\gamma(n) = \begin{cases} 1/2 & \text{if } n \equiv 1 \bmod 3, \\ 1 & \text{if } n \equiv 2 \bmod 3. \end{cases}$$

Proof. Here we prove only the case $A_{\ell,m}(n)$. Consider the case $n \equiv 1 \bmod 3$ first. Assume that $3 \nmid u$. We have $\Delta \equiv 0 \bmod 3$. If $3 \parallel \Delta$, then $3 \mid \Delta_0$ and the sum $\sum_{3|g}$ is empty, which yields

$$\sum_g H\Big(\frac{\Delta}{g^2}\Big) - \sum_{3|g} H\Big(\frac{\Delta}{g^2}\Big) = \Big(1 - \Big(\frac{\Delta_0}{3}\Big)\Big) \sum_g H\Big(\frac{\Delta}{g^2}\Big).$$

If $9 \mid \Delta$, we have

$$\begin{aligned}
&\sum_g H\Big(\frac{\Delta}{g^2}\Big) - 3\sum_{3|g} H\Big(\frac{\Delta}{g^2}\Big)\\
&\quad = \sum_h H(h^2\Delta_0) - 3\sum_{3|h} H\Big(\frac{h^2\Delta_0}{9}\Big)\\
&\quad = \sum_{(h,3)=1} H(h^2\Delta_0) + \sum_{3|h}\Big(H(h^2\Delta_0) - 3H\Big(\frac{h^2\Delta_0}{9}\Big)\Big)\\
&\quad = \sum_{(h,3)=1} H(h^2\Delta_0) - \sum_{3|h}\Big(\frac{h^2\Delta_0/9}{3}\Big)H\Big(\frac{h^2\Delta_0}{9}\Big)\\
&\quad = \sum_{(h,3)=1} H(h^2\Delta_0) - \sum_{(h,3)=1}\Big(\frac{h^2\Delta_0}{3}\Big)H(h^2\Delta_0)\\
&\quad = \Big(1 - \Big(\frac{\Delta_0}{3}\Big)\Big) \sum_{(h,3)=1} H(h^2\Delta_0).
\end{aligned}$$

Either way, we find that the first double sum in the definition of $A_{\ell,m}(n)$ is equal to

$$\sum_{3\nmid u}\Big(1 - \Big(\frac{\Delta_0}{3}\Big)\Big) \sum_{g,3\nmid\Delta/(g^2\Delta_0)} H\Big(\frac{\Delta}{g^2}\Big)P_k(1,n,u).$$

Now if $3 \mid u$, then Δ is not a multiple of 3, and hence

$$\Big(\frac{\Delta}{3}\Big) = \Big(\frac{\Delta_0}{3}\Big).$$

There is nothing to do in these cases. This proves the case $n \equiv 1 \bmod 3$.

Now assume that $n \equiv 2 \bmod 3$. We have $3 \nmid \Delta$. Thus, the sum $\sum_{3|g}$ is empty and

$$1 - \Big(\frac{u^2 - 4n}{3}\Big) = \begin{cases} 2 & \text{if } 3 \nmid u,\\ 0 & \text{if } 3 \mid u.\end{cases}$$

That is, the second sum in the definition of $A_{\ell,m}(n)$ vanishes. The assertion follows. ∎

3.4.3. Case $(t,24)=1$.

LEMMA 36. *For the case* $(t,24)=1$, *we have*

$$\sum_{(t,24)=1}\sum_{f}\sum_{\gamma\in\Gamma_{n,t,f}/\operatorname{SL}(2,\mathbb{Z})} J(\gamma)$$
$$=-\frac{n^{3/2-k}}{8}\begin{cases}(A_{0,0}(n)+C_{0,0}(n)) & \text{if } n\equiv 1 \bmod 3,\\ C_{0,0}(n) & \text{if } n\equiv 2 \bmod 3.\end{cases}$$

Here the outermost sums run over all integers t satisfying $t^2<4n^2$ and $(t,24)=1$, the middle sums run over positive integers f such that $(t^2-4n^2)/f^2$ is a discriminant, and the functions $A_{0,0}(n)$ and $C_{0,0}(n)$ are defined as in Notation 34.

Proof. Since $(t,24)=1$, the integers f are always odd. Let class representatives $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ of $\Gamma_{n,t,f}/\operatorname{SL}(2,\mathbb{Z})$ be chosen as per Lemma 26. By Lemma 29, we have

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}^*=\left(\begin{pmatrix}a&b\\c&d\end{pmatrix},n^{-k-1/2}\epsilon(a,b,c,d)^r(c\tau+d)^{k+1/2}\right),$$

where

$$\epsilon(a,b,c,d)=\left(\frac{d}{c}\right)e^{2\pi i(bd(1-c^2)+c(a+d-3))/24}.$$

If $3\nmid f$, then $(c,6)=1$ and $bd(1-c^2)\equiv 0 \bmod 24$. If $3\mid f$, then $3\mid b$ and $8\mid(1-c^2)$. Either way, we get $\epsilon(a,b,c,d)=(d/c)e^{2\pi ict/24}e^{-2\pi ic/8}$ and

$$J(\gamma)=\frac{n^{k+1/2}}{w_{n,t,f}}\left(\frac{d}{c}\right)e^{2\pi irc/8}e^{-2\pi irct/24}\frac{\rho^{1/2-k}}{\rho-\overline{\rho}}, \tag{40}$$

where $\rho=(t+\sqrt{t^2-4n^2})/2$.

Now for $3 \nmid f$, we have, by the formulas in Lemma 30,

$$
\begin{aligned}
&e^{2\pi i rc/8} e^{-2\pi i rct/24} \\
&\quad = \frac{1}{4}\Big(\Big(\frac{8}{rc}\Big) + i\Big(\frac{-8}{rc}\Big)\Big)\Big(\Big(\frac{8}{-rct}\Big) + \sqrt{3}\Big(\frac{24}{-rct}\Big) - i\Big(\frac{-8}{-rct}\Big) + i\sqrt{3}\Big(\frac{-24}{-rct}\Big)\Big) \\
&\quad = \frac{1}{4}\Big(\Big(\frac{8}{t}\Big) - \Big(\frac{-8}{t}\Big)\Big) + \frac{i}{4}\Big(\frac{-4}{rc}\Big)\Big(\Big(\frac{8}{t}\Big) + \Big(\frac{-8}{t}\Big)\Big) \\
&\qquad + \frac{\sqrt{3}}{4}\Big(\frac{12}{rc}\Big)\Big(\Big(\frac{24}{t}\Big) + \Big(\frac{-24}{t}\Big)\Big) + \frac{i\sqrt{3}}{4}\Big(\frac{-3}{rc}\Big)\Big(\Big(\frac{24}{t}\Big) - \Big(\frac{-24}{t}\Big)\Big) \\
&\quad = \frac{1}{2}\Big(\frac{8}{t}\Big)\Big(\delta_3(t) + i\delta_1(t)\Big(\frac{-4}{rc}\Big)\Big) + \frac{\sqrt{3}}{2}\Big(\frac{24}{t}\Big)\Big(\frac{12}{rc}\Big)\Big(\delta_1(t) + i\delta_3(t)\Big(\frac{-4}{rc}\Big)\Big),
\end{aligned}
$$

where

$$
\delta_j(t) = \begin{cases} 1 & \text{if } t \equiv j \bmod 4, \\ 0 & \text{otherwise.} \end{cases} \tag{41}
$$

For $3 \mid f$, we have $3 \mid c$ and

$$
\begin{aligned}
e^{2\pi i rc/8} e^{-2\pi i rct/24} &= \frac{1}{2}\Big(\Big(\frac{8}{rc}\Big) + i\Big(\frac{-8}{rc}\Big)\Big)\Big(\Big(\frac{8}{rct/3}\Big) - i\Big(\frac{-8}{rct/3}\Big)\Big) \\
&= \frac{1}{2}\Big(-\Big(\frac{8}{t}\Big) + \Big(\frac{-8}{t}\Big)\Big) - \frac{i}{2}\Big(\frac{-4}{rc}\Big)\Big(\Big(\frac{8}{t}\Big) + \Big(\frac{-8}{t}\Big)\Big) \\
&= -\delta_3(t)\Big(\frac{8}{t}\Big) - i\delta_1(t)\Big(\frac{8}{t}\Big)\Big(\frac{-4}{rc}\Big).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\sum_{(t,24)=1} \sum_{f} \sum_{\gamma \in \Gamma_{n,t,f}/\mathrm{SL}(2,\mathbb{Z})} J(\gamma) \\
&\quad = \frac{n^{k+1/2}}{2} \sum_{(t,24)=1} \Big(\delta_3(t)\Big(\frac{8}{t}\Big) L_{1,0}(n,t) \\
&\qquad + i\delta_1(t)\Big(\frac{-4}{r}\Big)\Big(\frac{8}{t}\Big) L_{-1,0}(n,t) \\
&\qquad + \sqrt{3}\delta_1(t)\Big(\frac{12}{r}\Big)\Big(\frac{24}{t}\Big) L_{3,0}(n,t) \\
&\qquad + i\sqrt{3}\delta_3(t)\Big(\frac{-3}{r}\Big)\Big(\frac{24}{t}\Big) L_{-3,0}(n,t)\Big),
\end{aligned} \tag{42}
$$

where $L_{e,\ell}(n,t)$ are defined by (24). (Note that the second sums in the definition of $L_{3,0}(n,t)$ and $L_{-3,0}(n,t)$ are empty.)

For the term $\delta_3(t)L_{1,0}(n,t)$, $\delta_3(t)\neq 0$ implies that $t+2n$ is a discriminant, and Lemma 33 applies. That is, $\delta_3(t)L_{1,0}(n,t)$ is nonzero only when $t+2n=u^2$ for some (odd) integer u. If this situation occurs, then $t+2n\equiv 1 \bmod 8$ and

$$\left(\frac{8}{t}\right)=(-1)^{(t^2-1)/8}=(-1)^{n(n-1)/2}=\left(\frac{-4}{n}\right). \tag{43}$$

Also observe that since $(t,24)=1$, if $n\equiv 2 \bmod 3$, any integer u such that $u^2=t+2n$ must be divisible by 3. This information, together with Lemma 33, yields

$$\begin{aligned}\sum_{(t,24)=1}\delta_3(t)\left(\frac{8}{t}\right)L_{1,0}(n,t)&=\frac{\lambda_1(n)}{2}\left(\frac{-4}{n}\right)\sum_{(u,6)=1}M_{1,0}(n,u)\\&\quad+\frac{1}{2}\left(\frac{-4}{n}\right)\sum_{(u,6)=3}\left(1+\left(\frac{n}{3}\right)\right)M_{1,0}(n,u),\end{aligned} \tag{44}$$

where the sums run over all positive integers u such that $u^2<4n$ satisfying the given conditions, $M_{e,\ell}(n,u)$ are defined by (25), and for $j=1,2$,

$$\lambda_j(n)=\begin{cases}1 & \text{if } n\equiv j \bmod 3,\\ 0 & \text{otherwise.}\end{cases} \tag{45}$$

Likewise, for the term $\delta_1(t)L_{-1,0}(n,t)$ in (42), $\delta_1(t)\neq 0$ implies that $-(t+2n)$ is a discriminant. Therefore, assuming that $\delta_1(t)=1$, Lemma 33 shows that $L_{-1,0}(n,t)$ is nonzero only when $2n-t=u^2$ is a square. In such cases, (43) continues to hold. Then Lemma 33 yields

$$\begin{aligned}\sum_{(t,24)=1}\delta_1(t)\left(\frac{8}{t}\right)L_{-1,0}(n,t)&=i\frac{\lambda_1(n)}{2}\left(\frac{-4}{rn}\right)\sum_{(u,6)=1}M'_{1,0}(n,u)\\&\quad+\frac{i}{2}\left(\frac{-4}{rn}\right)\sum_{(u,6)=3}\left(1+\left(\frac{n}{3}\right)\right)M'_{1,0}(n,u).\end{aligned} \tag{46}$$

The computations for the rest of terms in (42) are similar. We find that, for t with $\delta_1(t)=1$, $3(t+2n)$ is a discriminant. For such t, $L_{3,0}(n,t)$ is nonzero only when $3(t+2n)=(3u)^2$ is a square. Then $t+2n\equiv 3 \bmod 24$ and

$$\begin{aligned}\left(\frac{24}{t}\right)&=\left(\frac{-3}{t}\right)\left(\frac{-4}{t}\right)\left(\frac{8}{t}\right)=\left(\frac{-3}{n}\right)(-1)^{(t-1)/2+(t^2-1)/8}\\&=\left(\frac{-3}{n}\right)(-1)^{n(n-1)/2}=\left(\frac{12}{n}\right).\end{aligned}$$

Then by Lemma 33,

$$\sum_{(t,24)=1}\delta_1(t)\left(\frac{24}{t}\right)L_{3,0}(n,t)=\frac{1}{2}\left(\frac{12}{n}\right)\sum_{(u,2)=1}M_{3,0}(n,u),\tag{47}$$

where the sum runs over all *positive* integers u satisfying $9u^2<12n$ and the given conditions. By a similar computation and the same lemma, we also have

$$\sum_{(t,24)=1}\delta_3(t)\left(\frac{24}{t}\right)L_{-3,0}(n,t)=\frac{i}{2}\left(\frac{-4}{r}\right)\left(\frac{12}{n}\right)\sum_{(u,2)=1}M'_{3,0}(n,u).\tag{48}$$

Combining (42), (44), and (46)–(48), we get

$$\begin{aligned}&\sum_{(t,24)=1}\sum_{f}\sum_{\gamma\in\Gamma_{n,t,f}/\mathrm{SL}(2,\mathbb{Z})}J(\gamma)\\&=\frac{\lambda_1(n)n^{k+1/2}}{4}\left(\frac{-4}{n}\right)\sum_{(u,6)=1}\big(M_{1,0}(n,u)-M'_{1,0}(n,u)\big)\\&\quad+\frac{n^{k+1/2}}{4}\left(\frac{-4}{n}\right)\sum_{(u,6)=3}\left(1+\left(\frac{n}{3}\right)\right)\big(M_{1,0}(n,u)-M'_{1,0}(n,u)\big)\\&\quad+\frac{3^kn^{k+1/2}}{4}\left(\frac{12}{n}\right)\left(\frac{12}{r}\right)\sum_{(u,2)=1}\big(M_{3,0}(n,u)-M'_{3,0}(n,u)\big).\end{aligned}\tag{49}$$

Notice that for $\tau=(eu+\sqrt{e^2u^2-4en})/2$ we have

$$\frac{\tau^{1-2k}-\overline{\tau}^{1-2k}}{\tau-\overline{\tau}}=-(en)^{1-2k}\frac{\tau^{2k-1}-\overline{\tau}^{2k-1}}{\tau-\overline{\tau}}.$$

Now if $n\equiv 1 \bmod 3$, then $(-4/n)=(12/n)$ and

$$\sum_{(t,24)=1}\sum_{f}\sum_{\gamma}J(\gamma)=-\frac{n^{3/2-k}}{8}\big(A_{0,0}(n)+C_{0,0}(n)\big).$$

(Note that in the definition of $A_{\ell,m}(n)$, the integers u can be positive or negative, but in (49), the integers u are always positive. This explains the additional factor $1/2$ above.) If $n \equiv 2 \bmod 3$, then the factor $1+(n/3)$ in the middle sum in (49) is equal to 0, and we have

$$\sum_{(t,24)=1} \sum_{f} \sum_{\gamma} J(\gamma) = -\frac{n^{3/2-k}}{8} C_{0,0}(n).$$

This completes the proof. ∎

3.4.4. Case $(t,24)=3$.

Lemma 37. *We have*

$$\sum_{(t,24)=3} \sum_{f} \sum_{\gamma\in\Gamma_{n,t,f}/\,\mathrm{SL}(2,\mathbb{Z})} J(\gamma) = -\frac{n^{3/2-k}}{4} \begin{cases} 0 & \text{if } n\equiv 1 \bmod 3, \\ A_{0,0}(n) & \text{if } n\equiv 2 \bmod 3, \end{cases}$$

where $A_{0,0}(n)$ *is defined as in Notation 34.*

3.4.5. Case $(t,24)=4$.

Lemma 38. *We have*

$$\sum_{(t,24)=4} \sum_{f} \sum_{\gamma\in\Gamma_{n,t,f}/\,\mathrm{SL}(2,\mathbb{Z})} J(\gamma) = -\frac{n^{3/2-k}}{16} \begin{cases} D_0(n) & \text{if } n\equiv 1 \bmod 4, \\ 0 & \text{if } n\equiv 7 \bmod 12, \\ B_0(n) & \text{if } n\equiv 11 \bmod 12. \end{cases}$$

3.4.6. Case $(t,24)=8$.

Lemma 39. *We have*

$$\sum_{(t,24)=4} \sum_{f} \sum_{\gamma\in\Gamma_{n,t,f}/\,\mathrm{SL}(2,\mathbb{Z})} J(\gamma) = -\frac{n^{3/2-k}}{16} \begin{cases} 0 & \text{if } n\equiv 1 \bmod 12, \\ B_0(n) & \text{if } n\equiv 5 \bmod 12, \\ D_0(n) & \text{if } n\equiv 3 \bmod 4. \end{cases}$$

3.4.7. Case $(t,24)=12$.

Lemma 40. *We have*

$$\sum_{(t,24)=12} \sum_{f} \sum_{\gamma\in\Gamma_{n,t,f}/\,\mathrm{SL}(2,\mathbb{Z})} = -\frac{n^{3/2-k}}{8} \begin{cases} B_0(n) & \text{if } n\equiv 7 \bmod 12, \\ 0 & \text{otherwise}, \end{cases}$$

where $B_0(n)$ *is defined as in Notation 34.*

3.4.8. Case $(t,24)=24$.

LEMMA 41. *We have*

$$\sum_{(t,24)=24}\sum_{f}\sum_{\gamma\in\Gamma_{n,t,f}/\operatorname{SL}(2,\mathbb{Z})} J(\gamma) = -\frac{n^{3/2-k}}{8}\begin{cases} B_0(n) & \text{if } n\equiv 1 \bmod 12,\\ 0 & \text{otherwise,}\end{cases}$$

where $B_0(n)$ *is defined as in Notation 34.*

3.4.9. Case $(t,24)=2$ *and* $2\nmid f$.

LEMMA 42. *When* $(t,24)=2$ *and* f *is odd, we have*

$$\begin{aligned}&\sum_{(t,24)=2}\sum_{f\,\text{odd}}\sum_{\gamma\in\Gamma_{n,t,f}/\operatorname{SL}(2,\mathbb{Z})} J(\gamma)\\ &= -\frac{n^{3/2-k}}{16}\begin{cases} D_1^*(n) & \text{if } n\equiv 1 \bmod 3,\\ (B_1^*(n)+D_1^*(n)) & \text{if } n\equiv 2 \bmod 3.\end{cases}\end{aligned}$$

3.4.10. Case $(t,24)=2$ *and* $2\mid f$.

LEMMA 43. *Assume that* $2\parallel t$, *and write* $t=2t'$. *For class representatives* $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ *of* $\Gamma_{n,t,f}/\operatorname{SL}(2,\mathbb{Z})$ *given in Lemma 26, define* $\epsilon(a,b,c,d)$ *as in Lemma 29. Denote the odd part of* c *by* c'.

(1) *If* $2\parallel f$, *then*

$$\epsilon(a,b,c,d)=\left(\frac{2}{n}\right)\left(\frac{c'}{d}\right)e^{-2\pi i c't'/12}e^{2\pi i(t'-1)/8}.$$

(2) *If* $4\parallel f$, *then*

$$\epsilon(a,b,c,d)=-(-1)^{(t^2-4n^2)/64}\left(\frac{c'}{d}\right)e^{2\pi i c't'/3}e^{2\pi i(t'-1)/8}.$$

(3) *If* $2^v\parallel f$, $v\ge 3$, *then*

$$\epsilon(a,b,c,d)=\left(\frac{2}{t'}\right)^v\left(\frac{c'}{d}\right)e^{2\pi i 2^{v-2}c't'/3}e^{2\pi i(t'-1)/8}.$$

Proof. We have

$$\begin{aligned}&\big(ac(1-d^2)+d(b-c+3)-3\big)-\big(bd(1-c^2)+c(a+d)\big)\\ &= -(n^2+2)cd+3d-3\equiv -3cd+3d-3 \bmod 24.\end{aligned}$$

Thus,

$$\epsilon(a,b,c,d) = \left(\frac{c}{d}\right) e^{2\pi i(bd(1-c^2)+ct)/24} e^{2\pi i(-cd+d-1)/8}. \tag{50}$$

From now on, we let c', f', and t' be the odd parts of c, f, and t, respectively.

Consider the case $2 \parallel f$. By Lemma 26, we may assume that $2 \parallel c$ and that $d \equiv t' \bmod 8$. For such representatives, we have $a = t - d = 2t' - d \equiv t' \bmod 8$ and

$$t^2 - 4n^2 = (d-a)^2 + 4bc \equiv 4bc \bmod 64.$$

Now

$$t^2 - 4n^2 = 32\frac{(t')^2 - n^2}{8} \equiv \begin{cases} 32 \bmod 64 & \text{if } (\frac{2}{t'})(\frac{2}{n}) = -1, \\ 0 \bmod 64 & \text{if } (\frac{2}{t'})(\frac{2}{n}) = 1, \end{cases}$$

which shows that b is divisible by 4 and $8 \mid b$ if and only if $(2/t')(2/n) = 1$. Now if $3 \nmid f$, then $c^2 \equiv 4 \bmod 24$ so that

$$bd(1-c^2) \equiv \begin{cases} 12 \bmod 24 & \text{if } 4 \parallel b, \\ 0 \bmod 24 & \text{if } 8 \mid b. \end{cases}$$

The same congruences also hold when $3 \mid f$. Therefore, from (50), we have

$$\begin{aligned} \epsilon(a,b,c,d) &= \left(\frac{2}{d}\right)\left(\frac{c'}{d}\right)\left(\frac{2}{n}\right)\left(\frac{2}{t'}\right) e^{2\pi i c't'/6} e^{2\pi i(-ct'+t'-1)/8} \\ &= \left(\frac{2}{n}\right)\left(\frac{c'}{d}\right) e^{-2\pi i c't'/12} e^{2\pi i(t'-1)/8}. \end{aligned}$$

This proves statement (1) of the lemma.

We next consider the case $4 \parallel f$. By Lemma 26, we may assume that $d \equiv t' \bmod 16$. For such representatives, we also have $a = 2t' - d \equiv t' \bmod 16$. Thus, from $t^2 - 4n^2 = (d-a)^2 + 4bc$, we see that $4bc \equiv t^2 - 4n^2 \bmod 256$. That is,

$$\frac{b}{4}\frac{c}{4} \equiv \frac{t^2 - 4n^2}{64} \bmod 4,$$

from which we obtain $e^{2\pi i bd(1-c^2)/24} = (-1)^{b/4} = (-1)^{(t^2-4n^2)/64}$. It follows that

$$\begin{aligned} \epsilon(a,b,c,d) &= (-1)^{(t^2-4n^2)/64} \left(\frac{c}{d}\right) e^{2\pi i ct/24} e^{-2\pi i cd/8} e^{2\pi i(d-1)/8} \\ &= -(-1)^{(t^2-4n^2)/64} \left(\frac{c'}{d}\right) e^{2\pi i c't'/3} e^{2\pi i(t'-1)/8}. \end{aligned}$$

This proves statement (2).

For $8 \parallel f$, class representatives given in Lemma 26 satisfy either $d \equiv t' + 4 \bmod 8$ or $d \equiv t' \bmod 8$. In either case, we have

$$\left(\frac{2}{d}\right) e^{2\pi i(d-1)/8} = \left(\frac{2}{t'}\right) e^{2\pi i(t'-1)/8}.$$

Then from (50), we get

$$\epsilon(a,b,c,d) = \left(\frac{2}{d}\right)\left(\frac{c'}{d}\right) e^{2\pi i ct/24} e^{2\pi i(d-1)/8} = \left(\frac{2}{t'}\right)\left(\frac{c'}{d}\right) e^{4\pi i c't'/3} e^{2\pi i(t'-1)/8}.$$

This proves the case $8 \parallel f$. The proof of the case $16 \mid f$ is similar and is omitted. ∎

Lemma 44. *We have*

$$\sum_{(t,24)=2} \sum_{2 \parallel f} \sum_{\gamma \in \Gamma_{n,t,f}/\operatorname{SL}(2,\mathbb{Z})} J(\gamma)$$
$$= -\frac{n^{3/2-k}}{16} \begin{cases} A^*_{2,0}(n) + C_{1,0}(n) & \text{if } n \equiv 1 \bmod 12, \\ C_{1,0}(n) & \text{if } n \equiv 5 \bmod 12, \\ A_{1,0}(n) + C^*_{2,0}(n) & \text{if } n \equiv 7 \bmod 12, \\ C^*_{2,0}(n) & \text{if } n \equiv 11 \bmod 12. \end{cases}$$

Proof. Let class representatives $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be chosen as per Lemma 26. In particular, we have $d \equiv (t/2) \bmod 8$. Let c' and t' denote the odd parts of c and t, respectively. By (22) and Lemma 43, we have

$$J(\gamma) = \frac{n^{k+1/2}}{w_{n,t,f}} \left(\frac{2}{n}\right)\left(\frac{c'}{d}\right) e^{2\pi i r c't'/12} e^{-2\pi i r(t'-1)/8} \frac{\rho^{1/2-k}}{\rho - \overline{\rho}},$$

where $\rho = (t + \sqrt{t^2 - 4n^2})/2$. We check directly using the quadratic reciprocity law that

$$\left(\frac{c'}{d}\right) e^{-2\pi i r(t'-1)/8} = \left(\frac{d}{c'}\right)\left(\frac{8}{t'}\right) \begin{cases} 1 & \text{if } t' \equiv 1 \bmod 4, \\ i\left(\frac{-4}{rc'}\right) & \text{if } t' \equiv 3 \bmod 4. \end{cases} \tag{51}$$

If $3 \nmid f$, then

$$\begin{aligned}
&\Big(\frac{c'}{d}\Big) e^{2\pi i r c' t'/12} e^{-2\pi i r(t'-1)/8} \\
&\quad = \frac{1}{2}\Big(\frac{d}{c'}\Big)\Big(\frac{8}{t'}\Big)\Big(\sqrt{3}\Big(\frac{12}{rc't'}\Big) + i\Big(\frac{-4}{rc't'}\Big)\Big)\Big(\delta_1(t') + i\delta_3(t')\Big(\frac{-4}{rc'}\Big)\Big) \\
&\quad = \frac{1}{2}\Big(\frac{d}{c'}\Big)\Big(\frac{8}{t'}\Big)\Big(\delta_3(t') + i\delta_1(t')\Big(\frac{-4}{rc'}\Big)\Big) \\
&\qquad + \frac{\sqrt{3}}{2}\Big(\frac{d}{c'}\Big)\Big(\frac{24}{t'}\Big)\Big(\frac{12}{rc'}\Big)\Big(\delta_1(t') + i\delta_3(t')\Big(\frac{-4}{rc'}\Big)\Big),
\end{aligned}$$

where $\delta_1(t')$ and $\delta_3(t')$ are defined by (41). If $3 \mid f$, then $3 \mid c'$ and

$$\begin{aligned}
\Big(\frac{c'}{d}\Big) e^{2\pi i r c' t'/12} e^{-2\pi i r(t'-1)/8} &= i\Big(\frac{d}{c'}\Big)\Big(\frac{8}{t'}\Big)\Big(\frac{-4}{rc't'/3}\Big)\Big(\delta_1(t') + i\delta_3(t')\Big(\frac{-4}{rc'}\Big)\Big) \\
&= -\Big(\frac{d}{c'}\Big)\Big(\frac{8}{t'}\Big)\Big(\delta_3(t') + i\delta_1(t')\Big(\frac{-4}{rc'}\Big)\Big).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{(t,24)=2}\sum_{2\| f}\sum_{\gamma} J(\gamma) \\
&= \frac{n^{k+1/2}}{2}\Big(\frac{2}{n}\Big)\sum_{(t,24)=2}\Big(\Big(\frac{8}{t'}\Big)\Big(\delta_3(t')L_{1,1}(n,t) + i\delta_1(t')\Big(\frac{-4}{r}\Big)L_{-1,1}(n,t)\Big) \\
&\qquad + \sqrt{3}\Big(\frac{24}{t'}\Big)\Big(\frac{12}{r}\Big)\Big(\delta_1(t')L_{3,1}(n,t) + i\delta_3(t')\Big(\frac{-4}{r}\Big)L_{-3,1}(n,t)\Big)\Big).
\end{aligned} \tag{52}$$

By Lemma 33, $L_{1,1}(n,t)$ is nonzero only when $t+2n$ is a square. If $t+2n = u^2$ is indeed a square, then we have $u^2/2 = t' + n \equiv 0, 2 \bmod 8$. Then the condition $\delta_3(t') \neq 0$ forces u to satisfy

$$\begin{cases} 4 \mid u & \text{if } n \equiv 1 \bmod 4, \\ 2 \parallel u & \text{if } n \equiv 3 \bmod 4, \end{cases} \tag{53}$$

and also

$$\Big(\frac{8}{t'}\Big) = \Big(\frac{-4}{n}\Big)\Big(\frac{8}{n}\Big). \tag{54}$$

Furthermore, since $3 \nmid t$, when $n \equiv 2 \bmod 3$, we must have $3 \mid u$.

Similarly, $L_{-1,1}(n,t)$ is nonzero only when $2n-t$ is a square. If $2n-t=u^2$ is indeed a square, then the condition $\delta_1(t)=1$ forces (53) and (54) to hold. Furthermore, if $n\equiv 2 \bmod 3$, then we must have $3\mid u$. Thus, by Lemma 33, when $n\equiv 1 \bmod 4$,

$$\begin{aligned}
&\Big(\frac{2}{n}\Big)\sum_{(t,24)=2}\Big(\frac{8}{t'}\Big)\Big(\delta_3(t')L_{1,1}(n,t)+i\delta_1(t')\Big(\frac{-4}{r}\Big)L_{-1,1}(n,t)\Big)\\
&\quad=\frac{1}{4}\lambda_1(n)\sum_{4|u,3\nmid u}\big(M_{1,0}(n,u)-M'_{1,0}(n,u)\big)\\
&\qquad+\frac{1}{4}\sum_{12|u}\Big(1+\Big(\frac{n}{3}\Big)\Big)\big(M_{1,0}(n,u)-M'_{1,0}(n,u)\big),
\end{aligned}\tag{55}$$

and when $n\equiv 3 \bmod 4$,

$$\begin{aligned}
&\Big(\frac{2}{n}\Big)\sum_{(t,24)=2}\Big(\frac{8}{t'}\Big)\Big(\delta_3(t')L_{1,1}(n,t)+i\delta_1(t')\Big(\frac{-4}{r}\Big)L_{-1,1}(n,t)\Big)\\
&\quad=-\frac{1}{4}\lambda_1(n)\sum_{2\|u,3\nmid u}\big(M_{1,0}(n,u)-M'_{1,0}(n,u)\big)\\
&\qquad-\frac{1}{4}\sum_{2\|u,3|u}\Big(1+\Big(\frac{n}{3}\Big)\Big)\big(M_{1,0}(n,u)-M'_{1,0}(n,u)\big),
\end{aligned}\tag{56}$$

where the sums run over all positive integers u satisfying $u^2<4n$ and the specified conditions and $\lambda_1(n)$ is defined by (45).

Likewise, $L_{3,1}(n,t)$ (resp., $L_{-3,1}(n,t)$) is nonzero only when $(t+2n)/3$ (resp., $(2n-t)/3$) is a square. If $(t+2n)/3=u^2$ (resp., $(2n-t)/3=u^2$) is indeed a square, then $t'+n\equiv 0,6 \bmod 8$ (resp., $n-t'\equiv 0,6 \bmod 8$). The condition $\delta_1(t')\neq 0$ (resp., $\delta_3(t')\neq 0$) forces that

$$\begin{cases} 2\parallel u & \text{if } n\equiv 1 \bmod 4,\\ 4\mid u & \text{if } n\equiv 3 \bmod 4,\end{cases}$$

and also checking case by case, we find that

$$\Big(\frac{24}{t'}\Big)=\Big(\frac{24}{n}\Big).$$

Then by Lemma 33, for $n \equiv 1 \bmod 4$,

$$\begin{aligned}(57)\quad &\left(\frac{2}{n}\right) \sum_{(t,24)=2} \left(\frac{24}{t'}\right)\Big(\delta_1(t')L_{3,1}(n,t) + i\delta_3(t')\left(\frac{-4}{r}\right)L_{-3,1}(n,t)\Big)\\ &= \frac{1}{4}\left(\frac{12}{n}\right)\sum_{2\|u}\big(M_{3,0}(n,u) - M'_{3,0}(n,u)\big),\end{aligned}$$

and for $n \equiv 3 \bmod 4$,

$$\begin{aligned}(58)\quad &\left(\frac{2}{n}\right) \sum_{(t,24)=2} \left(\frac{24}{t'}\right)\Big(\delta_1(t')L_{3,1}(n,t) + i\delta_3(t')\left(\frac{-4}{r}\right)L_{-3,1}(n,t)\Big)\\ &= \frac{1}{4}\left(\frac{12}{n}\right)\sum_{4|u}\big(M_{3,0}(n,u) - M'_{3,0}(n,u)\big),\end{aligned}$$

where the sums run over all positive integers u such that $9u^2 < 12n$ and the specified conditions are met. Substituting (55)–(58) into (52) and simplifying, we obtain the claimed formula. ∎

3.4.11. Case $(t,24) = 2$ *and* $4 \parallel f$.

LEMMA 45. *We have*

$$\sum_{(t,24)=2}\sum_{4\|f}\sum_{\gamma\in\Gamma_{n,t,f}/\,\mathrm{SL}(2,\mathbb{Z})} J(\gamma)$$
$$= -\frac{n^{3/2-k}}{32}\begin{cases}(2A_{1,1}(n) - C_{2,0}(n) + C^*_{3,0}(n)) & \text{if } n \equiv 1 \bmod 24,\\ (C_{2,0}(n) - C^*_{3,0}(n)) & \text{if } n \equiv 5 \bmod 24,\\ (A_{2,0}(n) - A^*_{3,0}(n) + 2C_{1,1}(n)) & \text{if } n \equiv 7 \bmod 24,\\ 2C_{1,1}(n) & \text{if } n \equiv 11 \bmod 24,\\ (2A_{1,1}(n) + C_{2,0}(n) - C^*_{3,0}(n)) & \text{if } n \equiv 13 \bmod 24,\\ (-C_{2,0}(n) + C^*_{3,0}(n)) & \text{if } n \equiv 17 \bmod 24,\\ (-A_{2,0}(n) + A^*_{3,0}(n) + 2C_{1,1}(n)) & \text{if } n \equiv 19 \bmod 24,\\ 2C_{1,1}(n) & \text{if } n \equiv 23 \bmod 24.\end{cases}$$

3.4.12. Case $(t,24)=2$ and $8 \mid f$.

LEMMA 46. *Assume that $v \geq 3$. When $n \equiv 1 \bmod 4$,*

$$\sum_{(t,24)=2} \sum_{2^v \| f} \sum_{\gamma \in \Gamma_{n,t,f}/\,\mathrm{SL}(2,\mathbb{Z})} J(\gamma)$$
$$= n^{3/2-k} \Big(\frac{8}{n}\Big)^{v-1} \Big(\frac{\lambda_1(n)}{16} A_{1,v-1}(n) - \frac{(-1)^{v-1}}{2^{v+2}} C_{v-1,1}(n) - \frac{(-1)^{v-1}}{2^{v+3}} C^*_{v,0}(n) \Big),$$

and when $n \equiv 3 \bmod 4$,

$$\sum_{(t,24)=2} \sum_{2^v \| f} \sum_{\gamma \in \Gamma_{n,t,f}/\,\mathrm{SL}(2,\mathbb{Z})} J(\gamma)$$
$$= n^{3/2-k} \Big(\frac{8}{n}\Big)^{v-1} \Big(-\frac{\lambda_1(n)}{2^{v+2}} A_{v-1,1}(n) - \frac{\lambda_1(n)}{2^{v+3}} A^*_{v,0}(n) + \frac{(-1)^{v-1}}{16} C_{1,v-1}(n) \Big),$$

where $\lambda_1(n)$ is defined by (45).

3.4.13. Case $(t,24)=6$ and $2 \nmid f$.

LEMMA 47. *We have*

$$\sum_{(t,24)=6} \sum_{f \,\mathrm{odd}} \sum_{\gamma \in \Gamma_{n,t,f}/\,\mathrm{SL}(2,\mathbb{Z})} J(\gamma) = -\frac{n^{3/2-k}}{8} \begin{cases} B^*_1(n) & \text{if } n \equiv 1 \bmod 3, \\ 0 & \text{if } n \equiv 2 \bmod 3. \end{cases}$$

3.4.14. Case $(t,24)=6$ and $2 \,\|\, f$.

LEMMA 48. *We have*

$$\sum_{(t,24)=6} \sum_{2 \| f} \sum_{\gamma \in \Gamma_{n,t,f}/\,\mathrm{SL}(2,\mathbb{Z})} J(\gamma) = -\frac{n^{3/2-k}}{8} \begin{cases} 0 & \text{if } n \equiv 1 \bmod 3, \\ A^*_{2,0}(n) & \text{if } n \equiv 5 \bmod 12, \\ A_{1,0}(n) & \text{if } n \equiv 11 \bmod 12. \end{cases}$$

3.4.15. Case $(t,24)=6$ and $4 \,\|\, f$.

LEMMA 49. *We have*

$$\sum_{(t,24)=6} \sum_{4 \| f} \sum_{\gamma} J(\gamma) = -\frac{n^{3/2-k}}{16} \begin{cases} 0 & \text{if } n \equiv 1 \bmod 3, \\ 2A_{1,1}(n) & \text{if } n \equiv 5 \bmod 12, \\ (-A_{2,0}(n) + A^*_{3,0}(n)) & \text{if } n \equiv 11 \bmod 24, \\ (A_{2,0}(n) - A^*_{3,0}(n)) & \text{if } n \equiv 23 \bmod 24. \end{cases}$$

3.4.16. Case $(t,24)=6$ *and* $8 \mid f$.

LEMMA 50. *Let* $v \geq 3$ *be an integer. When* $n \equiv 1 \bmod 3$,

$$\sum_{(t,24)=6} \sum_{2^v \| f} \sum_{\gamma} J(\gamma) = 0.$$

When $n \equiv 5 \bmod 12$,

$$\sum_{(t,24)=6} \sum_{2^v \| f} \sum_{\gamma} J(\gamma) = n^{3/2-k} \left(\frac{8}{n}\right)^{v-1} \frac{A_{1,v-1}(n)}{8}.$$

When $n \equiv 11 \bmod 12$,

$$\sum_{(t,24)=6} \sum_{2^v \| f} \sum_{\gamma} J(\gamma) = -n^{3/2-k} \left(\frac{8}{n}\right)^{v-1} \left(\frac{A_{v-1,1}(n)}{2^{v+1}} + \frac{A_{v,0}(n)}{2^{v+2}}\right).$$

§4. Traces of Hecke operators on $S_{2k}^{\text{new}}(6)$

Let W_e, $e = 1, 2, 3, 6$, be the Atkin–Lehner involutions on $S_{2k}(6)$. For $\epsilon_2, \epsilon_3 \in \{\pm 1\}$, let $S_{2k}(6, \epsilon_2, \epsilon_3)$ be the Atkin–Lehner eigensubspace of $S_{2k}(6)$ with eigenvalues ϵ_2 and ϵ_3 for W_2 and W_3, respectively, and let $S_{2k}^{\text{new}}(6, \epsilon_2, \epsilon_3)$ be its newform subspace. We have

$$\begin{aligned}
&\operatorname{tr}\big(T_n \mid S_{2k}^{\text{new}}(6,\epsilon_2,\epsilon_3)\big) \\
&\quad = \operatorname{tr}\big(T_n \mid S_{2k}(6,\epsilon_2,\epsilon_3)\big) - \operatorname{tr}\big(T_n \mid S_{2k}(3,\epsilon_3)\big) \\
&\qquad - \operatorname{tr}\big(T_n \mid S_{2k}(2,\epsilon_2)\big) + \operatorname{tr}\big(T_n \mid S_{2k}(1)\big) \\
&\quad = \frac{1}{4}\Big(\operatorname{tr}\big(T_n \mid S_{2k}(6)\big) + \epsilon_2 \operatorname{tr}\big(T_n W_2 \mid S_{2k}(6)\big) + \epsilon_3 \operatorname{tr}\big(T_n W_3 \mid S_{2k}(6)\big) \\
&\qquad + \epsilon_6 \operatorname{tr}\big(T_n W_6 \mid S_{2k}(6)\big)\Big) - \frac{1}{2}\Big(\operatorname{tr}\big(T_n \mid S_{2k}(3)\big) \\
&\qquad + \epsilon_3 \operatorname{tr}\big(T_n W_3 \mid S_{2k}(3)\big)\Big) \\
&\qquad - \frac{1}{2}\Big(\operatorname{tr}\big(T_n \mid S_{2k}(2)\big) + \epsilon_2 \operatorname{tr}\big(T_n W_2 \mid S_{2k}(2)\big)\Big) + \operatorname{tr}\big(T_n \mid S_{2k}(1)\big),
\end{aligned} \tag{59}$$

where $\epsilon_6 = \epsilon_2\epsilon_3$. Thus, in order to obtain a formula for the trace of T_n on $S_{2k}^{\text{new}}(6, \epsilon_2, \epsilon_3)$, we need to evaluate $\operatorname{tr}(T_n W_e \mid S_{2k}(N))$ for various N and e.

The traces of $\operatorname{tr}(T_n \mid S_{2k}(N))$ have been computed by [5] and [12], while those of $\operatorname{tr}(T_n W_e \mid S_{2k}(N))$ are given by [27]. The formulas are summarized in the following proposition. Note that in [27], the numbers of optimal embeddings are expressed simply as the number of solutions to some congruence equations. Here we use the formula for the numbers of optimal embedding from [17, Section 1]. Note also that here we give the formulas only in cases relevant to our discussion; the general cases are much more complicated.

PROPOSITION 51. *Let N be a positive square-free integer, let e be a positive divisor of N, and let k be an integer greater than 1. For a discriminant $\Delta < 0$ of an imaginary quadratic order, we let Δ_0 denote the discriminant of the field $\mathbb{Q}(\sqrt{\Delta})$, and for a prime p, let*

$$\alpha(\Delta, p) = \begin{cases} 1 + (\frac{\Delta_0}{p}) & \text{if } p \nmid (\Delta/\Delta_0), \\ 2 & \text{if } p \mid (\Delta/\Delta_0). \end{cases}$$

Then for a positive divisor e of N and a positive integer n relatively prime to N, we have

$$\begin{aligned} &\operatorname{tr}\big(T_n W_e \bigm| S_{2k}(N)\big) \\ &\quad = -\frac{1}{2e^{k-1}} \sum_{\Delta = e^2u^2 - 4en < 0} \sum_{\substack{g^2 \mid (\Delta/\Delta_0), \\ (g,e)=1}} \prod_{p \mid (N/e)} \alpha(g^2\Delta_0, p) \\ &\qquad \cdot H(g^2\Delta_0) \frac{\tau^{2k-1} - \overline{\tau}^{2k-1}}{\tau - \overline{\tau}} \\ &\qquad - 2^{\omega(N)-1}\delta_1(e) \sum_{a \mid n} \min(a, n/a)^{2k-1} + \frac{2k-1}{12}\delta_2(e,n) n^{k-1} \prod_{p \mid N}(p+1), \end{aligned}$$

where $\tau = (eu + \sqrt{\Delta})/2$, $\omega(N)$ is the number of prime divisors of N and

$$\delta_1(e) = \begin{cases} 1 & \text{if } e = 1, \\ 0 & \text{otherwise}, \end{cases} \qquad \delta_2(e,n) = \begin{cases} 1 & \text{if } e = 1 \text{ and } n \text{ is a square}, \\ 0 & \text{else}. \end{cases}$$

Using this formula, we now compute the trace of T_n on $S_{2k}(6, \epsilon_2, \epsilon_3)$.

PROPOSITION 52. *Let n be a positive integer relatively prime to* 6. *Then for $\epsilon_2, \epsilon_3 \in \{\pm 1\}$, the trace of T_n on $S_{2k}^{\text{new}}(\Gamma_0(6), \epsilon_2, \epsilon_3)$ is*

$$
\begin{aligned}
-\frac{1}{8} &\sum_{u^2<4n,(g,6)=1} \Big(1 - \Big(\frac{\Delta_0}{2}\Big)\Big)\Big(1 - \Big(\frac{\Delta_0}{3}\Big)\Big) H(g^2\Delta_0) P_k(1,n,u) \\
&+ \frac{\epsilon_2}{8 \cdot 2^{k-1}} \sum_{4u^2<8n,(g,3)=1} \Big(1 - \Big(\frac{\Delta_0}{3}\Big)\Big) H(g^2\Delta_0) P_k(2,n,u) \\
&+ \frac{\epsilon_3}{8 \cdot 3^{k-1}} \sum_{9u^2<12n,(g,2)=1} \Big(1 - \Big(\frac{\Delta_0}{2}\Big)\Big) H(g^2\Delta_0) P_k(3,n,u) \\
&- \frac{\epsilon_2\epsilon_3}{8 \cdot 6^{k-1}} \sum_{36u^2<24n,g} H(g^2\Delta_0) P_k(6,n,u) + \frac{2k-1}{24}\delta(n) n^{k-1},
\end{aligned}
$$

where

$$
\delta(n) = \begin{cases} 1 & \text{if } n \text{ is a square}, \\ 0 & \text{otherwise.} \end{cases}
$$

Proof. We first consider the terms

$$
\begin{aligned}
&\frac{1}{4}\operatorname{tr}\big(T_n \mid S_{2k}(6)\big) - \frac{1}{2}\operatorname{tr}\big(T_n \mid S_{2k}(3)\big) \\
&\qquad - \frac{1}{2}\operatorname{tr}\big(T_n \mid S_{2k}(2)\big) + \operatorname{tr}\big(T_n \mid S_{2k}(1)\big)
\end{aligned} \tag{60}
$$

in (59). By Proposition 51, we have

$$
\begin{aligned}
\operatorname{tr}\big(T_n \mid S_{2k}(1)\big) = &-\frac{1}{2}\sum_u \sum_g H(g^2\Delta_0) P_k(1,n,u) \\
&- \frac{1}{2}\sum_{a|n} \min(a, n/a)^{2k-1} + \frac{2k-1}{12}\delta(n) n^{k-1},
\end{aligned} \tag{61}
$$

$$
\begin{aligned}
&\operatorname{tr}\big(T_n \mid S_{2k}(2)\big) \\
&\quad = -\frac{1}{2}\sum_u \Big(2\sum_{2|g} H(g^2\Delta_0) + \Big(1 + \Big(\frac{\Delta_0}{2}\Big)\Big)\sum_{2\nmid g} H(g^2\Delta_0)\Big) \\
&\qquad \times P_k(1,n,u) - \sum_{a|n} \min(a, n/a)^{2k-1} + \frac{2k-1}{4}\delta(n) n^{k-1},
\end{aligned} \tag{62}
$$

$$\begin{aligned}
&\operatorname{tr}\bigl(T_n \bigm| S_{2k}(3)\bigr) \\
&\quad = -\frac{1}{2}\sum_u \Bigl(2\sum_{3|g} H(g^2\Delta_0) + \Bigl(1+\Bigl(\frac{\Delta_0}{3}\Bigr)\Bigr)\sum_{3\nmid g} H(g^2\Delta_0)\Bigr) \\
&\qquad \times P_k(1,n,u) - \sum_{a|n} \min(a,n/a)^{2k-1} + \frac{2k-1}{3}\delta(n)n^{k-1},
\end{aligned} \tag{63}$$

$$\begin{aligned}
&\operatorname{tr}\bigl(T_n \bigm| S_{2k}(6)\bigr) \\
&\quad = -\frac{1}{2}\sum_u \Bigl(4\sum_{(g,6)=6} H(g^2\Delta_0) + 2\sum_{(g,6)=3}\Bigl(1+\Bigl(\frac{\Delta_0}{2}\Bigr)\Bigr)H(g^2\Delta_0) \\
&\qquad + 2\sum_{(g,6)=2}\Bigl(1+\Bigl(\frac{\Delta_0}{3}\Bigr)\Bigr)H(g^2\Delta_0) \\
&\qquad + \sum_{(g,6)=1}\Bigl(1+\Bigl(\frac{\Delta_0}{2}\Bigr)\Bigr)\Bigl(1+\Bigl(\frac{\Delta_0}{3}\Bigr)\Bigr)H(g^2\Delta_0)\Bigr)P_k(1,n,u) \\
&\qquad + 2\sum_{a|n}\min(a,n/a)^{2k-1} + (2k-1)\delta(n)n^{k-1},
\end{aligned} \tag{64}$$

where

$$\delta(n) = \begin{cases} 1 & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

Substituting (61)–(64) into (60) and simplifying, we get

$$\begin{aligned}
&\frac{1}{4}\operatorname{tr}\bigl(T_n \bigm| S_{2k}(6)\bigr) - \frac{1}{2}\operatorname{tr}\bigl(T_n \bigm| S_{2k}(3)\bigr) \\
&\qquad - \frac{1}{2}\operatorname{tr}\bigl(T_n \bigm| S_{2k}(2)\bigr) + \operatorname{tr}\bigl(T_n \bigm| S_{2k}(1)\bigr) \\
&\quad = -\frac{1}{8}\sum_u \sum_{(g,6)=1}\Bigl(1-\Bigl(\frac{\Delta_0}{2}\Bigr)\Bigr)\Bigl(1-\Bigl(\frac{\Delta_0}{3}\Bigr)\Bigr)H(g^2\Delta_0)P_k(1,n,u) \\
&\qquad + \frac{2k-1}{24}\delta(n)n^{k-1}.
\end{aligned} \tag{65}$$

We next consider the terms

$$\frac{\epsilon_2}{4}\operatorname{tr}\bigl(T_nW_2 \bigm| S_{2k}(6)\bigr) - \frac{\epsilon_2}{2}\operatorname{tr}\bigl(T_nW_2 \bigm| S_{2k}(2)\bigr)$$

in (59). We have, by Proposition 51,

$$\operatorname{tr}\big(T_nW_2 \mid S_{2k}(6)\big) = -\frac{1}{2\cdot 2^{k-1}} \sum_{4u^2<8n} \Big(2\sum_{3|g} H(g^2\Delta_0) + \sum_{3\nmid g}\Big(1+\Big(\frac{\Delta_0}{3}\Big)\Big)H(g^2\Delta_0)\Big)P_k(2,n,u)$$

and

$$\operatorname{tr}\big(T_nW_2 \mid S_{2k}(2)\big) = -\frac{1}{2\cdot 2^{k-1}} \sum_{4u^2<8n} \sum_{g} H(g^2\Delta_0)P_k(2,n,u).$$

Thus,

$$\begin{aligned} &\frac{\epsilon_2}{4}\operatorname{tr}\big(T_nW_2 \mid S_{2k}(6)\big) - \frac{\epsilon_2}{2}\operatorname{tr}\big(T_nW_2 \mid S_{2k}(2)\big) \\ &\quad = \frac{\epsilon_2}{8\cdot 2^{k-1}} \sum_{4u^2<8n} \sum_{(g,3)=1} \Big(1-\Big(\frac{\Delta_0}{3}\Big)\Big)H(g^2\Delta_0)P_k(2,n,u). \end{aligned} \tag{66}$$

Similarly, we have

$$\begin{aligned} &\frac{\epsilon_3}{4}\operatorname{tr}\big(T_nW_3 \mid S_{2k}(6)\big) - \frac{\epsilon_3}{2}\operatorname{tr}\big(T_nW_3 \mid S_{2k}(3)\big) \\ &\quad = \frac{\epsilon_2}{8\cdot 3^{k-1}} \sum_{9u^2<12n} \sum_{(g,2)=1} \Big(1-\Big(\frac{\Delta_0}{2}\Big)\Big)H(g^2\Delta_0)P_k(3,n,u) \end{aligned} \tag{67}$$

and

$$\frac{\epsilon_6}{4}\operatorname{tr}\big(T_nW_6 \mid S_{2k}(6)\big) = -\frac{\epsilon_6}{8\cdot 6^{k-1}} \sum_{36u^2<24n} \sum_{g} H(g^2\Delta_0)P_k(6,n,u). \tag{68}$$

Summarizing (59) and (65)–(68), we obtain the claimed formula. This proves the proposition. ∎

§5. Comparison of traces

In this section, we will prove

$$\operatorname{tr}\big(T_{n^2} \mid \mathcal{S}_{r,s}(1)\big) = \Big(\frac{12}{n}\Big)\operatorname{tr}\Big(T_n \;\Big|\; S_{2k}^{\mathrm{new}}\Big(6, -\Big(\frac{8}{r}\Big), -\Big(\frac{12}{r}\Big)\Big)\Big)$$

for positive integers n relatively prime to 6 and thereby establish Theorem 1. The verification is done case by case according to the residue of n modulo 24.

Here we work out only the cases $n \equiv 1 \bmod 24$ and $n \equiv 11 \bmod 24$ and omit the proof for the other cases.

Recall that the Hecke operator T_{n^2} on $\mathcal{S}_{r,s}(1)$ is defined by

$$T_{n^2} : f \longmapsto n^{k-3/2} \sum_{ad=n, a|d} af \mid [\mathscr{M}^*_{(d/a)^2}] = n^{k-3/2} \sum_{a|m} af \mid [\mathscr{M}^*_{(n/a^2)^2}],$$

where m is the positive integer such that n/m^2 is square-free and where the action of $[\mathscr{M}^*_{(n/a^2)^2}] = [\mathscr{M}_{(n/a^2)^2}(1)^*]$ is defined as in Lemma 9. We first consider the case $n \equiv 1 \bmod 24$. Write $\ell = n/a^2$. Note that we have $\ell \equiv 1 \bmod 24$. According to Propositions 15, 21, and 23 and Lemmas 24, 36–42, and 44–50, we have

$$\begin{aligned}
\operatorname{tr}[\mathscr{M}^*_{\ell^2}] = \frac{\sqrt{\ell}}{8} &\sum_{e=1,2,3,6} \left(\frac{-4e}{r}\right)\left(1 - \left(\frac{-e}{3}\right)\right)\big(H(-4e\ell) - H(-e\ell)\big) \\
&- \frac{\ell^{3/2-k}}{4}\big(A_{0,0}(\ell) + C_{0,0}(\ell)\big) - \frac{\ell^{3/2-k}}{8} D_0(\ell) - \frac{\ell^{3/2-k}}{4} B_0(\ell) \\
&- \frac{\ell^{3/2-k}}{8} D_1^*(\ell) - \frac{\ell^{3/2-k}}{8}\big(A^*_{2,0}(\ell) + C_{1,0}(\ell)\big) \\
&- \frac{\ell^{3/2-k}}{16}\big(2A_{1,1}(\ell) - C_{2,0}(\ell) + C^*_{3,0}(\ell)\big) \\
&+ \ell^{3/2-k} \sum_{v \geq 3} \left(\frac{1}{8} A_{1,v-1}(\ell) + \frac{(-1)^v}{2^{v+1}} C_{v-1,1}(\ell) + \frac{(-1)^v}{2^{v+2}} C^*_{v,0}(\ell)\right) \\
&- \frac{\ell^{3/2-k}}{4} B_1^*(\ell) + \delta_1(\ell)\frac{2k-1}{24},
\end{aligned} \tag{69}$$

where

$$\delta_1(\ell) = \begin{cases} 1 & \text{if } \ell = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For the first sum above, we observe that

$$P_k(e, \ell, 0) = (e\ell)^{k-1} \frac{i^{2k-1} - i^{1-2k}}{2i} = (-e\ell)^{k-1} = -\left(\frac{-4}{r}\right)(e\ell)^{k-1},$$

and the sum, including the factor $\sqrt{\ell}/8$ in front, can be written as

$$-\frac{\ell^{3/2-k}}{8} \sum_{e=1,2,3,6} \frac{1}{e^{k-1}} \left(\frac{e}{r}\right)\left(1 - \left(\frac{-e}{3}\right)\right)\big(H(-4e\ell) - H(-e\ell)\big) P_k(e, \ell, 0).$$

Now we have $H(-e\ell)=0$ except for $e=3$, in which case we have

$$H(-12\ell)-H(-3\ell)=\Big(1-\Big(\frac{-3\ell}{2}\Big)\Big)H(-3\ell).$$

Thus,

$$\begin{aligned}
&\frac{\sqrt{\ell}}{8}\sum_{e=1,2,3,6}\Big(\frac{-4e}{r}\Big)\Big(1-\Big(\frac{-e}{3}\Big)\Big)\big(H(-4e\ell)-H(-e\ell)\big)\\
&\quad=-\frac{\ell^{3/2-k}}{8}\bigg(\Big(1-\Big(\frac{-4\ell}{2}\Big)\Big)\Big(1-\Big(\frac{-4\ell}{3}\Big)\Big)H(-4\ell)P_k(1,\ell,0)\\
&\qquad+\frac{1}{2^{k-1}}\Big(\frac{8}{r}\Big)\Big(1-\Big(\frac{-8\ell}{3}\Big)\Big)H(-8\ell)P_k(2,\ell,0)\\
&\qquad+\frac{1}{3^{k-1}}\Big(\frac{12}{r}\Big)\Big(1-\Big(\frac{-3\ell}{2}\Big)\Big)H(-3\ell)P_k(3,\ell,0)\\
&\qquad+\frac{1}{6^{k-1}}\Big(\frac{24}{r}\Big)H(-24\ell)P_k(6,\ell,0)\bigg).
\end{aligned}\tag{70}$$

We next consider the terms $A_{i,j}(\ell)$ in (69). Here we remind the reader that the summations $\sum_u\sum_g$ in the subsequent discussion are all subject to the condition $(\ell,u,\Delta/g^2\Delta_0)=1$ inherited from the definition of $A_{i,j}(\ell)$.

For an integer u contributing to the sums $A_{i,j}(\ell)$, let $\Delta=u^2-4\ell$, and let Δ_0 be the discriminant of the field $\mathbb{Q}(\sqrt{\Delta})$. For $A_{0,0}(\ell)$, we have

$$\Big(\frac{\Delta_0}{2}\Big)=\Big(\frac{5}{2}\Big)=-1,$$

and by Lemma 35,

$$A_{0,0}(\ell)=\frac{1}{2}\sum_{2\nmid u}\Big(1-\Big(\frac{\Delta_0}{2}\Big)\Big)\Big(1-\Big(\frac{\Delta_0}{3}\Big)\Big)\sum_{(g,6)=1}H(g^2\Delta_0)P_k(1,\ell,u).\tag{71}$$

For $A^*_{2,0}(\ell)$, we have $4\mid\Delta_0$ and

$$A^*_{2,0}(\ell)=\sum_{4\mid u}\Big(1-\Big(\frac{\Delta_0}{2}\Big)\Big)\Big(1-\Big(\frac{\Delta_0}{3}\Big)\Big)\sum_{(g,6)=1}H(g^2\Delta_0)P_k(1,\ell,u).\tag{72}$$

For $A_{1,j}(\ell)$, we let $v=\mathrm{ord}_2(\Delta/\Delta_0)/2$, that is, the 2-adic valuation of the conductor of Δ. If $v=1$, then since $32\mid(u^2-4n)$ for u with $2\parallel u$, we have $8\mid\Delta_0$ and

$$-\sum_{2\parallel g}H\Big(\frac{\Delta}{g^2}\Big)=-\Big(1-\Big(\frac{\Delta_0}{2}\Big)\Big)\sum_{2\nmid g}H(g^2\Delta_0).\tag{73}$$

If $v \geq 2$, then

$$
\begin{aligned}
&-\sum_{2\|g} H\Big(\frac{\Delta}{g^2}\Big) + \sum_{4|g} H\Big(\frac{\Delta}{g^2}\Big) \\
&\quad = -\sum_{2^{v-1}\|g} H(g^2\Delta_0) + \sum_{j=0}^{v-2}\sum_{2^j\|g} H(g^2\Delta_0) \\
&\quad = \sum_{2\nmid g}\Big(-2^{v-2}\Big(2-\Big(\frac{\Delta_0}{2}\Big)\Big) + \sum_{j=1}^{v-2} 2^{j-1}\Big(2-\Big(\frac{\Delta_0}{2}\Big)\Big)+1\Big) H(g^2\Delta_0) \\
&\quad = -\sum_{2\nmid g}\Big(1-\Big(\frac{\Delta_0}{2}\Big)\Big) H(g^2\Delta_0).
\end{aligned} \tag{74}
$$

It follows from (73), (74), and Lemma 35 that

$$
\begin{aligned}
&-A_{1,1}(\ell) + \sum_{2\leq j<\infty} A_{1,j}(\ell) \\
&\qquad = -\sum_{2\|u}\sum_{(g,6)=1}\Big(1-\Big(\frac{\Delta_0}{2}\Big)\Big)\Big(1-\Big(\frac{\Delta_0}{3}\Big)\Big) H(g^2\Delta_0) P_k(1,\ell,u).
\end{aligned} \tag{75}
$$

In summary, from (71), (72), and (75), we get

$$
\begin{aligned}
&-\frac{1}{4}A_{0,0}(\ell) - \frac{1}{8}A^*_{2,0}(\ell) - \frac{1}{8}A_{1,1}(\ell) + \frac{1}{8}\sum_{j\geq 2} A_{1,j}(\ell) \\
&\qquad = -\frac{1}{8}\sum_{u\neq 0}\sum_{(g,6)=1}\Big(1-\Big(\frac{\Delta_0}{2}\Big)\Big)\Big(1-\Big(\frac{\Delta_0}{3}\Big)\Big) H(g^2\Delta_0) P_k(1,\ell,u),
\end{aligned} \tag{76}
$$

where the outer summation runs over all nonzero integers u with $u^2 < 4\ell$ and the inner summation runs over all positive integers g dividing the conductor of $\Delta = u^2 - 4\ell$ such that $(g,6) = 1$ and $(\ell, u, \Delta/g^2\Delta_0) = 1$.

The terms $B_j(\ell)$ in (69) are easy to deal with. By Lemma 35, we have

$$
\begin{aligned}
&-B_0(\ell) - B_1^*(\ell) \\
&\qquad = -\frac{1}{2\cdot 2^{k-1}}\Big(\frac{8}{r}\Big)\sum_{u\neq 0}\sum_{(g,3)=1}\Big(1-\Big(\frac{\Delta_0}{3}\Big)\Big) H(g^2\Delta_0) P_k(2,\ell,u).
\end{aligned} \tag{77}
$$

Here, as above, the outer summation runs over all nonzero integers u with $4u^2 < 8\ell$ and the inner summation runs over all positive integers g dividing the conductor of $\Delta = 4u^2 - 8\ell$ such that $(g,3) = 1$ and $(\ell, u, \Delta/g^2\Delta_0) = 1$.

We next consider $C_{i,j}(\ell)$ in (69). If $\Delta = 9u^2 - 12n$ with $2 \nmid u$, then $\Delta_0 \equiv 5 \bmod 8$, and we have

$$C_{0,0}(\ell) = \frac{1}{2 \cdot 3^{k-1}} \Big(\frac{12}{r}\Big) \sum_{2 \nmid u} \Big(1 - \Big(\frac{\Delta_0}{2}\Big)\Big) \sum_g H(g^2\Delta_0) P_k(3, \ell, u). \tag{78}$$

If $\Delta = 9u^2 - 12n$ with $2 \parallel u$, then $4 \mid \Delta_0$ and

$$C_{1,0}(\ell) = \frac{1}{2 \cdot 3^{k-1}} \Big(\frac{12}{r}\Big) \sum_{2 \parallel u} \Big(1 - \Big(\frac{\Delta_0}{2}\Big)\Big) \sum_g H(g^2\Delta_0) P_k(3, \ell, u). \tag{79}$$

If $\Delta = 9u^2 - 12n$ with $2^2 \parallel u$, then $\Delta_0 \equiv 1 \bmod 8$, and for any odd integer g, we have $H(4g^2\Delta_0) = H(g^2\Delta_0)$. Thus,

$$\begin{aligned} &C_{2,0}(\ell) - C_{2,1}(\ell) \\ &\quad = 0 \\ &\quad = \frac{1}{3^{k-1}} \Big(\frac{12}{r}\Big) \sum_{2^2 \parallel u} \Big(1 - \Big(\frac{\Delta_0}{2}\Big)\Big) \sum_{(g,2)=1} H(g^2\Delta_0) P_k(3, \ell, u). \end{aligned} \tag{80}$$

If $\Delta = 9u^2 - 12n$ with $8 \mid u$, then $\Delta_0 \equiv 5 \bmod 8$, and for any odd integer g, we have $H(4g^2\Delta_0) = 3H(g^2\Delta_0)$. It follows that $C_{j,0}(\ell) = 3C_{j,1}(\ell)$ and that

$$\begin{aligned} &-\frac{1}{16} C^*_{3,0}(\ell) - \sum_{3 \le j < \infty} \frac{(-1)^j}{2^{j+2}} C_{j,1}(\ell) + \sum_{3 \le j < \infty} \frac{(-1)^j}{2^{j+2}} C^*_{j,0}(\ell) \\ &\quad = -\frac{3}{16} C^*_{3,1}(\ell) - \sum_{3 \le j < \infty} \frac{(-1)^j}{2^{j+2}} C_{j,1}(\ell) + \sum_{3 \le j < \infty} \frac{3(-1)^j}{2^{j+2}} \sum_{j \le i < \infty} C_{i,1}(\ell) \\ &\quad = \sum_{3 \le i < \infty} C_{i,1}(\ell) \Big(-\frac{3}{16} - \frac{(-1)^i}{2^{i+2}} + \sum_{3 \le j \le i} \frac{3(-1)^j}{2^{j+2}}\Big) = -\frac{1}{4} \sum_{3 \le i < \infty} C_{i,1}(\ell) \\ &\quad = -\frac{1}{8 \cdot 3^{k-1}} \Big(\frac{12}{r}\Big) \sum_{8 \mid u, u \ne 0} \Big(1 - \Big(\frac{\Delta_0}{2}\Big)\Big) \sum_{(g,2)=1} H(g^2\Delta_0) P_k(3, \ell, u). \end{aligned} \tag{81}$$

Combining (78)–(81), we get

$$
\begin{aligned}
&-\frac{1}{4}C_{0,0}(\ell)-\frac{1}{8}C_{1,0}(\ell)+\frac{1}{16}C_{2,0}(\ell)-\frac{1}{16}C^*_{3,0}(\ell)\\
&\qquad+\sum_{v\ge 3}\Big(\frac{(-1)^v}{2^{v+1}}C_{v-1,1}(\ell)+\frac{(-1)^v}{2^{v+2}}C^*_{v,0}(\ell)\Big)\\
&\qquad=-\frac{1}{8\cdot 3^{k-1}}\Big(\frac{12}{r}\Big)\sum_{u\neq 0}\sum_{(g,2)=1}\Big(1-\Big(\frac{\Delta_0}{2}\Big)\Big)H(g^2\Delta_0)P_k(3,\ell,u).
\end{aligned}
\tag{82}
$$

Again, here the outer summation runs over all nonzero integers u with $9u^2<12\ell$, and the inner summation runs over all positive divisors of the conductor of Δ satisfying $(g,2)=1$ and $(\ell,u,\Delta/g^2\Delta_0)=1$.

The terms $D_j(\ell)$ in (69) are easy. We have

$$
-D_0(\ell)-D_1^*(\ell)=-\frac{1}{6^{k-1}}\Big(\frac{24}{r}\Big)\sum_{u\neq 0}\sum_{g}H(g^2\Delta_0)P_k(6,\ell,u). \tag{83}
$$

Here the summation over g is subject to the condition $(\ell,u,\Delta/g^2\Delta_0)=1$.

Substituting (70), (76), (77), (82), and (83) into (69), we get

$$
\begin{aligned}
\operatorname{tr}[\mathscr{M}_{\ell^2}]=&-\frac{\ell^{3/2-k}}{8}\sum_{u}\sum_{(g,6)=1}\Big(1-\Big(\frac{\Delta_0}{2}\Big)\Big)\Big(1-\Big(\frac{\Delta_0}{3}\Big)\Big)H(g^2\Delta_0)P_k(1,\ell,u)\\
&-\frac{\ell^{3/2-k}}{8\cdot 2^{k-1}}\Big(\frac{8}{r}\Big)\sum_{u}\sum_{(g,3)=1}\Big(1-\Big(\frac{\Delta_0}{3}\Big)\Big)H(g^2\Delta_0)P_k(2,\ell,u)\\
&-\frac{\ell^{3/2-k}}{8\cdot 3^{k-1}}\Big(\frac{12}{r}\Big)\sum_{u}\sum_{(g,2)=1}\Big(1-\Big(\frac{\Delta_0}{2}\Big)\Big)H(g^2\Delta_0)P_k(3,\ell,u)\\
&-\frac{\ell^{3/2-k}}{8\cdot 6^{k-1}}\Big(\frac{24}{r}\Big)\sum_{u}\sum_{g}H(g^2\Delta_0)P_k(6,\ell,u)+\delta_1(\ell)\frac{2k-1}{24}.
\end{aligned}
\tag{84}
$$

Here the summations $\sum_g$ are all subject to the condition $(\ell,u,\Delta/g^2\Delta_0)=1$. Now recall from the definition of T_{n^2} that

$$
\operatorname{tr}\big(T_{n^2}\mid\mathcal{S}_{r,s}(1)\big)=n^{k-3/2}\sum_{a\mid m}a\operatorname{tr}[\mathscr{M}_{(n/a^2)^2}],
$$

where m is the integer such that n/m^2 is square-free. Thus, we need to sum up (84) over all $\ell = n/a^2$. Observe that

$$P_k(e, n/a^2, u) = a^{2-2k} P_k(e, n, au).$$

Take the fourth sum in (84), for example. We find that

$$\begin{aligned} &n^{k-3/2} \sum_{a|m} a(n/a^2)^{3/2-k} \sum_u \sum_{(n/a^2, u, \Delta/g^2\Delta_0)=1} H(g^2\Delta_0) P_k(6, n/a^2, u) \\ &= \sum_{a|m} \sum_u \sum_{(n, au, ah)=a} H\Big(\frac{a^2u^2 - 24n}{(ah)^2}\Big) P_k(6, n, au) \\ &= \sum_u \sum_g H(g^2\Delta_0) P_k(6, n, u), \end{aligned}$$

where there is no longer any restriction to g in the inner sum. Similar formulas hold for other three sums in (84). Also,

$$n^{k-3/2} \sum_{a|m} a\delta_1(n/a^2) = \begin{cases} n^{k-1} & \text{if } n \text{ is a square}, \\ 0 & \text{otherwise}. \end{cases}$$

Therefore, we find that

$$\begin{aligned} &\operatorname{tr}\big(T_{n^2} \bigm| \mathcal{S}_{r,s}(1)\big) \\ &= -\frac{1}{8} \sum_u \sum_{(g,6)=1} \Big(1 - \Big(\frac{\Delta_0}{2}\Big)\Big)\Big(1 - \Big(\frac{\Delta_0}{3}\Big)\Big) H(g^2\Delta_0) P_k(1, n, u) \\ &\quad - \frac{1}{8 \cdot 2^{k-1}} \Big(\frac{8}{r}\Big) \sum_u \sum_{(g,3)=1} \Big(1 - \Big(\frac{\Delta_0}{3}\Big)\Big) H(g^2\Delta_0) P_k(2, n, u) \\ &\quad - \frac{1}{8 \cdot 3^{k-1}} \Big(\frac{12}{r}\Big) \sum_u \sum_{(g,2)=1} \Big(1 - \Big(\frac{\Delta_0}{2}\Big)\Big) H(g^2\Delta_0) P_k(3, n, u) \\ &\quad - \frac{1}{8 \cdot 6^{k-1}} \Big(\frac{24}{r}\Big) \sum_u \sum_g H(g^2\Delta_0) P_k(6, n, u) + \frac{2k-1}{24} \delta(n) n^{k-1}. \end{aligned}$$

Comparing the trace of T_n on $S_{2k}(6, \epsilon_2, \epsilon_3)$ given in Proposition 52, we find that

$$\operatorname{tr}\big(T_{n^2} \bigm| \mathcal{S}_{r,s}(1)\big) = \Big(\frac{12}{n}\Big) \operatorname{tr}\Big(T_n \mid S_{2k}\Big(6, -\Big(\frac{8}{r}\Big), -\Big(\frac{12}{r}\Big)\Big)\Big).$$

This proves the case $n \equiv 1 \bmod 24$. (Note that in this case $(12/n) = 1$.) We now consider the case $n \equiv 11 \bmod 24$.

Assume that $n \equiv 11 \bmod 24$. By Propositions 15, 21, and 23 and Lemmas 24, 36–42, and 44–50, we have

$$\begin{aligned}
\operatorname{tr}[\mathscr{M}^*_{\ell^2}] = \frac{\sqrt{\ell}}{8} \sum_{e=1,2,3,6} \Big(\frac{-4e}{r}\Big)\Big(1 - \Big(\frac{-4e\ell}{3}\Big)\Big)\big(H(-4e\ell) - H(-e\ell)\big) \\
+ \ell^{3/2-k}\Big(-\frac{1}{2}A_{0,0}(\ell) - \frac{1}{4}A_{1,0}(\ell) + \frac{1}{8}A_{2,0}(\ell) - \frac{1}{8}A^*_{3,0}(\ell) \\
+ \sum_{3\le v<\infty}\Big(\frac{(-1)^v}{2^v}A_{v-1,1}(\ell) + \frac{(-1)^v}{2^{v+1}}A_{v,0}(\ell)\Big) - \frac{1}{8}B^*_0(\ell) \\
- \frac{1}{4}C_{0,0}(\ell) - \frac{1}{8}C^*_{2,0}(\ell) - \frac{1}{8}C_{1,1}(\ell) + \frac{1}{8}\sum_{v\ge 2} C_{1,v}(\ell) - \frac{1}{8}D^*_0(\ell)\Big).
\end{aligned}$$

The computation for $A_{i,j}(\ell)$ (resp., $C_{i,j}(\ell)$) is parallel to that for $C_{i,j}(\ell)$ (resp., $A_{i,j}(\ell)$) in the case $n \equiv 1 \bmod 24$. The computation for $B^*_0(\ell)$ and $D^*_0(\ell)$ is almost the same as before. (The reader is reminded that there is a difference of $1/2$ between the case $n \equiv 1 \bmod 3$ and the case $n \equiv 2 \bmod 3$ in the formulas for $A_{i,j}(\ell)$ and $B_j(\ell)$ in Lemma 35.) For the first sum above, instead of (70), we have

$$\begin{aligned}
&\frac{\sqrt{\ell}}{8} \sum_{e=1,2,3,6} \Big(\frac{-4e}{r}\Big)\Big(1 - \Big(\frac{-e\ell}{3}\Big)\Big)\big(H(-4e\ell) - H(-e\ell)\big) \\
&= -\frac{\ell^{3/2-k}}{8}\Big(\Big(1 - \Big(\frac{-\ell}{2}\Big)\Big)\Big(1 - \Big(\frac{-\ell}{3}\Big)\Big)H(-\ell)P_k(1,\ell,0) \\
&\quad + \frac{1}{2^{k-1}}\Big(\frac{8}{r}\Big)\Big(1 - \Big(\frac{-8\ell}{3}\Big)\Big)H(-8\ell)P_k(2,\ell,0) \\
&\quad + \frac{1}{3^{k-1}}\Big(\frac{12}{r}\Big)\Big(1 - \Big(\frac{-12\ell}{2}\Big)\Big)H(-12\ell)P_k(3,\ell,0) \\
&\quad + \frac{1}{6^{k-1}}\Big(\frac{24}{r}\Big)H(-24\ell)P_k(6,\ell,0)\Big).
\end{aligned}$$

Therefore, (84) remains valid, from which we conclude that

$$\operatorname{tr}\big(T_{n^2} \mid \mathcal{S}_{r,s}(1)\big) = \operatorname{tr}\Big(T_n \Big| S_{2k}\Big(6, -\Big(\frac{8}{r}\Big), -\Big(\frac{12}{r}\Big)\Big)\Big).$$

This proves the case $n \equiv 11 \bmod 24$. We skip the proof of the other cases.

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