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THE NORM OF A REE GROUP

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Abstract. We give an explicit construction of the Ree groups of type G_2 as groups acting on mixed Moufang hexagons together with detailed proofs of the basic properties of these groups contained in the two fundamental papers of Tits on this subject (see [7] and [8]). We also give a short proof that the norm of a Ree group is anisotropic.

§1. Introduction

The finite Ree groups of type G_2 were introduced by Ree in [5]. In [8], Tits showed how to construct these groups over an arbitrary field K of characteristic 3 having an endomorphism whose square is the Frobenius endomorphism of K. His result can be summarized as follows.

THEOREM 1.1. *Let K be a field of characteristic 3, and suppose that K has an endomorphism θ such that*

$$x^{\theta^2} = x^3$$

for all $x \in K$. Let U denote the set $K \times K \times K$ endowed with the multiplication

$$(a,b,c) \cdot (x,y,z) = (a+x, b+y+ax^\theta, c+z+ay-bx-ax^{\theta+1}), \tag{1.2}$$

and let

$$H = \{h_t \mid t \in K^*\}, \tag{1.3}$$

where for each $t \in K^*$, h_t is the map from U to itself given by the formula

$$(a,b,c)^{h_t} = (ta, t^{\theta+1}b, t^{\theta+2}c).$$

Let

$$N(a,b,c) = -ac^\theta + a^{\theta+1}b^\theta - a^{\theta+3}b - a^2b^2 + b^{\theta+1} + c^2 - a^{2\theta+4} \tag{1.4}$$

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for all $(a,b,c) \in U$*, and let* X *denote the disjoint union of* U *and a symbol* ∞*. Then the following hold.*

(i) U *is a group with identity* $(0,0,0)$ *(which we denote by* 0*) and inverses given by*

$$(a,b,c)^{-1} = (-a, -b + a^{\theta+1}, -c),$$

and H *is a group of automorphisms of* U*.*

(ii) *The map* N *is anisotropic. This is to say,* $N(a,b,c) = 0$ *if and only if* $(a,b,c) = 0$*.*

(iii) *Let* ω *be the map from* X *to itself that interchanges* ∞ *and* 0 *and maps an arbitrary element* (a,b,c) *of* U^* *to*

$$(-v/w, -u/w, -c/w), \tag{1.5}$$

where $v = a^\theta b^\theta - c^\theta + ab^2 + bc - a^{2\theta+3}$*,* $u = a^2 b - ac + b^\theta - a^{\theta+3}$*, and* $w = N(a,b,c)$*. Let* U *be identified with the permutation group of* X *that fixes* ∞ *and acts on* $X\backslash\{\infty\}$ *by right multiplication. Let* H *be identified with the permutation group of* X *that fixes* ∞ *and acts on* $X\backslash\{\infty\}$ *by the formula (1.3) (and thus fixes also* 0*). Let* $K^\dagger$ *be the subgroup of* K^* *generated by* $\{N(a,b,c) \mid (a,b,c) \in U^*\}$*, and let*

$$H^\dagger = \{h_t \mid t \in K^\dagger\} \subset H. \tag{1.6}$$

Then ω *is a permutation of* X *of order 2, and the subgroup* G *of* $\mathrm{Sym}(X)$ *generated by* U *and* ω *has the following properties.*

(I) G *is a 2-transitive permutation group on* X*.*
(II) U *is a normal subgroup of the stabilizer* G_∞ *and* $G_\infty = UH^\dagger$*.*
(III) $G = \langle U, U^\omega \rangle$*.*
(IV) H *normalizes* G*.*
(V) ω *inverts every element of* H*.*
(VI) *If* $|K| > 3$*, then* G *is simple.*

Tits's proof of Theorem 1.1 in [8] is based on the standard embedding of the split Moufang hexagon in six-dimensional projective space (see also [10, Section 7.7]). The purpose of this note is to give an alternative proof of Theorem 1.1 in which we construct the set X inside the mixed hexagon defined over the pair (K, K^θ), which we construct directly without reference to projective space.

Our motivation is threefold. First, since the Ree groups of type G_2 continue to be the center of lively interest (see especially [2]), we want to give a proof of Theorem 1.1 in which many of the details left to the reader in [8] are filled in. We also want to provide independent confirmation of the accuracy of the formulas occurring in Theorem 1.1. (In fact, in [8] a θ is missing in the second term in the definition of the norm, and a minus sign is missing in front of the whole expression on page 12, where θ is called σ and the norm N is called w.) Second, we want to examine the fact that the map N, which we call the *norm* of G, is anisotropic. As in [8], this fact emerges "geometrically" in the course of our proof of Theorem 1.1; in Section 6, we give a short algebraic proof. Third, we hope that the method we use to prove Theorem 1.1 can serve as a model for other calculations in Moufang polygons and in more general types of buildings.

If $|K| = 3$, then the endomorphism θ is trivial and the group G is not simple; in fact, it is isomorphic to $\mathrm{Aut}(L_2(8))$ in this case and thus has a normal subgroup of index 3 (which is simple).

If K is finite, then $H^\dagger = H$ and thus $H \subset G$ (by [5, (8.4)]). It is not true in general, however, that $H = H^\dagger$. We say a few words about this in Section 7. (For another approach to the finite Ree groups, see [4].)

We mention that there are also Ree groups of type F_4. The canonical reference for these groups is [7].

We would also like to bring the reader's attention to Remark 3.11 below.

§2. The hexagon of mixed type

Let K be a field of characteristic 3, and let θ be a square root of the Frobenius endomorphism of K. We now begin our proof of Theorem 1.1 by constructing the mixed hexagon associated with the pair (K, θ). (See [9, (16.20) and (41.20)] for the definition of a mixed hexagon.) Let $U_1, U_2, \ldots, U_6$ be six groups isomorphic to the additive group of K, and for each $i \in [1, 6]$, let x_i be an isomorphism from K to U_i. Let U_+ be the group generated by the groups $U_1, U_2, \ldots, U_6$ subject to the commutator relations

$$\begin{aligned} &[x_1(s), x_5(t)] = x_3(-st), \\ &[x_2(s), x_6(t)] = x_4(st), \text{ and} \qquad (2.1) \\ &[x_1(s), x_6(t)] = x_2(-s^\theta t) x_3(-s^2 t^\theta) x_4(s^\theta t^2) x_5(s t^\theta) \end{aligned}$$

for all $s,t \in K$ and $[U_i,U_j]=1$ for all other pairs i,j such that $1 \le i < j \le 6$. (We are using the convention that $[a,b]=a^{-1}b^{-1}ab=(b^{-1})^a b$.) By Propositions 2.2 and 2.5 below and [9, (5.6)], every element of U_+ can be written uniquely as an element in the product $U_1U_2\cdots U_6$. It is easily checked that there is an automorphism ρ of U_+ interchanging $x_i(t)$ and $x_{7-i}(t)$ for all $i \in [1,6]$ and all $t \in K$. We will see below that the group U in Theorem 1.1 is the centralizer of ρ in U_+.

Let $U_{i,j}$ denote the subgroup $U_iU_{i+1}\cdots U_j$ of U_+ for all i,j such that $1 \le i \le j \le 6$ (so that $U_{i,i}=U_i$ for each i). For each $i \in [1,5]$, let W_i denote the set of right cosets in U_+ of $U_{1,6-i}$. For each $i \in [6,10]$, let W_i denote the set of right cosets in U_+ of $U_{12-i,6}$. Let W be the disjoint union of $W_1, W_2, \ldots, W_{10}$ together with two symbols $\bullet$ and $\star$. For each $i \in [1,9]$, let E_i be the set of pairs $\{x,y\}$ such that $x \in W_i$, $y \in W_{i+1}$ and the intersection of x and y is nonempty. Let E be the set of (unordered) 2-element subsets of W consisting of $\{\bullet,\star\}$, $\{\bullet,x\}$ for all $x \in W_1$, $\{\star,y\}$ for all $y \in W_{10}$ together with all the pairs in $E_1 \cup E_2 \cup \cdots \cup E_9$. Finally, let Γ be the graph with vertex set W and edge set E.

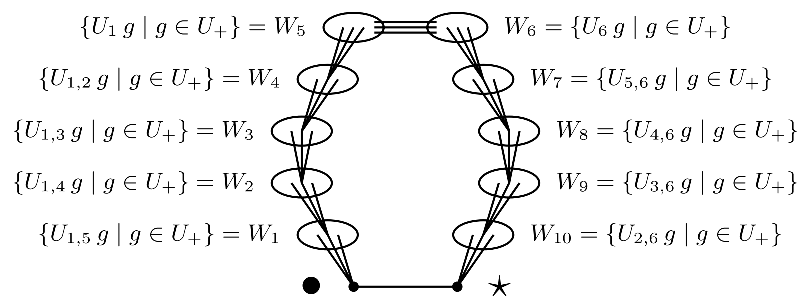

Figure 1: The graph Γ

Proposition 2.2. *The graph* Γ *is the Moufang hexagon associated with the hexagonal system* $(K/K^\theta)^\circ$ *as defined in* [9, (15.20) and (16.8)].

Proof. Let $\tilde U_+$ and $\tilde U_1,\ldots,\tilde U_6$ be the groups obtained by setting $F=K^\theta$, $J=K$, $T(a,b)=0$, $a^\#=a^2$, $N(a)=a^3$, and $a \times b = 2ab$ for all $a,b \in K$ in [9, (16.8)]. By [9, (8.13)], the maps $x_i(s) \mapsto x_i(s^\theta)$ for $i=2$, 4, and 6; $x_i(s) \mapsto x_i(-s)$ for $i=3$ and 5; and $x_1(s) \mapsto x_1(s)$ extend to an isomorphism ψ from U_+ to $\tilde U_+$ mapping U_i to $\tilde U_i$ for all $i \in [1,6]$. The graph Γ is precisely the graph called $\mathcal{G}(U_+,U_1,\ldots,U_6)$ in [9, (8.1)] and the Moufang hexagon associated with the hexagonal system $(K/K^\theta)^\circ$ is $\mathcal{G}(\tilde{U}_+,\tilde{U}_1,\dots,\tilde{U}_6)$ (see [9, Chapter 16, page 163]). Hence, the isomorphism ψ induces an isomorphism from Γ to this Moufang hexagon. □

NOTATION 2.3. Let $D=\mathrm{Aut}(\Gamma)$, and let $D^\dagger$ denote the subgroup of D generated by all the root groups of Γ.

From now on, we will write U_{ij} in place of $U_{i,j}$. The group U_+ acts faithfully by right multiplication on the elements of

$$W_1\cup\cdots\cup W_{10}$$

and maps the set E of edges of Γ to itself. This allows us to identify U_+ with a subgroup of the stabilizer $D_{\bullet,\star}$ (which we continue to denote by U_+). Just to fix notation, we observe, for example, that

$$U_{15}^{x_6(t)}=U_{15}x_6(t), \tag{2.4}$$

where the cosets U_{15} and $U_{15}x_6(t)$ are vertices in the set W_1 and the expression on the left means the image of the vertex U_{15} under the action of the element $x_6(t)\in U_+$.

PROPOSITION 2.5. *The groups $U_1,U_2,\dots,U_6$ are the root groups of Γ corresponding to the six roots of Σ that contain the edge $\{\bullet,\star\}$.*

Proof. This holds by [9, (8.2)]. □

We mention that by [9, (35.13) and (36.1)], the extension K/K^θ is an invariant of the quadrangle Γ, from which it follows that Γ is a split Moufang hexagon if and only if the field K is perfect.

§3. The automorphisms m_1 and m_6

Let Σ denote the apartment of Γ spanned by the vertices $\bullet$, $\star$, $U_{1,6-i}\in W_i$ for all $i\in[1,5]$ and $U_{12-i,6}\in W_i$ for all $i\in[6,10]$. Let

$$m_1=\mu\big(x_1(1)\big)\qquad\text{and}\qquad m_6=\mu\big(x_6(1)\big),$$

where the map μ is defined (with respect to the apartment Σ) as in [9, (6.1)]. Both of these elements are contained in the group $D^\dagger$, and both induce reflections on Σ; m_1 induces the reflection fixing $\star$ and U_1, and m_6 induces the reflection fixing $\bullet$ and U_6. By [9, (32.12)], we have

$$x_6(t)^{m_1}=x_2(t)\qquad\text{and}\qquad x_5(t)^{m_1}=x_3(t)$$

and

$$x_1(t)^{m_6} = x_5(-t) \qquad \text{and} \qquad x_2(t)^{m_6} = x_4(t)$$

for all $t \in K$. Thus the action of m_1 on the vertices in W_1 is given by

$$\big(U_{15}x_6(t)\big)^{m_1} = U_{15}^{x_6(t)m_1} = U_{15}^{m_1x_2(t)} = U_{36}x_2(t) \tag{3.1}$$

(see (2.4) above), and the action of m_6 on the vertices in W_{10} is given by

$$\big(U_{26}x_1(t)\big)^{m_6} = U_{26}^{x_1(t)m_6} = U_{26}^{m_6x_5(-t)} = U_{14}x_5(-t) \tag{3.2}$$

for all $t \in K$. Similarly, we have

$$\big(U_{14}x_5(t)\big)^{m_1} = U_{46}x_3(t) \tag{3.3}$$

and

$$\big(U_{36}x_2(t)\big)^{m_6} = U_{13}x_4(t) \tag{3.4}$$

for all $t \in K$.

Proposition 3.5. *The maps m_1 and m_6 are as in Tables 1 and 2. (For use in Section 4, we have also recorded the product m_1m_6 in Table 3.)*

Proof. Let ξ denote the permutation of W given in Table 1. We claim that ξ maps edges to edges and is thus an automorphism of Γ. To begin, we choose an edge e containing one vertex in W_5 and one vertex in W_6. Thus $e = \{U_1g, U_6g\}$ for some

$$g = x_1(s)x_2(t)x_3(r)x_4(u)x_5(v)x_6(w) \in U_+.$$

We have

$$U_1g = U_1x_2(t)x_3(r)x_4(u)x_5(v)x_6(w),$$

and hence

$$(U_1g)^\xi = U_1x_2(w)x_3(v)x_4(u+wt)x_5(-r)x_6(-t).$$

By (2.1), we have

$$\begin{aligned} U_6g &= U_6x_1(s)x_2(t)x_3(r)x_4(u)x_5(v)x_6(w) \\ &= U_6x_1(s)\cdot x_6(w)x_2(t)\cdot x_3(r)x_4(u+wt)x_5(v) \end{aligned}$$

Table 1: The action of m_1 on Γ

$$\star \mapsto \star$$

$$\bullet \mapsto U_{26}$$

$$U_{15}x_6(t) \mapsto U_{36}x_2(t)$$

$$U_{14}x_5(s)x_6(t) \mapsto U_{46}x_2(t)x_3(s)$$

$$U_{13}x_4(r)x_5(s)x_6(t) \mapsto U_{56}x_2(t)x_3(s)x_4(r)$$

$$U_{12}x_3(u)x_4(r)x_5(s)x_6(t) \mapsto U_6x_2(t)x_3(s)x_4(r)x_5(-u)$$

$$U_1x_2(v)x_3(u)x_4(r)x_5(s)x_6(t) \mapsto U_1x_2(t)x_3(s)x_4(r+vt)x_5(-u)x_6(-v)$$

$$U_6x_1(s)x_2(t)x_3(r)x_4(u)x_5(v) \xmapsto{s=0} U_{12}x_3(v)x_4(u)x_5(-r)x_6(-t)$$

$$\xmapsto{s\neq 0} U_6x_1(-s^{-1})x_2(-s^{-\theta}t)x_3(v+s^{-2}t^{\theta}) \cdot x_4(u-s^{-\theta}t^2)x_5(s^{-1}t^{\theta}-r)$$

$$U_{56}x_1(s)x_2(t)x_3(r)x_4(u) \xmapsto{s=0} U_{13}x_4(u)x_5(-r)x_6(-t)$$

$$\xmapsto{s\neq 0} U_{56}x_1(-s^{-1})x_2(-s^{-\theta}t)x_3(-s^{-1}r-s^{-2}t^{\theta}) \cdot x_4(u-s^{-\theta}t^2)$$

$$U_{46}x_1(s)x_2(t)x_3(r) \xmapsto{s=0} U_{14}x_5(-r)x_6(-t)$$

$$\xmapsto{s\neq 0} U_{46}x_1(-s^{-1})x_2(-s^{-\theta}t)x_3(-s^{-1}r-s^{-2}t^{\theta})$$

$$U_{36}x_1(s)x_2(t) \xmapsto{s=0} U_{15}x_6(-t)$$

$$\xmapsto{s\neq 0} U_{36}x_1(-s^{-1})x_2(-s^{-\theta}t)$$

$$U_{26}x_1(s) \xmapsto{s=0} \bullet$$

$$\xmapsto{s\neq 0} U_{26}x_1(-s^{-1})$$

Table 2: The action of m_6 on Γ

$$\begin{aligned}
\star &\mapsto U_{15} \\
\bullet &\mapsto \bullet \\
U_{15}x_6(w) &\overset{w=0}{\longmapsto} \star \\
&\overset{w\neq 0}{\longmapsto} U_{15}x_6(-w^{-1}) \\
U_{14}x_5(v)x_6(w) &\overset{w=0}{\longmapsto} U_{26}x_1(-v) \\
&\overset{w\neq 0}{\longmapsto} U_{14}x_5(-vw^{-\theta})x_6(-w^{-1}) \\
U_{13}x_4(u)x_5(v)x_6(w) &\overset{w=0}{\longmapsto} U_{36}x_1(-v)x_2(-u) \\
&\overset{w\neq 0}{\longmapsto} U_{13}x_4(-v^\theta w^{-2} - w^{-1}u)x_5(-vw^{-\theta}) \\
&\quad\ \cdot x_6(-w^{-1}) \\
U_{12}x_3(r)x_4(u)x_5(v)x_6(w) &\overset{w=0}{\longmapsto} U_{46}x_1(-v)x_2(-u)x_3(r) \\
&\overset{w\neq 0}{\longmapsto} U_{12}x_3(r - v^2w^{-\theta})x_4(-v^\theta w^{-2} - w^{-1}u) \\
&\quad\ \cdot x_5(-vw^{-\theta})x_6(-w^{-1}) \\
U_1x_2(t)x_3(r)x_4(u)x_5(v)x_6(w) &\overset{w=0}{\longmapsto} U_{56}x_1(-v)x_2(-u)x_3(r)x_4(t) \\
&\overset{w\neq 0}{\longmapsto} U_1x_2(v^\theta w^{-1} - u - tw)x_3(r - v^2w^{-\theta}) \\
&\quad\ \cdot x_4(-v^\theta w^{-2} - w^{-1}u)x_5(-vw^{-\theta}) \\
&\quad\ \cdot x_6(-w^{-1}) \\
U_6x_1(s)x_2(t)x_3(r)x_4(u)x_5(v) &\mapsto U_6x_1(-v)x_2(-u)x_3(r - sv)x_4(t)x_5(s) \\
U_{56}x_1(s)x_2(t)x_3(r)x_4(u) &\mapsto U_1x_2(-u)x_3(r)x_4(t)x_5(s) \\
U_{46}x_1(s)x_2(t)x_3(r) &\mapsto U_{12}x_3(r)x_4(t)x_5(s) \\
U_{36}x_1(s)x_2(t) &\mapsto U_{13}x_4(t)x_5(s) \\
U_{26}x_1(s) &\mapsto U_{14}x_5(s)
\end{aligned}$$

Table 3: The action of $m_1 m_6$ on Γ

$$\star \mapsto U_{15}$$

$$\bullet \mapsto U_{14}$$

$$U_{15}x_6(t) \mapsto U_{13}x_4(t)$$

$$U_{14}x_5(s)x_6(t) \mapsto U_{12}x_3(s)x_4(t)$$

$$U_{13}x_4(r)x_5(s)x_6(t) \mapsto U_1x_2(-r)x_3(s)x_4(t)$$

$$U_{12}x_3(u)x_4(r)x_5(s)x_6(t) \mapsto U_6x_1(u)x_2(-r)x_3(s)x_4(t)$$

$$U_1x_2(v)x_3(u)x_4(r)x_5(s)x_6(t) \xmapsto{v=0} U_{56}x_1(u)x_2(-r)x_3(s)x_4(t)$$

$$\xmapsto{v\neq 0} U_1x_2(u^\theta v^{-1} - r)x_3(s + u^2v^{-\theta}) \cdot x_4(u^\theta v^{-2} + v^{-1}r + t)x_5(-uv^{-\theta})x_6(v^{-1})$$

$$U_6x_1(s)x_2(t)x_3(r)x_4(u)x_5(v) \xmapsto{s=0,t=0} U_{46}x_1(r)x_2(-u)x_3(v)$$

$$\xmapsto{s=0,t\neq 0} U_{12}x_3(v + r^2t^{-\theta})x_4(r^\theta t^{-2} + t^{-1}u) \cdot x_5(-rt^{-\theta})x_6(t^{-1})$$

$$\xmapsto{s\neq 0} U_6x_1(r - s^{-1}t^\theta)x_2(s^{-\theta}t^2 - u) \cdot x_3(v - s^{-2}t^\theta - s^{-1}r)x_4(-s^{-\theta}t)x_5(-s^{-1})$$

$$U_{56}x_1(s)x_2(t)x_3(r)x_4(u) \xmapsto{s=0,t=0} U_{36}x_1(r)x_2(-u)$$

$$\xmapsto{s=0,t\neq 0} U_{13}x_4(r^\theta t^{-2} + t^{-1}u)x_5(-rt^{-\theta})x_6(t^{-1})$$

$$\xmapsto{s\neq 0} U_1x_2(-u + s^{-\theta}t^2)x_3(-s^{-1}r - s^{-2}t^\theta) \cdot x_4(-s^{-\theta}t)x_5(-s^{-1})$$

$$U_{46}x_1(s)x_2(t)x_3(r) \xmapsto{s=0,t=0} U_{26}x_1(r)$$

$$\xmapsto{s=0,t\neq 0} U_{14}x_5(-rt^{-\theta})x_6(t^{-1})$$

$$\xmapsto{s\neq 0} U_{12}x_3(-s^{-1}r - s^{-2}t^\theta)x_4(-s^{-\theta}t)x_5(-s^{-1})$$

(*continued*)

Table 3: (*continued*)

$$U_{36}x_1(s)x_2(t) \xmapsto{s=0,t=0} \star$$

$$\xmapsto{s=0,t\neq 0} U_{15}x_6(t^{-1})$$

$$\xmapsto{s\neq 0} U_{13}x_4(-s^{-\theta}t)x_5(-s^{-1})$$

$$U_{26}x_1(s) \xmapsto{s=0} \bullet$$

$$\xmapsto{s\neq 0} U_{14}x_5(-s^{-1})$$

$$\begin{aligned} &= U_6x_1(s)\cdot x_2(t-s^\theta w)x_3(r-s^2w^\theta) \\ &\quad \cdot x_4(u+wt+s^\theta w^2)x_5(v+sw^\theta). \end{aligned}$$

If $s=0$, then

$$\begin{aligned} (U_6g)^\xi &= \big(U_6x_2(t)x_3(r)x_4(u+wt)x_5(v)\big)^\xi \\ &= U_{12}x_3(v)x_4(u+wt)x_5(-r)x_6(-t), \end{aligned}$$

and thus $(U_1g)^\xi \subset (U_6g)^\xi$. Suppose, instead, that $s\neq 0$, and let

$$\hat{g} = x_1(-s^{-1})x_2(w)x_3(v)x_4(u+wt)x_5(-r)x_6(-t).$$

Note that $\hat{g}\in (U_1g)^\xi$. By (2.1) again, we have

$$\begin{aligned} U_6\hat{g} &= U_6x_1(-s^{-1})\cdot x_6(-t)x_2(w)\cdot x_3(v)x_4(u)x_5(-r) \\ &= U_6x_1(-s^{-1})\cdot x_2(w-s^{-\theta}t)x_3(v+s^{-2}t^\theta)x_4(u-s^{-\theta}t^2)x_5(-r+s^{-1}t^\theta). \end{aligned}$$

Therefore, $(U_6g)^\xi = U_6\hat{g}$ by Table 1 and a bit of calculation. Thus

$$\hat{g}\in (U_1g)^\xi \cap (U_6g)^\xi.$$

We conclude that $e^\xi = \{(U_1g)^\xi, (U_6g)^\xi\}$ is an edge of Γ whether $s=0$ or not. It is now an easy task to check in a similar fashion that ξ maps all the remaining edges to edges; we leave this to the reader.

Next we observe that the automorphism ξ induces the same reflection of the apartment Σ as does m_1, and it agrees with m_1 on the set of neighbors of $\bullet$ and on the set of neighbors of U_{15} by (3.1) and (3.3). By [9, (3.7)], it follows that $\xi = m_1$. (In fact, Table 1 was created by starting with the action of m_1 on Σ, the set of neighbors of $\bullet$, and the set of neighbors of U_{15} and working backward.) By (3.2), (3.4), and a similar argument, the claim holds for m_6. ∎

Now let ρ be the automorphism of U_+ mentioned above. Thus

$$x_i(t)^\rho = x_{7-i}(t) \tag{3.6}$$

for all $i \in [1,6]$ and all $t \in K$. By [9, (7.5)], there exists a unique automorphism of Γ that maps the apartment Σ to itself, interchanges $\bullet$ and $\star$, and induces ρ on U_+. We denote this automorphism of Γ also by ρ. Thus, in particular, $U_1^\rho = U_6$ and $U_6^\rho = U_1$.

From now on, we set

$$\omega = (m_1 m_6)^3. \tag{3.7}$$

PROPOSITION 3.8. *The automorphisms ρ and ω commute with each other, and both have order 2.*

Proof. Since ρ has order 2 as an automorphism of U_+, it also has order 2 as an automorphism of Γ. By [9, (6.9)], $\omega = (m_6 m_1)^3$, and by [9, (6.2)], $m_1^\rho = m_6$ and $m_6^\rho = m_1$. Thus $\omega^\rho = (m_6 m_1)^3 = \omega$. Let $d = m_1^2$ and $e = m_6^2$ (so that $[m_1, d] = [m_6, e] = 1$). Then d and e both act trivially on the apartment Σ, and by [9, (29.12)], d centralizes U_1 and U_4 and inverts every element of U_i for all other $i \in [1,6]$ and e centralizes U_3 and U_6 and inverts every element of U_i for all other $i \in [1,6]$. By [9, (6.7)], d and e are elements of order 2 (so $m_1^{-1} = dm_1$ and $m_6^{-1} = em_6$), and their product (in either order) is the unique element of D acting trivially on Σ that centralizes U_2 and U_5 and inverts every element of U_i for all other $i \in [1,6]$. Since $U_i^{m_1} = U_{8-i}$ for all $i \in [2,6]$ and $U_i^{m_6} = U_{6-i}$ for all $i \in [1,5]$, both e^{m_1} and d^{m_6} centralize U_2 and U_5 and invert every element of U_i for all other $i \in [1,6]$. Thus $e^{m_1} = ed = d^{m_6}$. It follows by repeated use of these relations that

$$(m_1^{-1} m_6^{-1})^3 = (dm_1 \cdot em_6)^3 = (m_1 m_6)^3,$$

and hence $\omega^{-1} = (m_6 m_1)^{-3} = \omega$. ∎

PROPOSITION 3.9. *Let φ be the map from U to U_+ given by*

$$\varphi(a,b,c) = x_1(a)x_2(b)x_3(c-ab+a^{\theta+2})x_4(c+ab)x_5(b-a^{\theta+1})x_6(a).$$

Then φ is an injective homomorphism whose image is the centralizer of ρ in U_+.

Proof. By (1.2) and (2.1) and a bit of calculation, φ is a homomorphism. It is clearly injective. Now choose $a,b,c,d,e,f \in K$, and let

$$g = x_1(a)x_2(b)x_3(c)x_4(d)x_5(e)x_6(f).$$

By (2.1) and (3.6), we have

$$\begin{aligned}
g^\rho &= x_6(a)x_5(b)x_4(c)x_3(d)x_2(e)x_1(f)\\
&= x_5(b)x_4(c-ae)x_3(d)x_2(e)\cdot x_6(a)x_1(f)\\
&= x_2(e)x_3(d)x_4(c-ae)x_5(b)\cdot x_1(f)x_6(a)\cdot x_2(af^\theta)x_3(a^\theta f^2)x_4(-a^2f^\theta)\\
&\quad \cdot x_5(-a^\theta f)\\
&= x_1(f)x_2(e)x_3(d+bf)x_4(c-ae)x_5(b)\cdot x_2(af^\theta)x_3(a^\theta f^2)x_4(a^2f^\theta)\\
&\quad \cdot x_5(-a^\theta f)x_6(a)\\
&= x_1(f)x_2(e+af^\theta)x_3(d+bf+a^\theta f^2)x_4(c-ae+a^2f^\theta)x_5(b-a^\theta f)x_6(a).
\end{aligned}$$

Thus $g^\rho = g$ if and only if $a = f$, $e = b - a^{\theta+1}$, and $c = d + ab + a^{\theta+2}$. We conclude that g commutes with ρ if and only if $g = \varphi(a,b,d-ab)$. □

From now on, we identify U with its image in U_+ under the map φ in Proposition 3.9.

PROPOSITION 3.10. *Let X be the set of edges of Γ fixed by ρ, let ∞ denote the edge $\{\bullet,\star\}$, and let $G = \langle U,\omega\rangle$, where ω is as in (3.7). Then the following hold:*

(i) *U acts regularly on $X\backslash\{\infty\}$;*
(ii) *G acts 2-transitively on X;*
(iii) *$G = B \cup B\omega B$, where $B = G_\infty$;*
(iv) *U is a normal subgroup of the stabilizer G_∞;*
(v) *G acts faithfully on X.*

Proof. Since ρ interchanges the vertices $\bullet$ and $\star$, all the edges in X other than $\infty=\{\bullet,\star\}$ are 2-element subsets containing a right coset of U_1 and a right coset of U_6. Since $U_1\cap U_6=1$, the intersection of a right coset of U_1 and a right coset of U_6 is either empty or consists of a unique element. It follows that

$$X=\big\{\{U_1g,U_6g\}\,\big|\,g\in U\big\}\cup\{\infty\}.$$

In particular, (i) holds, and we can identify U with $X\backslash\{\infty\}$ via the map that sends $g\in U$ to $\{U_1,U_6\}^g=\{U_1g,U_6g\}$. In particular, $0=(0,0,0)\in U$ now denotes the edge $\{U_1,U_6\}$ itself. By Proposition 3.8, ω acts on the set X. Since ω interchanges the edges ∞ and 0 (by Table 3) and U acts transitively on $X\backslash\{\infty\}$, we conclude that (ii) and (iii) hold. Since U_+ is normal in D_∞ (by [9, (4.7) and (5.3)]) and G is contained in the centralizer of ρ, (iv) also holds.

Note that ω maps each vertex of Σ to a vertex at distance 6 from itself. Since the elements of U all fix the vertex $\bullet$ and Γ is bipartite, it follows that the distance from x to x^g is even for every vertex x and every $g\in\langle U,\omega\rangle$. In particular, no element of G interchanges the two vertices of an edge.

For each $x\in X\backslash\{\infty\}$, there exists a unique apartment Σ_x of Γ containing the edges x and ∞. For each $(a,b,c)\in U$, we have $U_{15}\varphi(a,b,c)=U_{15}x_6(a)$ and $U_{26}\varphi(a,b,c)=U_{26}x_1(a)$ by Proposition 3.9. For each vertex u adjacent to $\bullet$ or $\star$, therefore, there exists an edge $x\in X\backslash\{\infty\}$ such that $u\in\Sigma_x$. If an element of G acts trivially on X, then it acts trivially on all these apartments; thus it also acts trivially on the set of all vertices adjacent to $\bullet$ or $\star$, and hence it is itself trivial by [9, (3.7)]. Thus (v) holds. ∎

REMARK 3.11. The permutation group on U obtained by letting U act on itself by right multiplication is of course the same as the permutation group on U obtained by letting $U^{\rm opp}$ act on itself by left multiplication. It follows that Theorem 1.1 is equivalent to the assertion obtained by replacing the multiplication on U defined in (1.2) by the opposite multiplication and, in part (iii), letting U act on $U=X\backslash\{\infty\}$ by left rather than right multiplication, and this "left-handed" version of Theorem 1.1 produces the same group G. We have chosen to work with right cosets and to let U_+ act by right multiplication in order to conform to [9] and to the recent literature on Moufang sets, whereas Tits [8] chose to work with left multiplication. This explains why the group U in Theorem 1.1 is the opposite of the group U in [8, Example 3, pages 210–215].

Proposition 3.12. *Let H be as in (1.3), let $D^\dagger$ be as in Notation 2.3, let D° denote the centralizer of ρ in $D^\dagger$, and let T denote the two-point stabilizer $D^\circ_{\infty,0}$. Then there is a canonical isomorphism π from H to T that is compatible with the map φ in Proposition 3.9.*

Proof. Let $g \in D^\dagger_{\infty,0}$. Thus g acts trivially on the apartment Σ. By [9, (15.20) and (33.16)] and the isomorphism described in the proof of Proposition 2.2, there exist $a, u \in K^*$ such that $x_1(s)^g = x_1(a^2u^{-\theta}s)$ and $x_6(s)^g = x_6(a^{-\theta}u^2s)$ for all $s \in K$. By [9, (33.5)], the centralizer of $\langle U_1, U_6\rangle$ in $D_{\infty,0}$ is trivial. By (3.6), therefore, g commutes with ρ (and hence is contained in T) if and only if $a^2u^{-\theta} = a^{-\theta}u^2$. Since the maps $x \mapsto x^{2+\theta}$ and $x \mapsto x^{2-\theta}$ from K^* to K^* are inverses of each other, we conclude that $a = u$ and that the map $g \mapsto a^{2-\theta}$ is an isomorphism from T to K^*. Now let $t = a^{2-\theta}$, so that $x_1(s)^g = x_1(ts)$ and $x_6(s)^g = x_6(ts)$ for all $s \in K$. By the commutator relations (2.1), it follows that $x_2(s)^g = x_2(t^{\theta+1}s)$, $x_3(t)^g = x_3(t^{\theta+2}s)$, $x_4(s)^g = x_4(t^{\theta+2}s)$, and $x_5(s)^g = x_5(t^{\theta+1}s)$. By Proposition 3.9, therefore, $(a, b, c)^g = (a, b, c)^{h_t}$, where h_t is as in (1.3). □

From now on we identify H with the two-point stabilizer T via the map π in Proposition 3.12.

§4. The formula (1.5)

In this section we show that the norm N defined in (1.4) is anisotropic and that the automorphism ω satisfies (1.5). We do this by computing explicitly the action of ω on X using Table 3.

For each $g = (a, b, c) \in U$, we have

$$U_1g = U_1x_2(b)x_3(c - ab + a^{\theta+2})x_4(c + ab)x_5(b - a^{\theta+1})x_6(a) \tag{4.1}$$

by Proposition 3.9 and

$$U_1g \cap U = \{g\} \tag{4.2}$$

by Proposition 3.10(i).

Lemma 4.3. *Suppose that $U_1x_2(\ddot{v})x_3(\ddot{u})x_4(\ddot{r})s_5(\ddot{s})x_6(\ddot{t}) = U_1g$ for some $g \in U$. Then $g = (\ddot{t}, \ddot{v}, \ddot{r} - \ddot{v}\ddot{t})$.*

Proof. This holds by (4.1) and (4.2). □

We now fix $g=(a,b,c)\in U^*$, and let u, v, and $w=N(a,b,c)$ be as in Theorem 1.1(iii). Observe first that the following curious identity holds:

$$w=av+bu+c^2. \tag{4.4}$$

Let $m=m_1m_6$ (so that $\omega=m^3$), let α denote the vertex U_1g, let $\beta=\alpha^m$, and let $\gamma=\beta^m$. Our goal is to show that $w\neq 0$ and that

$$(a,b,c)^\omega=(-v/w,-u/w,-c/w). \tag{4.5}$$

LEMMA 4.6. *Suppose that $w\neq 0$ and that*

$$\alpha^\omega=U_1x_2(\ddot{v})x_3(\ddot{u})x_4(\ddot{r})x_5(\ddot{s})x_6(\ddot{t}).$$

Then (4.5) holds if and only if

$$\ddot{t}=-v/w, \tag{4.7}$$

$$\ddot{v}=-u/w,\ \textit{and} \tag{4.8}$$

$$\ddot{r}=-c/w+(-v/w)(-u/w). \tag{4.9}$$

Proof. Since ω maps $X\backslash\{\infty,0\}$ to itself, we have $\alpha^\omega=U_1e$ for some $e\in U^*$. The claim holds, therefore, by Lemma 4.3. ∎

To begin, we assume that

$$b\neq 0, \tag{4.10}$$

so by Table 3 applied to (4.1), we have

$$\beta=\alpha^m=U_1x_2(\hat{v})x_3(\hat{u})x_4(\hat{r})x_5(\hat{s})x_6(\hat{t}),$$

where

$$\begin{aligned}
\hat{v}&=b^{-1}c^\theta-a^\theta b^{\theta-1}+a^{2\theta+3}b^{-1}-c-ab,\\
\hat{u}&=b-a^{\theta+1}+b^{-\theta}c^2+a^2b^{-\theta+2}+a^{2\theta+4}b^{-\theta}\\
&\quad+ab^{-\theta+1}c-a^{\theta+2}b^{-\theta}c+a^{\theta+3}b^{-\theta+1},\\
\hat{r}&=b^{-2}c^\theta-a^\theta b^{\theta-2}+a^{2\theta+3}b^{-2}+b^{-1}c-a,\\
\hat{s}&=-b^{-\theta}c+ab^{-\theta+1}-a^{\theta+2}b^{-\theta},\ \text{and}\\
\hat{t}&=b^{-1}.
\end{aligned} \tag{4.11}$$

It is straightforward to check that the following identities hold:

$$w = b\hat{u}^\theta - \hat{v}(\hat{v} - c), \tag{4.12}$$

$$b\hat{r} = \hat{v} - c, \tag{4.13}$$

$$b\hat{s}^\theta = -a - b^{-1}(\hat{v} + c), \text{ and} \tag{4.14}$$

$$\hat{v} = -b^{-1}v. \tag{4.15}$$

Next we assume that

$$v \neq 0. \tag{4.16}$$

Thus also $\hat{v} \neq 0$ (by (4.15)), so by a second application of Table 3, we have

$$\gamma = \beta^m = U_1 x_2(\tilde{v}) x_3(\tilde{u}) x_4(\tilde{r}) x_5(\tilde{s}) x_6(\tilde{t}),$$

where

$$\tilde{v} = \hat{u}^\theta \hat{v}^{-1} - \hat{r}, \tag{4.17}$$

$$\tilde{u} = \hat{s} + \hat{u}^2 \hat{v}^{-\theta}, \tag{4.18}$$

$$\tilde{r} = \hat{u}^\theta \hat{v}^{-2} + \hat{v}^{-1}\hat{r} + \hat{t}, \tag{4.19}$$

$$\tilde{s} = -\hat{u}\hat{v}^{-\theta}, \text{ and}$$

$$\tilde{t} = \hat{v}^{-1}. \tag{4.20}$$

Note that

$$\begin{aligned}
\tilde{v} &= \hat{u}^\theta \hat{v}^{-1} - \hat{r} && \text{by (4.17),} \\
&= \hat{v}^{-1} b^{-1} \cdot (b\hat{u}^\theta - b\hat{r}\hat{v}) \\
&= \hat{v}^{-1} b^{-1} \cdot \big(b\hat{u}^\theta - \hat{v}(\hat{v} - c)\big) && \text{by (4.13),} \\
&= \hat{v}^{-1} b^{-1} \cdot w && \text{by (4.12), and} \\
&= -w/v && \text{by (4.15).}
\end{aligned}$$

In particular, we have

$$\tilde{v} = b^{-1}\hat{v}^{-1} w \tag{4.21}$$

as well as

$$\tilde{v} = -wv^{-1} \tag{4.22}$$

and $\hat{u}^{\theta}\hat{v}^{-1} = \hat{r} + b^{-1}\hat{v}^{-1}w$, so

$$b^2\hat{u}^{2\theta}\hat{v}^{-2} = b^2\hat{r}^2 - b\hat{r}\hat{v}^{-1}w + \hat{v}^{-2}w^2. \tag{4.23}$$

Moreover,

$$\begin{aligned} \tilde{r} &= \hat{u}^{\theta}\hat{v}^{-2} + \hat{v}^{-1}\hat{r} + \hat{t} && \text{by (4.19)}, \\ &= \hat{u}^{\theta}\hat{v}^{-2} + \hat{v}^{-1}\hat{r} + b^{-1} && \text{by (4.11)}, \\ &= (\tilde{v} - \hat{r})\hat{v}^{-1} + b^{-1} && \text{by (4.17)}; \end{aligned}$$

hence,

$$\tilde{r} = b^{-1}\hat{v}^{-2}w - \hat{r}\hat{v}^{-1} + b^{-1} \quad \text{by (4.21)}, \tag{4.24}$$

and thus

$$b\tilde{r}w = \hat{v}^{-2}w^2 - b\hat{r}\hat{v}^{-1}w + w. \tag{4.25}$$

We record also that

$$b^2\hat{s}^{\theta}\hat{v} = -ab\hat{v} - \hat{v}^2 - \hat{v}c \quad \text{by (4.14)}. \tag{4.26}$$

The vertex $\alpha^{\omega} = \gamma^m$ lies on an edge contained in X. Hence, $\alpha^{\omega} \in W_5$ (where W_5 is as in Figure 1). It follows that $\tilde{v} \neq 0$, since otherwise $\gamma^m \in W_7$ by Table 3. By (4.22), we conclude that

$$w \neq 0,$$

and by a final application of Table 3, we have

$$\alpha^{\omega} = \gamma^m = U_1 x_2(\ddot{v}) x_3(\ddot{u}) x_4(\ddot{r}) x_5(\ddot{s}) x_6(\ddot{t}),$$

where

$$\begin{aligned} \ddot{v} &= \tilde{u}^{\theta}\tilde{v}^{-1} - \tilde{r}, \\ \ddot{u} &= \tilde{s} + \tilde{u}^2\tilde{v}^{-\theta}, \\ \ddot{r} &= \tilde{u}^{\theta}\tilde{v}^{-2} + \tilde{v}^{-1}\tilde{r} + \tilde{t}, \\ \ddot{s} &= -\tilde{u}\tilde{v}^{-\theta}, \text{ and} \\ \ddot{t} &= \tilde{v}^{-1}. \end{aligned}$$

We now observe that $\ddot{t} = \tilde{v}^{-1} = -v/w$ by (4.22), so (4.7) holds. Furthermore,

$$\begin{aligned} -b\ddot{v}w &= -b(\tilde{u}^\theta\tilde{v}^{-1} - \tilde{r})w \\ &= -b^2\tilde{u}^\theta\hat{v} + b\tilde{r}w && \text{by (4.21)} \\ &= -b^2(\hat{s} + \hat{u}^2\hat{v}^{-\theta})^\theta\hat{v} + b\hat{r}w && \text{by (4.18)} \\ &= -b^2\hat{s}^\theta\hat{v} - b^2\hat{u}^{2\theta}\hat{v}^{-2} + b\tilde{r}w. \end{aligned}$$

Applying (4.23), (4.25), and (4.26) to the three terms in this last expression, we find that

$$\begin{aligned} -b\ddot{v}w &= ab\hat{v} + \hat{v}^2 + c\hat{v} - b^2\hat{r}^2 + w \\ &= ab\hat{v} + \hat{v}^2 + c\hat{v} - (\hat{v} - c)^2 + w && \text{by (4.13)} \\ &= -av - c^2 + w && \text{by (4.15)} \\ &= bu && \text{by (4.4).} \end{aligned}$$

Thus (4.8) holds. Finally, we have

$$\begin{aligned} w\ddot{r} &= w(\tilde{u}^\theta\tilde{v}^{-2} + \tilde{v}^{-1}\tilde{r} + \tilde{t}) \\ &= w\tilde{v}^{-1}(\ddot{v} - \tilde{r}) + w\tilde{t} \\ &= -\tilde{v}^{-1}(u + w\tilde{r}) + w\tilde{t} && \text{by (4.8)} \\ &= uvw^{-1} + v\tilde{r} + w\tilde{t} && \text{by (4.22)} \\ &= uvw^{-1} + v\tilde{r} + w\hat{v}^{-1} && \text{by (4.20)} \\ &= uvw^{-1} + v(b^{-1}\hat{v}^{-2}w - \hat{v}^{-1}\hat{r} + b^{-1}) + w\hat{v}^{-1} && \text{by (4.24)} \\ &= uvw^{-1} + (-\hat{v}^{-1}w + b\hat{r} - \hat{v}) + w\hat{v}^{-1} && \text{by (4.15)} \\ &= uvw^{-1} - c && \text{by (4.13),} \end{aligned}$$

so (4.9) also holds. By Lemma 4.6, it follows that (4.5) holds. We conclude that $w \neq 0$ and that the identity (1.5) holds for all "generic" points in U^*, that is, for all $g = (a, b, c)$ in U^* satisfying (4.10) and (4.16).

Next we consider the case that $b \neq 0$ but $v = 0$. By (4.15), we have $\hat{v} = 0$ as well, and hence

$$\beta = \alpha^m = U_1 x_3(\hat{u}) x_4(\hat{r}) x_5(\hat{s}) x_6(\hat{t}).$$

It follows from Table 3 that

$$\gamma = \beta^m = U_{56} x_1(\hat{u}) x_2(-\hat{r}) x_3(\hat{s}) x_4(\hat{t}).$$

If $\hat{u} = 0$, it would follow from Table 3 that $\alpha^\omega = \gamma^m \in W_3 \cup W_9$. This is impossible since the vertex α^ω lies on an edge contained in X. We conclude that $\hat{u} \neq 0$. It follows from (4.12) (with $\hat{v} = 0$) that $w \neq 0$. From Table 3 we now obtain

$$\alpha^\omega = \gamma^m = U_1 x_2(\ddot{v}) x_3(\ddot{u}) x_4(\ddot{r}) x_5(\ddot{s}),$$

where

$$\begin{aligned}
\ddot{v} &= -\hat{t} + \hat{u}^{-\theta} \hat{r}^2, \\
\ddot{u} &= -\hat{u}^{-1} \hat{s} + \hat{u}^{-2} \hat{r}^\theta, \\
\ddot{r} &= \hat{u}^{-\theta} \hat{r}, \text{ and} \\
\ddot{s} &= -\hat{u}^{-1}.
\end{aligned}$$

Remembering that $v = \hat{v} = 0$, we calculate that

$$\begin{aligned}
\ddot{r} &= \hat{u}^{-\theta} \hat{r} \\
&= w^{-1} b \cdot \hat{r} \quad \text{by (4.12)} \\
&= -w^{-1} c \quad \text{by (4.13)}
\end{aligned}$$

and that

$$\begin{aligned}
\ddot{v} &= -\hat{t} + \hat{u}^{-\theta} \hat{r}^2 \\
&= -b^{-1} + w^{-1} b \cdot (-b^{-1} c)^2 \quad \text{by (4.11)–(4.13)} \\
&= -b^{-1} w^{-1} \cdot (w - c^2) \\
&= -b^{-1} w^{-1} \cdot bu \quad \text{by (4.4)} \\
&= -w^{-1} u.
\end{aligned}$$

By Lemma 4.6, we conclude that (4.5) holds.

We can thus assume from now on that $b = 0$, so

$$\alpha = U_1 x_3(c + a^{\theta+2}) x_4(c) x_5(-a^{\theta+1}) x_6(a),$$

as well as

$$v = -c^\theta - a^{2\theta+3} = -z^\theta, \tag{4.27}$$

$$u = -ac - a^{\theta+3}, \text{ and} \tag{4.28}$$

$$w = -ac^\theta + c^2 - a^{2\theta+4} = c^2 - az^\theta, \tag{4.29}$$

where

$$z = c + a^{\theta+2}.$$

From Table 3 we now obtain

$$\beta = \alpha^m = U_{56}x_1(z)x_2(-c)x_3(-a^{\theta+1})x_4(a).$$

Note that a and c cannot both be 0, since otherwise $g = (a, 0, c) = 0 \in U$.

Suppose that $c = -a^{\theta+2}$ or, equivalently, that $z = 0$. Then $a \neq 0$, and Table 3 tells us that

$$\begin{aligned} \beta = \alpha^m &= U_{56}x_2(a^{\theta+2})x_3(-a^{\theta+1})x_4(a), \\ \gamma = \beta^m &= U_{13}x_5(a^{-\theta-2})x_6(a^{-\theta-2}), \\ \alpha^\omega = \gamma^m &= U_1x_3(a^{-\theta-2})x_4(a^{-\theta-2}). \end{aligned}$$

By (4.27)–(4.29), we have $v = 0$, $u = 0$, and $w = a^{2\theta+4} \neq 0$, and by Lemma 4.6, we conclude once again that (4.5) holds.

Suppose, finally, that $c \neq -a^{\theta+2}$ or, equivalently, that $z \neq 0$. From Table 3 we obtain

$$\gamma = \beta^m = U_1x_2(\tilde{v})x_3(\tilde{u})x_4(\tilde{r})x_5(\tilde{s}),$$

where

$$\tilde{v} = -a + z^{-\theta}c^2, \tag{4.30}$$

$$\tilde{u} = z^{-1}a^{\theta+1} + z^{-2}c^\theta, \tag{4.31}$$

$$\tilde{r} = z^{-\theta}c, \text{ and} \tag{4.32}$$

$$\tilde{s} = -z^{-1}.$$

It follows from (4.29) and (4.30) that

$$w = z^\theta\tilde{v}. \tag{4.33}$$

Observe that $\tilde{v} \neq 0$, since it would otherwise follow from Table 3 again that $\alpha^\omega = \gamma^m \in W_7$, which is impossible. Therefore, $w \neq 0$ also in this last case. By one final application of Table 3, we obtain

$$\alpha^\omega = \gamma^m = U_1 x_2(\ddot{v}) x_3(\ddot{u}) x_4(\ddot{r}) x_5(\ddot{s}) x_6(\ddot{t}),$$

where

$$\begin{aligned}
\ddot{v} &= \tilde{u}^\theta \tilde{v}^{-1} - \tilde{r}, \\
\ddot{u} &= \tilde{s} + \tilde{u}^2 \tilde{v}^{-\theta}, \\
\ddot{r} &= \tilde{u}^\theta \tilde{v}^{-2} + \tilde{v}^{-1}\tilde{r}, \\
\ddot{s} &= -\tilde{u}\tilde{v}^{-\theta}, \text{ and} \\
\ddot{t} &= \tilde{v}^{-1}.
\end{aligned}$$

By (4.27) and (4.33), we have

$$\ddot{t} = \tilde{v}^{-1} = z^\theta / w = -v/w. \tag{4.34}$$

Furthermore,

$$\begin{aligned}
\ddot{v} &= \tilde{u}^\theta \tilde{v}^{-1} - \tilde{r} \\
&= (z^{-1}a^{\theta+1} + z^{-2}c^\theta)^\theta \cdot (-v/w) - z^{-\theta}c && \text{by (4.31), (4.32),} \\
& && \text{and (4.34)} \\
&= w^{-1} \cdot \big((z^{-\theta}a^{\theta+3} + z^{-2\theta}c^3) \cdot z^\theta - z^{-\theta}cw\big) && \text{by (4.27)} \\
&= w^{-1} \cdot (a^{\theta+3} + z^{-\theta}c^3 - z^{-\theta}c^3 + ac) && \text{by (4.29)} \\
&= -u/w && \text{by (4.28),} \qquad (4.35)
\end{aligned}$$

and

$$\begin{aligned}
\ddot{r} &= \tilde{u}^\theta \tilde{v}^{-2} + \tilde{v}^{-1}\tilde{r} \\
&= (-v/w) \cdot (\tilde{u}^\theta \tilde{v}^{-1} + \tilde{r}) && \text{by (4.34)} \\
&= (-v/w) \cdot \big((-u/w) - \tilde{r}\big) && \text{by (4.35)} \\
&= (-v/w)(-u/w) + (v/w) \cdot z^{-\theta}c && \text{by (4.32)} \\
&= -c/w + (-v/w)(-u/w) && \text{by (4.27).}
\end{aligned}$$

By Lemma 4.6, we conclude that (4.5) holds also in this last case.

This completes the proof that $w \neq 0$ and that the identity (1.5) holds for every $g = (a, b, c)$ in U^*. ∎

§5. Properties (I)–(VI)

By Proposition 3.8, ω is a permutation of X of order 2. To conclude our proof of Theorem 1.1, it thus remains only to show that (I)–(VI) hold. By Proposition 3.10(ii),(v), (I) holds. For each $x \in X$, there exists $g \in G$ mapping ∞ to x; let $U_x = U^g$. If g_1, g_2 are two elements of G mapping ∞ to the same element of X, then $g_1 g_2^{-1} \in G_\infty$ and thus $U^{g_1} = U^{g_2}$ (by Proposition 3.10(iv)). By Proposition 3.10(i), it follows that $(X, (U_x)_{x\in X})$ is a Moufang set (as defined, e.g., in [1, Section 2.1]). Let $G^\dagger = \langle U_x \mid x \in X\rangle$, and let μ be as in [1, Definition 3.1]. Thus for each $a \in U^*$, $\mu(a)$ is the unique element of $U_0 a U_0 = U^\omega a U^\omega$ that interchanges ∞ and 0. (Note that this is not the same μ as in the definition of m_1 and m_6 at the beginning of Section 3 above.) By [1, Theorem 3.1(ii)], we have

$$G^\dagger_\infty = U \cdot \langle \mu(a)\mu(b) \mid a, b \in U^*\rangle. \tag{5.1}$$

Proposition 5.2. *The following hold.*

(i) $G^\dagger_\infty = UH^\dagger$, *where* $H^\dagger$ *is as defined in (1.6).*

(ii) $\omega \in \langle U, U^\omega\rangle$ *(so* $G = G^\dagger$*).*

Proof. We have $\langle \mu(a)\mu(b) \mid a, b \in U^*\rangle = H^\dagger$ by [3, Proposition 6.12(ii)], whose proof depends only on knowing that the norm N is anisotropic. By (5.1), therefore, (i) holds. At the conclusion of the proof of [3, Proposition 6.12(ii)], it is observed that $\omega = \mu(0, 0, 1)$. Hence, (ii) holds. □

By Propositions 3.10(iv) and 5.2, (II) and (III) hold. Let

$$t \cdot (a, b, c) = (a, b, c)^{h_t} \tag{5.3}$$

for each $(a, b, c) \in U$ and each $t \in K^*$. By (1.5), we have

$$\omega\big(t \cdot (a, b, c)\big) = t^{-1} \cdot \omega(a, b, c) \tag{5.4}$$

for all $(a, b, c) \in U$ and all $t \in K^*$. Thus (V) holds. Since H normalizes U, it follows that H also normalizes U^ω. Hence, (IV) follows from (III).

Suppose, finally, that $|K| > 3$. Let $K^\dagger$ be as in (1.6). Thus, in particular, $(K^*)^2 = N(0, 0, K^*) \subset K^\dagger$. Since $|K| > 3$, it follows that we can choose $t \in K^\dagger$ such that $t^{\theta+1} \neq 1$. Thus $t \neq 1$, so also $t^{\theta+2} \neq 1$. We have

$$[h_t, (a, 0, 0)] = \big((1 - t)a, (t - 1)t^\theta a^{\theta+1}, 0\big),$$

$$[h_t, (0, b, 0)] = \big(0, (1 - t^{\theta+1})b, 0\big), \text{ and}$$

$$[h_t, (0,0,c)] = \big(0,0,(1-t^{\theta+2})c\big)$$

for all $a, b, c \in K$. Hence, $U \subset [G,G]$. By Proposition 3.10(iii), $(G_\infty, \langle\omega\rangle)$ is a BN-pair (as defined in [6, Definition 2.1]). The group U is nilpotent. By [6, Proposition 2.8] and Proposition 3.10(iv),(v), it follows that G is simple. Thus (VI) holds.

§6. A more elementary reason that the norm is anisotropic

In this section we give a short algebraic proof that the norm N defined in (1.4) is anisotropic. Let

$$\Omega(a,b,c) = (-v, -uw^\theta, -cw^{\theta+1}) \tag{6.1}$$

for all $(a,b,c) \in U$, where, as in (1.4) and (1.5),

$$v = a^\theta b^\theta - c^\theta + ab^2 + bc - a^{2\theta+3},$$

$$u = a^2b - ac + b^\theta - a^{\theta+3},$$

and $w = N(a,b,c) = -ac^\theta + a^{\theta+1}b^\theta - a^{\theta+3}b - a^2b^2 + b^{\theta+1} + c^2 - a^{2\theta+4}$. Note that

$$N\big(t\cdot(a,b,c)\big) = t^{2\theta+4}N(a,b,c) \tag{6.2}$$

for all $(a,b,c) \in U$, where $t\cdot(a,b,c)$ is as in (5.3), and

$$N\big((a,b,c)^{-1}\big) = N(a,b,c) \tag{6.3}$$

for all $(a,b,c) \in U^*$, where $(a,b,c)^{-1}$ is as in Theorem 1.1(i).

Our proof rests on the observation that

$$N\big(\Omega(a,b,c)\big) = N(a,b,c)^{2\theta+3} \tag{6.4}$$

for all $(a,b,c) \in U$. This can be checked simply by plugging the definitions of v, u, and w into (6.1). (That this identity *ought* to hold follows from [3, (6.18)] and (5.4).)

Now fix $(a,b,c) \in U^*$ such that $w = 0$.

LEMMA 6.5. $v = 0$.

Proof. By (6.1) and (6.4), we have

$$N(-v,0,0) = N\big(\Omega(a,b,c)\big) = 0.$$

By (1.4), on the other hand, $N(-v,0,0) = -v^{2\theta+4}$. ∎

LEMMA 6.6. $a \neq 0$.

Proof. Suppose that $a = 0$. Since $(a, b, c) \neq 0$ and $w = 0$, we have $c \neq 0$. By (6.2), the norm of $c^{\theta-2} \cdot (0, b, c)$ is zero. We can thus assume that $c = 1$. It follows by (1.4) that $b \neq 1$, but Lemma 6.5 implies that $b = 1$. ∎

By (6.2) and Lemma 6.6, we can assume from now on that $a = 1$. Hence, $v = 0$ means that

$$b^\theta - c^\theta + b^2 + bc - 1 = 0, \tag{6.7}$$

and $w - v = 0$ means that

$$b^{\theta+1} + b^2 - b - bc + c^2 = 0. \tag{6.8}$$

By (6.3) and Lemma 6.5, we also have $v(-1, -b+1, -c) = v((1, b, c)^{-1}) = 0$, and thus

$$b^\theta + c^\theta - b^2 - b - 1 + bc - c = 0. \tag{6.9}$$

Adding (6.7) and (6.9), we find that

$$b^\theta + b - 1 = -bc - c. \tag{6.10}$$

Multiplying this last equation by b and comparing with (6.8), we obtain

$$c(c - b^2 + b) = 0. \tag{6.11}$$

Assume first that $c = 0$. Then by (6.7), we have $b^\theta + b^2 - 1 = 0$, whereas by (6.10), we have $b^\theta + b - 1 = 0$. We find that $b^2 = b$ and thus $b \in \{0, 1\}$, contradicting the equality $b^\theta + b - 1 = 0$.

Hence, $c \neq 0$, and it follows from (6.11) that $c = b^2 - b$. By (6.7), we now obtain

$$b^{2\theta} = b^3 - 1 - b^\theta;$$

from (6.10), on the other hand, we get

$$b^3 - 1 = -b^\theta.$$

Combining the last two equations, we obtain $b^{2\theta} = b^\theta$, but then $c^\theta = 0$, and hence $c = 0$ after all. With this contradiction, we conclude that the norm N is anisotropic.

§7. The subgroup $H^\dagger$

If K is finite, then $|K|$ is an odd power of 3, from which it follows that K^* is generated by $(K^*)^2 = N(0,0,K^*)$ and $-1 = N(0,1,1)$, so $K^\dagger = K^*$ and $H^\dagger = H$. (This is [5, (8.4)].) It is not necessarily true, however, that $H^\dagger = H$ if K is infinite. In this section we illustrate this with an example. As Tits suggests in [7, Section 1.12], we need to modify what he does there only slightly.

Let F be an odd degree extension of the field with three elements, and let K be the field of quotients of the polynomial ring $F[s,t]$ in two variables s and t. Since $|F|$ is an odd power of 3, there exists a unique endomorphism θ of K mapping F to F, t to s, and s to t^3 whose square is the Frobenius endomorphism. (In what follows, the reader may wish to think of s as being formally equal to $t^{\sqrt{3}}$.)

Proposition 7.1. *The group $K^\dagger \cap F(t)$ is generated by $(F(t)^*)^2$ and all irreducible polynomials in $F[t]$ of even degree.*

Proof. Since F is finite, we have $F^* \subset K^\dagger$. Let $f \in F[t]$ be an irreducible polynomial of even degree over F, and let α be a root of f in some splitting field L. Then $L = F(\alpha)$ and $[L:F] = \deg(f) = 2d$ for some d. Thus L contains an element β whose square is -1. Since $[L:F(\beta)] = d$, there are nonzero polynomials $p, q \in F[t]$ of degree at most d such that $p + \beta q$ is the minimal polynomial of α over $F(\beta)$. Thus $p + \beta q$ divides f. Hence, $p - \beta q$ also divides f. Since the polynomial $p + \beta q$ is irreducible over $F(\beta)$, it follows that it is relatively prime to the polynomial $p - \beta q$. Thus f/e equals the product of these two polynomials for some $e \in F^*$. Hence,

$$f = e(p^2 + q^2) = eN(0, p^{\theta - 1}, q) \in K^\dagger.$$

Since $h^{-1} = h \cdot h^{-2}$ for all $h \in F[t]^*$ and $(K^*)^2 \subset K^\dagger$, it will now suffice to show that no product in $F[t]$ of distinct irreducible polynomials of odd degree is contained in $K^\dagger$. Let $g \in F[t]$ be such a product, let F_1 be the splitting field of g over F, and let $K_1 = F_1(s,t)$. The extension F_1/F is of odd degree by the choice of g, so θ has a unique extension to an endomorphism of K_1 (which we continue to call θ) whose square is the Frobenius map. Let c be an arbitrary root of g in F_1, and let $d = c^\theta$. We define a valuation ν on K_1 with values in $\mathbb{Z}[\sqrt{3}]$. First, we declare the degree of a monomial $e(s-d)^m(t-c)^n$ (for $e \in F_1^*$) to be $n + m\sqrt{3}$. If $p \in F_1[s,t]^*$, we write p as a sum of monomials in the variables $t - c$ and $s - d$ and define $\nu(p)$ to be the minimum of the degrees of these monomials (minimum with respect to the natural ordering of $\mathbb{Z}[\sqrt{3}]$ as a subset of $\mathbb{R}$). Finally, we set $\nu(p/q) = \nu(p) - \nu(q)$ for all $p, q \in F_1[s,t]^*$. Then ν is a well-defined valuation on K_1. Since g is a product of distinct irreducibles, c is a simple root of g. Since the variable s does not occur in g, we conclude that $\nu(g) = 1$. Since

$$\begin{aligned}\big(e(s-d)^m(t-c)^n\big)^\theta &= e^\theta(t^3-c^3)^m(s-d)^n\\ &= e^\theta(t-c)^{3m}(s-d)^n\end{aligned}$$

for all $e \in F_1$ and all $m, n \geq 0$, it follows that $\nu(u^\theta) = \sqrt{3}\cdot\nu(u)$ for all $u \in K_1^*$.

Now let $w = N(a,b,c)$ for $a,b,c \in K_1$. By [3, (9.3)] (whose proof depends only on the fact that the norm is anisotropic), $\nu(w)$ is equal to the minimum of $(2\sqrt{3}+4)\nu(a)$, $(\sqrt{3}+1)\nu(b)$, and $2\nu(c)$. Since $(\sqrt{3}+1)^2 = 2\sqrt{3}+4$ and $(\sqrt{3}+1)(\sqrt{3}-1) = 2$, it follows that $\nu(K_1^\dagger) = (\sqrt{3}+1)\mathbb{Z}[\sqrt{3}]$. Since $\nu(g) = 1 \notin (\sqrt{3}+1)\mathbb{Z}[\sqrt{3}]$, we conclude that $g \notin K_1^\dagger$. Hence, $g \notin K^\dagger$. ∎

COROLLARY 7.2. *$K^*/K^\dagger$ is infinite.*

Proof. There are infinitely many pairwise nonproportional irreducible polynomials of odd degree in $F[t]$. By Proposition 7.1, these polynomials have pairwise distinct images in $K^*/K^\dagger$. ∎

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