Nagoya Math. J. **212** (2013), 1–17
DOI 10.1215/00277630-2324129

SHARP ESTIMATES OF THE POTENTIAL KERNEL FOR THE HARMONIC OSCILLATOR WITH APPLICATIONS

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Abstract. We prove qualitatively sharp estimates of the potential kernel for the harmonic oscillator. These bounds are then used to show that the $L^p - L^q$ estimates of the associated potential operator obtained recently by Bongioanni and Torrea are in fact sharp.

§1. Introduction

The study of the potential theory for the d-dimensional harmonic oscillator

$$\mathcal{H} = -\Delta + \|x\|^2$$

has recently been initiated by Bongioanni and Torrea [2]. The multidimensional Hermite functions h_k are eigenfunctions of $\mathcal{H}$, and we have $\mathcal{H}h_k = (2|k| + d)h_k$. The operator $\mathcal{H}$ has a natural self-adjoint extension, here still denoted by $\mathcal{H}$, whose spectral decomposition is given by h_k.

The integral kernel $G_t(x, y)$ of the Hermite semigroup $\{\exp(-t\mathcal{H}) : t > 0\}$ is known explicitly to be

$$\begin{aligned} G_t(x,y) &= \sum_{n=0}^{\infty} e^{-(2n+d)t} \sum_{|k|=n} h_k(x)h_k(y) \\ &= \big(2\pi \sinh(2t)\big)^{-d/2} \exp\Big(-\frac{1}{4}\big[\tanh(t)\|x+y\|^2 + \coth(t)\|x-y\|^2\big]\Big). \end{aligned}$$

(See [6] for this symmetric variant of the formula.)

Received November 15, 2011. Accepted May 10, 2012.
First published online June 20, 2013.
2010 Mathematics Subject Classification. Primary 47G40; Secondary 31C15.
The work of both authors was partially supported by Ministry of Science and Higher Education (Poland) grant N N201 417839.

Given $\sigma > 0$, consider the negative power $\mathcal{H}^{-\sigma}$, which is a contraction on $L^2(\mathbb{R}^d)$. It is easily seen that $\mathcal{H}^{-\sigma}$ coincides in $L^2(\mathbb{R}^d)$ with the *potential operator*

$$\mathcal{I}^\sigma f(x) = \int_{\mathbb{R}^d} \mathcal{K}^\sigma(x,y) f(y)\, dy, \tag{1}$$

where the *potential kernel* is given by

$$\mathcal{K}^\sigma(x,y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty G_t(x,y) t^{\sigma-1}\, dt. \tag{2}$$

Note that all the spaces $L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, are contained in the natural domain of $\mathcal{I}^\sigma$ consisting of those functions f for which the integral in (1) converges x almost everywhere (see [4, Section 2]).

The main result of this paper, Theorem 2.4, provides qualitatively sharp estimates of the potential kernel (2). As an application of this result, we prove sharpness of the $L^p - L^q$ estimates for the potential operator (1) obtained recently by Bongioanni and Torrea [2, Theorem 8] (see Theorem 3.1 below).

Recall that an operator T defined on $L^p(\mathbb{R}^d)$ for some $1 \le p \le \infty$, with values in the space of measurable functions on $\mathbb{R}^d$, is said to be of weak type (p,q), $1 \le q < \infty$, provided that

$$\big|\big\{x \in \mathbb{R}^d \colon \big|Tf(x)\big| > \lambda\big\}\big| \le C\big(\|f\|_p/\lambda\big)^q, \tag{3}$$

with $C > 0$ independent of $f \in L^p(\mathbb{R}^d)$ and $\lambda > 0$. The restricted weak type (p,q) of T means that (3) holds for $f = \chi_E$, where E is any measurable subset of $\mathbb{R}^d$ of finite measure. By definition, weak type (p,∞) coincides with strong type (p,∞); that is, the estimate $\|Tf\|_\infty \le C\|f\|_p$, $f \in L^p(\mathbb{R}^d)$. In terms of Lorentz spaces, the weak type (p,q) is equivalent to the boundedness from $L^p(\mathbb{R}^d)$ to $L^{q,\infty}(\mathbb{R}^d)$, and the restricted weak type (p,q) is characterized by the boundedness from $L^{p,1}(\mathbb{R}^d)$ to $L^{q,\infty}(\mathbb{R}^d)$ (see [1, Chapter 4, Section 4]). Strong type (p,q) means, of course, the L^p-L^q boundedness.

The notation $X \lesssim Y$ will be used to indicate that $X \le CY$ with a positive constant C independent of significant quantities; we will write $X \simeq Y$ when simultaneously $X \lesssim Y$ and $Y \lesssim X$. We will also use the notation $X \simeq\simeq Y \exp(-cZ)$ to indicate that there exist positive constants C, c_1 and c_2, independent of significant quantities, such that

$$C^{-1} Y \exp(-c_1 Z) \le X \le CY \exp(-c_2 Z).$$

Further, in a number of places, we will use natural and self-explanatory generalizations of the "$\simeq\simeq$" relation, for instance, in connection with certain integrals involving exponential factors. In such cases, the exact meaning will be clear from the context. By convention, "$\simeq\simeq$" is understood as "$\simeq$" whenever no exponential factors are involved.

We write $\log^+$ for the positive part of the logarithm, and we write $\vee, \wedge$ for the operations of taking maximum and minimum, respectively.

§2. Estimates of the potential kernel

We begin with two technical results describing the behavior of the integrals

$$I_A(T) = \int_T^\infty t^A \exp(-t)\, dt, \quad T > 0,$$

$$J_A(T,S) = \int_T^S t^A \exp(-t)\, dt, \quad 0 < T < S < \infty.$$

Notice that $I_A(T)$ dominates $J_A(T,S)$. The following lemma is a refinement of [4, Lemma 2.1] (see also [5, Lemma 1.1]).

LEMMA 2.1. *Let $A \in \mathbb{R}$ and $\gamma > 0$ be fixed. Then*

$$I_A(\gamma T) \simeq T^A \exp(-\gamma T), \quad T \geq 1, \tag{4}$$

and for $0 < T < 1$,

$$I_A(\gamma T) \simeq \begin{cases} T^{A+1}, & A < -1, \\ \log(2/T), & A = -1, \\ 1, & A > -1. \end{cases}$$

Proof. We assume that $\gamma = 1$. From the proof it will be clear that the estimates are true for any $\gamma > 0$. The case $0 < T < 1$ was treated in the proof of [4, Lemma 2.1], so we consider $T \geq 1$ and focus on showing (4). The lower bound in (4) is straightforward; we have

$$\begin{aligned} I_A(T) &> \int_T^{2T} t^A e^{-t}\, dt \\ &\gtrsim T^A \int_T^{2T} e^{-t}\, dt = T^A(e^{-T} - e^{-2T}) \gtrsim T^A e^{-T}, \quad T \geq 1. \end{aligned}$$

It remains to prove the upper bound,

$$\int_T^\infty t^A e^{-t}\, dt \lesssim T^A e^{-T}, \quad T \geq 1, \tag{5}$$

and here we assume that $A>0$, since for $A\le 0$ we have $t^A\le T^A$, $t>T\ge 1$, and the conclusion is trivial. Choosing T_A such that for $T\ge T_A$ one has

$$\int_{2T}^{\infty} t^A e^{-t}\,dt \le \frac{1}{2}\int_T^{\infty} t^A e^{-t}\,dt,$$

we can write

$$\begin{aligned}\int_T^{\infty} t^A e^{-t}\,dt &\le \int_T^{2T} t^A e^{-t}\,dt + \int_{2T}^{\infty} t^A e^{-t}\,dt\\ &\le C T^A e^{-T} + \frac{1}{2}\int_T^{\infty} t^A e^{-t}\,dt, \quad T\ge T_A.\end{aligned}$$

This implies (5) for $T\ge T_A$ and, consequently, for all $T\ge 1$. ∎

LEMMA 2.2. *Let $A\in\mathbb{R}$ and $\gamma>0$ be fixed. Then for $0<T<S\le 2T$ we have*

$$T^A(S-T)\exp(-2\gamma T) \lesssim J_A(\gamma T,\gamma S) \lesssim T^A(S-T)\exp(-\gamma T), \tag{6}$$

while for $S>2T>0$ we have $J_A(\gamma T,\gamma S)\simeq I_A(\gamma T)$ when $S\ge 2$, and

$$J_A(\gamma T,\gamma S)\simeq \begin{cases} T^{A+1}, & A<-1,\\ \log(S/T), & A=-1,\\ S^{A+1}, & A>-1,\end{cases}$$

when $0<S<2$.

Proof. As in the proof of Lemma 2.1, it is enough to deal with the case $\gamma=1$. The bounds for $T<S\le 2T$ follow since then $\int_T^S t^A e^{-t}\,dt \simeq T^A\int_T^S e^{-t}\,dt$ and

$$(S-T)e^{-2T} \le \int_T^S e^{-t}\,dt \le (S-T)e^{-T}.$$

Assume now that $S>2T$. Clearly, $J_A(T,S)<I_A(T)$. On the other hand, if $T\ge 1$, then

$$J_A(T,S) > \int_T^{2T} t^A e^{-t}\,dt \gtrsim T^A\int_T^{2T} e^{-t}\,dt \gtrsim T^A e^{-T} \gtrsim I_A(T),$$

the last estimate being a consequence of (4). When $0<T<1$, we distinguish two subcases. If $S\ge 2$, then again, $J_A(T,S)\gtrsim \int_T^2 t^A\,dt \gtrsim I_A(T)$. If $2T<S<2$, then $J_A(T,S)\simeq\int_T^S t^A\,dt$, and evaluating the last integral, we arrive at the claimed bounds for $J_A(T,S)$. ∎

We note that (4) and (6) may be written slightly less precisely as

$$I_A(\gamma T) \simeq\simeq \exp(-cT), \quad T \geq 1,$$
$$J_A(\gamma T, \gamma S) \simeq\simeq T^A (S-T) \exp(-cT), \quad 0 < T < S \leq 2T,$$

respectively. This fact will be used again without further mention.

We now apply Lemmas 2.1 and 2.2 to prove qualitatively sharp estimates of the integral

$$E_A(T,S) = \int_0^1 t^A \exp(-Tt^{-1} - St)\, dt, \quad 0 < T, S < \infty.$$

The following result provides, in particular, a refinement and generalization of [3, Lemma 2.4].

LEMMA 2.3. *Let $A \in \mathbb{R}$ be fixed. Then*

$$E_A(T,S) \simeq\simeq \exp\big(-c\sqrt{T(T \vee S)}\big) \times \begin{cases} T^{A+1}, & A < -1, \\ 1 + \log^+ \frac{1}{T(T\vee S)}, & A = -1, \\ (S \vee 1)^{-A-1}, & A > -1, \end{cases}$$

uniformly in $T, S > 0$.

Proof. We first estimate $E_A(T,S)$ in terms of the integrals I_A and J_A. For $0 < S \leq 2T$, we have

$$E_A(T,S) \simeq\simeq \int_0^1 t^A \exp(-cTt^{-1})\, dt \simeq T^{A+1} \int_{cT}^\infty u^{-A-2} e^{-u}\, du$$
$$= T^{A+1} I_{-A-2}(cT),$$

where the second relation follows by the change of variable $t = cT/u$. When $S > 2T$, we change the variable $t = u\sqrt{T/S}$ and we get

$$E_A(T,S) = \Big(\frac{T}{S}\Big)^{(A+1)/2} \int_0^{\sqrt{S/T}} u^A \exp\big(-\sqrt{TS}(u+u^{-1})\big)\, du \equiv \mathcal{J}_1 + \mathcal{J}_2,$$

where $\mathcal{J}_1$ and $\mathcal{J}_2$ come from splitting the integration over the intervals $(0,1)$ and $(1, \sqrt{S/T})$, respectively. Then

$$\mathcal{J}_1 \simeq\simeq \Big(\frac{T}{S}\Big)^{(A+1)/2} \int_0^1 u^A \exp(-c\sqrt{TS}u^{-1})\, du \simeq T^{A+1} \int_{c\sqrt{TS}}^\infty z^{-A-2} e^{-z}\, dz$$
$$= T^{A+1} I_{-A-2}(c\sqrt{TS})$$

and

$$\mathcal{J}_2 \simeq\simeq \Big(\frac{T}{S}\Big)^{(A+1)/2} \int_1^{\sqrt{S/T}} u^A \exp(-c\sqrt{TS}u)\, du \simeq S^{-A-1} \int_{c\sqrt{TS}}^{cS} z^A e^{-z}\, dz$$
$$= S^{-A-1} J_A(c\sqrt{TS}, cS).$$

Summing up, we have

$$E_A(T,S) \simeq\simeq T^{A+1} I_{-A-2}\big(c\sqrt{T(T\vee S)}\big) + \chi_{\{S>2T\}} S^{-A-1} J_A(c\sqrt{TS}, cS),$$

uniformly in $S, T > 0$. In the next step, we describe the behavior of the two terms here by means of Lemmas 2.1 and 2.2.

From Lemma 2.1 it follows that

$$T^{A+1} I_{-A-2}\big(c\sqrt{T(T\vee S)}\big) \simeq\simeq T^{A+1} \exp\big(-c\sqrt{T(T\vee S)}\big), \quad T(T\vee S) \ge 1$$

(here, and also in analogous places below, c on the left-hand side should be understood as a *given* constant) and that

$$T^{A+1} I_{-A-2}\big(c\sqrt{T(T\vee S)}\big) \simeq \begin{cases} T^{A+1}, & A<-1, \\ \log(\frac{4}{T(T\vee S)}), & A=-1, \\ (\frac{T}{T\vee S})^{(A+1)/2}, & A>-1. \end{cases} \quad T(T\vee S)\le 1,$$

The term $S^{-A-1}J_A(c\sqrt{TS}, cS)$ comes into play when $S > 2T$, and in this case we use Lemma 2.2 to write the bounds

$$S^{-A-1} J_A(c\sqrt{TS}, cS) \simeq \chi_{\{S\ge 2\}}\Phi_1 + \chi_{\{S<2\}}\Phi_2,$$

where

$$\Phi_1 = S^{-A-1} I_A(c\sqrt{TS}), \qquad \Phi_2 = \begin{cases} (T/S)^{(A+1)/2}, & A<-1, \\ \log(\frac{S}{T}), & A=-1, \\ 1, & A>-1. \end{cases}$$

By Lemma 2.1,

$$\Phi_1 \simeq\simeq S^{-A-1}\exp(-c\sqrt{TS}), \quad TS \geq 1,$$

$$\Phi_1 \simeq \begin{cases} (T/S)^{(A+1)/2}, & A < -1, \\ \log(\frac{4}{TS}), & A = -1, \quad TS \leq 1, \\ S^{-A-1}, & A > -1. \end{cases}$$

To proceed, it is convenient to consider each of the cases $A < -1$, $A = -1$, and $A > -1$ separately.

If $A < -1$, then

$$\begin{aligned} E_A(T,S) \simeq\simeq\ & \chi_{\{2>S>2T\}}\Big(\frac{T}{S}\Big)^{(A+1)/2} \\ & + \begin{cases} T^{A+1}\exp(-c\sqrt{T(T\vee S)}), & T(T\vee S) \geq 1, \\ T^{A+1}, & T(T\vee S) < 1, \end{cases} \\ & + \chi_{\{S>2T\}}\chi_{\{S\geq 2\}} \begin{cases} T^{A+1}\exp(-c\sqrt{TS}), & TS \geq 1, \\ (\frac{T}{S})^{(A+1)/2}, & TS < 1. \end{cases} \end{aligned}$$

Here the first and third terms are insignificant compared with the second one. In case of the third summand, this is because $A < -1$ and $(T/S)^{(A+1)/2} < T^{A+1}$ for $TS < 1$. A similar argument is used for the first one. The required estimates of $E_A(T,S)$ follow.

If $A = -1$, then

$$\begin{aligned} E_{-1}(T,S) \simeq\simeq\ & \chi_{\{2>S>2T\}}\log\frac{S}{T} + \begin{cases} \exp(-c\sqrt{T(T\vee S)}), & T(T\vee S) \geq 1, \\ \log(\frac{4}{T(T\vee S)}), & T(T\vee S) < 1, \end{cases} \\ & + \chi_{\{S>2T\}}\chi_{\{S\geq 2\}} \begin{cases} \exp(-c\sqrt{TS}), & TS \geq 1, \\ \log(\frac{4}{TS}), & TS < 1. \end{cases} \end{aligned}$$

Similar to the case of $A < -1$, here also the first and third terms are insignificant compared with the second one. This is clear for the third summand, and for the first one this is because $\log(S/T) < \log(4/(TS))$ when $S < 2$. Thus, the desired bounds of $E_{-1}(T,S)$ also follow.

Finally, we consider the case $A > -1$, which is less direct than the previous two. We have

$$E_A(T,S) \simeq\simeq \chi_{\{2>S>2T\}} + \begin{cases} T^{A+1}\exp(-c\sqrt{T(T\vee S)}), & T(T\vee S)\ge 1,\\ (\frac{T}{T\vee S})^{(A+1)/2}, & T(T\vee S)<1, \end{cases}$$
$$+ \chi_{\{S>2T\}}\chi_{\{S\ge 2\}} \begin{cases} T^{A+1}\exp(-c\sqrt{TS}), & TS\ge 1,\\ S^{-A-1}, & TS<1. \end{cases}$$

Observe that here the relation $\simeq\simeq$ remains valid if the sum of the first and the third terms is replaced by the comparable (in the sense of $\simeq$) expression

$$\chi_{\{S>2T\}} \begin{cases} T^{A+1}\exp(-c\sqrt{TS}), & TS\ge 1,\\ (S\vee 1)^{-A-1}, & TS<1. \end{cases}$$

Taking into account that $T^{A+1}\exp(-c\sqrt{TS}) \simeq\simeq S^{-A-1}\exp(-c\sqrt{TS})$ for $TS\ge 1$, we conclude that

$$E_A(T,S) \simeq\simeq \begin{cases} (T\vee S)^{-A-1}\exp(-c\sqrt{T(T\vee S)}), & T(T\vee S)\ge 1,\\ (\frac{T}{T\vee S})^{(A+1)/2}, & T(T\vee S)<1, \end{cases}$$
$$+ \chi_{\{S>2T\}} \begin{cases} S^{-A-1}\exp(-c\sqrt{TS}), & TS\ge 1,\\ (S\vee 1)^{-A-1}, & TS<1. \end{cases}$$

Now, if $T\ge S$ and $T(T\vee S)=T^2<1$, then $(T/(T\vee S))^{1/2}=1\simeq 1/(S\vee 1)$, while for $T<S$ and $T(T\vee S)=TS<1$, we have $(T/(T\vee S))^{1/2} = (T/S)^{1/2}<1/(S\vee 1)$. Therefore,

$$E_A(T,S) \simeq\simeq \begin{cases} (T\vee S)^{-A-1}\exp(-c\sqrt{T(T\vee S)}), & T(T\vee S)\ge 1,\\ (S\vee 1)^{-A-1}, & T(T\vee S)<1. \end{cases}$$

We claim that this implies that

$$E_A(T,S) \simeq\simeq (S\vee 1)^{-A-1}\exp\big(-c\sqrt{T(T\vee S)}\big),$$

which are precisely the required estimates.

To justify the claim, it is enough to recall that $A>-1$ and to observe that if $T\ge S$ and $T(T\vee S)=T^2\ge 1$, then

$$\begin{aligned} (T\vee S)^{-A-1}\exp\big(-c\sqrt{T(T\vee S)}\big) &= T^{-A-1}\exp(-cT)\\ &\simeq (T\vee 1)^{-A-1}\exp(-cT)\\ &\simeq\simeq (S\vee 1)^{-A-1}\exp(-cT), \end{aligned}$$

while if $T<S$ and $T(T\vee S)=TS\geq 1$ (this forces $S>1$), then

$$(T\vee S)^{-A-1}\exp\big(-c\sqrt{T(T\vee S)}\big)=S^{-A-1}\exp(-c\sqrt{TS})\\ \simeq (S\vee 1)^{-A-1}\exp(-c\sqrt{TS}).$$

The proof is finished. ∎

We are now in a position to prove qualitatively sharp estimates of the potential kernel.

THEOREM 2.4. *For $\sigma>0$, we have*

$$\mathcal{K}^\sigma(x,y)\simeq\simeq \exp\big(-c\|x-y\|\big(\|x\|+\|y\|\big)\big)\\ \times\begin{cases}\|x-y\|^{2\sigma-d}, & \sigma<d/2,\\ 1+\log^+\frac{1}{\|x-y\|(\|x\|+\|y\|)}, & \sigma=d/2,\\ (1+\|x+y\|)^{d-2\sigma}, & \sigma>d/2,\end{cases}$$

uniformly in $x,y\in\mathbb{R}^d$.

Proof. We decompose

$$\Gamma(\sigma)\mathcal{K}^\sigma(x,y)=\int_0^1 G_t(x,y)t^{\sigma-1}\,dt+\int_1^\infty G_t(x,y)t^{\sigma-1}\,dt\\ \equiv \mathcal{J}_0^\sigma(x,y)+\mathcal{J}_\infty^\sigma(x,y).$$

For $0<t<1$, we have $\tanh t\simeq t$, $\coth t\simeq t^{-1}$, $\sinh 2t\simeq t$, and therefore,

$$\mathcal{J}_0^\sigma(x,y)\simeq\simeq E_{\sigma-d/2-1}\big(c\|x-y\|^2,c\|x+y\|^2\big).$$

This combined with Lemma 2.3 shows that the estimates from the statement hold with $\mathcal{K}^\sigma(x,y)$ replaced by $\mathcal{J}_0^\sigma(x,y)$. Further, taking into account that $\tanh t\simeq 1\simeq\coth t$ for $t>1$, we see that

$$\mathcal{J}_\infty^\sigma(x,y)\simeq\simeq\exp\big(-c\big(\|x\|^2+\|y\|^2\big)\big).$$

Thus, $\mathcal{J}_0^\sigma(x,y)$ dominates $\mathcal{J}_\infty^\sigma(x,y)$ in the above decomposition, in the sense that

$$\mathcal{J}_\infty^\sigma(x,y)\lesssim E_{\sigma-d/2-1}\big(c\|x-y\|^2,c\|x+y\|^2\big)$$

for a sufficiently small constant $c>0$. The conclusion follows. ∎

§3. Sharpness of the L^p-L^q boundedness of the potential operator

Given $0<\sigma<d/2$, define the region

$$R=\Big\{\Big(\frac{1}{p},\frac{1}{q}\Big): 0\le \frac{1}{p}\le 1 \text{ and } 0\vee\Big(\frac{1}{p}-\frac{2\sigma}{d}\Big)\le \frac{1}{q}\le 1\wedge\Big(\frac{1}{p}+\frac{2\sigma}{d}\Big)\Big\}$$
$$\setminus\Big(\Big\{\Big(\frac{1}{p},\frac{1}{q}\Big): 0\le\frac{1}{p}\le 1-\frac{2\sigma}{d} \text{ and } \frac{1}{q}=\frac{1}{p}+\frac{2\sigma}{d}\Big\}$$
$$\cup\Big\{\Big(\frac{2\sigma}{d},0\Big),\Big(1,1-\frac{2\sigma}{d}\Big)\Big\}\Big)$$

contained in the unit $(1/p,1/q)$-square $[0,1]^2$ (see Figure 1).

The following result enhances [2, Theorem 8] (see also [4, Theorem 2.3]).

THEOREM 3.1. *Let $d\ge 1$, let $0<\sigma<d/2$, and let $1\le p,q\le\infty$. Then $\mathcal{I}^\sigma\colon L^p(\mathbb{R}^d)\to L^q(\mathbb{R}^d)$ boundedly if and only if $(1/p,1/q)$ lies in the region R.*

On the other hand, $\mathcal{I}^\sigma$ is not even of restricted weak type (p,q) when $(1/p,1/q)$ is not in the closure of R. Moreover, $\mathcal{I}^\sigma$ is of weak type (p,q)

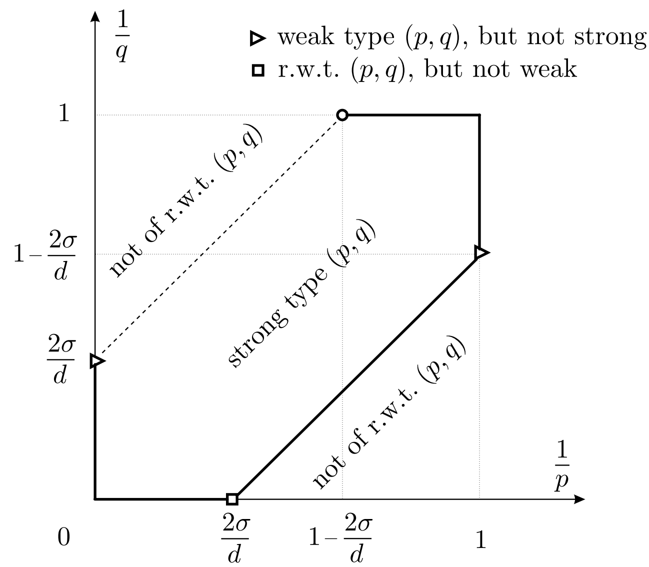

Figure 1: Mapping properties of $\mathcal{I}^\sigma$ for $0<\sigma<d/2$
(r.w.t. = restricted weak type).

for $(1/p,1/q)=(0,2\sigma/d)$ *and* $(1/p,1/q)=(1,1-2\sigma/d)$. *For* $(1/p,1/q)=(2\sigma/d,0)$, *the restricted weak type is true, whereas the weak type fails.*

Before giving the proof, we present a short argument showing [2, (21) and (41)], the result we will apply in a moment.

LEMMA 3.2. *Given* $\sigma>0$,

$$\big\|\mathcal{K}^\sigma(x,\cdot)\big\|_1 \simeq \big(1\vee\|x\|\big)^{-2\sigma}, \quad x\in\mathbb{R}^d.$$

Proof. Using the identity

$$\begin{aligned}\exp(-t\mathcal{H})\mathbf{1}(x) &= \int_{\mathbb{R}^d} G_t(x,y)\,dy\\ &= (\cosh 2t)^{-d/2}\exp\Big(-\frac{1}{2}\tanh(2t)\|x\|^2\Big), \quad x\in\mathbb{R}^d\end{aligned}$$

(see [6, Proposition 3.3]), we may write

$$\begin{aligned}\int_{\mathbb{R}^d}\mathcal{K}^\sigma(x,y)\,dy &= \frac{1}{\Gamma(\sigma)}\int_0^\infty\int_{\mathbb{R}^d} G_t(x,y)\,dy\,t^{\sigma-1}\,dt\\ &= \frac{1}{\Gamma(\sigma)}\int_0^\infty (\cosh 2t)^{-d/2}\exp\Big(-\frac{1}{2}\tanh(2t)\|x\|^2\Big)t^{\sigma-1}\,dt.\end{aligned}$$

Here we split the integration to the intervals $(0,1)$ and $(1,\infty)$ and we denote the resulting integrals by $\mathcal{J}_0$ and $\mathcal{J}_\infty$, respectively. Then, uniformly in $x\in\mathbb{R}^d$,

$$\begin{aligned}\mathcal{J}_0 \simeq\simeq \int_0^1 \exp\big(-ct\|x\|^2\big)t^{\sigma-1}\,dt &= \|x\|^{-2\sigma}\int_0^{\|x\|^2} e^{-ct}s^{\sigma-1}\,dt\\ &\simeq \|x\|^{-2\sigma}\big(\|x\|^{2\sigma}\wedge 1\big)\end{aligned}$$

and

$$\mathcal{J}_\infty \simeq\simeq \int_1^\infty e^{-td}\exp\big(-c\|x\|^2\big)t^{\sigma-1}\,dt = C_{d,\sigma}\exp\big(-c\|x\|^2\big).$$

The conclusion follows. ◻

Proof of Theorem 3.1. We first focus on strong-type inequalities. Then, in view of [2, Theorem 8], what remains to prove are the following two items.

(a) $\mathcal{I}^\sigma$ is not L^p-L^q bounded for $2\sigma/d<1/p<1$ and $0<1/q<1/p-2\sigma/d$.

(b) $\mathcal{I}^\sigma$ is not $L^p - L^q$ bounded for $0 < 1/p < 1 - 2\sigma/d$ and $1/p + 2\sigma/d \le 1/q < 1$.

To justify (a), we fix p and q satisfying the assumed conditions, and we define

$$f(y) = \chi_{\{\|y\|<1\}} \|y\|^{-2\sigma - d/q}.$$

This function is in $L^p(\mathbb{R}^d)$ since $-(2\sigma + d/q)p + d > 0$. However, $\mathcal{I}^\sigma f \notin L^q(\mathbb{R}^d)$. Indeed, considering x such that $\|x\| < 1$ and using the lower bound from Theorem 2.4, we get

$$\begin{aligned}\mathcal{I}^\sigma f(x) &\gtrsim \int_{\|y\|<\|x\|/2} \|x-y\|^{2\sigma-d} \|y\|^{-2\sigma-d/q}\, dy \\ &\gtrsim \|x\|^{2\sigma-d} \int_{\|y\|<\|x\|/2} \|y\|^{-2\sigma-d/q}\, dy = C\|x\|^{-d/q},\end{aligned}$$

and the function $x \mapsto \chi_{\{\|x\|<1\}}\|x\|^{-d/q}$ does not belong to $L^q(\mathbb{R}^d)$.

Proving (b), we may assume that $(1/p, 1/q)$ lies on the critical segment $1/q = 1/p + 2\sigma/d$, $0 < 1/p < 1 - 2\sigma/d$. The case when $1/q > 1/p + 2\sigma/d$ is contained below, in the negative result concerning the restricted weak-type estimate. Define

$$f(y) = \chi_{\{\|y\|>e\}} \|y\|^{-d/p} \big(\log \|y\|\big)^{-1/p - 2\sigma/d}.$$

We have

$$\int_{\mathbb{R}^d} \big|f(y)\big|^p\, dy = C_d \int_e^\infty r^{-1} (\log r)^{-1-2\sigma p/d}\, dr < \infty,$$

so $f \in L^p(\mathbb{R}^d)$. We claim that $\mathcal{I}^\sigma f \notin L^q(\mathbb{R}^d)$. Assuming that $\|x\| > 2e$ and using the lower bound from Theorem 2.4, we write

$$\begin{aligned}\mathcal{I}^\sigma f(x) &\gtrsim \int_{\|x\|/2<\|y\|<\|x\|} \|x-y\|^{2\sigma-d} \\ &\qquad \times \exp\big(-c\|x-y\|\big(\|x\|+\|y\|\big)\big) \|y\|^{-d/p} \big(\log\|y\|\big)^{-1/p-2\sigma/d}\, dy \\ &\gtrsim \|x\|^{-d/p} \big(\log\|x\|\big)^{-1/p-2\sigma/d} \\ &\qquad \times \int_{\|x\|/2<\|y\|<\|x\|} \|x-y\|^{2\sigma-d} \exp\big(-2c\|x-y\|\|x\|\big)\, dy.\end{aligned}$$

As we will see in a moment, the last integral is comparable with $\|x\|^{-2\sigma}$. Thus,

$$\begin{aligned}\mathcal{I}^\sigma f(x) &\gtrsim \|x\|^{-d/p-2\sigma}\big(\log\|x\|\big)^{-1/p-2\sigma/d}\\ &= \|x\|^{-d/q}\big(\log\|x\|\big)^{-1/q}, \quad \|x\|>2e,\end{aligned}$$

and the claim follows.

It remains to analyze the last integral, which we denote by $\mathcal{J}$. Changing the variable $y = x - z/\|x\|$, we get

$$\mathcal{J} = \|x\|^{-2\sigma}\int_{D_x}\|z\|^{2\sigma-d}e^{-2c\|z\|}\,dz,$$

where the set of integration is $D_x = \{z\in\mathbb{R}^d : \|x\|^2/2 < \|x\|x\| - z\| < \|x\|^2\}$. We now observe that D_x contains the ball $B_x = \{z\in\mathbb{R}^d : \|x\|x\|/4 - z\| < \|x\|^2/4\}$. Indeed, if $z\in B_x$, then

$$\begin{aligned}\frac{\|x\|^2}{2} &< \left|\left\|\frac{x\|x\|}{4} - z\right\| - \left\|\frac{3}{4}x\|x\|\right\|\right|\\ &\le \big\|x\|x\| - z\big\| \le \left\|\frac{x\|x\|}{4} - z\right\| + \left\|\frac{3}{4}x\|x\|\right\| < \|x\|^2.\end{aligned}$$

Thus, we have

$$\|x\|^{-2\sigma}\int_{B_x}\|z\|^{2\sigma-d}e^{-2c\|z\|}\,dz \le \mathcal{J} \le \|x\|^{-2\sigma}\int_{\mathbb{R}^d}\|z\|^{2\sigma-d}e^{-2c\|z\|}\,dz.$$

Clearly, the integral over $\mathbb{R}^d$ here is finite. The integral over B_x depends on x only through $\|x\|$. Since the balls B_x are increasing in the sense of $\subset$ when x is moved away from the origin along a fixed line passing through the origin, we see that the integral over B_x is an increasing function of $\|x\|$, which is positive and finite. We conclude that $\mathcal{J}\simeq\|x\|^{-2\sigma}$, $\|x\|>1$, as desired.

We pass to weak-type and restricted weak-type inequalities. Consider first the three "corners" of the boundary of R from the statement of Theorem 3.1. If $(1/p,1/q) = (1, 1-2\sigma/d)$, then the weak type $(1, d/(d-2\sigma))$ holds by [4, Theorem 2.3]. Notice that this property can be expressed in terms of Lorentz spaces by saying that $\mathcal{I}^\sigma$ is *bounded* from $L^1(\mathbb{R}^d)$ to $L^{d/(d-2\sigma),\infty}(\mathbb{R}^d)$. Then $(\mathcal{I}^\sigma)^*$ (the adjoint operator in the Banach space sense) maps boundedly

$(L^{d/(d-2\sigma),\infty}(\mathbb{R}^d))^*$ into $(L^1(\mathbb{R}^d))^* = L^\infty(\mathbb{R}^d)$. Further, the associate space of $L^{d/(d-2\sigma),\infty}(\mathbb{R}^d)$ in the sense of [1, Chapter 1, Definition 2.3] is $L^{d/(2\sigma),1}(\mathbb{R}^d)$ (see [1, Chapter 4, Theorem 4.7]), and by [1, Chapter 1, Theorem 2.9] it can be regarded as a subspace of the dual of $L^{d/(d-2\sigma),\infty}(\mathbb{R}^d)$. Since $(\mathcal{I}^\sigma)^* = \mathcal{I}^\sigma$ by symmetry of the kernel, we infer that $\mathcal{I}^\sigma$ is of restricted weak type $d/(2\sigma), \infty$. On the other hand, the weak type $d/(2\sigma), \infty$ coincides, by definition, with the strong type, so $\mathcal{I}^\sigma$ is not of weak type $d/(2\sigma), \infty$ in view of the strong-type results we already know. This clarifies the situations related to the "corners" $(1, 1-(2\sigma/d))$ and $2\sigma/d, 0$.

Taking into account $(1/p, 1/q) = (0, 2\sigma/d)$, we will show that $\mathcal{I}^\sigma$ is of weak type $(\infty, d/(2\sigma))$. To do that, it is enough to verify the estimate

$$\big|\big\{x \in \mathbb{R}^d : \big|\mathcal{I}^\sigma f(x)\big| > \lambda\big\}\big| \lesssim \Big(\frac{\|f\|_\infty}{\lambda}\Big)^{d/(2\sigma)}, \quad \lambda > 0, f \in L^\infty(\mathbb{R}^d). \tag{7}$$

But this is immediate in view of the bound (see Lemma 3.2)

$$\big\|\mathcal{K}^\sigma(x,\cdot)\big\|_1 \le C\|x\|^{-2\sigma}, \quad x \in \mathbb{R}^d,$$

since then it follows that $|\mathcal{I}^\sigma f(x)| \le C\|x\|^{-2\sigma}\|f\|_\infty$ and, consequently,

$$\big\{x \in \mathbb{R}^d : \big|\mathcal{I}^\sigma f(x)\big| > \lambda\big\} \subset \Big\{x \in \mathbb{R}^d : \|x\| < \Big(C\frac{\|f\|_\infty}{\lambda}\Big)^{1/(2\sigma)}\Big\}.$$

This inclusion leads directly to (7).

Finally, we disprove the restricted weak type in the two triangles (see Figure 1). In the lower triangle we use an *au contraire* argument involving an extension of the Marcinkiewicz interpolation theorem for Lorentz spaces due to Stein and Weiss (see [1, Chapter 4, Theorem 5.5]). Indeed, if $\mathcal{I}^\sigma$ were of restricted weak type (p,q) for some p and q such that $1/q < 1/p - 2\sigma/d$, then by interpolation with a strong-type pair satisfying $1/q = 1/p - 2\sigma/d$, $p > 1$, $q < \infty$, $\mathcal{I}^\sigma$ would be of strong type $(\widetilde{p}, \widetilde{q})$ for some $\widetilde{p}$ and $\widetilde{q}$ corresponding to a point in the lower triangle, a contradiction with (a) above.

To treat the upper triangle, we will give an explicit counterexample. Let for large r

$$f_r(y) = \chi_{\{\|y\|<r\}}.$$

Clearly, we have $\|f_r\|_p \simeq r^{d/p}$. Estimating as in the proof of (b) above, we get

$$
\begin{aligned}
\mathcal{I}^\sigma f_r(x) &\gtrsim \int_{\|x\|/2<\|y\|<\|x\|} \|x-y\|^{2\sigma-d} \\
&\qquad \times \exp\big(-c\|x-y\|\big(\|x\|+\|y\|\big)\big)\chi_{\{\|y\|<r\}}\,dy \\
&\geq \chi_{\{\|x\|<r\}} \int_{\|x\|/2<\|y\|<\|x\|} \|x-y\|^{2\sigma-d}\exp\big(-2c\|x-y\|\|x\|\big)\,dy \\
&\gtrsim \chi_{\{1<\|x\|<r\}}\|x\|^{-2\sigma},
\end{aligned}
$$

uniformly in large r and $x \in \mathbb{R}^d$. Consequently,

$$
\big|\big\{x\in\mathbb{R}^d : \mathcal{I}^\sigma f_r(x) > \lambda\big\}\big| \geq \big|\big\{1<\|x\|<r : \|x\| < (C\lambda)^{-1/(2\sigma)}\big\}\big|
$$

for some $C>0$ independent of r and $\lambda>0$. Taking $\lambda = r^{-2\sigma}$, we conclude that the weak-type (p,q) estimate for $\mathcal{I}^\sigma$ implies that $r^d \lesssim r^{dq/p+2\sigma q}$. This bound, however, fails when $1/q > 1/p + 2\sigma/d$ and $r\to\infty$.

The proof is finished. ∎

For completeness, we remark that in the context of Theorem 3.1, the question of weak/restricted weak type (p,q) inequalities related to the segment $1/q = 1/p + 2\sigma/d$, $1 \le q < 2\sigma/d$, is more subtle and remains open. Considering the case $\sigma > d/2$, the operator $\mathcal{I}^\sigma$ is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ for every $1\le p,q\le\infty$ (see [4, Theorem 2.3]). The behavior of $\mathcal{I}^\sigma$ in the limiting case $\sigma = d/2$ is described by the theorem below. This result enhances [4, Theorem 2.3] when $\sigma = d/2$.

THEOREM 3.3. *Let $d \ge 1$, and let $1 \le p,q \le \infty$. Then $\mathcal{I}^{d/2}$ is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ except for $(p,q)=(\infty,1)$ and $(p,q)=(1,\infty)$. Considering the two singular cases, we have the following.*

(i) *$\mathcal{I}^{d/2}$ is of weak type $(\infty,1)$ but not of strong type $(\infty,1)$.*
(ii) *$\mathcal{I}^{d/2}$ is not of restricted weak type $(1,\infty)$.*

Proof. The L^p-L^q boundedness is contained in [4, Theorem 2.3]. To show (i), we observe that the weak type $(\infty,1)$ holds true since the proof of (7) covers also the case $\sigma = d/2$. The strong type $(\infty,1)$ fails because $\mathcal{I}^{d/2}\mathbf{1} \notin L^1(\mathbb{R}^d)$, as easily seen by means of Lemma 3.2.

It remains to verify (ii). For $0<\varepsilon<1/e$, let $f_\varepsilon(x)=\chi_{\{\|x\|<\varepsilon\}}$. By the lower bound of Theorem 2.4, it follows that

$$\mathcal{I}^{d/2}f_\varepsilon(x)\gtrsim\int_{\|y\|<\varepsilon}\log\frac{1}{\|x-y\|(\|x\|+\|y\|)}\,dy,\quad \|x\|<1/e,$$

uniformly in $\varepsilon<1/e$. Therefore,

$$\begin{aligned}\|\mathcal{I}^{d/2}f_\varepsilon\|_\infty &\gtrsim\int_{\|y\|<\varepsilon}-\log\|y\|\,dy\\ &=C_d\int_0^\varepsilon -r^{d-1}\log r\,dr\gtrsim\varepsilon^d\log\frac{1}{\varepsilon},\quad 0<\varepsilon<1/e,\end{aligned}$$

and we conclude that

$$\frac{\|\mathcal{I}^{d/2}f_\varepsilon\|_\infty}{\|f_\varepsilon\|_1}\gtrsim\log\frac{1}{\varepsilon},\quad 0<\varepsilon<1/e.$$

Letting $\varepsilon\to0^+$, we see that $\mathcal{I}^{d/2}$ is not of restricted weak type $(1,\infty)$. ∎

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