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ON LOW-DIMENSIONAL RICCI LIMIT SPACES

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Abstract. We call a Gromov–Hausdorff limit of complete Riemannian manifolds with a lower bound of Ricci curvature a *Ricci limit space*. Furthermore, we prove that any Ricci limit space has integral Hausdorff dimension, provided that its Hausdorff dimension is not greater than 2. We also classify 1-dimensional Ricci limit spaces.

§1. Introduction

In this article, we study a pointed metric space (Y, y) that is a pointed Gromov–Hausdorff limit of a sequence of complete, pointed, connected n-dimensional Riemannian manifolds, $\{(M_i, m_i)\}_i$, with $\mathrm{Ric}_{M_i} \geq -(n-1)$; we call such a pointed metric space (Y, y) a *Ricci limit space*. The structure theory was much developed by Cheeger and Colding (see [3], [4], [5]) and has many important applications to Riemannian manifolds. The main purpose of this paper is to study low-dimensional Ricci limit spaces by using Cheeger and Colding's theory and several results of [12]. We first give the classification of Ricci limit spaces with Hausdorff dimension smaller than 2.

THEOREM 1.1. *Let (Y, y) be a Ricci limit space. Assume that Y is not a single point. Then the following conditions are equivalent:*

(1) $1 \leq \dim_H Y < 2$ *holds,*
(2) $\mathcal{R}_i = \emptyset$ *holds for every* $i \geq 2$*,*
(3) $\upsilon(\mathcal{R}_i) = 0$ *holds for every* $i \geq 2$*,*
(4) Y *is isometric to* $\mathbf{R}$*, to* $\mathbf{R}_{\geq 0}$*, to* $\mathbf{S}^1(r) = \{x \in \mathbf{R}^2 \mid |x| = r\}$ *for some* $r > 0$*, or to* $[0, l]$ *for some* $l > 0$.

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Here, $\mathcal{R}_i$ is the i-dimensional regular set of Y, $\dim_H Y$ is the Hausdorff dimension of Y, υ is a limit measure on Y, and $\dim_H Y < 1$ holds if and only if Y is a single point. Therefore, Theorem 1.1 gives the isometric classification of Ricci limit spaces whose Hausdorff dimension is smaller than 2. As a corollary of Theorem 1.1, we hold that, if $\dim_H Y \le 2$ holds, then $\dim_H Y$ is an integer (see also [19]).

We will give an another characterization of low-dimensional points under an additional assumption. For that, we define the *local Hausdorff dimension* $\dim_H^{\mathrm{loc}} x$ around a point $x \in Y$ by

$$\dim_H^{\mathrm{loc}} x = \lim_{r \to 0} \dim_H B_r(x).$$

Put $Y(\alpha) = \{x \in Y \mid \dim_H^{\mathrm{loc}} x = \alpha\}$ for $\alpha \ge 0$. Note that if Y is not a single point, then $\dim_H^{\mathrm{loc}} x \ge 1$ holds for every $x \in Y$. Next, we define the notion of the Alexandrov point. For a proper geodesic space X and a point $x \in X$, we say that x is an *Alexandrov point (in X)* if there exist an open neighborhood U of x and a negative number $K < 0$ satisfying the following properties: for every $x_1, x_2, x_3 \in U$ and for every $x_4 \in X$ with $\overline{x_1, x_4} + \overline{x_4, x_2} = \overline{x_1, x_2}$, there exist points $y_1, y_2, y_3, y_4 \in \mathbb{H}^2(K)$ such that $\overline{x_1, x_2} = \overline{y_1, y_2}, \overline{x_2, x_3} = \overline{y_2, y_3}, \overline{x_3, x_1} = \overline{y_3, y_1}, \overline{x_1, x_4} = \overline{y_1, y_4}$, $\overline{y_1, y_4} + \overline{y_4, y_2} = \overline{y_1, y_2}$, and $\overline{x_3, x_4} \ge \overline{y_3, y_4}$. Here, $\mathbb{H}^2(K)$ is the 2-dimensional space form with the sectional curvature $K_{\mathbf{H}^2(K)} \equiv K$, and $\overline{x_1, x_2}$ is the distance between x_1 and x_2.

Denote by $\mathrm{Alex}(X)$ the set of Alexandrov points in X. Roughly speaking, an Alexandrov point on a metric space means that there exists a lower bound of sectional curvature around the point in the sense of Alexandrov geometry. Therefore, by the definition, all points in every Alexandrov space are Alexandrov points. We next state another characterization of low-dimensional points in Ricci limit spaces.

THEOREM 1.2. *Let (Y, y) be a Ricci limit space. Assume that $\mathcal{R}_1 \neq \emptyset$. Then, we have* $\mathrm{Alex}(Y) = \bigcup_{\alpha < 2} Y(\alpha) = Y(1)$.

Note that this theorem is stronger than Theorem 1.1. The proof of $\bigcup_{\alpha<2} Y(\alpha) \subset \mathrm{Alex}(Y)$ is the same as the proof of Theorem 1.1. A main idea of the proof of $\mathrm{Alex}(Y) \subset Y(1)$ is to compare a measure-theoretic property of a point in $\mathcal{R}_1$ and one of an Alexandrov point by using [12, Theorem 1.1]. We give some application to Theorem 1.2 in the following.

Fix a sufficiently small positive number $\epsilon > 0$. Let Z be the completion of the 5-dimensional Riemannian manifold $(\mathbf{R}_{>0} \times \mathbf{S}^4, dr^2 + (r^{1+\epsilon}/2)^2 g_{\mathbf{S}^4})$, where $g_{\mathbf{S}^4}$ is the standard Riemannian metric on a 4-dimensional unit sphere in $\mathbb{R}^5$. It is known that this space is a Ricci limit space (see [3, Example 8.77]). On the other hand, for every $\tau > 0$, let Z_τ be the space obtained by adjoining the segment $[-\tau, 0]$ to Z at their origins. Cheeger and Colding showed that for every $\tau > 0$, Z_τ is *not* a Ricci limit space as a corollary of [4, Theorem 5.1]. This nonexistence result also follows from Theorem 1.2 directly. This is simply an alternative proof.

Let Z_1 and Z_2 be copies of Z (namely, Z_1 and Z_2 are both isometric to Z), and let $\hat{Z}$ be the space obtained by adjoining Z_1 to Z_2 at their origins. It follows directly from Theorem 1.2 that $\hat{Z}$ is not a Ricci limit space. Note that the nonexistence of $\hat{Z}$ as a Ricci limit space does not follow from [4, Theorems 3.7, 5.1] (see also Proposition 4.7 [20, Theorem 1.3]).

Theorem 1.2 implies that it is very difficult to construct a Ricci limit space whose 1-dimensional regular set is not empty and whose Hausdorff dimension is not 1. In fact, by using the results of this article, we can prove that if $\mathcal{R}_1 \neq \emptyset$, then $\dim_H Y = 1$ in [13]. As more nonexistence results, we also get that $(M \times Z_\tau, (m, 0))$ is not a Ricci limit space for every $\tau > 0$ and for every pointed-connected complete k-dimensional Riemannian manifold (M, m) (see Remark 5.8).

The organization of this article is the following. In Section 2, we introduce several notions on metric spaces that we will subsequently need. The proof of Theorem 1.1 is based on several results on regular sets due to Cheeger and Colding's work; in Section 3, we recall those results. In Section 4, we study a local structure around given low-dimensional points. Theorem 1.1 follows directly from the local structure properties (see Theorems 4.3, 4.5). The main idea of the proof is a geometric rescaling argument based on several properties of regular sets from Section 3. We study the conditions under which a limit measure υ is locally equivalent to the 1-dimensional Hausdorff measure H^1. Here, for a topological space X, a point $x \in X$, and Borel measures υ, μ on X, we say that υ is *locally equivalent to* μ *at* $x \in X$ if there exist a positive number $C > 1$ and an open neighborhood U of x such that $C^{-1}\mu(A) \le \upsilon(A) \le C\mu(A)$ for every Borel set $A \subset U$. We give a necessary and sufficient condition that υ is locally equivalent to H^1 at a point (see Theorem 4.8). The proof is based on Theorem 1.1 and [12, Theorem 1.1], essentially. Roughly speaking, Theorem 4.8 implies a characterization of the local structure around a low-dimensional point in a Ricci limit space as a metric measure space. In Section 5, we study several properties of the Alexandrov set in a Ricci limit space. A main result in

Section 5 is Theorem 5.4. As a corollary, we give a proof of Theorem 1.2. In Sections 6 and 7, we also study the problem of whether the Hausdorff dimension of a Ricci limit space is an integer. We prove—especially, under the assumption that $2 \le \dim_H Y < 3$, by using part of the proof of Theorem 1.1—that $\dim_H(Y \setminus C_x) \le 2$ holds for every $x \in Y$. Here, C_x is the cut locus of x, defined by $C_x = \{z \in X \mid \overline{x,z} + \overline{z,w} - \overline{x,w} > 0$ for every $w \in X \setminus \{z\}\}$ if X is not a single point and defined by $C_x = \emptyset$ if otherwise (see Corollary 6.4). Cheeger and Colding defined the polarity of a Ricci limit space, which is a sufficient condition for a Ricci limit space to have integral Hausdorff dimension. We can rewrite the condition by using properties of cut locus on iterated tangent cones. Actually, it is easy to check that a Ricci limit space (Y, y) is polar if and only if $C_x = \emptyset$ holds for every iterated tangent cone (X, x) of Y. Menguy [14] showed that there exists a nonpolar Ricci limit space whose Hausdorff dimension is an integer. We will give an another sufficient condition for a Ricci limit space to have integral Hausdorff dimension that is a weaker condition than the polarity. Actually, in Section 8, we prove that if $\dim_H(X \setminus C_x) = \dim_H X$ holds for every iterated tangent cone (X, x) of Y, then $\dim_H B_r(z) \in \mathbf{Z}$ holds for every $z \in Y$ and every $r > 0$. We say that a Ricci limit space is *weakly polar* if the space satisfies the condition (for details, see Theorem 7.2). It is unknown whether there exists a nonweakly polar Ricci limit space. In fact, note that the nonpolar Ricci limit space in the example in [14] is weakly polar. We also study several properties of a weakly polar limit space (see Corollary 7.7).

§2. Notation

We recall some fundamental notions on metric spaces and Ricci limit spaces.

DEFINITION 2.1. We say that a metric space X is *proper* if every bounded closed subset of X is compact. A metric space X is said to be a *geodesic space* if for all points $x_1, x_2 \in X$ there exists an isometric embedding $\gamma : [0, \overline{x_1, x_2}] \to X$ such that $\gamma(0) = x_1$ and $\gamma(\overline{x_1, x_2}) = x_2$ hold. We say that γ is a *minimal geodesic from* x_1 *to* x_2.

For a proper geodesic space X, $x \in X$, $A \subset X$, and $r > 0$, put $B_r(x) = \{z \in X \mid \overline{x,z} < r\}$, put $\overline{B}_r(x) = \{z \in X \mid \overline{x,z} \le r\}$, put $\partial B_r(x) = \{z \in X \mid \overline{x,z} = r\}$, and put $C_x(A) = \{z \in X \mid$ there exists $w \in A$ such that $\overline{x,z} + \overline{z,w} = \overline{x,w}$ holds$\}$. Throughout this article, we fix a positive integer $n > 0$.

DEFINITION 2.2. Let (Y, y) be a pointed proper geodesic space, and let K be a real number. We say that (Y, y) is an (n, K)-*Ricci limit space* (of $\{(M_i, m_i)\}_i$) if there exist sequences of real numbers $\{K_i\}_i$ and of pointed, complete, connected n-dimensional Riemannian manifolds $\{(M_i, m_i)\}_i$ with $\mathrm{Ric}_{M_i} \geq K_i(n-1)$ such that K_i converges to K and such that (M_i, m_i) converges to (Y, y) as $i \to \infty$ in the sense of pointed Gromov–Hausdorff topology.

We recall the definition of pointed Gromov–Hausdorff convergence. For a sequence of pointed proper geodesic spaces $\{(X_i, x_i)\}_i$, we say that (X_i, x_i) *converges to a pointed proper geodesic space* (X_∞, x_∞) *in the sense of Gromov–Hausdorff topology* if there exist sequences of positive numbers $\{\epsilon_i\}_i, \{R_i\}_i$ and of maps $\phi_i : (B_{R_i}(x_i), x_i) \to (B_{R_i}(x_\infty), x_\infty)$ such that $\epsilon_i \to 0$, $R_i \to \infty$, and $B_{\epsilon_i}(\mathrm{Image}(\phi_i)) \supset B_{R_i}(x_\infty)$ and such that $|\overline{z_i, w_i} - \overline{\phi_i(z_i), \phi_i(w_i)}| < \epsilon_i$ for every $z_i, w_i \in B_{R_i}(x_i)$. For the sake of simplicity, we denote this by $(X_i, x_i) \to (X_\infty, x_\infty)$. Moreover, for a sequence of points $z_i \in B_{R_i}(x_i)$, we say that z_i *converges to* $z_\infty \in X_\infty$ if $\phi_i(z_i) \to z_\infty$. For simplicity, we denote this by $z_i \to z_\infty$ (see [8], [9], and [10]).

Note that, for every $K \neq 0$ and for every (n, K)-Ricci limit space (Y, y), by suitable rescaling there exists a sequence of complete, connected, n-dimensional Riemannian manifolds $\{(M_i, m_i)\}_i$ with $\mathrm{Ric}_{M_i} \geq K(n-1)$ such that $(M_i, m_i) \to (Y, y)$. Throughout this paper, (Y, y) *is always a fixed* $(n, -1)$-*Ricci limit space of* $\{(M_i, m_i)\}$, *and it is not reduced to a single point.* We will say that such a (Y, y) is a *Ricci limit space* for the sake of simplicity.

DEFINITION 2.3. Let (W, w), (Z, z) be pointed proper geodesic spaces. We say that (W, w) is *a tangent cone at* $z \in Z$ if there exists a sequence of positive numbers $\{r_i\}_i$ with $r_i \to 0$ such that $(Z, r_i^{-1} d_Z, z) \to (W, w)$, where d_Z is the distance function on Z.

Note that by Gromov's compactness theorem, for every $x \in Y$ there exists a tangent cone $(T_x Y, 0_x)$ at x; however, in general, it is not unique (see, e.g., [16]). Note that $(T_x Y, 0_x)$ is an $(n, 0)$-Ricci limit space for every tangent cone $(T_x Y, 0_x)$ at x.

Next, we give several fundamental notions on Ricci limit spaces due to Cheeger and Colding [3]. Throughout this article, *for every metric space* X_1, X_2, *the metric on* $X_1 \times X_2$ *is always* $\sqrt{d_{X_1}^2 + d_{X_2}^2}$.

Definition 2.4. Let Z be a proper geodesic space. Assume that for every $\alpha \in Z$ there exists a tangent cone $(T_\alpha Z, 0_\alpha)$ at α. Then for every $k \geq 0$ and for every $\epsilon > 0$, put

(1) $\mathcal{WE}_k(Z) = \{x \in Z \mid$ there exists a tangent cone $(T_x Z, 0_x)$ at x and a proper geodesic space W such that $T_x Z$ is isometric to $\mathbf{R}^k \times W\}$;
(2) $\mathcal{E}_k(Z) = \{x \in Z \mid$ for every tangent cone $(T_x Z, 0_x)$ at x, there exists a proper geodesic space W such that $T_x Z$ is isometric to $\mathbf{R}^k \times W\}$;
(3) $\underline{\mathcal{WE}}_k(Z) = \{x \in Z \mid$ there exist a tangent cone $(T_x Z, 0_x)$ at x and a proper geodesic space W such that W is not a single point and such that $T_x Z$ is isometric to $\mathbf{R}^k \times W\}$;
(4) $\mathcal{R}_k(Z) = \{x \in Z \mid$ every tangent cone $(T_x Z, 0_x)$ at x is isometric to $(\mathbf{R}^k, 0_k)\}$;
(5) $(\mathcal{WE}_k)_\epsilon(Z) = \{x \in Z \mid$ there exist $0 < r < \epsilon$ and a proper geodesic space (W, w) such that $d_{\mathrm{GH}}\big((\overline{B}_r(x), x), (\overline{B}_r((0_k, w)), (0_k, w))\big) < \epsilon r$ for $\overline{B}_r((0_k, w)) \subset \mathbf{R}^k \times W\}$;
(6) $(\mathcal{E}_k)_\epsilon(Z) = \{x \in Z \mid$ there exists $r > 0$ such that, for every $0 < t < r$, there exists a proper geodesic space (W, w) such that $d_{\mathrm{GH}}\big((\overline{B}_t(x), x), (\overline{B}_t((0_k, w)), (0_k, w))\big) < \epsilon t$ holds for $\overline{B}_r((0_k, w)) \subset \mathbf{R}^k \times W\}$, where d_{GH} is the Gromov–Hausdorff distance between pointed compact metric spaces.

For simplicity, we will denote (Y, y) by $\mathcal{WE}_k = \mathcal{WE}_k(Y)$, $\mathcal{E}_k = \mathcal{E}_k(Y)$, and so forth. We call the set $\mathcal{R}_k$ the *k-dimensional regular set of* Y, and we will call the set $\mathcal{R} = \bigcup_k \mathcal{R}_k$ *the regular set of* Y (see [15], [16] for interesting examples).

Remark 2.5. It is easy to check the following:

(1) $(\mathcal{WE}_k)_\epsilon$ is open,
(2) $\mathcal{WE}_k = \bigcap_{\epsilon > 0} (\mathcal{WE}_k)_\epsilon$, $\mathcal{E}_k = \bigcap_{\epsilon > 0} (\mathcal{E}_k)_\epsilon$,
(3) $\mathcal{WE}_k = \mathcal{E}_k = \mathcal{R}_k = \emptyset$ for every $k \geq n + 1$.

We end this section by giving the definition of limit measure. The measure is a useful tool to study Ricci limit spaces.

Definition 2.6. Let υ be a Borel measure on Y. We say that υ is the *limit measure of* $\{(M_j, m_j, \mathrm{vol} / \mathrm{vol}\, B_1(m_j))\}_j$ if

$$\frac{\mathrm{vol}\, B_r(x_j)}{\mathrm{vol}\, B_1(m_j)} \to \upsilon\big(B_r(x)\big)$$

as $j \to \infty$ for every $r > 0$, every $x \in Y$, and every sequence $x_j \in M_j$ with $x_j \to x$. Then, we say that $(M_j, m_j, \mathrm{vol}/\mathrm{vol}\, B_1(m_j))$ *converges to* (Y, y, υ) *in the sense of measured Gromov–Hausdorff topology, or* (Y, y, υ) *is the Ricci limit space of* $\{(M_j, m_j, \mathrm{vol}/\mathrm{vol}\, B_1(m_j))\}_j$. We denote this by $(M_j, m_j, \mathrm{vol}/\mathrm{vol}\, B_1(m_j)) \to (Y, y, \upsilon)$ for simplicity's sake (see also [6]).

By taking a subsequence $\{(M_{i(j)}, m_{i(j)})\}_j$ of $\{(M_i, m_i)\}_i$ there exists the limit measure on Y of $\{(M_{i(j)}, m_{i(j)}, \mathrm{vol}/\mathrm{vol}\, B_1(m_{i(j)}))\}_j$ (see, e.g., [3, Theorems 1.6 and 1.10], [7]). Therefore, throughout this article, υ *is always the limit measure on* Y *of* $\{(M_j, m_j, \mathrm{vol}/\mathrm{vol}\, B_1(m_j))\}_j$.

§3. Some properties of regular sets

One of the important results on regular sets due to Cheeger and Colding (see [3, Theorem 2.1]) is that $\upsilon(Y \setminus \mathcal{R}) = 0$. We need more detailed properties of regular sets to study low-dimensional Ricci limit spaces in the following sections. Cheeger and Colding's articles do not state these results in the form we need here, but our results in this article are essentially direct consequences of their work. Note that the following proposition is not a direct consequence of $\upsilon(Y \setminus \mathcal{R}) = 0$.

PROPOSITION 3.1. *We have that* $\upsilon(B_r(x) \cap (\bigcup_{j \ge k} \mathcal{R}_j)) > 0$ *for every* $x \in \mathcal{WE}_k$ *and for every* $r > 0$.

Proof. By [5, Theorem 3.3], we have $\upsilon(B_r(x) \cap \mathcal{E}_k) > 0$ for every $r > 0$. If $\upsilon(B_r(x) \cap \mathcal{R}_k) > 0$, then we have the claim. Assume that $\upsilon(B_r(x) \cap \mathcal{R}_k) = 0$. Then, since $\upsilon(B_r(x) \cap \mathcal{E}_k) \le \upsilon(B_r(x) \cap \mathcal{R}_k) + \upsilon(B_r(x) \cap \underline{\mathcal{WE}}_k)$, we have $\upsilon(B_r(x) \cap \underline{\mathcal{WE}}_k) > 0$. By [3, Lemmas 2.5 and 2.6], we have $\upsilon(B_r(x) \cap \mathcal{E}_{k+1}) > 0$. The iteration stops since $\mathcal{E}_l = \emptyset$ for any $l > n$ by the Hausdorff dimension argument. By iterating this argument, we have the assertion. ∎

PROPOSITION 3.2. *We have that* $\upsilon(B_r(x) \cap (\bigcup_{j \ge k+1} \mathcal{R}_j)) > 0$ *for every* $x \in \underline{\mathcal{WE}}_k$ *and for every* $r > 0$.

Proof. First, note that, for every $\epsilon > 0, \delta > 0$, and $x \in \underline{\mathcal{WE}}_k$, there exists $s > 0$ with $s < \epsilon$ such that

$$\frac{\upsilon(B_s(x) \setminus (\mathcal{WE}_{k+1})_\delta)}{\upsilon(B_s(x))} < \epsilon$$

(see [3, (2.42)] for the proof). Note that this statement does not follow directly from the result $\upsilon(\underline{\mathcal{WE}}_k \setminus \mathcal{WE}_{k+1}) = 0$. Fix a sequence of positive numbers $\{\epsilon_i\}_i$ with $\epsilon_i \to 0$. Then there exists a sequence $x_i \in (\mathcal{WE}_{k+1})_{\epsilon_i}$ with $x_i \to x$. By [5, Theorem 3.3] and the definition of $(\mathcal{WE}_{k+1})_\epsilon$, there exists a sequence of positive numbers $\{\delta_i\}_i$ with $\delta_i \to 0$ such that $\upsilon(B_{\delta_i}(x_i) \cap \mathcal{E}_{k+1}) > 0$. Since $B_{\delta_i}(x_i) \subset B_r(x)$ for every sufficiently large i, we have $\upsilon(B_r(x) \cap \mathcal{E}_{k+1}) > 0$. By an argument similar to the proof of Proposition 3.1, we have the assertion. $\square$

We will use the following two corollaries in the following sections.

Corollary 3.3. *We have* $\underline{\mathcal{WE}}_k \subset \bigcup_{i \geq k+1} \overline{\mathcal{R}}_i$ *for every* $k \geq 1$.

Corollary 3.4. *Let* $i \geq 1$. *Then we have the following.*

(1) *If* $\upsilon(\mathcal{R}_j) = 0$ *for every* $j \geq i$, *then we have* $\mathcal{WE}_j = \phi$ *for every* $j \geq i$. *In particular, we have* $\mathcal{R}_j = \emptyset$ *for every* $j \geq i$.
(2) *If* $\upsilon(\mathcal{R}_j) = 0$ *for every* $j \geq i+1$, *then we have* $\underline{\mathcal{WE}}_j = \emptyset$ *for every* $j \geq i$.

§4. Local structure around low-dimensional points

In this section, we exhibit a local structure around a low-dimensional point in a Ricci limit space, which, as a corollary, gives Theorem 1.1.

4.1. Local metric structure around low-dimensional points

We say that a point $x \in Y$ is an *interior point* on a minimal geodesic $\gamma : [0, l] \to Y$ $(l > 0)$ if $x \in \gamma((0, l))$ holds.

Proposition 4.1. *Let* x *be a point in* $\mathcal{R}_1$. *Then,* x *is an interior point on a minimal geodesic.*

Proof. This proof is done by contradiction. Assume that the assertion is false. Let $\{r_i\}_i$ be a sequence of positive numbers with $r_i \to 0$ such that $(Y, r_i^{-1} d_Y, x) \to (\mathbf{R}, 0)$. Then there exist sequences of points $\{x_i^-\}_i, \{x_i^+\}_i \in Y$, and of positive numbers $\{\epsilon_i\}_i$ such that $\epsilon_i \to 0$, $|\overline{x_i^-, x} - r_i| < \epsilon_i r_i$, $|\overline{x_i^+, x} - r_i| < \epsilon_i r_i$, and $\overline{x_i^-, x} + \overline{x_i^+, x} - \overline{x_i^-, x_i^+} < \epsilon_i r_i$. Fix a minimal geodesic $\gamma_i : [0, \overline{x_i^-, x_i^+}] \to Y$ from x_i^- to x_i^+, and put $s_i = \overline{x, \mathrm{Image}(\gamma_i)}$. By the assumption, we have $s_i > 0$. By the triangle inequality, we have $s_i \to 0$. By Gromov's compactness theorem, without loss of generality, we can assume that $(Y, x, s_i^{-1} d_Y)$ converges to a tangent cone $(T_x Y, 0_x)$ at x. By the construction, there exist $z \in \partial B_1(0_x)$ and an isometric embedding $L : \mathbf{R} \to T_x Y$ such that $z \in \mathrm{Image}(L)$ and $0_x \notin \mathrm{Image}(L)$. By applying the splitting theorem to $(T_x Y, z)$ (see [2, Theorem 6.64]), there exists a proper geodesic space W such that W is not a single point and such that $T_x Y$ is isometric to $\mathbf{R} \times W$. This contradicts the assumption that $x \in \mathcal{R}_1$. $\square$

REMARK 4.2. By the proof of Proposition 4.1, we have that every $x \in \mathcal{R}_1$ is an interior point on a limit minimal geodesic. Here we say that a minimal geodesic $\gamma : [0,l] \to Y$ is *a limit minimal geodesic (of* $\{(M_i, m_i)\}_i$*)* if there exists a sequence of minimal geodesics $\gamma_i : [0, l_i] \to M_i$ such that $l_i \to l$ and $\gamma_i \to \gamma$ in the sense of Gromov–Hausdorff topology. This result is essentially used in [13].

THEOREM 4.3. *Let* $x \in Y \setminus \bigcup_{i\geq 2} \overline{\mathcal{R}}_i$. *Then there exists* $\epsilon > 0$ *such that* $(B_\epsilon(x), x)$ *is isometric either to* $((-\epsilon, \epsilon), 0)$ *or to* $([0, \epsilon), 0)$.

Proof.
(1) *The case* $x \in \mathcal{R}_1$.
By Proposition 4.1, there exist $r > 0$, $x_-, x_+ \in Y$, and a minimal geodesic $\gamma : [0, \overline{x_-, x_+}] \to Y$ from x_- to x_+ such that $\overline{x_-, x} = \overline{x_+, x} = 100r$, $x \in \mathrm{Image}(\gamma)$ and such that $\overline{B}_{100r}(x) \subset Y \setminus \bigcup_{i\geq 2} \overline{\mathcal{R}}_i$. It suffices to check that $\overline{B}_{10r}(x) \setminus \mathrm{Image}(\gamma) = \emptyset$. Assume that $\overline{B}_{10r}(x) \setminus \mathrm{Image}(\gamma) \neq \emptyset$. Let $z \in \overline{B}_{10r}(x) \setminus \mathrm{Image}(\gamma)$, and let $w \in \mathrm{Image}(\gamma)$ with $\overline{z, w} = \overline{z, \mathrm{Image}(\gamma)} > 0$. Note that $w \in B_{50r}(x)$. Fix a minimal geodesic $\gamma_1 : [0, \overline{z, w}] \to Y$ from z to w. For every $\epsilon > 0$ with $\epsilon \ll \overline{z, \mathrm{Image}(\gamma)}$, let $w(\epsilon) \in \mathrm{Image}(\gamma_1)$, and let $x_-(\epsilon)$, $x_+(\epsilon) \in \mathrm{Image}(\gamma)$ with $\overline{w, w(\epsilon)} = \overline{x_-(\epsilon), w} = \overline{x_+(\epsilon), w} = \epsilon$. Then we have $\overline{x_-(\epsilon), w(\epsilon)} = \overline{x_-(\epsilon), w(\epsilon)} + \overline{w(\epsilon), z} - \overline{w(\epsilon), z} \geq \overline{z, w} - \overline{w(\epsilon), z} = \epsilon$. Similarly, we have $\overline{x_+(\epsilon), w(\epsilon)} \geq \epsilon$. Therefore, for every tangent cone $(T_wY, 0_w)$ at w, there exists a proper geodesic space W such that W is not a single point and such that T_wY is isometric to $\mathbf{R} \times W$. Thus, we have $w \in \underline{\mathcal{WE}}_1$. By Corollary 3.3, we have $w \in \bigcup_{i\geq 2} \overline{\mathcal{R}}_i$. This contradicts the assumption that $\mathrm{Image}(\gamma) \subset Y \setminus \bigcup_{i\geq 2} \overline{\mathcal{R}}_i$.
(2) *The case* $x \in Y \setminus \mathcal{R}_1$.
There exist $r > 0$, $x_+ \in Y$, and a minimal geodesic $\gamma : [0, \overline{x, x_+}] \to Y$ from x to x_+ such that $\overline{x, x_+} = 100r$ and $B_{100r}(x) \subset Y \setminus \bigcup_{i\geq 2} \overline{\mathcal{R}}_i$. It suffices to check that $B_{10r}(x) \setminus \mathrm{Image}(\gamma) = \emptyset$. Assume that $B_{10r}(x) \setminus \mathrm{Image}(\gamma) \neq \emptyset$. Let $z \in B_{10r}(x) \setminus \mathrm{Image}(\gamma)$, and let $w \in \mathrm{Image}(\gamma)$ with $\overline{z, w} = \overline{z, \mathrm{Image}(\gamma)} > 0$. Note that $w \in B_{50r}(x)$. If $w \neq x$, then, by case 1, there exists $\epsilon > 0$ such that $(B_\epsilon(w), w)$ is isometric to $((-\epsilon, \epsilon), 0)$. This contradicts the fact that $\overline{z, w} = \overline{z, \mathrm{Image}(\gamma)}$. Thus, we have $w = x$. Fix $\epsilon > 0$ with $\epsilon \ll 100r$, $x_+(\epsilon) \in \mathrm{Image}(\gamma)$ with $\overline{x, x_+(\epsilon)} = \epsilon$, and fix a minimal geodesic $\gamma_\epsilon : [0, \overline{z, x_+(\epsilon)}] \to Y$ from z to $x_+(\epsilon)$.

CLAIM 4.4. *We have*

$$x \in \mathrm{Image}(\gamma_\epsilon).$$

This proof is done by contradiction. Assume that the assertion is false. Put $t=\inf\{\overline{z,m} \mid m \in \mathrm{Image}(\gamma_\epsilon) \cap \mathrm{Image}(\gamma)\} > 0$. By the definition, we have that $\gamma_\epsilon(t) \in \mathrm{Image}(\gamma)$ and that $\gamma_\epsilon(s) \notin \mathrm{Image}(\gamma)$ for every $s<t$. On the other hand, by the assumption, we have $\gamma_\epsilon(t) \in \mathcal{E}_1$. Since $\gamma_\epsilon(t) \notin \underline{\mathcal{WE}_1}$, we have $\gamma_\epsilon(t) \in \mathcal{R}_1$. By case 1, there exists $\tau>0$ such that $(B_\tau(\gamma_\epsilon(t)), \gamma_\epsilon(t))$ is isometric to $((-\tau,\tau),0)$. This contradicts the fact that $\gamma_\epsilon(s) \notin \mathrm{Image}(\gamma)$ for every $s<t$. Therefore, we have Claim 4.4.

By Claim 4.4, we have $x \in \mathcal{E}_1$. Since $x \notin \underline{\mathcal{WE}_1}$, we have $x \in \mathcal{R}_1$. This contradicts the assumption that $x \in Y \setminus \mathcal{R}_1$. ∎

THEOREM 4.5. *Let x be a point in Y. Then, $1 \le \dim_H^{\mathrm{loc}} x < 2$ holds if and only if $x \in Y \setminus \bigcup_{i\ge 2} \overline{\mathcal{R}}_i$ holds.*

Proof. By Theorem 4.3, if $x \in Y \setminus \bigcup_{i\ge 2} \overline{\mathcal{R}}_i$, then $1 \le \dim_H^{\mathrm{loc}} x < 2$. Let $i \ge 2$, and let $x \in \overline{\mathcal{R}}_i$. For every $s>0$, take $z_s \in B_s(x) \cap \mathcal{R}_i$. By [4, Corollary 1.36], we have that $\dim_H B_t(z_s) \ge 2$ for every $s,t>0$. In particular, we have that $\dim_H B_s(x) \ge i \ge 2$ for every $s>0$. Therefore, we have $\dim_H^{\mathrm{loc}} x \ge i \ge 2$. ∎

Theorem 1.1 follows directly from Corollary 3.4, Theorem 4.3, and Theorem 4.5. Put $A_Y(1)=\{x \in Y \mid \liminf_{r\to 0} \upsilon(B_r(x))/r > 0\}$ (called the *Ahlfors one regular set of* (Y,y,υ); see [12, Section 6] for the definition of the set $A_Y(\alpha)$ for a real number $1 \le \alpha \le n$). Note that the subset $A_Y(1)$ is *one dimension* in some sense. Actually, υ and the 1-dimensional Hausdorff measure H^1 are mutually absolutely continuous on $A_Y(1)$. We end this section by giving the following corollary.

COROLLARY 4.6. *Assume that $\upsilon(Y \setminus A_Y(1))=0$. Then we have $\dim_H Y = 1$.*

Proof. By [5, Theorems 3.23 and 4.6], we have $\upsilon\big(\mathcal{R}_i \setminus (\mathcal{R}_i \cap A_Y(i))\big)=0$ for every i. Therefore, by the assumption, we have $\upsilon(\mathcal{R}_i)=0$ for every $i \ge 2$. Thus, the assertion follows directly from Theorem 1.1. ∎

4.2. Local measure structure around low-dimensional points

In this section, we study local equivalence between a limit measure υ and the 1-dimensional Hausdorff measure H^1. Note that it follows from the Bishop–Gromov inequality for υ that $\upsilon_{-1}(\{x\}) \le \liminf_{r\to 0} \upsilon(B_r(x))/r \le C(n)\upsilon_{-1}(\{x\})$ for every $x \in Y$ (see [4], [12] for the definition of the measure υ_{-1} on Y).

PROPOSITION 4.7. *Let x be a point in $\mathcal{R}_1$. Then we have $\liminf_{r\to 0} \upsilon(B_r(x))/r > 0$.*

Proof. The proof is done by contradiction. Assume that the assertion is false. Hence, we have $\upsilon_{-1}(\{x\}) = 0$. Then, by [4, Theorem 3.7], for every $x_1, x_2 \in Y \setminus \{x\}$ and every $\epsilon > 0$, there exist $y_1, y_2 \in Y$ and a minimal geodesic $\gamma : [0, \overline{y_1, y_2}] \to Y$ from y_1 to y_2 such that $\overline{x_1, y_1} \le \epsilon$, $\overline{x_2, y_2} \le \epsilon$, and $x \notin \mathrm{Image}(\gamma)$. Then, by an argument similar to the proof of Proposition 4.1, there exist a tangent cone $(T_x Y, 0_x)$ at x and a proper geodesic space W such that W is not a single point and such that $T_x Y$ is isometric to $\mathbf{R} \times W$. This contradicts the assumption that $x \in \mathcal{R}_1$. □

The next theorem is the main result in this section. This is a characterization of local equivalence between a limit measure and H^1.

THEOREM 4.8. *Let x be a point in Y. The following conditions are equivalent:*

(1) *a limit measure υ and the* 1*-dimensional Hausdorff measure H^1 are locally equivalent at x;*
(2) $\liminf_{r\to 0} \upsilon(B_r(x))/r > 0$ *and* $1 \le \dim_H^{\mathrm{loc}} x < 2$ *hold.*

Proof. If υ is locally equivalent to H^1 at x, then it follows from Theorems 4.3 and 4.5 that $\dim_H^{\mathrm{loc}} x = 1$ and $\liminf_{r\to 0} \upsilon(B_r(x))/r > 0$. Assume that $\liminf_{r\to 0} \upsilon(B_r(x))/r > 0$ and that $1 \le \dim_H^{\mathrm{loc}} x < 2$. Then, by Theorems 4.3 and 4.5, there exists $\epsilon > 0$ such that $(B_{2\epsilon}(x), x)$ is isometric either to $((-2\epsilon, 2\epsilon), 0)$ or to $([0, 2\epsilon), 0)$. It follows from [12, Theorem 1.1] that there exists $d \ge 1$ such that $d^{-1} \le \liminf \upsilon(B_r(y))/r \le \limsup \upsilon(B_r(y))/r \le d$ for every $y \in B_\epsilon(x)$. For every $a \in B_\epsilon(x)$, there exists $r_a > 0$ such that $d^{-1}/2 \le \upsilon(B_r(a))/r \le 2d$ for every $r < r_0$. It follows from the standard covering lemma (see [18, Chapter 1]) that there exists $C(d, n) \ge 1$ such that $C(d, n)^{-1} H^1(A) \le \upsilon(A) \le C(d, n) H^1(A)$ for every Borel subset A of $B_\epsilon(x)$. □

Note that there exist two limit measures υ_1, υ_2 on a $(2, 0)$-Ricci limit space $[0, 1]$ such that υ_1 is locally equivalent to H^1 at 0 and υ_2 is not locally equivalent to H^1 at 0 (see [3, Example 1.24]).

§5. Alexandrov set

In this section, we study the Alexandrov set in a Ricci limit space (Y, y). In particular, we give a proof of Theorem 1.2, and we show several nonexistence results for metric spaces as a Ricci limit space (e.g., $Z_\tau, \hat{Z}$ in Section 1).

5.1. A proof of Theorem 1.2

Note that the next proposition is a direct consequence of the facts that the rescaled pointed proper geodesic space $(Y, r^{-1}d_Y, x)$ is a Ricci limit space for every $0 < r \le 1$ and every $x \in Y$ and that the measure $\upsilon_r = \upsilon/\upsilon(B_r(x))$ is a limit measure of it.

PROPOSITION 5.1. *For every $0 < r < 1$ and every $x \in Y$, there exists a limit measure υ_r on $(Y, r^{-1}d_Y, x)$ such that $\upsilon_r(B_{s_1}^{r^{-1}d_Y}(x_1))\upsilon(B_{s_2 r}(x_2)) = \upsilon_r(B_{s_2}^{r^{-1}d_Y}(x_2))\upsilon(B_{s_1 r}(x_1))$ for every $x_1, x_2 \in Y$ and every $s_1, s_2 > 0$. In particular, for every tangent cone $(T_xY, 0_x)$ at x, there exist a limit measure υ_∞ on $(T_xY, 0_x)$ and a sequence of positive numbers $\{r_i\}_i$ with $r_i \to 0$ such that $\upsilon(B_{sr_i}(x))/\upsilon(B_{r_i}(x)) \to \upsilon_\infty(B_s(0_x))$ for every $s > 0$.*

We will give a proof of the next proposition in the appendix.

PROPOSITION 5.2. *Let (W, w) be a pointed proper geodesic space, and let $d \ge 1$ with $d^{-1} \le \operatorname{diam} W \le d$. Assume that $(\mathbf{R}^k \times W, (0_k, w))$ is an $(n, 0)$-Ricci limit space. Then, for every limit measure υ on $\mathbf{R}^k \times W$, there exists a Borel measure υ_W on W such that $\upsilon = H^k \times \upsilon_W$ and that $\limsup_{\delta \to 0} \upsilon_W(B_\delta(z))/\delta \le C(n, d, R) < \infty$ for every $R > 0$ and every $z \in B_R(w)$.*

Compare the following proposition and Proposition 4.7.

PROPOSITION 5.3. *Let x be a point in $\underline{\mathcal{WE}}_1$. Then we have $\liminf_{r \to 0} \upsilon(B_r(x))/r = 0$.*

Proof. The proof is done by contradiction. Assume that the assertion is false. There exist a tangent cone $(T_xY, 0_x)$ at x and a proper geodesic space W such that W is not a single point and that T_xY is isometric to $\mathbf{R} \times W$. Let υ_∞ be a limit measure on T_xY as in Proposition 5.1. Then it follows from [12, Proposition 4.3] that $(\upsilon_\infty)_{-1}(\{0_x\}) > 0$. This contradicts Proposition 5.2. ∎

The following theorem is the main result in this section.

THEOREM 5.4. *Let x be a point in Y, and let w, z be points in $Y \setminus \{x\}$. Assume that $\overline{x,w} + \overline{w,z} = \overline{x,z}$, $\upsilon(C_w(\{z\})) > 0$, and $\dim_H^{\mathrm{loc}} x > 1$. Then, x is not an Alexandrov point.*

Proof. This proof is done by contradiction. Assume that x is an Alexandrov point. Fix a sufficiently small $r > 0$, and fix a minimal geodesic γ :

$[0,\overline{x,z}] \to Y$ from x to z. Without loss of generality, we can assume that $B_r(x) \subset \mathrm{Alex}(Y)$. Put $\alpha = \gamma(r)$, and put $w = \gamma(r/2)$.

CLAIM 5.5. *Let $\hat{\gamma} : [0,\overline{w,z}] \to Y$ be a minimal geodesic from w to z. Then, we have $\alpha \in \mathrm{Image}(\hat{\gamma})$.*

The proof is done by contradiction. Assume that the assertion is false. Then there exists $s \in [0,\overline{w,z}]$ such that $\hat{\gamma}(s) \in \partial B_r(x)$ and $\hat{\gamma}(s) \neq \alpha$. Put $\hat{\alpha} = \hat{\gamma}(s)$. Then, we have that $0 \le \overline{x,w} + \overline{w,\hat{\alpha}} - \overline{x,\hat{\alpha}} = \overline{x,w} + (\overline{w,\hat{\alpha}} + \overline{\hat{\alpha},z}) - (\overline{x,\hat{\alpha}} + \overline{\hat{\alpha},z}) \le \overline{x,w} + \overline{w,z} - \overline{x,z} = 0$. Therefore, there exists a minimal geodesic $\Gamma : [0,\overline{x,\hat{\alpha}}] \to Y$ from x to $\hat{\alpha}$ such that $w \in \mathrm{Image}(\Gamma)$. This contradicts the assumption that $B_r(x) \subset \mathrm{Alex}(Y)$. Thus, we have the assertion.

By Claim 5.5, for every sufficiently small $t > 0$, there exists $\alpha_t \in Y$ such that $\partial B_t(w) \cap C_w(\{z\}) = \{\alpha_t\}$. By the assumption of $\upsilon(C_w(\{z\})) > 0$ and [12, Theorem 4.6], we have $\upsilon_{-1}(\{\alpha_t\}) > 0$. On the other hand, for the tangent cone $(T_{\alpha_t}Y, 0_{\alpha_t})$ at α_t, there exists a proper geodesic space W such that $T_{\alpha_t}Y$ is isometric to $\mathbf{R} \times W$. By the assumption of $\dim_H^{\mathrm{loc}} x > 1$ and $\alpha_t \in \mathrm{Alex}(Y)$, we have that W is not a single point. Therefore, by Proposition 5.3, we have $\upsilon_{-1}(\{\alpha_t\}) = 0$. This is a contradiction. ∎

We end this section by giving a proof of Theorem 1.2.

Proof of Theorem 1.2. It suffices to check that $\mathrm{Alex}(Y) \subset Y(1)$. Let $x \in \mathrm{Alex}(Y)$, and let $z \in \mathcal{R}_1$. If $z = x$, then it follows from the fact $x \in \mathrm{Alex}(Y)$ that there exists $\epsilon > 0$ such that $(B_\epsilon(x), x)$ is isometric to $((-\epsilon,\epsilon), 0)$. In particular, we have $\dim_H^{\mathrm{loc}} x = 1$. Hence, assume that $x \neq z$ below. Let r be a sufficiently small positive number, and let $w \in B_r(x) \setminus \{x\} \subset \mathrm{Alex}(Y)$ with $\overline{x,w} + \overline{w,z} = \overline{x,z}$. By Proposition 4.7 and [12, Corollary 5.7], we have $\upsilon(C_w(\{z\})) > 0$. Thus, by Theorem 5.4, we have $\dim_H^{\mathrm{loc}} x = 1$. Therefore, we have Theorem 1.2. ∎

5.2. Alexandrov set in tangent cones

In this section, we give an analogous statement to Theorem 1.2 for tangent cones by using the measure contraction argument (see, e.g., [3, Appendix 2] or [17] for the measure contraction property).

THEOREM 5.6. *Let (X, x) be a proper geodesic space, and let k be a non-negative integer. Assume that $(\mathbf{R}^k \times X, (0_k, x))$ is an $(n, 0)$-Ricci limit space and that $X(1) \neq \emptyset$. Then we have $\mathrm{Alex}(X) = X(1)$.*

Proof. Let $w \in \mathrm{Alex}(X)$, and let $z \in X(1)$. Assume that $\dim_H^{\mathrm{loc}} w > 1$ holds. By Corollary 3.3 and an argument similar to the proof of Theorem 4.5, there exists an open neighborhood U of z such that $U \cap \underline{\mathcal{WE}}_1(X) = \emptyset$. Then, by an argument similar to the proof of Theorem 4.3, there exists a sufficiently small $\epsilon > 0$ such that $(B_\epsilon(z), z)$ is isometric either to $((-\epsilon,\epsilon),0)$ or to $([0,\epsilon\gamma : [0,\overline{z,w}] \to X)$ from z to w. Put $\hat{z} = \gamma(\epsilon/2)$, put $\hat{w} = \gamma(\overline{z,w} - \epsilon)$, and put $\alpha = \gamma(\overline{z,w} - 2\epsilon)$.

Claim 5.7. *We have*

$$C_{(0_k,\hat{w})}\big(B_\tau(0_k,\hat{z})\big) \cap \big(B_{\epsilon+\tau}(0_k,\hat{w}) \setminus B_\epsilon(0_k,\hat{w})\big) \subset B_{3\tau}(0_k,\alpha).$$

The proof is as follows. Let $g \in C_{(0_k,\hat{w})}(B_\tau(0_k,\hat{z})) \cap (B_{\epsilon+\tau}(0_k,\hat{w}) \setminus B_\epsilon(0_k,\hat{w}))$. There exist $(v,\hat{x}) \in B_\tau(0_k,\hat{z})$ and a minimal geodesic Γ from $(v,\hat{x})$ to $(0_k,\hat{w})$ such that $\Gamma(t_0) = g$ for some t_0. Denote $\Gamma(t) = (a(t),\hat{\gamma}(t))$, and denote $\Phi(s) = \hat{\gamma}(\overline{(v,\hat{x}),(0_k,\hat{w})}s/\overline{\hat{x},\hat{w}})$ for $0 \le s \le \overline{\hat{x},\hat{w}}$. Note that $|a(t)| \le \tau$ for every t and that $\Phi(s)$ is a minimal geodesic from $\hat{x}$ to $\hat{w}$. By an argument similar to the proof of Claim 5.5, we have $\alpha \in \mathrm{Image}(\hat{\gamma})$. On the other hand, since $g \in B_{\epsilon+\tau}(0_k,\hat{w}) \setminus B_\epsilon(0_k,\hat{w})$, we have $\hat{\gamma}(t_0) \in B_{\epsilon+\tau}(\hat{w}) \setminus B_{\epsilon-\tau}(\hat{w})$. Since $\alpha \in \mathrm{Image}(\hat{\gamma}) \cap B_{\epsilon+\tau}(\hat{w}) \setminus B_{\epsilon-\tau}(\hat{w})$, we have $\overline{\hat{\gamma}(t_0),\alpha} \le 2\tau$. Therefore, we have $\overline{g,(0_k,\alpha)} \le |a(t_0)| + \overline{\hat{\gamma}(t_0),\alpha} \le 3\tau$.

Therefore, by the Bishop–Gromov inequality for υ, we have $\upsilon(B_\tau(0_k,\hat{z})) \le C(\epsilon,n,\overline{z,x})\upsilon(B_{2\tau}(0_k,\alpha))$. Since the ball $B_\tau(0_k,\hat{z})$ is Euclidean (or half a Euclidean ball), by [5, Theorem 4.6], we have $\liminf_{\tau\to 0}\upsilon(B_\tau(0_k,\hat{z}))/\tau^{k+1} > 0$. Therefore, we have $\liminf_{\tau\to 0}\upsilon(B_\tau(0_k,\alpha))/\tau^{k+1} > 0$. Thus, by Propositions 5.1 and 5.2, there exists $C > 1$ such that $C^{-1}\tau^{k+1} \le \upsilon(B_\tau(0_k,\alpha)) \le C\tau^{k+1}$ for every $0 < \tau < 1$. Therefore, there exist a pointed proper geodesic space (Z_1,z_1), a tangent cone $T_{(0_k,\alpha)}(\mathbf{R}^k \times X)$, a limit measure $\hat{\upsilon}$ on $T_{(0_k,\alpha)}(\mathbf{R}^k \times X)$, and a Borel measure υ_{Z_1} on Z_1 such that $T_{(0_k,\alpha)}(\mathbf{R}^k \times X)$ is isometric to $\mathbf{R}^{k+1} \times Z_1$, $\hat{\upsilon} = H^{k+1} \times \upsilon_{Z_1}$, and $\liminf_{\tau\to 0}\hat{\upsilon}(B_\tau(0_k,z_1))/\tau^{k+1} > 0$. On the other hand, since $\alpha \in \mathrm{Alex}(X)$ and $\dim_H^{\mathrm{loc}} w > 1$, we have that Z_1 is not a single point. Therefore, by Proposition 5.2, we have $\liminf_{\tau\to 0}\hat{\upsilon}(B_\tau(0_k,z_1))/\tau^{k+1} = 0$. This is a contradiction. Therefore, we have $\mathrm{Alex}(X) \subset X(1)$.

Let $\beta \in X(1)$, and let $\delta > 0$ with $\dim_H B_\delta(\beta) < 2$. By Corollary 3.3 and an argument similar to the proof of Theorem 4.5, we have $B_\delta(\beta) \cap \underline{\mathcal{WE}}_1(X) = \emptyset$. Thus, by an argument similar to the proof of Theorem 4.3, there exists $r > 0$ such that $(B_r(\beta),\beta)$ is isometric either to $((-r,r),0)$ or to $([0,r),0)$. In particular, we have $\beta \in \mathrm{Alex}(X)$. □

Remark 5.8. Let (X,x) be a pointed proper geodesic space. For an open subset U of X, we say that U has *k-dimensional C^∞-Riemannian* *structure* if for every $x \in U$ there exist an open neighborhood V of x and a k-dimensional (not necessary complete) Riemannian manifold N such that V is isometric to N. Assume that there exist open sets U_1, U_2 of X such that U_1 has 1-dimensional C^∞-Riemannian structure and such that U_2 has $(k \geq 2)$-dimensional C^∞-Riemannian structure. Let (M, m) be a pointed l-dimensional complete C^∞-Riemannian manifold. Then, by an argument similar to the proof of Theorem 5.6, we have that $(M \times X, (m, x))$ is not a Ricci limit space, especially, that $(M \times Z_\tau, (m, 0))$ is not a Ricci limit space.

We say that a proper geodesic space X is *nonbranching* if, for every $x \in X$ and every $y \in X \setminus C_x$, there exists a unique minimal geodesic from x to y.

THEOREM 5.9. *Assume that $\mathcal{R}_1 \neq \emptyset$ and that Y is nonbranching. Then we have* $\dim_H Y = 1$.

Proof. Let $x \in \mathcal{R}_1$. First, we will show that $Y \setminus C_x \subset A_Y(1)$. Let $z \in Y \setminus C_x$. There exists $w \in Y \setminus C_x$ such that $z \neq w$ and $\overline{x,z} + \overline{z,w} = \overline{x,w}$ hold. By the assumption of nonbranching, there exists a unique minimal geodesic $\gamma : [0, \overline{x,w}] \to Y$ from x to w that satisfies $z \in \mathrm{Image}(\gamma)$. By Proposition 4.7 and [12, Theorem 1.1], we have $\upsilon_{-1}(\{z\}) > 0$. Therefore, we have $Y \setminus C_x \subset A_Y(1)$. It follows from [12, Theorem 3.2] that $\upsilon(Y \setminus A_Y(1)) = 0$. By Corollary 4.6, we have the assertion. $\square$

Note that it is unknown whether a branching Ricci limit space exists. However, if we drop the nonbranching assumption in the theorem above, then we get the same conclusion (see [13]).

§6. The case $2 \leq \dim_H Y < 3$

In this section, we study the Hausdorff dimension of a Ricci limit space (Y, y) with $2 \leq \dim_H Y < 3$. The main result in this section is Corollary 6.4.

PROPOSITION 6.1. *Let $s \geq 1$, let U be an open subset of Y with $\dim_H U \leq s$, $x \in U$, and let $(T_xY, 0_x)$ be a tangent cone at x. Assume that there exists a proper geodesic space W such that T_xY is isometric to $\mathbf{R}^{[s]-1} \times W$. Then, W is isometric to a single point, to $\mathbf{R}$, to $\mathbf{R}_{\geq 0}$, to $\mathbf{S}^1(r)$ for some $r > 0$, or to $[0, l]$ for some $l > 0$, where $[s] = \max\{k \in \mathbf{Z} \mid k \leq s\}$.*

Proof. By an argument similar to the proof of Theorem 4.3, it suffices to check that $\underline{\mathcal{WE}}_1(W) = \emptyset$. Assume that $\underline{\mathcal{WE}}_1(W) \neq \emptyset$. Then we have $\underline{\mathcal{WE}}_{[s]}(T_xY) \neq \emptyset$. Thus, by Corollary 3.3, we have $\mathcal{WE}_{[s]+1}(T_xY) \neq \emptyset$. Hence, we have that $(\mathcal{WE}_{[s]+1})_\epsilon \cap U \neq \emptyset$ for every $\epsilon > 0$. Thus, by [5, Theorem 3.3] and Corollary 3.3, there exists $i \geq [s]+1$ such that $\mathcal{R}_i \cap U \neq \emptyset$. Therefore, by [4, Corollary 1.36], we have that $\dim_H U \geq i \geq [s]+1 > s$. This is a contradiction. Therefore, we have $\underline{\mathcal{WE}}_1(W) = \emptyset$. ∎

Corollary 6.2. *Let $s \geq 1$, and let U be an open subset of Y with $\dim_H U \leq s$. Then, we have $\dim_H(\mathcal{E}_{[s]-1} \cap U) \leq [s]$.*

Proof. First, we will show the following.

Claim 6.3. *Let X be a proper geodesic space, let $A \subset X$, and let $s > 0$. Assume that the following hold.*

(1) *For every $x \in X$ and every sequence of positive numbers $\{r_i\}_i$ with $r_i \to 0$, there exist a subsequence $\{r_{i(j)}\}_j$ and a tangent cone $(T_x X, 0_x)$ at x such that $(X, r_{i(j)}^{-1} d_X, x) \to (T_x X, 0_x)$.*
(2) *$\dim_H T_\alpha X \leq s$ holds for every $\alpha \in A$ and for every tangent cone $(T_\alpha X, 0_\alpha)$ at α.*

Then, we have $\dim_H A \leq s$.

This proof is done by contradiction. Assume that $\dim_H A > s$. Fix $\epsilon > 0$ with $\dim_H A > s + \epsilon$. Then it is not difficult to check that there exist $\alpha \in A$ and a sequence of positive numbers $\{r_i\}_i$ with $r_i \to 0$ such that $\lim_{i\to\infty}(H_\infty^{s+\epsilon}(A \cap \overline{B}_{r_i}(\alpha))/{r_i}^{s+\epsilon}) > 0$ (see [4, (1.39)] for the definition of the $(s+\epsilon)$-dimensional spherical Hausdorff content $H_\infty^{s+\epsilon}$). By the first assumption, without loss of generality, we can assume that there exists a tangent cone $(T_\alpha X, 0_\alpha)$ at α such that $(X, r_i^{-1} d_X, \alpha) \to (T_\alpha X, 0_a)$. By the construction, it is not difficult to see that $H^{s+\epsilon}(\overline{B}_1(0_\alpha)) > 0$. In particular, we have that $\dim_H T_\alpha X \geq s + \epsilon > s$. This is a contradiction. Therefore, we have Claim 6.3.

By Proposition 6.1, for every $x \in \mathcal{E}_{[s]-1} \cap U$ and every tangent cone $(T_x Y, 0_x)$ at x, we have $\dim_H T_x Y \leq [s]$. Therefore, Corollary 6.2 follows directly from Claim 6.3. ∎

We end this section by giving the following.

Corollary 6.4. *Assume that $2 \leq \dim_H Y < 3$. Then we have that $\dim_H(Y \setminus C_x) \leq 2$ for every $x \in Y$.*

Proof. This is proved by $Y \setminus C_x \subset \mathcal{E}_1$ and Corollary 6.2. ∎

Remark 6.5. It seems that $\dim_H(Z \setminus C_z) = \dim_H Z$ holds for every Ricci limit space (Y, y) and every tangent cone (Z, z) at every $x \in Y$. If this is true, then we can prove that $\dim_H Y \in \mathbf{Z}$ holds for every Ricci limit space (Y, y) (see Section 7).

§7. Hausdorff dimension of Ricci limit spaces

In this section, we study a weakly polar Ricci limit space (Y, y).

DEFINITION 7.1. A pointed proper geodesic space (X, x) is called an *iterated tangent cone of* Y if there exists a sequence of pointed proper geodesic spaces $\{(X_i, x_i)\}_{i=0}^N$ such that $X_0 = Y$ and $(X_N, x_N) = (X, x)$ and such that (X_{i+1}, x_{i+1}) is a tangent cone at a point in X_i for every i.

Recall that a Ricci limit space (Y, y) is weakly polar if $\dim_H X = \dim_H(X \setminus C_x)$ holds for every iterated tangent cone (X, x) of Y.

THEOREM 7.2. *Assume that* Y *is weakly polar. Then we have that* $\dim_H B_R(z) \in \mathbf{Z}$ *for every* $z \in Y$ *and every* $R > 0$*. In particular, we have that* $\dim_H Y \in \mathbf{Z}$ *and* $\dim_H^{\mathrm{loc}} z \in \mathbf{Z}$*.*

Proof. Fix an integer $k > 0$ with $\dim_H B_R(z) < k+1$. It suffices to check that $\dim_H B_R(z) \le k$. By Claim 6.3, it suffices to see that $\dim_H T_z Y \le k$ holds for every $z \in Y$ and every tangent cone $(T_z Y, 0_z)$ at z. Fix a tangent cone $(T_z Y, 0_z)$, and put $(Y_1, y_1) = (T_z Y, 0_z)$. By the assumption and Claim 6.3, it suffices to see that $\dim_H T_{z_1} Y_1 \le k$ holds for every $z_1 \in Y_1 \setminus C_{y_1}$ and every tangent cone $(T_{z_1} Y_1, 0_{z_1})$ at z_1. We also fix a tangent cone $(T_{z_1} Y_1, 0_{z_1})$, and we put $(Y_2, y_2) = (T_{z_1} Y_1, 0_{z_1})$. By the construction, there exists a pointed proper geodesic space (W_2, w_2) such that (Y_2, y_2) is isometric to $(\mathbf{R} \times W_2, (0, w_2))$. Without loss of generality, we can assume that W_2 is not a single point. Note the following.

CLAIM 7.3. *We have that* $C_{(0_k, w)} = \mathbf{R}^k \times C_w$ *in* $\mathbf{R}^k \times W$ *for every* $k \ge 1$ *and every pointed proper geodesic space* (W, w)*.*

This claim is a direct consequence of the fact that every minimal geodesic in a product of geodesic spaces is a product of minimal geodesics of the factors (see, e.g., [1]).

By the assumption of weak polarity, Claim 7.3, and [11, Corollary 5.4], we have $\dim_H(W_2 \setminus C_{w_2}) \ge \dim_H C_{w_2}$. Thus, it suffices to see that $\dim_H T_{\hat{w}_2} W_2 \le k-1$ for every $\hat{w}_2 \in W_2 \setminus C_{w_2}$ and every tangent cone $(T_{\hat{w}_2} W_2, 0_{\hat{w}})$ at $\hat{w}_2$. Fix a tangent cone $(T_{\hat{w}_2} W_2, 0_{\hat{w}})$, and put $(W_3, w_3) = (T_{\hat{w}_2} W_2, 0_{\hat{w}_2})$. By the construction, there exists a pointed proper geodesic space (W_4, w_4) such that (W_3, w_3) is isometric to $(\mathbf{R} \times W_4, (0, w_4))$. By Claim 6.3, without loss of generality, we can assume that W_4 is not a single point. Since $(\mathbf{R}^2 \times W_4, (0_2, w_4))$ is an iterated tangent cone of Y, by the assumption of weak polarity and Claim 7.3, we have $\dim_H(W_4 \setminus C_{w_4}) \ge$

$\dim_H C_{w_4}$. Therefore, it suffices to see that $\dim_H T_{\hat{w}_4} W_4 \le k-2$ for every $\hat{w}_4 \in W_4 \setminus C_{w_4}$ and every tangent cone $(T_{\hat{w}_4} W_4, 0_{\hat{w}_4})$ at $\hat{w}_4$.

Continue this argument, and construct a pointed proper geodesic space (W_{2k}, w_{2k}) as above. Then, it suffices to see that $\dim_H W_{2k} \le 0$; that is, W_{2k} is a single point. Assume that W_{2k} is not a single point. Then, by the construction, there exist an iterated tangent cone (X, x) of $B_R(z)$ and a proper geodesic space L such that X is isometric to $\mathbf{R}^{k+1} \times L$. Therefore, we have that $(\mathcal{WE}_{k+1})_\epsilon \cap B_R(z) \neq \emptyset$ for every $\epsilon > 0$. Thus, by Corollary 3.3 and [5, Theorem 3.3], there exists $i \ge k+1$ such that $\mathcal{R}_i \cap B_R(z) \neq \emptyset$. Therefore, by [4, Corollary 1.36], we have that $\dim_H B_R(z) \ge i \ge k+1$. This is a contradiction. Therefore, we have $\dim_H B_R(z) \le k$. ∎

Remark 7.4. By an argument similar to the proof of Theorem 7.2, if $\dim_H (X \setminus \mathcal{WD}_0(x)) \ge \dim_H \mathcal{WD}_0(x)$ holds for every iterated tangent cone (X, x) of Y, then we have the same conclusion to Theorem 7.2 (see [3, Definition 2.10] for the definition of $\mathcal{WD}_0(x)$).

Remark 7.5. Recall that we say that Y is *polar* if for every iterated tangent cone (X, x) of Y and every $z \in Z \setminus \{x\}$, there exists an isometric embedding γ from $\mathbf{R}_{\ge 0}$ to X such that $\gamma(0) = x$ and $\gamma(\overline{x, z}) = z$ (see [3]). It is not difficult to see that Y is polar if and only if $C_x = \emptyset$ for every iterated tangent cone (X, x) of Y.

Theorem 7.6. *Let $R > 0$, let $k \ge 1$, and let $z \in Y$. Assume that Y is weakly polar and that $\dim_H B_R(z) \ge k$ holds. Then, we have $\upsilon(B_R(z) \cap (\bigcup_{i \ge k} \mathcal{R}_i)) > 0$.*

Proof. Fix a sufficiently small $\epsilon > 0$. By the assumption, we have $H^{k-\epsilon}(B_R(z)) = \infty$. Hence, by an argument similar to the proof of Claim 6.3, there exist $x \in B_R(z)$ and a tangent cone $(T_x Y, 0_x)$ at x such that $H^{k-\epsilon}(T_x Y) > 0$ holds. Fix a tangent cone $(T_x Y, 0_x)$, and put $(Y_1, y_1) = (T_x Y, 0_x)$. Since $\dim_H Y_1 \ge k - \epsilon > k - 2\epsilon > 0$ and $\dim_H (Y_1 \setminus C_{y_1}) = \dim_H Y_1$, we have $H^{k-2\epsilon}(Y_1 \setminus C_{y_1}) = \infty$. Similarly, there exist $x_1 \in Y_1 \setminus C_{y_1}$ and a tangent cone $(T_{x_1} Y_1, 0_{x_1})$ at x_1 such that $H^{k-2\epsilon}(T_{x_1} Y_1) > 0$ holds. Put $(Y_2, y_2) = (T_{x_1} Y_1, 0_{x_1})$. By the construction, there exists a pointed proper geodesic space (X_2, x_2) such that (Y_2, y_2) is isometric to $(\mathbf{R} \times X_2, (0, x_2))$. Thus, we have that $\dim_H X_2 \ge k - 1 - 2\epsilon > k - 1 - 3\epsilon > 0$. Therefore, since $\dim_H X_2 = \dim_{\mathcal{H}} (X_2 \setminus C_{x_2})$, we have $H^{k-1-3\epsilon}(X_2 \setminus C_{x_2}) = \infty$. By an argument similar to that above, there exist $\hat{x}_2 \in X_2$ and a tangent cone $(T_{\hat{x}_2} X_2, 0_{\hat{x}_2})$ at $\hat{x}_2$ such that $H^{k-1-3\epsilon}(T_{\hat{x}_2} X_2) > 0$. Put $(X_3, x_3) =$

$(T_{\hat{x}_2}X_2, 0_{\hat{x}_2})$. By the construction, there exists a pointed proper geodesic space (X_4, x_4) such that (X_3, x_3) is isometric to $(\mathbf{R} \times X_4, (0, x_4))$. Since $(\mathbf{R}^2 \times X_4, (0_2, x_4))$ is an iterated tangent cone of $B_R(z)$, by the assumption, we have $\dim_H X_4 = \dim_H(X_4 \setminus C_{x_4})$ and $\dim_H X_4 \geq k - 2 - 3\epsilon > k - 2 - 4\epsilon$.

Continue this argument, and construct a pointed proper geodesic space $(X_{2(k-1)}, x_{2(k-1)})$ as above. By the construction, $(\mathbf{R}^{k-1} \times X_{2(k-1)}, (0_k, x_{2(k-1)}))$ is an iterated tangent cone of $B_R(z)$. We have $\dim_H X_{2(k-1)} \geq k - (k-1) - 2(k-2)\epsilon > 1 - 2(k-1)\epsilon > 0$. Since $X_{2(k-1)}$ is a geodesic space, we have $\dim_H X_{2(k-1)} \geq 1$. Therefore, there exists a pointed proper geodesic space (W, w) such that $(\mathbf{R}^k \times W, (0_k, w))$ is an iterated tangent cone of $B_R(z)$. Thus, we have that $(\mathcal{WE}_k)_\epsilon \cap B_R(z) \neq \emptyset$ holds for every $\epsilon > 0$. Therefore, by Corollary 3.3 and [5, Theorem 3.3], we have $\upsilon(B_R(z) \cap (\bigcup_{i \geq k} \mathcal{R}_i)) > 0$. □

The main result in this section is the following.

COROLLARY 7.7. *Assume that Y is weakly polar. Let $k \geq 1$, satisfying that $\mathcal{R}_k \neq \emptyset$ and that $\mathcal{R}_i = \emptyset$ for every $i > k$. Then we have that $\dim_H Y = k$, $H^k(\mathcal{R}^k) > 0$, and $\upsilon(\mathcal{R}^k) > 0$.*

Proof. By [4, Corollary 1.36], we have $\dim_H Y \geq k$. Assume that $\dim_H Y \geq k+1$. Then, by Theorem 7.6, there exists $i \geq k+1$ such that $\mathcal{R}_i \neq \emptyset$. This contradicts the assumption. Thus, we have $\dim_H Y < k+1$. By Theorem 7.2, we have $\dim_H Y = k$. Next, assume that $\upsilon(\mathcal{R}_k) = 0$. Then we have that $\upsilon(\bigcup_{i \geq k} \mathcal{R}_i) = \upsilon(\mathcal{R}_k) = 0$. This contradicts Proposition 3.1. Thus, we have $\upsilon(\mathcal{R}_k) > 0$. By [5, Theorems 3.23 and 4.6], we have $H^k(\mathcal{R}^k) > 0$. □

Appendix A. Proof of Proposition 5.2

First, we give the following lemma without the proof because it follows directly from easy calculation.

LEMMA A.1. *Let (X, x) be a pointed metric space, let $R \geq 1$, let $\delta, \epsilon > 0$, let $v_\alpha, v_\beta \in \overline{B}_1(0_k) \subset \mathbf{R}^k$, and let $x_\alpha, x_\beta \in \overline{B}_R(x) \backslash B_{R^{-1}}(x)$. Assume that $\overline{x_\alpha, x_\beta} \leq \delta$ and that $\overline{(0_k, x), (v_\alpha, x_\alpha)} + \overline{(v_\alpha, x_\alpha), (v_\beta, x_\beta)} - \overline{(0_k, x), (v_\beta, x)} \leq \epsilon$ holds in $\mathbf{R}^k \times X$. Then, we have that $\overline{(v_\alpha, x_\alpha), (v_\beta, x_\beta)} \leq C(r, R)(\delta + \epsilon)$.*

Proof of Proposition 5.2. Without loss of generality, we can assume that $z \in \overline{B}_R(w) \setminus B_{d^{-1}}(w)$. By the assumption, there exist a sequence of complete, pointed, connected n-dimensional Riemannian manifolds $\{(M_j, m_j)\}_j$ and a sequence of positive numbers $\{\epsilon_j\}_j$ with $\epsilon_j \to 0$ such that $\mathrm{Ric}_{M_j} \geq -\epsilon_j$

and $(M_j, m_j, \mathrm{vol}/\mathrm{vol}\,B_1(m_j)) \to (\mathbf{R}^k \times W, (0_k, w), \upsilon)$. Fix a sufficiently small $\delta > 0$. Let $\{(t_i, x_i)\}_{i=1}^N$ be a maximal δ-separated subset of $[0,1]^k \times \overline{B}_\delta(z)$, let $z \in \overline{B}_R(w) \setminus B_r(w)$, and let $y_j^i \in M_j$ with $y_j^i \to (t_i, x_i)$ as $j \to \infty$. Note that $\{\overline{B}_{\delta/3}(y_j^i)\}_i$ is pairwise disjoint for every sufficiently large j. Put $r = d^{-1}$, put $X_j = \bigcup_i \overline{B}_{\delta/3}(y_j^i)$, put $S_{m_j}M_j = \{u \in T_{m_j}M_j \mid |u| = 1\}$, put $t(u) = \sup\{t \in \mathbf{R}_{>0} \mid \exp_{m_j} su \in M_j \setminus C_{m_j}$ for every $0 < s < t\}$ for $u \in S_{m_j}M_j$, put $\hat{S}_{m_j}M_j = \{u \in S_{m_j}M_j \mid$ there exists $0 < t < t(u)$ such that $\exp_{m_j} tu \in X_j$ holds$\}$, and put $A_j(u) = \{t \in (0, t(u)) \mid \exp_{m_j} tu \in X_j\}$ for $u \in \hat{S}_{m_j}M_j$ and $\theta(t,u) = t^{n-1}\sqrt{\det(g_{ij}|_{\exp_{m_j} tu})}$, where $g_{ij} = g(\partial/\partial x_i, \partial/\partial x_j)$ for a normal coordinate $(x_1, x_2, \ldots, x_n)$ around m_j. Then, by the Laplacian comparison theorem, we have

$$\begin{aligned}
\mathrm{vol}\,X_j &= \int_{\hat{S}_{m_j}M_j}\int_{A_j(u)} \theta(t,u)\,dt\,du \\
&\le \int_{\hat{S}_{m_j}}\int_{A_j(u)} \sinh^{n-1}(t)\frac{\theta\left(\frac{r}{2},u\right)}{\sinh^{n-1}\left(\frac{r}{2}\right)}\,dt\,du \\
&\le \int_{\hat{S}_{m_j}M_j} \frac{\theta\left(\frac{r}{2},u\right)}{\sinh^{n-1}\left(\frac{r}{2}\right)}\int_{A_j(u)} \sinh^{n-1}(2R+10)\,dt\,du \\
&\le C(n,r,R)\int_{\hat{S}_{m_j}M_j} \theta\Big(\frac{r}{2},u\Big)H^1\big(A_j(u)\big)\,du.
\end{aligned}$$

Put $a_j(u) = \inf A_j(u)$, and put $b_j(u) = \sup A_j(u)$ for $u \in \hat{S}_{m_j}M_j$. Then, by Lemma A.1, we have that $b_j(u) - a_j(u) \le C(r,R)\delta$ for every sufficiently large j. Thus $\underline{\mathrm{vol}}\,X_j \le C(r,R)\delta\,\underline{\mathrm{vol}}(\partial B_{\frac{r}{2}}(m_j) \setminus C_{m_j})$, where $\underline{\mathrm{vol}} = \mathrm{vol}/\mathrm{vol}\,B_1(m_j)$. By the Bishop–Gromov inequality, we have $\mathrm{vol}(\partial B_{r/2}(m_j) \setminus C_{m_j})/\mathrm{vol}\,B_{r/2}(m_j) \le \mathrm{vol}\,\partial B_{r/2}(\underline{p})/\mathrm{vol}\,B_{r/2}(\underline{p})$, where $\underline{p}$ is a point in the n-dimensional space form whose sectional curvature is equal to -1. Thus, we have

$$\sum_{i=1}^N \upsilon\big(B_{\frac{\delta}{3}}(t_i, x_i)\big) \le C(n,r,R)\delta.$$

By [3, Proposition 1.35], there exists a Borel measure υ_W on W such that $\upsilon = H^k \times \upsilon_W$. Therefore, by the Bishop–Gromov inequality for υ, we have

$$\begin{aligned}\upsilon_W\big(B_\delta(w)\big) = \upsilon\big([0,1]^k \times B_\delta(w)\big) &\le \sum_{i=1}^{N} \upsilon\big(B_\delta(t_i, x_i)\big)\\ &\le C(n)\sum_{i=1}^{N} \upsilon\big(B_{\frac{\delta}{3}}(t_i, x_i)\big)\\ &\le C(n,r,R)\delta.\end{aligned}$$

Therefore, we have Proposition 5.2. ∎

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References

[1] D. Burago, Y. Burago, and S. Ivanov, *A Course in Metric Geometry*, Grad. Stud. Math. **33**, Amer. Math. Soc., Providence, 2001. MR 1835418.

[2] J. Cheeger and T. H. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. of Math. (2) **144** (1996), 189–237. MR 1405949. DOI 10.2307/2118589.

[3] ———, *On the structure of spaces with Ricci curvature bounded below, I*, J. Differential Geom. **46** (1997), 406–480. MR 1484888.

[4] ———, *On the structure of spaces with Ricci curvature bounded below, II*, J. Differential Geom. **54** (2000), 13–35. MR 1815410.

[5] ———, *On the structure of spaces with Ricci curvature bounded below, III*, J. Differential Geom. **54** (2000), 37–74. MR 1815411.

[6] T. H. Colding, *Ricci curvature and volume convergence*, Ann. of Math. (2) **145** (1997), 477–501. MR 1454700. DOI 10.2307/2951841.

[7] K. Fukaya, *Collapsing of Riemannian manifolds and eigenvalues of Laplace operator*, Invent. Math. **87** (1987), 517–547. MR 0874035. DOI 10.1007/BF01389241.

[8] ———, "Hausdorff convergence of Riemannian manifolds and its applications" in *Recent Topics in Differential and Analytic Geometry*, Adv. Stud. Pure Math. **18-I**, Academic Press, Boston, 1990, 143–238. MR 1145256.

[9] ———, "Metric Riemannian geometry" in *Handbook of Differential Geometry, Vol. 2*, Elsevier/North-Holland, Amsterdam, 2006. MR 2194670. DOI 10.1016/S1874-5741(06)80007-5.

[10] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*, with appendices by M. Katz, P. Pansu, and S. Semmes, Birkhäuser, Boston, 1999. MR 1699320.

[11] S. Honda, *Ricci curvature and almost spherical multi-suspension*, Tohoku Math. J. (2) **61** (2009), 499–522. MR 2598247. DOI 10.2748/tmj/1264084497.

[12] ———, *Bishop–Gromov type inequality on Ricci limit spaces*, J. Math. Soc. Japan **63** (2011), 419–442. MR 2793106.

[13] ———, *A note on one-dimensional regular sets*, in preparation.

[14] X. Menguy, *Examples of nonpolar limit spaces*, Amer. J. Math. **122** (2000), 927–937. MR 1781925.

[15] ———, *Noncollapsing examples with positive Ricci curvature and infinite topological type*, Geom. Funct. Anal. **10** (2000), 600–627. MR 1779615. DOI 10.1007/PL00001632.

[16] ———, *Examples of strictly weakly regular points*, Geom. Funct. Anal. **11** (2001), 124–131. MR 1829644. DOI 10.1007/PL00001667.

[17] S. Ohta, *On the measure contraction property of metric measure spaces*, Comment. Math. Helv. **82** (2007), 805–828. MR 2341840. DOI 10.4171/CMH/110.

[18] L. Simon, *Lectures on Geometric Measure Theory*, Proc. Centre Math. Anal. Austral. Nat. Univ. **3**, Australian National University, Canberra, 1983. MR 0756417.

[19] C. Sormani, *The almost rigidity of manifolds with lower bounds on Ricci curvature and minimal volume growth*, Comm. Anal. Geom. **8** (2000), 159–212. MR 1730892.

[20] M. Watanabe, *Local cut points and metric measure spaces with Ricci curvature bounded below*, Pacific J. Math. **233** (2007), 229–256. MR 2366375. DOI 10.2140/pjm.2007.233.229.

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