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VARIATION FORMULAS FOR PRINCIPAL FUNCTIONS, II: APPLICATIONS TO VARIATION FOR HARMONIC SPANS

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To Professor Mitsuru Nakai,
on the occasion of his 77th birthday

Abstract. A domain $D \subset \mathbb{C}_z$ admits the circular slit mapping $P(z)$ for $a, b \in D$ such that $P(z) - 1/(z-a)$ is regular at a and $P(b) = 0$. We call $p(z) = \log|P(z)|$ the L_1-*principal function* and $\alpha = \log|P'(b)|$ the L_1-*constant*, and similarly, the radial slit mapping $Q(z)$ implies the L_0-principal function $q(z)$ and the L_0-constant β. We call $s = \alpha - \beta$ the *harmonic span* for (D, a, b). We show the geometric meaning of s. Hamano showed the variation formula for the L_1-constant $\alpha(t)$ for the moving domain $D(t)$ in $\mathbb{C}_z$ with $t \in B := \{t \in \mathbb{C} : |t| < \rho\}$. We show the corresponding formula for the L_0-constant $\beta(t)$ for $D(t)$ and combine these to prove that, if the total space $\mathcal{D} = \bigcup_{t \in B}(t, D(t))$ is pseudoconvex in $B \times \mathbb{C}_z$, then $s(t)$ is subharmonic on B. As a direct application, we have the subharmonicity of $\log\cosh d(t)$ on B, where $d(t)$ is the Poincaré distance between a and b on $D(t)$.

§1. Introduction

Let R be a bordered Riemann surface with boundary $\partial R = C_1 + \cdots + C_\nu$ in a larger Riemann surface $\widetilde{R}$, where C_j is a C^ω smooth contour in $\widetilde{R}$. Fix two points a, b with local coordinates $U_a : |z| < r_0$ and $U_b : |z - \xi| < r_1$, where a and b correspond to 0 and ξ, respectively (where U_a and U_b have no relations). Among all harmonic functions u on $R \setminus \{a, b\}$ with logarithmic singularity $\log(1/|z|)$ at a and $\log|z - \xi|$ at b normalized $\lim_{z \to 0}(u(z) - \log(1/|z|)) = 0$, we have two special functions p and q with the boundary conditions that, for each C_j, p satisfies $p(z) = \text{constant } c_j$ on C_j and

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$\int_{C_j} \frac{\partial p(z)}{\partial n_z} ds_z = 0$ (where $\frac{\partial}{\partial n_z}$ is the outer normal derivative and ds_z is the arc length element at z of C_j), while q satisfies $\frac{\partial q(z)}{\partial n_z} = 0$ on C_j. We consider the constant terms $\alpha := \lim_{z\to\xi}(p(z) - \log|z-\xi|)$ and $\beta := \lim_{z\to\xi}(q(z) - \log|z-\xi|)$. We call $p(z)$ the L_1-*principal function* and α the L_1-*constant* for (R, a, b) with respect to local coordinates U_a and U_b or, simply, for $(R, 0, \xi)$, and similarly, we call $q(z)$ the L_0-*principal function* and β the L_0-*constant* (see, [1, Chapter III, Section 3]). Now let $B = \{t \in \mathbb{C} : |t| < \rho\}$, and let $\mathcal{R} : t \in B \to R(t) \Subset \widetilde{R}$ be a smooth variation of Riemann surfaces $R(t)$ in $\widetilde{R}$ with $t \in B$ such that $\partial R(t)$ is C^ω smooth in $\widetilde{R}$ and $R(t), t \in B$ contains $z = 0$ in U_a and $\xi(t)$, which vary holomorphically in U_b. Then each $R(t), t \in B$ admits the L_1-principal function $p(t, z)$ and L_1-constant $\alpha(t)$ for $(R(t), 0, \xi(t))$ and, similarly, the L_0-principal function $q(t, z)$ and the L_0-constant $\beta(t)$.

Hamano [9] showed the variation formula of the second order for $\alpha(t)$ (see Lemma 2.1 below), which implies that, if the total space $\mathcal{R} = \bigcup_{t\in B}(t, R(t))$ is a pseudoconvex domain in $B \times \widetilde{R}$, then $\alpha(t)$ is subharmonic on B. Continuing on [9], we show the variation formula for $\beta(t)$ (see Lemma 2.2 below), which continues on [10]. To prove the formula for $\beta(t)$, we add a new idea to Hamano's proof for $\alpha(t)$. In fact, the formula for $\alpha(t)$ does not concern the genus of $R(t)$, but the formula for $\beta(t)$ does concern it. The formula for $\beta(t)$ implies that, if $\mathcal{R}$ is pseudoconvex in $B \times \widetilde{R}$ and if $R(t), t \in B$ is planar, then $\beta(t)$ is superharmonic on B. This contrast between the subharmonicity of $\alpha(t)$ and the superharmonicity of $\beta(t)$ is unified with the notion of the harmonic span $s(t) := \alpha(t) - \beta(t)$ for $(R(t), 0, \xi(t))$ introduced by Nakai (see, [13, Chapter II, Section 3]): *if $\mathcal{R}$ is pseudoconvex in $B \times \widetilde{R}$ and $R(t), t \in B$ is planar, then $s(t)$ is subharmonic on B*; this implies Corollary 4.1. Assume, moreover, that each $R(t), t \in$ is simply connected. Let $\xi_i := \bigcup_{t\in B}(t, \xi_i(t)), i = 1, 2$ be two holomorphic sections of $\mathcal{R}$ over B, and let $d(t)$ denote the Poincaré distance between $\xi_1(t)$ and $\xi_2(t)$ on $R(t)$. Then $\delta(t) := \log\cosh d(t)$ is subharmonic on B. Further, $\delta(t)$ is harmonic on B if and only if $\mathcal{R}$ is fiber-preserving biholomorphic to the product $B \times R(0)$.

§2. Variation formulas for L_0-principal functions

Let $B = \{t \in \mathbb{C} : |t| < \rho\}$, and let $\pi : \widetilde{\mathcal{R}} \to B$ be a holomorphic family such that $\widetilde{\mathcal{R}}$ is a complex 2-dimensional manifold, π is a holomorphic projection from $\widetilde{\mathcal{R}}$ onto B, and each fiber $\widetilde{R}(t)=\pi^{-1}(t), t\in B$ is irreducible and non-singular in $\widetilde{\mathcal{R}}$. We put $\widetilde{\mathcal{R}}=\bigcup_{t\in B}(t,\widetilde{R}(t))$, and we call $\widetilde{R}(t)$ the *fiber of* $\widetilde{\mathcal{R}}$ *over* $t\in B$. Let $\mathcal{R}=\bigcup_{t\in B}(t,R(t))$ be a subdomain in $\widetilde{\mathcal{R}}$ such that we have the following conditions:

(1) $\widetilde{R}(t)\Supset R(t)\neq\emptyset$, $t\in B$, and $R(t)$ is a connected Riemann surface of genus $g\geq 0$ such that $\partial R(t)$ in $\widetilde{R}(t)$ consists of a finite number of C^ω smooth contours $C_j(t), j=1,\dots,\nu$;

(2) the boundary $\partial\mathcal{R}=\bigcup_{t\in B}(t,\partial R(t))$ of $\mathcal{R}$ in $\widetilde{\mathcal{R}}$ is C^ω smooth and $\partial\mathcal{R}$ is transverse to each fiber $\widetilde{R}(t), t\in B$.

Note that g and ν are independent of $t\in B$. Each $C_j(t)$ is oriented by $\partial R(t)=C_1(t)+\cdots+C_\nu(t)$. We regard the complex manifold $\mathcal{R}$ as a variation of Riemann surfaces $R(t)$ with parameter $t\in B$,

$$\mathcal{R}: t\in B\to R(t)\Subset\widetilde{R}(t).$$

We denote by $\Gamma(B,\mathcal{R})$ the set of all holomorphic sections of $\mathcal{R}$ over B. Assume that there exist $\Xi_0,\Xi_\xi\in\Gamma(B,\mathcal{R})$ with $\Xi_0\cap\Xi_\xi=\emptyset$ such that there exist π-local coordinates $U_0:=B\times\{|z|<r_0\}$ and $U_\xi:=B\times\{|z-\xi(t)|<r_1\}$ of neighborhoods V_0 of Ξ_0 and V_ξ of Ξ_ξ in $\mathcal{R}$ such that Ξ_0 corresponds to $z=0$ and Ξ_ξ corresponds to $z=\xi(t), t\in B$. Let $t\in B$ be fixed. Then $R(t)$ admits the functions $p(t,z)$ and $q(t,z)$ such that both functions are continuous on $\overline{R(t)}$ and harmonic on $R(t)\setminus\{0,\xi(t)\}$ with poles $\log(1/|z|)$ at $z=0$ and $\log|z-\xi(t)|$ at $z=\xi(t)$ normalized $\lim_{z\to 0}(p(t,z)-\log(1/|z|))=\lim_{z\to 0}(q(t,z)-\log(1/|z|))=0$ at $z=0$, and $p(t,z)$ and $q(t,z)$ satisfy the following boundary conditions (L_1) and (L_0), respectively: for $j=1,\dots,\nu$,

$$(L_1)\quad p(t,z)=\text{constant } c_j(t)\quad\text{on } C_j(t)\qquad\text{and}\qquad \int_{C_j(t)}\frac{\partial p(t,z)}{\partial n_z}\,ds_z=0;$$

$$(L_0)\quad \frac{\partial q(t,z)}{\partial n_z}=0\quad\text{on } C_j(t).$$

We have

$$\begin{aligned} p(t,z)&=\log\frac{1}{|z|}+0+h_0(t,z)\quad\text{on } U_0(t),\\ q(t,z)&=\log\frac{1}{|z|}+0+\mathfrak{h}_0(t,z)\quad\text{on } U_0(t),\end{aligned}\tag{2.1}$$

where $h_0(t,z)$, $\mathfrak{h}_0(t,z)$ are harmonic for z on $U_0(t)$ such that $h_0(t,0),\mathfrak{h}_0(t,0)\equiv 0$ on B, and

$$
\begin{aligned}
p(t,z) &= \log|z-\xi(t)| + \alpha(t) + h_\xi(t,z) \quad \text{on } U_\xi(t), \\
q(t,z) &= \log|z-\xi(t)| + \beta(t) + \mathfrak{h}_\xi(t,z) \quad \text{on } U_\xi(t),
\end{aligned}
\tag{2.2}
$$

where $\alpha(t), \beta(t)$ are the constant terms and where $h_\xi(t,z), \mathfrak{h}_\xi(t,z)$ are harmonic for z on $U_\xi(t)$ such that $h_\xi(t,\xi(t)), \mathfrak{h}_\xi(t,\xi(t)) \equiv 0$ on B. We call $p(t,z)$ the L_1-*principal function*, or simply L_1-*function*, and α the L_1-*constant* for $(R(t), 0, \xi(t))$, and similarly, we call $q(t,z)$ the L_0-*function* and $\beta(t)$ the L_0-*constant*.

The following variation formula is for the second order for $\alpha(t)$.

Lemma 2.1 ([9, Lemma 1.3]). *We have*

$$
\frac{\partial^2 \alpha(t)}{\partial t \partial \bar{t}} = \frac{1}{\pi} \int_{\partial R(t)} k_2(t,z) \Big| \frac{\partial p(t,z)}{\partial z} \Big|^2 ds_z + \frac{4}{\pi} \iint_{R(t)} \Big| \frac{\partial^2 p(t,z)}{\partial \bar{t} \partial z} \Big|^2 dx\, dy.
$$

Here

$$
k_2(t,z) = \Big(\frac{\partial^2 \varphi}{\partial t \partial \bar{t}} \Big| \frac{\partial \varphi}{\partial z} \Big|^2 - 2\Re \Big\{ \frac{\partial^2 \varphi}{\partial \bar{t} \partial z} \frac{\partial \varphi}{\partial t} \frac{\partial \varphi}{\partial \bar{z}} \Big\} + \Big| \frac{\partial \varphi}{\partial t} \Big|^2 \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} \Big) \Big/ \Big| \frac{\partial \varphi}{\partial z} \Big|^3
$$

on $\partial \mathcal{R}$, where $\varphi(t,z)$ is a C^2 defining function of $\partial \mathcal{R}$.

Note that $k_2(t,z)$ on $\partial \mathcal{R}$ does not depend on the choice of defining functions $\varphi(t,z)$ of $\partial \mathcal{R}$. We call $k_2(t,z)$ the *Levi curvature* for $\partial \mathcal{R}$ (see [11, (1.3)], [12, (7)]).

We give the variation formulas for $\beta(t)$. In the case where $R(t)$ is of genus $g \geq 1$, we need the following consideration, which was not necessary for the variation formulas for $\alpha(t)$. We draw, as usual, A, B cycles $\{A_k(t), B_k(t)\}_{1 \leq k \leq g}$ on $R(t)$, which vary continuously in $\mathcal{R}$ with $t \in B$ without passing through $0, \xi(t)$:

$$
\begin{aligned}
&A_k(t) \cap B_l(t) = \emptyset \quad \text{for } k \neq l, \qquad A_k \times B_k = 1 \quad \text{for } k = 1, \dots, g, \\
&A_k(t) \cap A_l(t) = B_k(t) \cap B_l(t) = \emptyset \quad \text{for } k \neq l.
\end{aligned}
\tag{2.3}
$$

Here $A_k(t) \times B_k(t) = 1$ means that $A_k(t)$ crosses $B_k(t)$ once from the right-hand side to the left-hand side of the direction $B_k(t)$. On $R(t), t \in B$ we put $*dq(t,z) = -\frac{\partial q(t,z)}{\partial y} dx + \frac{\partial q(t,z)}{\partial x} dy$, the conjugate differential of $dq(t,z)$.

LEMMA 2.2. *We have*

$$\frac{\partial^2\beta(t)}{\partial t\partial\bar{t}} = -\frac{1}{\pi}\int_{\partial R(t)} k_2(t,z)\Big|\frac{\partial q(t,z)}{\partial z}\Big|^2 ds_z - \frac{4}{\pi}\iint_{R(t)}\Big|\frac{\partial^2 q(t,z)}{\partial\bar{t}\partial z}\Big|^2 dx\,dy$$
$$-\frac{2}{\pi}\Im\sum_{k=1}^{g}\Big(\frac{\partial}{\partial t}\int_{A_k(t)} * dq(t,z)\Big)\cdot\Big(\frac{\partial}{\partial\bar{t}}\int_{B_k(t)} * dq(t,z)\Big).$$

Proof. We divide the proof into two steps.

STEP 1. The formula does not depend on the choice of either π-biholomorphic mappings or π-local coordinates.

In fact, let $\widetilde{\pi}:\widetilde{\mathcal{D}}\to B$ be a holomorphic family, and let a subdomain $\mathcal{D}$ of $\widetilde{\mathcal{D}}$ satisfy conditions (1) and (2). We write $\widetilde{\pi}^{-1}(t)=\widetilde{D}(t)$ and $\mathcal{D}=\bigcup_{t\in B}(t,D(t))$, where $D(t)\Subset\widetilde{D}(t)$. Assume that there exists a π-biholomorphic mapping

$$T:(t,z)\in\widetilde{\mathcal{R}}\to(t,w)=\big(t,F(t,z)\big)\in\widetilde{\mathcal{D}}$$

such that $T(\mathcal{R})=\mathcal{D}$. Thus, $R(t)$ and $D(t)$ are equivalent as Riemann surfaces. We write $\widetilde{\Xi}_0,\widetilde{\Xi}_{\widetilde{\xi}}\in\Gamma(B,\mathcal{D})$, which correspond to $\Xi_0,\Xi_\xi\in\Gamma(B,\mathcal{R})$ by T. We put $\widetilde{A}_k(t)=F(t,A_k(t))$, and we put $\widetilde{B}_k(t)=F(t,B_k(t))$ on $D(t)$. Since $\int_{A_k(t)} * dq(t,z)=\int_{\widetilde{A}_k(t)} * d\widetilde{q}(t,w)$, we have

$$(i)\ \frac{\partial}{\partial t}\int_{\widetilde{A}_k(t)} * d\widetilde{q}(t,w)=\frac{\partial}{\partial t}\int_{A_k(t)} * dq(t,z)\qquad\text{for } t\in B,$$

and similarly for $\widetilde{B}_k(t)$ and $B_k(t)$. Let $\widetilde{\pi}$-local coordinates $\widetilde{U}_0:=B\times\{|w|<\rho_0\}$ and $\widetilde{U}_{\widetilde{\xi}}:=B\times\{|w-\widetilde{\xi}(t)|<\rho_1\}$ of neighborhoods $\widetilde{V}_0$ of $\widetilde{\Xi}_0$ and $\widetilde{V}_{\widetilde{\xi}}$ of $\widetilde{\Xi}_{\widetilde{\xi}}$ in $\mathcal{D}$. Each $D(t), t\in B$ admits the L_0-function $\widetilde{q}(t,w)$ and the L_0-constant $\widetilde{\beta}(t)$ for $(D(t),0,\widetilde{\xi})$. We have

$$\widetilde{q}(t,w)=\log\frac{1}{|w|}+0+\widetilde{\mathfrak{h}}_0(t,w)\quad\text{on }\big\{|w|<\rho_0\big\},$$
$$\widetilde{q}(t,w)=\log|w-\widetilde{\xi}(t)|+\widetilde{\beta}(t)+\widetilde{\mathfrak{h}}_{\widetilde{\xi}}(t,w)\quad\text{on }\big\{|w-\widetilde{\xi}(t)|<\rho_1\big\},$$

where $\widetilde{\mathfrak{h}}_0(t,w)$ is harmonic on $\{|w|<\rho_0\}$ such that $\widetilde{\mathfrak{h}}_0(t,0)\equiv 0$ on B, and $\widetilde{h}_{\widetilde{\xi}}(t,z)$ is harmonic on $\{|w-\widetilde{\xi}(t)|<\rho_1\}$ such that $\widetilde{\mathfrak{h}}_{\widetilde{\xi}}(t,\xi(t))\equiv 0$ on B. Then we have the biholomorphic mappings $T_0:(t,z)\in U_0\to(t,w)=(t,f_0(t,z))\in\widetilde{U}_0$ such that $f_0(t,0)=0$, and $T_\xi:(t,z)\in U_\xi\to(t,w)=(t,f_\xi(t,z))\in\widetilde{U}_{\widetilde{\xi}}$ such that $f_\xi(t,\xi(t))=\widetilde{\xi}(t)$. For $t\in B$, we put $a_0(t):=\frac{\partial f_0(t,z)}{\partial z}|_{z=0}$ and $a_\xi(t):=$

$\frac{\partial f_\xi(t,z)}{\partial z}|_{z=\xi(t)}$, so that $a_0(t), a_\xi(t)$ are nonvanishing holomorphic functions on B. We have

$$\beta(t) = \widetilde{\beta}(t) - \log|a_0(t)| + \log|a_\xi(t)| \quad \text{on } B,$$

which implies that

$$(ii)\ \frac{\partial^2 \beta(t)}{\partial t \partial \bar{t}} = \frac{\partial^2 \widetilde{\beta}(t)}{\partial t \partial \bar{t}} \quad \text{for } t \in B.$$

If we write $\widetilde{k}_2(t,w)$ for the Levi curvature for $\partial\mathcal{D}$, then we have

$$\widetilde{k}_2(t,w) = k_2(t,z)\Big|\frac{\partial F(t,z)}{\partial z}\Big|$$

for $w = F(t,z)$ and $(t,z) \in \partial\mathcal{R}$, and hence

$$\widetilde{k}_2(t,w)\Big|\frac{\partial \widetilde{q}(t,w)}{\partial w}\Big|^2 |dw| = k_2(t,z)\Big|\frac{\partial q(t,z)}{\partial z}\Big|^2 |dz|,$$

which implies that

$$(iii)\ \int_{\partial D(t)} \widetilde{k}_2(t,w)\Big|\frac{\partial \widetilde{q}}{\partial w}(t,w)\Big|^2 ds_w = \int_{\partial R(t)} k_2(t,z)\Big|\frac{\partial q}{\partial z}(t,z)\Big|^2 ds_z$$

for $t \in B$. Since

$$(iv)\ \iint_{D(t)}\Big|\frac{\partial^2 \widetilde{q}}{\partial \bar{t}\partial w}(t,w)\Big|^2 du\,dv = \int_{R(t)}\Big|\frac{\partial^2 q}{\partial \bar{t}\partial z}(t,z)\Big|^2 dx\,dy$$

for $t \in B$, equations (i)–(iv) imply Step 1.

STEP 2. Lemma 2.2 is true.

In fact, it suffices to prove the lemma at $t = 0$. If necessary, take a smaller disk B of center 0. Then by the standard use of the immersion theorem for the open Riemann surfaces due to Nishimura [14] (see also [7]), we have a π-biholomorphic mapping from $\widetilde{\mathcal{R}}$ to an unramified (Riemann) domain $\widetilde{\mathcal{D}}$ over $B \times \mathbb{C}_w$ such that, if we write $T(\mathcal{R}) = \mathcal{D}$, then the holomorphic sections Ξ_0 and Ξ_ξ of $\mathcal{R}$ over B correspond to the constant sections $\Xi_0 := B \times \{w = 0\}$ and $\Xi_1 := B \times \{w = 1\}$ of $\mathcal{D}$ over B. By Step 1, it suffices to show the lemma for the unramified domain $\mathcal{D}$ over $B \times \mathbb{C}_w$ and the sections $\Xi_0, \Xi_1 \in \Gamma(B, \mathcal{D})$. For the sake of convenience, we use anew the notation $\widetilde{\mathcal{R}}$ and $\mathcal{R}$ for $\widetilde{\mathcal{D}}$ and $\mathcal{D}$. By condition (1), the boundary $\partial\mathcal{R}$ of $\mathcal{R}$ in $\widetilde{\mathcal{R}}$ is C^ω smooth, and each $\widetilde{R}(t), t \in B$ is a Riemann surface sheeted over $\mathbb{C}_z$ without ramification points. By condition (2), $R(t)$ is a relatively compact subdomain of $\widetilde{R}(t)$ with C^ω smooth boundary $\partial R(t)$ and $R(t) \ni 0, 1$. We have $\mathcal{R} = \bigcup_{t\in B}(t, R(t))$ and $\partial\mathcal{R} = \bigcup_{t\in B}(t, \partial R(t))$, which is transverse to each fiber $\widetilde{R}(t)$. Under these situations we find a neighborhood $V = \bigcup_{j=1}^{\nu} V_j$ (disjoint union) of $\partial R(0) = \bigcup_{j=1}^{\nu} C_j(0)$ such that $(B \times V) \cap (\Xi_0 \cup \Xi_1) = \emptyset$; V_j is a thin tubular neighborhood of $C_j(0)$ with $V_j \supset C_j(t)$ for $t \in B$, and $q(t, z)$ is harmonic on $(R(0) \cup V) \setminus \{0, 1\}$. We write $\widehat{R}(0) := R(0) \cup V$, so that $q(t, z)$ is defined in the product $B \times \widehat{R}(0)$. Then, (2.2) becomes

$$q(t, z) = \log|z-1| + \beta(t) + \mathfrak{h}_1(t, z) \quad \text{on } U_1(t), \tag{2.4}$$

where $\mathfrak{h}_1(t, 1) \equiv 0$ on B. For $t \in B$ we put $u(t, z) := q(t, z) - q(0, z)$ on $\widehat{R}(0) \setminus \{0, 1\}$. By putting $u(t, 0) = 0$ and $u(t, 1) = \beta(t) - \beta(0)$, $u(t, z)$ is harmonic on $\widehat{R}(0)$.

Let $0 < \varepsilon \ll 1$, let $\gamma_\varepsilon(0) = \{|z| < \varepsilon\}$, and let $\gamma_\varepsilon(1) = \{|z-1| < \varepsilon\}$. Then,

$$\int_{\partial R(0) - \partial\gamma_\varepsilon(0) - \partial\gamma_\varepsilon(1)} u(t, z)\frac{\partial q(0, z)}{\partial n_z}\, ds_z - q(0, z)\frac{\partial u(t, z)}{\partial n_z}\, ds_z = 0.$$

Letting $\varepsilon \to 0$, we have from $\frac{\partial q(0,z)}{\partial n_z} = 0$ on $C_j(0), j = 1, \ldots, \nu$,

$$\beta(t) - \beta(0) = \frac{-1}{2\pi}\sum_{j=1}^{\nu}\int_{C_j(0)} q(0, z)\frac{\partial q(t, z)}{\partial n_z}\, ds_z =: \frac{-1}{2\pi}\sum_{j=1}^{\nu} I_j(t). \tag{2.5}$$

We take a point $z_j^0(t)$ on each $C_j(t), t \in B$ such that $z_j^0(t)$ continuously moves in $\partial\mathcal{R}$ with $t \in B$, and we choose a harmonic conjugate function $q_j^*(t, z)$ of $q(t, z)$ in V_j such that $q_j^*(t, z_j^0(t)) = 0$. Since $\frac{\partial q(t,z)}{\partial n_z} = 0$ on $C_j(t)$, $q_j^*(t, z)$ is single valued in V_j and

$$q_j^*(t, z) = 0 \quad \text{for } z \in C_j(t). \tag{2.6}$$

Since $dq_j^*(t, z) = \frac{\partial q(t,z)}{\partial n_z} ds_z$, $dq(0, z) = -\frac{\partial q_j^*(0,z)}{\partial n_z} ds_z$ along $C_j(0)$, we have

$$\begin{aligned} I_j(t) &= \int_{C_j(0)} q(0, z)\, dq_j^*(t, z) = \int_{C_j(0)} d\big(q(0, z) q_j^*(t, z)\big) - q_j^*(t, z)\, dq(0, z) \\ &= \int_{C_j(0)} q_j^*(t, z)\frac{\partial q_j^*(0, z)}{\partial n_z}\, ds_z. \end{aligned}$$

Differentiating both sides by t and $\overline{t}$ at $t=0$, we have

$$\frac{\partial^2 I_j}{\partial t \partial \overline{t}}(0) = \int_{C_j(0)} \frac{\partial^2 q_j^*}{\partial t \partial \overline{t}}(0,z) \frac{\partial q_j^*(0,z)}{\partial n_z} ds_z. \tag{2.7}$$

We recall the following.

PROPOSITION 2.1 ([9, (1.2)]). *Let $u(t,z)$ be a C^2 function for (t,z) in a neighborhood $\mathcal{V}_j = \bigcup_{t\in B}(t, V_j(t))$ of $\mathcal{C}_j = \bigcup_{t\in B}(t, C_j(t))$ over $B\times \mathbb{C}_z$ such that $u(t,z), t\in B$ is harmonic for z in $V_j(t)$ and $u(t,z) =$ a certain constant $c_j(t)$ on $C_j(t)$. Then*

$$\begin{aligned}\frac{\partial^2 u}{\partial t\partial \overline{t}} \frac{\partial u}{\partial n_z} ds_z = 2k_2(t,z)\Big|\frac{\partial u}{\partial z}\Big|^2 ds_z + \frac{\partial^2 c_j(t)}{\partial t \partial \overline{t}} \frac{\partial u}{\partial n_z} ds_z \\ + 4\Im\Big\{\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial \overline{t}\partial z} dz\Big\} - 4\Im\Big\{\frac{\partial c_j(t)}{\partial t}\frac{\partial^2 u}{\partial \overline{t}\partial z} dz\Big\} \quad \textit{along } C_j(t).\end{aligned}$$

We apply this for $u(t,z) = q_j^*(t,z)$ with (2.6) to (2.7) and obtain

$$\frac{\partial^2 I_j}{\partial t \partial \overline{t}}(0) = 2\int_{C_j(0)} k_2(0,z)\Big|\frac{\partial q_j^*(0,z)}{\partial z}\Big|^2 ds_z + 4\Im \int_{C_j(0)} \frac{\partial q_j^*}{\partial t}(0,z)\frac{\partial^2 q_j^*}{\partial \overline{t}\partial z}(0,z)\, dz.$$

We put

$$\mathbf{a}_k(t) = \int_{A_k(t)} *dq(t,z), \qquad \mathbf{b}_k(t) = \int_{B_k(t)} *dq(t,z).$$

We fix a point $z^0 (\neq 0,1)$ such that $B\times\{z^0\}\subset \mathcal{R}$. On $R(t), t\in B$ we choose a branch $q^*(t,z)$ of a harmonic conjugate function of $q(t,z)$ on $\widehat{R}(0)\setminus\{0,1\}$ such that $q^*(t,z^0)=0$. Since $\int_{C_j(0)} *dq(t,z)=0$, we have

$$q^*(t,z') = q^*(t,z'') \quad \mathrm{mod}\big\{2\pi, \mathbf{a}_k(t), \mathbf{b}_k(t) (k=1,\dots,g)\big\}$$

for any z', z'' over the same point $z\in \widehat{R}(0)\setminus\{0,1\}$. We also have $q_j^*(t,z) - q^*(t,z) = c_j(t)$ on V_j, where $c_j(t)$ is a certain constant for $z\in V_j$. It follows that

$$\begin{aligned}&\int_{C_j(0)} \frac{\partial q_j^*}{\partial t}(0,z)\frac{\partial^2 q_j^*}{\partial \overline{t}\partial z}(0,z)\, dz \\ &\quad = \int_{C_j(0)} \frac{\partial q^*}{\partial t}(0,z)\frac{\partial^2 q^*}{\partial \overline{t}\partial z}(0,z)\, dz + \frac{\partial c_j}{\partial t}(0)\int_{C_j(0)} \frac{\partial^2 q^*}{\partial \overline{t}\partial z}(0,z)\, dz.\end{aligned}$$

The function $f(t,z) := q^*(t,z) - iq(t,z)$ belongs to $C^\omega(B\times V_j)$, and $f(t,z)$, $t\in B$ is a single-valued holomorphic function in V_j. Hence,

$$\int_{C_j(0)} \frac{\partial^2 q^*}{\partial\bar t\partial z}(0,z)\,dz = \frac{1}{2}\Big[\frac{\partial}{\partial\bar t}\,\Big(\int_{C_j(0)} f'_z(t,z)\,dz\Big)\Big]_{t=0} = 0.$$

Therefore,

$$\frac{\partial^2 I_j}{\partial t\partial\bar t}(0) = 2\int_{C_j(0)} k_2(0,z)\Big|\frac{\partial q^*(0,z)}{\partial z}\Big|^2 ds_z + 4\Im\Big\{\int_{C_j(0)} \frac{\partial q^*}{\partial t}(0,z)\frac{\partial^2 q^*}{\partial\bar t\partial z}(0,z)\,dz\Big\}.$$

It follows from (2.5) that

$$\frac{\partial^2 \beta}{\partial t\partial\bar t}(0) = -\frac{1}{\pi}\int_{\partial R(0)} k_2(0,z)\Big|\frac{\partial q^*(0,z)}{\partial z}\Big|^2 ds_z - \frac{2}{\pi}\Im\Big\{\int_{\partial R(0)} \frac{\partial q^*}{\partial t}(0,z)\frac{\partial^2 q^*}{\partial\bar t\partial z}(0,z)\,dz\Big\}.$$

We divide the proof into two cases.

Case 1: $R(t)$ is planar (i.e., $g=0$). In this case, each $q^*(t,z), t\in B$ is determined up to additive constants mod 2π. By (2.1) and (2.4), $\frac{\partial q^*(t,z)}{\partial t}, t\in B$ is a single-valued harmonic function on $\widehat R(0)$, and $\frac{\partial^2 q^*(t,z)}{\partial\bar t\partial z}, t\in B$ is a single-valued holomorphic function on $\widehat R(0)$. Then,

$$\int_{\partial R(0)} \frac{\partial q^*}{\partial t}(0,z)\frac{\partial^2 q^*}{\partial\bar t\partial z}(0,z)\,dz = 2i\iint_{R(0)}\Big|\frac{\partial^2 q^*}{\partial\bar t\partial z}(0,z)\Big|^2 dx\,dy.$$

Therefore,

$$\frac{\partial^2 \beta}{\partial t\partial\bar t}(0) = -\frac{1}{\pi}\int_{\partial R(0)} k_2(0,z)\Big|\frac{\partial q(0,z)}{\partial z}\Big|^2 ds_z - \frac{4}{\pi}\iint_{R(0)}\Big|\frac{\partial^2 q}{\partial\bar t\partial z}(0,z)\Big|^2 dx\,dy,$$

which is desired.

Case 2: $R(t)$ is of genus $g\geq 1$. We put $R'(0) = R(0)\setminus\bigcup_{k=1}^g(A_k(0)\cup B_k(0))$, and we put $\widehat R'(0) = R'(0)\cup V$, so that $R'(0)$ and $\widehat R'(0)$ are planar Riemann surfaces such that

$$\partial R'(0) = \partial R(0) + \sum_{k=1}^g\big(A_k^+(0) + A_k^-(0)\big) + \sum_{k=1}^g\big(B_k^+(0) + B_k^-(0)\big).$$

Here $A_k^+(0)$ is of the same direction of $A_k(0)$, $A_k^-(0)$ is of the opposite direction, and $B_k^+(0)$ and $B_k^-(0)$ are similar. For $t \in B$, if we restrict the branch $q^*(t,z)$ with $q^*(t,z^0)=0$ to $R'(0)\setminus\{0,1\}$, then $q^*(t,z')=q^*(t,z'') \bmod 2\pi$ for z', z'' over the same point $z \in \widehat{R}'(0)$. Hence, $\frac{\partial q^*}{\partial t}(0,z)$ and $\frac{\partial^2 q^*}{\partial \bar{t}\partial z}(0,z)$ are single-valued harmonic functions on $\widehat{R}'(0)$, so that

$$\begin{aligned}
&\int_{\partial R(0)} \frac{\partial q^*}{\partial t}(0,z)\frac{\partial^2 q^*}{\partial \bar{t}\partial z}(0,z)\,dz \\
&= \iint_{R'(0)} d\Big(\frac{\partial q^*}{\partial t}(0,z)\frac{\partial^2 q^*}{\partial \bar{t}\partial z}(0,z)\,dz\Big) \\
&\quad -\sum_{k=1}^{g}\int_{A_k^\pm(0)+B_k^\pm(0)} \frac{\partial q^*}{\partial t}(0,z)\frac{\partial^2 q^*}{\partial \bar{t}\partial z}(0,z)\,dz \\
&=: J_1 - J_2.
\end{aligned}$$

Since $\frac{\partial q^*}{\partial \bar{t}\partial z}(0,z)$ is holomorphic on $R'(0)$, we have

$$\begin{aligned}
J_1 &= 2i\iint_{R(0)}\Big|\frac{\partial^2 q}{\partial t\partial \bar{z}}(0,z)\Big|^2 dx\,dy; \\
J_2(A_k) &:= \int_{A_k^\pm(0)} \frac{\partial q^*}{\partial t}(0,z)\frac{\partial^2 q^*}{\partial \bar{t}\partial z}(0,z)\,dz \\
&= \int_{A_k(0)}\Big(\frac{\partial q^*}{\partial t}(0,z^+)-\frac{\partial q^*}{\partial t}(0,z^-)\Big)\frac{\partial^2 q^*}{\partial \bar{t}\partial z}(0,z)\,dz.
\end{aligned}$$

By (2.3) and $\int_{C_j(0)} *dq(t,z)=0$, it holds that, for $z^\pm$ over any $z\in A_k(0)$,

$$q^*(t,z^+)-q^*(t,z^-)=\int_{B_k(0)} *dq(t,\zeta) \quad \bmod 2\pi.$$

Therefore,

$$\frac{\partial q^*}{\partial t}(t,z^+)-\frac{\partial q^*}{\partial t}(t,z^-)=\frac{\partial}{\partial t}\int_{B_k(0)} *dq(t,\zeta),$$

independent of $z\in A_k(0)$. By $\frac{\partial q^*(t,z)}{\partial z}dz=(1/2)(*dq(t,z)-i\,dq(t,z))$,

$$\begin{aligned}
J_2(A_k) &= \Big[\frac{\partial}{\partial t}\Big(\int_{B_k(0)} *dq(t,\zeta)\Big)\Big]_{t=0}\cdot\Big[\frac{\partial}{\partial \bar{t}}\Big(\int_{A_k(0)}\frac{\partial q^*(t,z)}{\partial z}\,dz\Big)\Big]_{t=0} \\
&= \frac{1}{2}\frac{\partial \mathbf{b}_k}{\partial t}(0)\cdot\frac{\partial \mathbf{a}_k}{\partial \bar{t}}(0).
\end{aligned}$$

By $B_k(0)\times A_k(0)=-1$, it similarly holds that $J_2(B_k)=-(1/2)\frac{\partial \mathbf{a}_k}{\partial t}(0)\cdot\frac{\partial \mathbf{b}_k}{\partial \bar t}(0)$, so that $J_2(A_k)+J_2(B_k)=-i\Im\{\frac{\partial \mathbf{a}_k}{\partial t}(0)\cdot\frac{\partial \mathbf{b}_k}{\partial \bar t}(0)\}$. Therefore,

$$\Im\Big\{\int_{\partial R(0)}\frac{\partial q^*}{\partial t}(0,z)\frac{\partial^2 q^*}{\partial \bar t\partial z}(0,z)\,dz\Big\}=\Im\Big\{J_1-\sum_{k=1}^{g}\big(J_2(A_k)+J_2(B_k)\big)\Big\}$$

$$=2\iint_{R(0)}\Big|\frac{\partial^2 q}{\partial \bar t\partial z}(0,z)\Big|^2\,dx\,dy+\Im\Big\{\sum_{k=1}^{g}\frac{\partial \mathbf{a}_k}{\partial t}(0)\cdot\frac{\partial \mathbf{b}_k}{\partial \bar t}(0)\Big\}.$$

This completes Step 2. □

As noted in [9], since $\mathcal{R}$ is pseudoconvex in $\widetilde{\mathcal{R}}$ if and only if $k_2(t,z)\geq 0$ on $\partial\mathcal{R}$, Lemma 2.1 implies that, if $\mathcal{R}$ is pseudoconvex in $\widetilde{\mathcal{R}}$, then the L_1-constant $\alpha(t)$ for $(R(t),0,\xi(t))$ is C^ω subharmonic on B, while Lemma 2.2 makes the following contrast with it.

Theorem 2.1. *If $\mathcal{R}$ is pseudoconvex in $\widetilde{\mathcal{R}}$ and $R(t),t\in B$ is planar, then the L_0-constant $\beta(t)$ for $(R(t),0,\xi(t))$ is C^ω superharmonic on B.*

Remark 2.1. There are examples of $\pi:\mathcal{R}\to B$ such that $R(t),t\in B$ is not planar and $\beta(t)$ is not superharmonic on B.

In fact, let $\pi:\widehat{\mathcal{R}}\to B$ be a holomorphic family such that $\widehat{R}(t)=\pi^{-1}(t),t\in B$ is a compact Riemann surface of genus $g\geq 1$, and $\widehat{R}(t)$ is irreducible and nonsingular in $\widehat{\mathcal{R}}$ (where $\widehat{\mathcal{R}}$ may be the trivial $B\times\widehat{R}(0)$). Let $\Xi_0,\Xi_\xi\in\Gamma(B,\widehat{\mathcal{R}})$, and use the same notation U_0,U_ξ as in the proof of Lemma 2.2. The compact Riemann surface $\widehat{R}(t),t\in B$ admits a harmonic function $\widehat{p}(t,z)$ with poles $\log(1/|z|)$ at $z=0$ and $\log|z-\xi(t)|$ at $z=\xi(t)$ normalized $\lim_{z\to 0}(\widehat{p}(t,z)-\log(1/|z|))=0$. We put

$$\widehat{p}(t,z)=\log|z-\xi(t)|+\widehat{\alpha}(t)+\widehat{h}(t,z)\quad\text{on } U_\xi(t),$$

where $\widehat{h}(t,\xi(t))\equiv 0$ on B. Then $\frac{\partial\widehat{p}(t,z)}{\partial z}dz$ is a meromorphic differential of the third kind on $R(t)$ with poles $-1/z$ at $z=0$ and $1/(z-\xi(t))$ at $z=\xi(t)$. If necessary, take a slightly different $\Xi_\xi\in\Gamma(B,\widehat{\mathcal{R}})$. Then, since $R(t)$ is of genus $g\geq 1$, $\frac{\partial\widehat{p}(t,z)}{\partial z}dz$ is not holomorphic for $t\in B$; that is, $\frac{\partial^2\widehat{p}(t,z)}{\partial\bar t\partial z}dz\not\equiv 0$ on $R(t)$. We choose an $\eta\in\Gamma(B,\widehat{\mathcal{R}})$ with $\eta\cap(\Xi_0\cup\Xi_\xi)=\emptyset$ and a π-local coordinate $U_\eta:=B\times\{|w|<r_2\}$ of a neighborhood V_η of η in $\widehat{\mathcal{R}}$ such that $V_\eta\cap(V_0\cup V_\xi)=\emptyset$. For integer $n\geq 1$ with $1/n<r_2$, we put

$$\mathcal{R}_n=\widehat{\mathcal{R}}\setminus(B\times\{|w|\leq 1/n\}).$$

Then $\pi : \mathcal{R}_n \to B$ is a holomorphic family with conditions (1) and (2). Each $R_n(t), t \in B$ admits the L_1-function $p_n(t,z)$ and the L_1-constant $\alpha_n(t)$ for $(R_n(t), 0, \xi(t))$, and the L_0-function $q_n(t,z)$ and the L_0-constant $\beta_n(t)$. Since the Levi curvature $k_{n2}(t,z)$ for $\partial\mathcal{R}_n = B \times \{|w| = 1/n\}$ vanishes on $\partial\mathcal{R}_n$, Lemmas 2.1 and 2.2 reduce to

$$\begin{aligned}
\frac{\partial^2 \alpha_n(t)}{\partial t \partial \bar{t}} &= \frac{4}{\pi} \iint_{R_n(t)} \Big| \frac{\partial^2 p_n(t,z)}{\partial \bar{t} \partial z} \Big|^2 dx\, dy, \\
\frac{\partial^2 \beta_n(t)}{\partial t \partial \bar{t}} &= -\frac{4}{\pi} \iint_{R_n(t)} \Big| \frac{\partial^2 q_n(t,z)}{\partial \bar{t} \partial z} \Big|^2 dx\, dy \\
&\qquad - \frac{2}{\pi} \Im \sum_{k=1}^{q} \Big(\frac{\partial}{\partial t} \int_{A_k(t)} * dq_n(t,z) \Big) \cdot \Big(\frac{\partial}{\partial \bar{t}} \int_{B_k(t)} * dq_n(t,z) \Big).
\end{aligned}$$

It is known (see [1, Chapter III, Section 2]) that, for $t \in B$, both Dirichlet integrals $\|d(p_n(t,z) - \widehat{p}(t,z)\|^2_{R_n(t)}$ and $\|d(q_n(t,z) - \widehat{p}(t,z)\|^2_{R_n(t)}$ converge to 0 as $n \to \infty$, so that both $p_n(t,z)$ and $q_n(t,z)$ locally uniformly converge to $\widehat{p}(t,z)$ in $\widehat{R}(t) \setminus \{0, \xi(t)\}$, and hence both $\alpha_n(t)$ and $\beta_n(t)$ converge to $\widehat{\alpha}(t)$. We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\partial^2 \alpha_n(t)}{\partial t \partial \bar{t}} &= \lim_{n\to\infty} \frac{\partial^2 \beta_n(t)}{\partial t \partial \bar{t}} = \frac{\partial^2 \widehat{\alpha}(t)}{\partial t \partial \bar{t}}, \\
\frac{\partial^2 \widehat{\alpha}(t)}{\partial t \partial \bar{t}} &= \frac{4}{\pi} \iint_{\widehat{R}(t)} \Big| \frac{\partial^2 \widehat{p}(t,z)}{\partial \bar{t} \partial z} \Big|^2 dx\, dy \\
&= + \frac{2}{\pi} \Im \sum_{k=1}^{q} \Big(\frac{\partial}{\partial t} \int_{A_k(t)} * d\widehat{p}(t,z) \Big) \cdot \Big(\frac{\partial}{\partial \bar{t}} \int_{B_k(t)} * d\widehat{p}(t,z) \Big),
\end{aligned}$$

which implies that $\frac{\partial^2 \widehat{\alpha}(t)}{\partial t \partial \bar{t}} > 0$ on B and $\frac{\partial^2 \beta_n(t)}{\partial t \partial \bar{t}} > 0$ on B for sufficiently large n. Thus, $\pi : \mathcal{R}_n \to B$ is a desired example.

We show the following variation formulas of $\alpha(t)$ and $\beta(t)$ of the first-order $\frac{\partial \alpha(t)}{\partial t}$ and $\frac{\partial \beta(t)}{\partial t}$ under the same situations for the unramified domain $\mathcal{R}$ over $B \times \mathbb{C}_z$ as in Step 2 in the proof of Lemma 2.2 for general $\Xi_\xi : t \in B \to \xi(t) \in R(t)$ instead of $\Xi_1 := B \times \{z = 1\}$.

Lemma 2.3. *We have*

$$\frac{\partial \alpha(t)}{\partial t} = \frac{1}{\pi} \int_{\partial R(t)} k_1(t,z) \Big| \frac{\partial p(t,z)}{\partial z} \Big|^2 ds_z + 2 \frac{\partial h_\xi}{\partial z} \Big|_{(t,\xi(t))} \cdot \xi'(t),$$

$$\frac{\partial \beta(t)}{\partial t} = -\frac{1}{\pi}\int_{\partial R(t)} k_1(t,z)\Big|\frac{\partial q(t,z)}{\partial z}\Big|^2 ds_z + 2\frac{\partial \mathfrak{h}_\xi}{\partial z}\Big|_{(t,\xi(t))}\cdot \xi'(t).$$

Here

$$k_1(t,z) = \frac{\partial \varphi}{\partial t}\Big/\Big|\frac{\partial \varphi}{\partial z}\Big| \quad on\ \partial\mathcal{R},$$

and $\varphi(t,z)$ is a C^2 defining function of $\partial\mathcal{R}$.

The function $k_1(t,z)$ on $\partial\mathcal{R}$ is due to Hadamard. We note that $k_1(t,z)$ on $\partial\mathcal{R}$ as well as $k_2(t,z)$ does not depend on the choice of the defining functions $\varphi(t,z)$ for $\partial\mathcal{R}$. Contrary to the cases of $\frac{\partial^2\beta(t)}{\partial t\partial\bar t}$ and $k_2(t,z)|\frac{\partial q(t,z)}{\partial z}|^2 ds_z$, $\frac{\partial \alpha(t)}{\partial t}$ and $k_1(t,z)|\frac{\partial p(t,z)}{\partial z}|^2 ds_z$ (and similar to $\beta(t)$) depend on the π-biholomorphic mappings and π-local coordinates.

Proof. Since the proofs for $\alpha(t)$ and $\beta(t)$ are similar, we give the proof for $\beta(t)$. We divide it into two steps.

STEP 1. Lemma 2.3 is true in the case where Ξ_ξ is a constant section on B.

In fact, we simply put $\Xi_\xi := B\times\{z=1\}$. Similar to (2.7), we have

$$\frac{\partial I_j}{\partial t}(0) = \int_{C_j(0)} \frac{\partial q_j^*}{\partial t}(0,z)\frac{\partial q_j^*(0,z)}{\partial n_z}\,ds_z. \tag{2.8}$$

Under the same notation $u(t,z)$ and $C_j(t)$ as in Proposition 2.1, we similarly have

$$\frac{\partial u}{\partial t}\frac{\partial u}{\partial n_z}\,ds_z = 2k_1(t,z)\Big|\frac{\partial u}{\partial z}\Big|^2 ds_z + \frac{\partial c_j(t)}{\partial t}\frac{\partial u}{\partial n_z}\,ds_z \quad \text{along } C_j(t).$$

We apply this for $u(t,z) = q_j^*(t,z)$ with (2.6) to (2.8) and obtain

$$\frac{\partial I_j}{\partial t}(0) = 2\int_{C_j(0)} k_1(0,z)\Big|\frac{\partial q_j^*(0,z)}{\partial z}\Big|^2 ds_z.$$

Therefore,

$$\frac{\partial \beta}{\partial t}(0) = -\frac{1}{\pi}\int_{\partial R(0)} k_1(0,z)\Big|\frac{\partial q(0,z)}{\partial z}\Big|^2 ds_z \quad \text{by (2.5)},$$

which proves Step 1.

STEP 2. Lemma 2.3 is true for general Ξ_ξ on B.

In fact, it suffices to prove Lemma 2.2 at $t=0$. If necessary, take a smaller disk B of center 0. Then we find a biholomorphism $T:(t,z)\in B\times \mathbb{P}_z\mapsto (t,w)=(t,f(t,z))\in B\times\mathbb{P}_w$ such that $f(t,z)$ is a linear transformation for z, $f(t,0)=0$, $\frac{\partial f}{\partial z}(t,0)=1$, $f(t,\xi(t))=$ constant c for $t\in B$, and $\mathcal{D}:=T(\mathcal{R})$ is an unramified domain over $B\times\mathbb{C}_w$. We write $D(t)=f(t,R(t)), t\in B$, so that $\mathcal{D}=\bigcup_{t\in B}(t,D(t))$ and $\mathcal{D}$ has two constant sections $\Theta_0:=B\times\{w=0\}$ and $\Theta_c:=B\times\{w=c\}$. Thus, $\mathcal{D}:t\in B\to D(t)$ is a case in Step 1. For $t\in B$, we have the L_0-function $\widetilde{q}(t,w)$ and the L_0-constant $\widetilde{\beta}(t)$ for $(D(t),0,c)$, so that

$$\widetilde{q}(t,w)=\log\frac{1}{|w|}+\widetilde{\mathfrak{h}}_0(t,w)\quad\text{in } U_0(t),$$
$$\widetilde{q}(t,w)=\log|w-c|+\widetilde{\beta}(t)+\widetilde{\mathfrak{h}}_c(t,w)\quad\text{in } U_c(t),$$

where $\widetilde{\mathfrak{h}}_0(t,0),\widetilde{\mathfrak{h}}_c(t,c)\equiv 0$ on B. We put $\widetilde{A}_k(t)=f(t,A_k(t))$ and $\widetilde{B}_k(t)=f(t,B_k(t))$ on $D(t)$, which continuously vary in $\mathcal{D}$ with $t\in B$ without passing through $w=0,c$. Since

$$w=f(t,z)=\begin{cases} z+b_2(t)z^2+\cdots & \text{at } z=0,\\ c+a_1(t)(z-\xi(t))+a_2(t)(z-\xi(t))^2+\cdots & \text{at } z=\xi(t),\end{cases}$$

where $a_1(t)\neq 0, a_2(t),\dots,b_2(t),\dots$ are holomorphic on B, we have $q(t,z)=\widetilde{q}(t,f(t,z))$ in $\mathcal{R}$; namely,

$$q(t,z)=\log|f(t,z)-c|+\widetilde{\beta}(t)+\widetilde{\mathfrak{h}}_c\big(t,f(t,z)\big)\quad\text{at } z=\xi(t).$$

Therefore,

$$\beta(t)=\widetilde{\beta}(t)+\log|a_1(t)|,$$
$$\mathfrak{h}_\xi(t,z)=\widetilde{\mathfrak{h}}_c\big(t,f(t,z)\big)+\log\Big|1+\frac{a_2(t)}{a_1(t)}\big(z-\xi(t)\big)+\cdots\Big|.$$

Let $\psi(t,w)$ be a C^ω defining function of $\partial\mathcal{D}$. Then $\varphi(t,z):=\psi(t,f(t,z))$ is that of $\partial\mathcal{R}$, so that we have for $w=f(t,z)$

$$k_1(t,z)=\frac{\frac{\partial\varphi(t,z)}{\partial t}}{|\frac{\partial\varphi(t,z)}{\partial z}|}=\frac{\widetilde{k}_1(t,w)}{|\frac{\partial f(t,z)}{\partial z}|}+\frac{\frac{\partial f(t,z)}{\partial t}}{|\frac{\partial f(t,z)}{\partial z}|}\cdot\frac{\frac{\partial\psi}{\partial w}(t,w)}{|\frac{\partial\psi}{\partial w}(t,w)|},\quad (t,z)\in\partial\mathcal{R}.$$

Therefore,

$$\begin{aligned}\int_{\partial R(0)} & k_1(0,z)\Big|\frac{\partial q(0,z)}{\partial z}\Big|^2 ds_z \\ &= \int_{\partial R(0)} \frac{\widetilde{k}_1(0,w)}{|\frac{\partial f(0,z)}{\partial z}|}\Big|\frac{\partial q(0,z)}{\partial z}\Big|^2 ds_z \\ &\quad + \int_{\partial R(0)} \frac{\frac{\partial f}{\partial t}(0,z)}{|\frac{\partial f(0,z)}{\partial z}|}\cdot\frac{\frac{\partial \psi}{\partial w}(0,w)}{|\frac{\partial \psi}{\partial w}(0,w)|}\Big|\frac{\partial q(0,z)}{\partial z}\Big|^2 ds_z \\ &=: J_1 + J_2.\end{aligned}$$

Since $\frac{\partial \widetilde{q}(0,w)}{\partial w}\frac{f(0,z)}{dz} = \frac{\partial q(0,z)}{\partial z}$, we have, by Step 1,

$$J_1 = \int_{\partial D(0)} \widetilde{k}_1(0,w)\Big|\frac{\partial \widetilde{q}(0,w)}{\partial w}\Big|^2 ds_w = -\pi\frac{\partial \widetilde{\beta}}{\partial t}(0) = -\pi\Big(\frac{\partial \beta}{\partial t}(0) - \frac{1}{2}\frac{a_1'(0)}{a_1(0)}\Big).$$

If we put $z = g(t,w) := f^{-1}(t,w), t \in B$; $\widetilde{C}_j(0) = f(0, C_j(0))$; and $\widetilde{V}_j = f(0, V_j)$, then we have the single-valued conjugate harmonic function $\widetilde{q}_j^*(0,w)$ of $\widetilde{q}(0,w)$ in $\widetilde{V}_j$ that vanishes on $\widetilde{C}_j(0)$, and hence a function $k(w) \in C^\omega(V_j)$ such that $\widetilde{q}_j^*(0,w) = k(w)\psi(0,w)$ in $\widetilde{V}_j$, so that

$$\begin{aligned}J_2 &= -\sum_{j=1}^{\nu}\int_{\widetilde{C}_j(0)} \frac{\frac{\partial g}{\partial t}(0,w)}{\frac{\partial g(0,w)}{\partial w}}\frac{\frac{\partial \psi(0,w)}{\partial w}}{|\frac{\partial \psi(0,w)}{\partial w}|}\Big|\frac{\partial \widetilde{q}_j^*(0,w)}{\partial w}\Big|^2 ds_w \\ &= i\int_{\partial D(0)} \frac{\frac{\partial g}{\partial t}(0,w)}{\frac{\partial g(0,w)}{\partial w}}\Big(\frac{\partial \widetilde{q}^*(0,w)}{\partial w}\Big)^2 dw.\end{aligned}$$

By the residue theorem,

$$\begin{aligned}J_2 &= 2\pi \operatorname{Res}_{w=0,c}\Big\{\frac{\frac{\partial g}{\partial t}(0,w)}{\frac{\partial g(0,w)}{\partial w}}\Big(\frac{\partial \widetilde{q}(0,w)}{\partial w}\Big)^2\Big\} \\ &= 2\pi\Big(\frac{\partial \mathfrak{h}_\xi}{\partial z}(0,\xi(0))\xi'(0) - \frac{1}{4}\frac{a_1'(0)}{a_1(0)}\Big).\end{aligned}$$

Thus, $J_1 + J_2 = -\pi\big(\frac{\partial \beta}{\partial t}(0) - 2\frac{\partial \mathfrak{h}_\xi}{\partial z}(0,\xi(0))\xi'(0)\big)$, which is identical with the formula in Lemma 2.3. ∎

§3. Harmonic span and its geometric meaning

We recall the slit mapping theory in one complex variable. Let R be a planar Riemann surface sheeted over $\mathbb{C}_z$ bounded by a finite number of smooth contours $C_j, j=1,\dots,\nu$. Let $a\in R$, and let $U_a:=\{|z|<r_0\}$ be a local coordinate of a neighborhood V_a of a in R. We denote by $\mathcal{U}(R)$ the set of all univalent functions f on R such that $f(z)-1/z$ is regular at 0. For $w=f(z)\in\mathcal{U}(R)$ we consider the Euclidean area $E(f)$ of $\mathbb{C}_w\setminus f(R)$ and put

$$\mathcal{E}(R)=\sup\bigl\{E(f):f\in\mathcal{U}(R)\bigr\}.$$

Koebe (see [5, Chapter X]) constructed two special $f_i(z), i=1,0$ in $\mathcal{U}(R)$ such that $f_1(R)$ is a vertical slit domain in $\mathbb{P}_w$ and $f_0(R)$ is a horizontal slit domain. Grunsky [6, pp. 139–140] considered the function

$$g:=\frac{1}{2}(f_1+f_0)\quad\text{on } R$$

and showed that each $K_j:=-g(C_j), j=1,\dots,\nu$ bounds an unramified domain G_j over $\mathbb{C}_w$ such that, if we denote by $E_j(g)$ the Euclidean (multivalent) area of G_j and put $E(g)=\sum_{j=1}^{\nu}E_j(g)$, then $E(g)\geq\mathcal{E}(R)$. Then, Schiffer [16, p. 209] introduced the quantity $S(R)$, called the *span* for R,

$$S(R):=\Re\{a_1-b_1\},$$

where a_1 and b_1 are the coefficients of z (the first degree) of the Taylor expansions of $f_1(z)-1/z$ and $f_0(z)-1/z$ at 0, respectively, and showed the following beautiful results (see [16, p. 216]): $g\in\mathcal{U}(R)$, *each G_j is a convex domain in $\mathbb{C}_w$, and*

$$E(g)=\mathcal{E}(R)=\frac{\pi}{2}S(R).$$

His proofs were rather intuitive and short. The precise proofs are found in [1, Chapter III, Section 12].

Let $b\in R, a\neq b$, and let $U_b:=\{|z-\xi|<r_1\}$ be a local coordinate of a neighborhood V_b of b in R. We denote by $\mathcal{S}(R)$ the set of all univalent functions f on R such that $f(z)-1/z$ is regular at 0 and $f(\xi)=0$, say,

$$f(z)=c_1(z-\xi)+c_2(z-\xi)^2+\cdots\quad\text{at }\xi.$$

We put $c(f)=c_1$ $(\neq 0)$. We draw a simple curve l on R from ξ to 0. Let $w=f(z)\in\mathcal{S}(R)$. Then $f(l)$ is a simple curve from 0 to ∞ in $\mathbb{P}_w$, and each branch of $\log f(z)$ on $R\setminus l$ is single valued and univalent. Fix one of them, say, $\tau=\log f(z)$. Consider the Euclidean area $E_{\log}(f)(\geq 0)$ of the complement of $\log f(R\setminus l)$ in $\mathbb{C}_\tau$, and put

$$\mathcal{E}_{\log}(R)=\sup\bigl\{E_{\log}(f): f\in\mathcal{S}(R)\bigr\}.$$

Let $p(z)$ and α be the L_1-function and the L_1-constant for $(R,0,\xi)$, and similarly, let $q(z)$ and β be the L_0-function and the L_0-constant. We choose the harmonic conjugate $p^*(z)$ on R such that, if we put $P(z)=e^{p(z)+ip^*(z)}$ on R, then $P(z)-1/z$ is regular at 0. Then $P\in\mathcal{S}(R)$, and $w=P(z)$ is a circular slit mapping with $\log|c(P)|=\alpha$ and $E_{\log}(P)=0$. Similarly, $w=Q(z)=e^{q(z)+iq^*(z)}$ is the radial slit mapping with $\log|c(Q)|=\beta$ and $E_{\log}(Q)=0$. We see in [1, Chapter III, Section 4] that P *maximizes* $2\pi\log|c(f)|+E_{\log}(f)$, *while* Q *minimizes* $2\pi\log|c(f)|-E_{\log}(f)$ *among* $\mathcal{S}(R)$.

Nakai (see [13, Chapter II, Section 3]) expected that the quantity

$$s(R):=\alpha-\beta \tag{3.1}$$

will be important as Schiffer span $S(R)$ and named $s(R)$ the *harmonic span for* $(R,0,\xi)$. We show that $s(R)$ has some significant properties not only in one complex variable but in the several complex variables.

We write

$$\begin{aligned} P(z)&=e^{\alpha+i\theta_1}(z-\xi)+\sum_{n=2}^{\infty}a_n(z-\xi)^n \quad \text{at } \xi,\\ Q(z)&=e^{\beta+i\theta_0}(z-\xi)+\sum_{n=2}^{\infty}b_n(z-\xi)^n \quad \text{at } \xi, \end{aligned} \tag{3.2}$$

where θ_1,θ_0 are certain constants. We put

$$D_1:=P(R)=\mathbb{P}_w\setminus\bigcup_{j=1}^{\nu}P(C_j)=\mathbb{P}_w\setminus\bigcup_{j=1}^{\nu}\operatorname{arc}\{A_j^{(1)},A_j^{(2)}\},$$

$$D_0:=Q(R)=\mathbb{P}_w\setminus\bigcup_{j=1}^{\nu}Q(C_j)=\mathbb{P}_w\setminus\bigcup_{j=1}^{\nu}\operatorname{segment}\{B_j^{(1)},B_j^{(2)}\}.$$

Here

$$\begin{aligned} \operatorname{arc}\{A_j^{(1)},A_j^{(2)}\}&=\{r_je^{i\theta}:\theta_j^{(1)}\leq\theta\leq\theta_j^{(2)}\},\\ \operatorname{segment}\{B_j^{(1)},B_j^{(2)}\}&=\{re^{i\theta_j}:0<r_j^{(1)}\leq r\leq r_j^{(2)}<\infty\}, \end{aligned} \tag{3.3}$$

where $0<\theta_j^{(2)}-\theta_j^{(1)}<2\pi$ and $r_j,\theta_j^{(k)},\theta_j,r_j^{(k)}(j=1,\dots,\nu;k=1,2)$ are constants. We take the points $a_j^{(k)}, b_j^{(k)} \in C_j$ such that

$$P(a_j^{(k)}) = A_j^{(k)}, \qquad Q(b_j^{(k)}) = B_j^{(k)}. \tag{3.4}$$

Then, $\sqrt{P(z)Q(z)}$ consists of two single-valued branches $H(z)$ and $-H(z)$ on R, where $H(z)$ has only one pole at $z = 0$ such that $H(z) - 1/z$ is regular at 0, and $H(z)$ has 0 only at $z = \xi$. We write

$$H(z) = \sqrt{P(z)Q(z)} \quad \text{on } R.$$

Each branch of $\log P(z)$ and $\log Q(z)$ is also single-valued and univalent on $R \setminus l$, while $\log H(z)$ is single-valued but not univalent so far. We choose three branches in $R \setminus l$ such that

$$\tau = \log H(z) = \frac{1}{2}\big(\log P(z) + \log Q(z)\big).$$

We fix a tubular neighborhood V_j of each contour C_j with $V_i \cap V_j = \emptyset$ $(i \neq j)$ and $V_j \not\ni 0, \xi$, so that $\log H(z)$ on V_j is single valued.

Then we have the following geometric meaning of $s(R)$.

THEOREM 3.1. *We have the following.*

(1) *Each* $-(\log H)(C_j), j = 1, \ldots, \nu$ *is a convex curve in* $\mathbb{C}_\tau$, *and* $-H(C_j)$ *is a simple closed curve in* $\mathbb{C}_w$.
(2) $H \in \mathcal{S}(R)$, *and* $E_{\log}(H) = \mathcal{E}_{\log}(R) = (\pi/2)s(R)$.
(3) *Assume that* R *is simply connected, and let* $d(0, \xi)$ *denote the Poincaré distance between* 0 *and* ξ *on* R. *Then*

$$s(R) = 4 \log \cosh d(0, \xi).$$

The proofs of Schiffer's results (see [1, Chapter III, Section 12]) do not seem to be available to prove (1) and (2) in Theorem 3.1. We prove them by use of the Schottky double (compact) Riemann surface $\widehat{R}$ of R, which is also useful to prove Corollary 4.1 for the variation of Riemann surfaces.

Proof of Theorem 3.1. Similarly to $F := \frac{df_1}{df_0}$ used in [1, p. 182] (see [16, (25)]), we consider the function

$$W = F(z) := \frac{d \log Q}{d \log P}, \quad z \in R \cup \partial R, \tag{3.5}$$

which is a single-valued meromorphic function on R such that $\Re F = 0$ on ∂R, since $\log P(C_j)$ is a vertical segment and $\log Q(C_j)$ is a horizontal segment in $\mathbb{C}_\tau$. It follows from the Schwarz reflexion principle that F is meromorphically extended to the Schottky double Riemann surface $\widehat{R} = R \cup \partial R \cup R^*$ of R such that $F(z^*) = -\overline{F(z)}$, where $z^* \in R^*$ is the reflexion point of $z \in R$. Fix $C_j, j = 1, \dots, \nu$. Since $\Re \log P(z) = p(z)$ and $\Re \log Q(z) = q(z)$ on R, we have

$$\text{(3.6)} \quad \log P(z) = u_1(z) + iv_1(z), \qquad \log Q(z) = u_0(z) + iv_0(z), \quad z \in V_j,$$

where $u_1(z) =$ constant c_1 and $v_0(z) =$ constant c_0 on C_j. Then $\mathfrak{C}_j := \log H(C_j)$ is a closed (not necessarily simple so far) curve in $\mathbb{C}_\tau$

$$\text{(3.7)} \qquad \tau = \frac{1}{2}\big(c_1 + u_0(z)\big) + \frac{i}{2}\big(c_0 + v_1(z)\big), \quad z \in C_j.$$

Using notation (3.4), we show that

(i) $\{a_j^{(k)}, b_j^{(k)}\}_{k=1,2}$ are four distinct points, which necessarily line cyclically, for example, $(a_j^{(1)}, b_j^{(1)}, a_j^{(2)}, b_j^{(2)})$ on C_j;
(ii) the zeros of $F(z)$ are $\{b_j^{(k)}\}_{j=1,\dots,\nu;k=1,2}$ of order 1, and the poles are $\{a_j^{(k)}\}_{j=1,\dots,\nu;k=1,2}$ of order 1;
(iii) the curve $\mathfrak{C}_j$ is locally nonsingular in $\mathbb{C}_\tau$;
(iv) $\Re F(z) > 0$ on R;
(v) at any $\tau \in \mathfrak{C}_j$, the curvature $1/(\rho_j(\tau))$ of $\mathfrak{C}_j$ is negative.

We divide the proof into two steps.

Step 1. If we admit (i), then (ii)–(v) hold.

In fact, (i) clearly implies (iii). Since $P(z)$ is a circular slit mapping on R, and $Q(z)$ is a radial slit mapping on R, we have $F(z) \neq 0, \infty$ on $R \cup R^*$ and $F(z)$ has zeros at most $b_j^{(k)}$ and poles at most $a_j^{(k)}$, of order 1. It follows that (i) implies (ii). Further, (i) implies that $W = F(z)$ is locally one-to-one in a neighborhood of at any $z \in C_j$ even at $a_j^{(k)}, b_j^{(k)}$ $(k = 1, 2)$, so that F is a meromorphic function on $\widehat{R}$ of degree 2ν. Hence, for a fixed $j = 1, \dots, \nu$, if z travels C_j all once, then $F(z)$ travels the imaginary axis all just *twice*. It follows that $F(\widehat{R})$ is a 2ν sheeted compact Riemann surface over $\mathbb{P}_W$ with $2(2\nu + g - 1)$ branch points lying on $\mathbb{P}_W \setminus \{\Re W = 0\}$, and hence $F(\widehat{R})$ is divided by ν closed curves $F(C_j)$ into two connected parts over $\Re W > 0$ and $\Re W < 0$. Since $F(0) = 1$, we have $\Re F(z) > 0$ on R and $\Re F(z) < 0$ on R^*, which is (iv). To prove (v), fix $p_0 \in C_j$, and take a local parameter

$z = x + iy$ of a neighborhood V of p_0 such that p_0 corresponds to $z = 0$ and the oriented arc $C_j \cap V$ corresponds to $I := (-\rho, \rho)$ on the x-axis. Using this parameter, we see from $\Re F(z) > 0$ on R that

$$\Im F'(x) = \Im \frac{\partial F(x)}{\partial x} < 0 \quad \text{on } I. \tag{3.8}$$

By (3.7), the subarc $\Gamma_j := \log H(I)$ of $\mathfrak{C}_j$ in $\mathbb{C}_\tau$ is of the form

$$\tau = u(x) + iv(x) = \frac{1}{2}\big[\big(c_1 + u_0(x)\big) + i\big(c_0 + v_1(x)\big)\big], \quad x \in I.$$

Since the arc Γ_j is locally nonsingular by (iii), we calculate the curvature $1/\rho_j(x)$ at the point $(u(x), v(x))$ of Γ_j:

$$\frac{1}{\rho_j(x)} = \frac{v''(x)u'(x) - v'(x)u''(x)}{(v'(x)^2 + u'(x)^2)^{3/2}} = \frac{v_1''(x)u_0'(x) - v_1'(x)u_0''(x)}{(v_1'(x)^2 + u_0'(x)^2)^{3/2}}.$$

On the other hand, by (3.6) we have, for $x \in I \subset C_j$,

$$\Im F'(x) = \Im \left\{ \frac{d}{dx} \left(\frac{\frac{du_0(x)}{dx} + i\frac{dc_0}{dx}}{\frac{dc_1}{dx} + i\frac{dv_1(x)}{dx}} \right) \right\} = \frac{v_1''(x)u_0'(x) - v_1'(x)u_0''(x)}{v_1'(x)^2}.$$

Therefore,

$$\frac{1}{\rho_j(x)} = \frac{v_1'(x)^2}{(v_1'(x)^2 + u_0'(x)^2)^{3/2}} \cdot \Im F'(x).$$

Since $v_1'(0) = 0$ if and only if $x = a_j^{(k)}$, (3.8) proves (v) for $p_0 \neq a_j^{(k)}$. For $p_0 = a_j^{(k)}$, since $v_1'(0) = 0$ and $v_1''(0), u_0'(0) \neq 0$ under (i), $v_1'(x)^2 \cdot \Im F'(x)$ is regular and $\neq 0$. Hence, $1/\rho_j(p_0) < 0$, which proves (v).

Step 2. Item (i) is true.

In fact, assume that R does not satisfy (i). It does not occur $\{a_j^{(1)}, a_j^{(2)}\} = \{b_j^{(1)}, b_j^{(2)}\}$ for any j, so that $\{a_j^{(1)}, a_j^{(2)}\} \cap \{b_j^{(1)}, b_j^{(2)}\}$ consists of one point for some j, say, $j = 1, \ldots, \nu' (\leq \nu)$. We denote by o_j such a point on C_j. Hence, each $\mathfrak{C}_j := \log H(C_j), j = 1, \ldots, \nu'$ is a closed curve in $\mathbb{C}_\tau$ with only one singular point at $\mathfrak{o}_j := \log H(o_j)$, and F is a meromorphic function of degree $2\nu - \nu'$ on $\widehat{R}$. By the same reasoning as in Step 1, if z travels $C_j, j = 1, \ldots, \nu'$ all once, then $F(z)$ travels the imaginary axis all just *once* in $\mathbb{C}_\tau$, and $\Re F(z) > 0$ on R and $\Re F(z) < 0$ on R^*. This fact implies that

$1/\rho_j(\tau) < 0$ for $\tau \in \mathfrak{C}_j \setminus \{\mathfrak{o}_j\}$. To reach a contradiction, we focus to C_1. We may assume that $\mathfrak{o}_1 = 0$ of $\mathfrak{C}_1(\subset \mathbb{C}_\tau)$ and that $a_1^{(1)} = b_1^{(1)} = o_1$ on $C_1(\subset \mathbb{C}_z)$. If we take a small subarc C_1' centered at o_1 of C_1 and identify C_1' with $I = (-r, r)$ on the x-axis such that o_1 corresponds to $0 \in I$, then the subarc $\Gamma := \log H(C_1')$ of $\mathfrak{C}_1$ is written

$$\tau = \frac{1}{2}[(a_2x^2 + a_3x^3 + \cdots) + i(b_2x^2 + b_3x^3 + \cdots)], \quad x \in I,$$

where all a_k, b_k are real and $a_2, b_2 \neq 0$. The other cases being similar, we assume that $a_2, b_2 > 0$. We put $\Gamma' = \{\log H(x) \in \Gamma : x \text{ travels from } 0 \text{ to } r\}$, and similarly, we put Γ'' from 0 to $-r$, so that $\Gamma = -\Gamma'' + \Gamma'$. Since $1/\rho_1(\tau) < 0$ for $\tau \in \mathfrak{C}_1 \setminus \{\mathfrak{o}_1\}$, $\mathfrak{C}_1$ has a cusp singularity at $\mathfrak{o}_1$ such that Γ' starts at $\mathfrak{o}_1$ whose tangent decreases from $b_2/a_2 > 0$ as x travels from 0 to r, and similarly for Γ''. We put $\mathfrak{a} = \log H(a_1^{(2)})$, and we put $\mathfrak{b} = \log H(b_1^{(2)})$. Since the tangent $T(\tau)$ of $\mathfrak{C}_1$ at $\tau = \log H(z)$ is $T(\tau) = v_1'(z)/u_0'(z)$, we have $T(\mathfrak{a}) = 0, |T(\mathfrak{b})| = \infty$ and vice versa. This contradicts that $\mathfrak{C}_1$ is a closed curve with $1/\rho_1(\tau) < 0$ for any $\tau \in \mathfrak{C}_1 \setminus \{\mathfrak{o}_1\}$, which proves (i).

The first assertion in Theorem 3.1(1) follows (v). Using notation (3.3), we have

$$\mathrm{Max}_{z\in C_j}\{\Im \log H(z)\} - \mathrm{Min}_{z\in C_j}\{\Im \log H(z)\} \leq \frac{1}{2}(\theta_j^{(2)} - \theta_j^{(1)}) < \pi.$$

It follows that the first assertion implies the second assertion in Theorem 3.1(1). To prove (2), given $w' \in \mathbb{C}_w \setminus \bigcup_{j=1}^{\nu} H(C_j)$, we write $N(w')$ for the number of z in R such that $H(z) = w'$. If we denote by $W_j(w')$ the winding number of $H(C_j)$ about w', then we have $W_j(w') \leq 0$ by the second assertion in (1). Since $H(z)$ has only one pole at $z = 0$ of order 1 on R, we have by the argument principle

$$N(w') - 1 = \sum_{j=1}^{\nu} W_j(w') \leq 0,$$

so that $N(w') = 0$ or 1. Hence, $H(z)$ is univalent on R, which is the first assertion in (2). To prove the other assertions in (2), let $f \in \mathcal{S}(R)$. We put $u(z) := \log|f(z)|$, and we put $h(z) := \log|H(z)| = (1/2)(p(z) + q(z))$. Then $u(z) - h(z)$ is harmonic on the whole R, and its Dirichlet integral $D_R(u-h) := \|d(u-h)\|_R^2 \geq 0$ is written

$$D_R(u-h) = \int_{\partial R} u\, du^* - \int_{\partial R} u\, dh^* - \int_{\partial R} h\, du^* + \int_{\partial R} h\, dh^*.$$

By $\int_{C_j} du^* = 0$ and the boundary conditions for $p(z)$ and $q(z)$, we have

$$\int_{\partial R} u\,dh^* = \frac{1}{2}\int_{\partial R} u\,dp^* - p\,du^* = \pi\big(\log|c(f)| - \alpha\big),$$
$$\int_{\partial R} h\,du^* = \frac{1}{2}\int_{\partial R} q\,du^* - u\,dq^* = \pi\big(\beta - \log|c(f)|\big).$$

Therefore,

$$D_R(u-h) = \int_{\partial R} u\,du^* + \pi(\alpha-\beta) + \int_{\partial R} h\,dh^*.$$

We put $u = h$, in particular, to obtain $E_{\log}(H) = -\int_{\partial R} h\,dh^* = (\pi/2)(\alpha - \beta) = (\pi/2)s(R), E_{\log}(H) - E_{\log}(f) = D_R(u-h) \geq 0$, which are desired.

To prove Theorem 3.1(3), we first prove it in the case where R is the disk $D = \{|z| < r\}$ in $\mathbb{C}_z$. Let $\xi \in D$. We denote by $p(z)$ and α the L_1-function and the L_1-constant for $(D, 0, \xi)$, and similarly for $q(z)$ and β. We write $P(z)$ and $Q(z)$ the corresponding circular and radial slit mappings on D, so that $p(z) = \log|P(z)|$ and $q(z) = \log|Q(z)|$. We have (see [9, Section 5])

$$P(z) = \frac{-1}{\xi}\cdot\frac{z-\xi}{z}\cdot\Big(1 - \frac{z}{r}\frac{\overline{\xi}}{r}\Big)^{-1}, \quad z \in D,$$
$$\alpha = \log\Big|\frac{dP}{dz}(\xi)\Big| = -2\log|\xi| - \log\bigg(1 - \Big(\frac{|\xi|}{r}\Big)^2\bigg).$$

Putting $\theta_\xi = \arg\xi$, we have

$$Q(z) = \frac{1}{re^{\theta_\xi}}\Big[\Big(\frac{z}{re^{i\theta_\xi}} + \frac{re^{i\theta_\xi}}{z}\Big) - \Big(\frac{|\xi|}{r} + \frac{r}{|\xi|}\Big)\Big] = \frac{-1}{\xi}\cdot\frac{z-\xi}{z}\cdot\Big(1 - \frac{z}{r}\frac{\overline{\xi}}{r}\Big),$$
$$\beta = \log\Big|\frac{dQ}{dz}(\xi)\Big| = -2\log|\xi| + \log\bigg(1 - \Big(\frac{|\xi|}{r}\Big)^2\bigg).$$

Hence, the harmonic span $s(D) = \alpha - \beta$ for $(D, 0, \xi)$ is

$$s(D) = -2\log\big(1 - (|\xi|/r)^2\big). \tag{3.9}$$

Since the Poincaré distance $d(0,\xi)$ between 0 and ξ in D is equal to $(1/2)\times \log(1+|\xi|/r)/(1-|\xi|/r)$, we have $s(D) = 4\log\cosh d(0,\xi)$.

For the general R, although α and β depend on the choice of local coordinates $U_a := \{|z| < r_0\}$ and $U_b := \{|w-\xi| < r_1\}$ about a and b, *the harmonic span $s(R) = \alpha - \beta$ as well as Poincaré distance does not depend on it.* Hence, the first case $R = D$ and the Riemann's mapping theorem imply (3). ∎

EXAMPLE 3.1. We check Theorem 3.1(1), (2) in the case where $D = \{|z| < r\}$ and $\xi \in D$. By the above formulas,

$$H(z) = \sqrt{P(z)Q(z)} = 1/z - 1/\xi, \quad z \in D.$$

Thus, $H(z)$ is univalent on D. Since $C := \partial D = \{re^{i\theta} : 0 \le \theta \le 2\pi\}$, the closed curve $-H(C) = \{(e^{i\theta}/r) - 1/\xi : 0 \le \theta \le 2\pi\}$ is simple and $-\log H(C)$ is a convex curve. Further, we have $E_{\log}(H) = -\pi \log(1 - |\xi/r|^2)$. In fact, we prove it for $r = 1$ and $|\xi| < 1$. Since each branch of $\log(1/z - 1/\xi)$ is single valued and holomorphic in $\mathbb{C}_z \setminus D$, we have by $z\overline{z} = 1$ on C,

$$\begin{aligned} E_{\log}(H) &= \frac{i}{2} \int_{-C} \log(1/z - 1/\xi)\, d\overline{\log(1/z - 1/\xi)} \\ &= \frac{-i}{2} \int_{C} \log(1/z - 1/\xi) \frac{dz}{z - 1/\overline{\xi}} = -\pi \log(1 - |\xi|^2), \end{aligned}$$

which is desired. By (3.9), we have $E_{\log}(H) = (\pi/2)s(D)$.

REMARK 3.1. (1) Let $R_i, i = 1, 2$ be a planar Riemann surface such that $R_i \ni 0, \xi$. If we denote by s_i the harmonic span for $(R_i, 0, \xi)$, then we have by Theorem 3.1(2) that $R_1 \subset R_2$ induces $s_1 \ge s_2$, even when R_1 and R_2 are not homeomorphic to each other.

(2) Let R be a planar Riemann surface. As noted in the proof of Theorem 3.1(3), the harmonic span $s_R(\xi, \eta)$ is a positive function for $(\xi, \eta) \in (R \times R) \setminus \bigcup_{\xi \in R}(\xi, \xi)$. Further, $s_R(\xi, \eta) = s_R(\eta, \xi)$, and for a fixed $\xi_0 \in R$, $\lim_{\eta \to \partial R} s_R(\xi_0, \eta) = +\infty$. If we put $s_R(\xi, \xi) = 0$ for $\xi \in R$, then $s_R(\xi, \eta)$ is a C^2 function on $R \times R$, which satisfies, for a fixed $\xi_0 \in R$, that there exist $K > 0$ and $\delta > 0$ such that

$$|\eta - \xi_0|^2/K \le s(\xi_0, \eta) \le K|\eta - \xi_0|^2 \quad \text{for } |\eta - \xi_0| < \delta. \tag{3.10}$$

In fact, we may assume that R is a bounded domain in $\mathbb{C}_z$ and that $\xi_0 = 0 \in R$. We take $D_a := \{|z| < a\} \Subset R \Subset \{|z| < b\} := D_b$ in $\mathbb{C}_z$. By Remark 3.1(1) and (3.9), we have, for $\eta \in D_a$,

$$-\log(1 - |\eta/b|^2) = \frac{s_{D_b}(0, \eta)}{2} \le \frac{s_R(0, \eta)}{2} \le \frac{s_{D_a}(0, \eta)}{2} = -\log(1 - |\eta/a|^2),$$

which implies (3.10).

We call the function $s_R(\xi, \eta)$ on $R \times R$ the *S-function for R*.

§4. Variation formulas for the harmonic spans

We return to the variation of Riemann surfaces $\mathcal{R} : t \in B \to R(t) (\Subset \widetilde{R}(t))$ in $\widetilde{\mathcal{R}} = \bigcup_{t\in B}(t, \widetilde{R}(t))$ with conditions (1) and (2) in Section 2. Then Lemmas 2.1 and 2.2 immediately imply the following variation formulas of the harmonic span $s(t)$.

Lemma 4.1. *We have*

$$\frac{\partial s(t)}{\partial t} = \frac{1}{\pi}\int_{\partial R(t)} k_1(t,z)\Big(\Big|\frac{\partial p(t,z)}{\partial z}\Big|^2 + \Big|\frac{\partial q(t,z)}{\partial z}\Big|^2\Big)\, ds_z,$$

$$\begin{aligned}\frac{\partial^2 s(t)}{\partial t\partial \bar t} &= \frac{1}{\pi}\int_{\partial R(t)} k_2(t,z)\Big(\Big|\frac{\partial p(t,z)}{\partial z}\Big|^2 + \Big|\frac{\partial q(t,z)}{\partial z}\Big|^2\Big)\, ds_z \\ &\quad + \frac{4}{\pi}\iint_{R(t)}\Big(\Big|\frac{\partial^2 p(t,z)}{\partial \bar t\partial z}\Big|^2 + \Big|\frac{\partial^2 q(t,z)}{\partial \bar t\partial z}\Big|^2\Big)\, dx\, dy \\ &\quad + \frac{2}{\pi}\Im\sum_{k=1}^{g}\Big(\frac{\partial}{\partial t}\int_{A_k(t)} *dq(t,z)\Big)\cdot\Big(\frac{\partial}{\partial \bar t}\int_{B_k(t)} *dq(t,z)\Big).\end{aligned}$$

We say, in general, that $\mathcal{R} : t \in B \to R(t)$ is *equivalent to a trivial variation* if there exists a π-biholomorphism from the total space $\mathcal{R}$ onto a product space $B \times R_0$ (where R_0 is a Riemann surface).

In the case where $R(t)$ is planar, following (3.2), on $R(t), t \in B$ we have the circular and radial slit mappings

$$P(t,z) = e^{p(t,z)+ip(t,z)^*} \qquad \text{and} \qquad Q(t,z) = e^{q(t,z)+iq(t,z)^*}$$

such that $P(t,z) - 1/z$ and $Q(t,z) - 1/z$ are regular at $z = 0$. We put $D_1(t) = P(t, R(t))$, and we put $D_0(t) = Q(t, R(t))$, so that

$$D_1(t) = \mathbb{P}_w \setminus \bigcup_{j=1}^{\nu} P\big(t, C_j(t)\big) = \mathbb{P}_w \setminus \bigcup_{j=1}^{\nu} \mathrm{arc}\big\{A_j^{(1)}(t), A_j^{(2)}(t)\big\},$$

$$D_0(t) = \mathbb{P}_w \setminus \bigcup_{j=1}^{\nu} Q\big(t, C_j(t)\big) = \mathbb{P}_w \setminus \bigcup_{j=1}^{\nu} \mathrm{segment}\big\{B_j^{(1)}(t), B_j^{(2)}(t)\big\}.$$

Theorem 4.1. *Assume that* $\mathcal{R} = \bigcup_{t\in B}(t, R(t))$ *is pseudoconvex in* $\widetilde{\mathcal{R}}$ *and that each* $R(t), t \in B$ *is planar. Then*

(1) *$s(t)$ is subharmonic on B;*

(2) *if* $s(t)$ *is harmonic on* B*, then*
   (i) $s(t)$ *is constant on* B*, and*
   (ii) $\mathcal{R} : t \in B \to R(t)$ *is equivalent to a trivial variation. More concretely,* $\mathcal{R}$ *is* π*-biholomorphic to the product domain* $B \times \widetilde{D}_1$*, where* $\widetilde{D}_1$ *is a circular slit domain in* $\mathbb{P}_w$ *such that* $\widetilde{D}_1 = \mathbb{P}_w \setminus \bigcup_{j=1}^{\nu} \{\widetilde{A}_j e^{i\theta} : 0 \le \theta \le \Theta_j\}$*, where* $\widetilde{A}_1 = 1$ *and each* $\widetilde{A}_j$ $(\neq 0), j = 2, \ldots, \nu$ *is constant, by the holomorphic transformation* $T_0 : (t,z) \in \mathcal{R} \mapsto (t,w) = (t, \widetilde{P}(t,z)) \in B \times \widetilde{D}_1$*, where* $\widetilde{P}(t,z) = P(t,z)/A_1^{(1)}(t)$*.*

Proof. Lemma 4.1 implies (1). To prove (2), we may assume that $\mathcal{R} = \bigcup_{t \in B}(t, R(t))$ is an unramified domain over $B \times \mathbb{C}_z$ such that each $R(t), t \in B$ is contained in an unramified planar domain $\widetilde{R}$ over $\mathbb{C}_z$, and Ξ_0, Ξ_ξ are constant sections $B \times \{z = 0\}$, $B \times \{z = 1\}$, respectively. Assume that $s(t)$ is harmonic on B. By Lemma 4.1, we have

(a) $k_2(t,z) \equiv 0$ on $\partial\mathcal{R}$, that is, $\partial\mathcal{R}$ is a Levi flat surface over $B \times \mathbb{C}_z$;
(b) both $\frac{\partial p(t,z)}{\partial z}$ and $\frac{\partial q(t,z)}{\partial z}$ are holomorphic for $t \in B$.

By (b) and the normalization at $z = 0$, both $w = P(t,z)$ and $w = Q(t,z)$ are holomorphic for two complex variables (t,z) in $\mathcal{R}$ except $B \times \{0\}$. We put $D_1(t) = P(t, R(t)) \subset \mathbb{P}_w$ for $t \in B$, and $\mathcal{D}_1 = \bigcup_{t \in B}(t, D_1(t))$. Since $\mathcal{D}_1$ as well as $\mathcal{R}$ over $B \times \mathbb{C}_z$ is a pseudoconvex (univalent) domain in $B \times \mathbb{P}_w$, it follows from [3, p. 352] that each edge point $A_j^{(k)}(t)$ is holomorphic for $t \in B$ and that $A_j^{(2)}(t) = A_j^{(1)}(t) e^{i\Theta_j}$, where Θ_j is constant for $t \in B$. We consider the map $(t,w) \in \mathcal{D}_1 \mapsto (t, \widetilde{w}) = (t, L(t,w)) \in B \times \mathbb{P}_{\widetilde{w}}$, where $L(t,w) = w/A_1^{(1)}(t)$, and we put $\widetilde{\mathcal{D}}_1 = \bigcup_{t \in B}(t, \widetilde{D}_1(t))$, where $\widetilde{D}_1(t) = L(t, D_1(t))$. Each $\widetilde{D}_1(t), t \in B$ is a circular slit domain in $\mathbb{P}_{\widetilde{w}} \setminus \bigcup_{j=1}^{\nu} \widetilde{C}_j(t)$ such that the first circular slit $\widetilde{C}_1(t) = \{e^{i\theta} : 0 \le \theta \le \Theta_1\}$ is independent of $t \in B$, say, $\widetilde{C}_1 := \widetilde{C}_1(t)$. Since $\mathcal{R}$ is π-biholomorphic to $\widetilde{\mathcal{D}}_1$, and each $\widetilde{D}_1(t), t \in B$ has no ramification points, it suffices for (2)(ii) to prove that the edge point $\widetilde{A}_j^{(1)}(t) := A_j^{(1)}(t)/A_1^{(1)}(t)$ of each arc $\widetilde{C}_j^{(1)}(t), j = 2, \ldots, \nu$ does not depend on $t \in B$.

In fact, we see from (b) that the function $F(t,z)$ defined in (3.5),

$$W = F(t,z) = \frac{d_z \log Q(t,z)}{d_z \log P(t,z)}, \quad z \in R(t) \cup \partial R(t),$$

is holomorphic for $t \in B$ such that $F(t,0) = 1$ and $\Re F(t,z) = 0$ on $\partial R(t)$; that is, $F(t,z)$ is a meromorphic function for two complex variables $(t,z) \in \mathcal{R}$ such that $\Re F(t,z) = 0$ on $\partial\mathcal{R}$. We put $K_j(t) = F(t, C_j(t))$ in $\mathbb{P}_W$. In

Step 1 of the proof in Theorem 3.1(1) we proved that $K_j(t)$ rounds just twice on the imaginary axis in $\mathbb{P}_W$. We put $W(t)=F(t,R(t))$, and we put $\mathcal{W}=\bigcup_{t\in B}(t,W(t))$, so that $\partial\mathcal{W}=\bigcup_{t\in B}(t,\bigcup_{j=1}^{\nu}K_j(t))$, and $\mathcal{R}\approx\mathcal{W}$ (π-biholomorphic) by $T:(t,z)\in\mathcal{R}\mapsto(t,W)=(t,F(t,z))\in\mathcal{W}$. Thus, $W(t)$ has $6\nu-4$ ramification points. Consider the following π-biholomorphic mapping $(t,W)\in\mathcal{W}\to(t,\widetilde{w})=(t,\widetilde{G}(t,W))\in\widetilde{\mathcal{D}}_1$, where $\widetilde{G}(t,W):=L\big(t,P(t,F^{-1}(t,W))\big)$. We use the following elementary fact.

($*$) *Let $B=\{|t|<\rho\}$ in $\mathbb{C}_t$, and let $E=\{|z|<r\}\cap\{\Re z\geq 0\}$ in $\mathbb{C}_z$. If $f(t,z)$ is a holomorphic function for two complex variables (t,z) on $B\times E$ such that $|f(t,z)|=1$ on $B\times(E\cap\{\Re z=0\})$, then $f(t,z)$ does not depend on $t\in B$.*

We choose a point W_0 on $\partial K_1(0)\subset\partial\mathcal{W}$ such that $\widetilde{G}(0,W_0)=e^{i\theta_0}\in\widetilde{C}_1$ with $0<\theta_0<\Theta_1$ and the direction of $\widetilde{C}_1$ at $e^{i\theta_0}$ follows as θ_0 increases. Then we have a small disk $B_0\subset B$ of center 0 and a small half-disk $E=\{|W-W_0|<r\}\cap\{\Re W\geq 0\}$ in $\mathbb{C}_W$ such that $|\widetilde{G}(t,W)|\leq 1$ on $B_0\times E$ and $|\widetilde{G}(t,W)|=1$ on $B_0\times(E\cap\{\Re W=0\})$. By ($*$), $\widetilde{G}(t,W)$ for $W\in E\cap\{\Re W\geq 0\}$ does not depend on $t\in B_0$. By the analytic continuation, $\widetilde{G}(t,W)$ on $\mathcal{W}\cup\partial\mathcal{W}$ does not depend on $t\in B$.

Now assume that some $\widetilde{A}_j^{(1)}(t)$, $2\leq\exists j\leq\nu$ is not constant for $t\in B$. We take a point $W_0\in\mathbb{C}_W$ with $\Re W_0=0$. Since the component $K_j(t)$ of $\partial W(t)$ winds twice around the imaginary axis in $\mathbb{P}_W$, for each $t\in B$ we find four points of $K_j(t)$ over W_0. We fix one of them, say, $W_0(t)\in K_j(t)$, where the corresponding point $z_j(t)\in C_j(t)$ continuously varies in $\partial\mathcal{R}$ with $t\in B$. Since $\widetilde{C}_j(t)=\widetilde{G}(t,K_j(t))=\{\widetilde{A}_j^{(1)}(t)e^{i\theta}:0\leq\theta\leq\Theta_j\}$, where Θ_j is constant for $t\in B$, we have $\widetilde{G}(t,W_0)=\widetilde{A}_j^{(1)}(t)e^{i\theta(t)}$, where $\theta(t)$ $(0<\theta(t)<\Theta_j)$ continuously varies with $t\in B$. Since $|\widetilde{A}_j^{(1)}(t)|$ as well as $\widetilde{A}_j^{(1)}(t)$ is not constant for $t\in B$, $\widetilde{G}(t,W_0)$ *does* depend on $t\in B$, a contradiction, and (2)(ii) is proved.

From Remark 3.1(2), the harmonic span $s(t)$ for $(R(t),0,1)$ is equal to that for $(\widetilde{D}_1(t),\infty,0)$. Since $\widetilde{D}_1(t)=\widetilde{D}_1(0)$ for $t\in B$, $s(t)$ is constant on B, which proves (2)(i). □

For Theorem 4.1(2)(ii), we cannot replace the condition of the harmonicity of $s(t)$ on B by that of $\alpha(t)$ or $\beta(t)$ on B, in general. However, when $R(t),t\in B$ is simply connected, such replacement is possible by the proof of (2)(ii).

Theorem 4.1 and Theorem 3.1(3) directly imply the following.

Corollary 4.1. *Assume that* $\mathcal{R} = \bigcup_{t\in B}(t, R(t))$ *is pseudoconvex in* $\widetilde{\mathcal{R}}$ *and that* $R(t), t \in B$ *is simply connected. Let* $\xi_i \in \Gamma(B, \mathcal{R}), i = 1, 2,$ *and let* $d(t)$ *denote the Poincaré distance between* $\xi_1(t)$ *and* $\xi_2(t)$ *on* $R(t)$*. Then* $\delta(t) := \log\cosh d(t)$ *is subharmonic on* B*. Moreover,* $\delta(t)$ *is harmonic on* B *if and only if* $\mathcal{R}$ *is equivalent to the trivial variation.*

Brunella [4, p. 139] said that he could prove the stronger fact, that "$\operatorname{Log} d(t)$ *is subharmonic on* B," using [2] by the same idea.

Corollary 4.2. *Assume that* $\mathcal{R} = \bigcup_{t\in B}(t, R(t))$ *is pseudoconvex in* $\widetilde{\mathcal{R}}$ *and that each* $R(t), t \in B$ *is planar. Then the* S*-function* $s(t, \xi, \eta)$ *for* $R(t)$*,* $t \in B$ *is* C^2 *plurisubharmonic on* $\mathcal{R}^2 := \bigcup_{t\in B}(t, R(t) \times R(t))$*. In particular, for a fixed* $t_0 \in B$*, we simply put* $R(t_0) = R$ *and* $s(t_0, \xi, \eta) = s(\xi, \eta)$*. Then* $s(\xi, \eta)$ *is* C^2 *plurisubharmonic on* $R \times R$ *such that, for any complex line* l *except* $\xi = \eta$ *in* $R \times R$*, the restriction of* $s(\xi, \eta)$ *on* $l \cap (R \times R)$ *is strictly subharmonic.*

Proof. We may assume that $\widetilde{\mathcal{R}}$ as well as $\mathcal{R}$ is an unramified domain over $B \times \mathbb{C}_z$. Let $t \in B \to (\xi(t), \eta(t)) \in R(t) \times R(t)$ be any holomorphic mapping from B into $\mathcal{R}^2$. We put $s(t) := s(t, \xi(t), \eta(t))$ for $t \in B$, and we put $B' = B \setminus \{t \in B : \xi(t) = \eta(t)\}$. Consider the translation $T : (t, z) \in \mathcal{R} \mapsto (t, w) = (t, z - \eta(t))$ for $t \in B'$, and put $\mathcal{R}_1 := T(\mathcal{R})$ and $\xi_1 = T\xi$. Then $\mathcal{R}_1$ is pseudoconvex over $B' \times \mathbb{C}_w$ and $\xi_1 \in \Gamma(B', \widetilde{\mathcal{R}})$. By Theorem 4.1, the harmonic span $s_1(t)$ for $(R_1(t), 0, \xi_1(t))$ is C^ω subharmonic on B', and so is $s(t)$ on B'. It follows from (3.10) that $s(t)$ is C^2 subharmonic on B. By the same argument, we can prove the latter part under the second variation formula in Lemma 4.1 and (3.10). Thus we have the corollary. ∎

In conditions (1) and (2), if we replace C^ω smooth by C^∞ smooth, then the results in Sections 2 and 3 hold by replacing C^ω by C^∞. In fact, Lemmas 2.1 and 2.2 hold for the C^∞ category by not essentially changing the proofs for the C^ω category (see [11, Section 2] and [17, Section 3]).

§5. Approximation theorem for general variations of planar Riemann surfaces

In this section we consider the general variation of Riemann surfaces $\mathcal{R} : t \in \Delta \to R(t)$ with the conditions that (a) Δ is an open or a compact Riemann surface; (b) $\pi : \mathcal{R} \to \Delta$ is a 2-dimensional holomorphic family such that each fiber $R(t) = \pi^{-1}(t), t \in \Delta$ is irreducible and nonsingular in $\mathcal{R}$;

(c) each $R(t), t \in B$ is *planar*; and (d) for every $t \in \Delta$ there exists a neighborhood $B \subset \Delta$ of t such that $\pi^{-1}(B)$ is Stein.

In general, $R(t)$ might be infinite ideal boundary components and $\mathcal{R}: t \in \Delta \to R(t)$ might not be topologically trivial. For the approximation condition for these variations $\mathcal{R}$, we make the following.

Preparation

Let $\pi: \mathcal{R} \to \Delta$ be as above, and let $B \subset \Delta$ be a disk such that $\pi^{-1}(B)$ is Stein. For the sake of convenience we write anew $\Delta := B$ and $\mathcal{R} := \pi^{-1}(B)$. Due to Oka-Grauert (see [15, Theorem 8.22]), $\mathcal{R}$ admits a C^ω strictly plurisubharmonic exhaustion function $\psi(t,z)$. Let $\xi: t \in B \to \xi(t) \in R(t)$ and $\eta: t \in B \to \eta(t) \in R(t)$ be holomorphic sections of $\mathcal{R}$ over B such that $\xi \cap \eta = \emptyset$. Let $B \Subset \Delta$ be a small disk such that we find a continuous curve $g(t)$ connecting $\xi(t)$ and $\eta(t)$ on $R(t), t \in B$ which continuously varies in $\mathcal{R}$ with $t \in B$. We put $\mathcal{R}|_B = \bigcup_{t \in B}(t, R(t))$; $\xi|_B = \bigcup_{t \in B}(t, \xi(t))$; $\eta|_B = \bigcup_{t \in B}(t, \eta(t))$, and $g|_B = \bigcup_{t \in B}(t, g(t))$. We take so large $a \gg 1$ that $\mathcal{R}(a)|_B := \{(t,z) \in \mathcal{R}|_B : \psi(t,z) < a\} \supset g|_B$. Then we find a sequence $\{a_n\}_n$ with $a_n > a$ and $\lim_{n \to \infty} a_n = \infty$ such that

$$\mathcal{R}_n := \text{the connected component of } \mathcal{R}(a_n)|_B \text{ that contains } g|_B \tag{5.1}$$

satisfies (1) each $\mathcal{R}_n$ is a connected domain with real 3-dimensional C^ω surfaces $\partial \mathcal{R}_n$ in $\mathcal{R}|_B$ (but each $R_n(t), t \in B$ is not always connected); (2) if we consider the set $\mathcal{L}$ of points $t \in B$ such that there exists a point $(t, z(t)) \in \partial \mathcal{R}_n$ with $\frac{\partial \psi}{\partial z}(t, z(t)) = 0$, then $\mathcal{L}$ consists of two kinds of families $\mathcal{L}', \mathcal{L}''$ of finite C^ω arcs in B

$$\mathcal{L}' = \{l'_1, \ldots, l'_m\}, \qquad \mathcal{L}'' = \{l''_1, \ldots, l''_\mu\},$$

which have the following property.

For $\mathcal{L}'$: for $t_0 \in \mathcal{L}'$, except a finite set at which some l'_i and l'_j or l'_i itself intersects transversally, say, $t_0 \in l'_i$, $\partial R_n(t_0)$ (consisting of a finite number of closed curves) has only one singular point at $z(t_0)$, and we find a bidisk $B_0 \times V$ of center $(t_0, z(t_0))$ in $\mathcal{R}_{n+1}$ such that $B_0 \Subset B$ and $l'_i \cap B_0$ divides B_0 into two domains B'_0 and B''_0 in the manner that

(i) each $\partial R_n(t), t \in B'_0 \cup B''_0$ has no singular points;
(ii) each $\partial R_n(t), t \in l'_i \cap B_0$ has one singular point $z(t)$ at which two subarcs of $\partial R_n(t)$ transversally intersect;
(iii) each $R_n(t) \cap V, t \in B'_0 \cup (l'_i \cap B_0)$ consists of two (connected) domains, while each $R_n(t) \cap V, t \in B''_0$ consists of one domain;

For $\mathcal{L}''$: for $t_0 \in \mathcal{L}''$, except a finite point set, say, $t_0 \in l_i''$, we find a unique point $(t_0, z(t_0)) \in \partial\mathcal{R}_n$ with $\frac{\partial\psi}{\partial z}(t_0, z(t_0)) = 0$, and a bidisk $B_0 \times V$ of center $(t_0, z(t_0))$ in $\mathcal{R}_{n+1}$ such that $B_0 \Subset B$ and $l_i'' \cap B_0$ divides B_0 into two domains B_0' and B_0'' and $\exists\, C^\omega$ mapping $\mathfrak{z}$: $t \in l_i'' \cap B_0 \to z(t)$ such that $(t, z(t)) \in \partial\mathcal{R}_n$ with $\frac{\partial\psi}{\partial z}(t, z(t)) = 0$ in the manner that

(i) $[R_n(t) \cup \partial R_n(t)] \cap V = \emptyset$ for $t \in B_0' \cup (l_i'' \cap B_0)$;

(ii) $R_n(t) \cap V$ for $t \in B_0''$ is a simply connected domain $\delta_n(t)$ such that, for a given $t^0 \in l_i'' \cap B_0$, $\delta_n(t)$ shrinkingly approaches the point $z(t^0)$ as $t \in B_0'' \to t^0$.

For the singular point $z(t), t \in l_i' \subset \mathcal{L}'$, we have the connected component $C(t)$ of $\partial R_n(t)$ passing through $z(t)$. Then $C(t)$ consists of one closed curve, or two closed curves $C_i(t), i = 1, 2$, such that $C(t) = C_1(t) \cup C_2(t)$ and $C_1(t) \cap C_2(t) = z(t)$. For example, in (FIII) below, $C(t)$ consists of one closed curve, and in (FI) and (FII), $C(t)$ consists of two closed curves.

For the singular $z(t), t \in l_i'' \subset \mathcal{L}''$, $(t, z(t)) \in \partial\mathcal{R}_n$ but $z(t) \notin \partial R_n(t)$.

Fix $t \in B$ and $n \geq 1$, and consider the connected component $R_n'(t)$ of $R_n(t)$ that contains $g(t)$. We put $\mathcal{R}_n' = \bigcup_{t\in B}(t, R_n'(t))$, and we put $\partial\mathcal{R}_n' = \bigcup_{t\in B}(t, \partial R_n'(t))$. The variation

$$\mathcal{R}_n' : t \in B \to R_n'(t)$$

is no longer a smooth variation of $R_n'(t)$ with $t \in B$; that is, $\mathcal{R}_n'$ satisfies neither corresponding condition (1) nor (2) of $\mathcal{R}$ in Section 2. Since $R(t)$ is irreducible in $\mathcal{R}$, we have $R_n'(t) \Subset R_{n+1}'(t)$, $\lim_{n\to\infty} \mathcal{R}_n' = \mathcal{R}|_B$, and $\lim_{n\to\infty} R_n'(t) = R(t)$ for $t \in B$. By (i) and (ii) for $\mathcal{L}''$, there exists a neighborhood $\mathcal{V}$ of $\bigcup_{t\in\mathcal{L}''}(t, z(t))$ in $\mathcal{R}_{n+1}$ such that $[\mathcal{R}_n' \cup \partial\mathcal{R}_n'] \cap \mathcal{V} = \emptyset$, so that $\mathcal{L}''$ does not give any influence for the variation $\mathcal{R}_n'$ (contrary to that for the variation $\mathcal{R}_n$). Each $R(t), t \in \Delta$ is assumed to be *planar*. We separate the singular point $z(t)$ of $\partial R_n(t), t \in l_i' \subset \mathcal{L}'$ such that $z(t) \in \partial R_n'(t)$ into the following two cases: let $C(t)$ denote the connected component of $\partial R_n(t)$ passing through $z(t)$; then

(c1) $C(t)$ consists of two closed curves $C_i(t), i = 1, 2$, and one of them, say, $C_1(t)$, is one of the boundary components of $R_n'(t)$, so that $(C_2(t) \setminus \{z(t)\}) \cap \partial R_n'(t) = \emptyset$;

(c2) $C(t)$ is one of the boundary components of $R_n'(t)$, so that two distinct points of $\partial R_n'(t)$ lie over $z(t)$.

For example, if the shadowed part below is $R_n'(t)$, then the singular point $z(t)$ is of case (c1) for (FI), and of case (c2) for (FII) or (FIII).

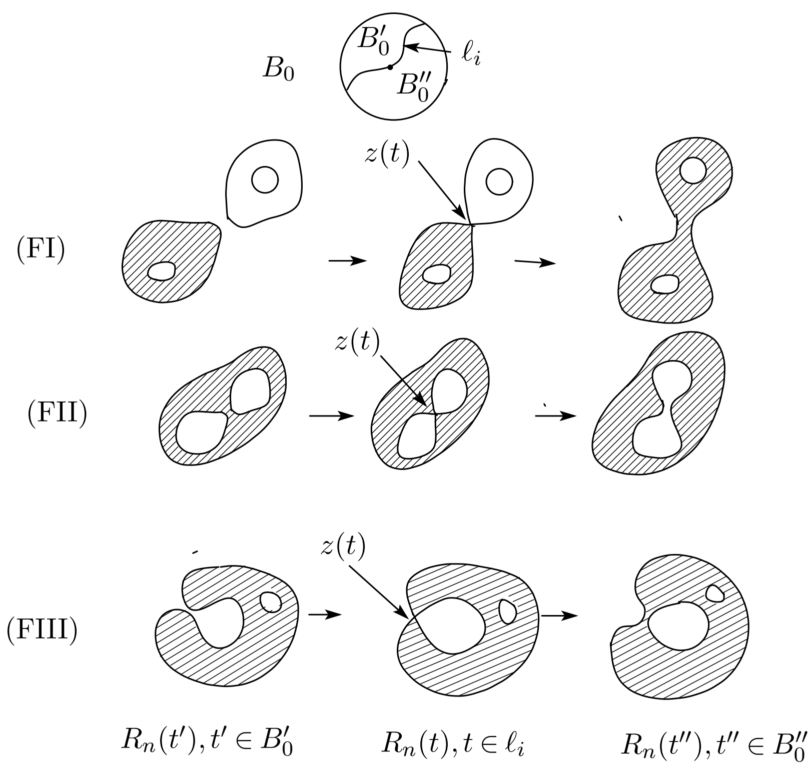

Figure 1: Variation $\mathcal{R}'_n : t \in B_0 \to R'_n(t)$

For $t \in B$ we consider the L_1-function $p_n(t,z)$, the L_0-function $q_n(t,z)$, and the harmonic span $s_n(t)$ for $(R'_n(t), \xi(t), \eta(t))$.

Lemma 5.1 (Hamano [8]). *Let $\mathcal{R}$ be a Stein manifold, and each $R(t), t \in \Delta$ is planar. Then we have the following.*

(1) *$p_n(t,z)$ and $q_n(t,z)$ are continuous for (t,z) in $\mathcal{R}'_n$, and $s_n(t)$ is continuous on B.*
(2) *Assume that at each singular point $z(t)$ of $\partial R_n(t), t \in l'_i \subset \mathcal{L}'$ such that $z(t) \in \partial R'_n(t)$, case (c1) only occurs. Then*
  (i) *$p_n(t,z)$ and $q_n(t,z)$ are of class C^1 for (t,z) on $\mathcal{R}'_n \setminus \{\xi, \eta\}$;*
  (ii) *$s_n(t)$ is C^1 subharmonic on B.*
(3) *In general, (2) does not hold in case (c2).*

As an example of (FI) of $\mathcal{L}'$, let $B = \{|t| < 1/10\}$, let $D = \{|z| < 2\}$, let $\psi_1 = (e^{-100+|t|^2}/|z-1|^2) - 1$, let $\psi_2 = |z^2 - 1| - (1 - 2\Re t - |t|^2)$, let $\psi_3 =$

$(e^{-100+|t|^2}/|z+1|^2)-1$, and let $\mathcal{R}=\{(t,z)\in B\times D:\psi_1<0,\psi_2<0,\psi_3<0\}$. Then $\mathcal{R}$ is pseudoconvex in $B\times D$, and the arc $l'=\{t\in B:2\Re t+|t|^2=0\}$ divides B into two domains $B'\cup B''$ such that $\partial R(t),t\in l'$ consists of two circles $\psi_1(t,z)=0$, $\psi_3(t,z)=0$ and the leminiscate $C:|z^2-1|=1$ with singular point $z(t)=0$. We similarly have examples (FII), (FIII) of $\mathcal{L}'$.

As an example of $\mathcal{L}''$, let B,D be the same as above. Let $\psi(t,z):=|z-t|^2+|t|^2+2\Re t$, and put $\mathcal{R}=\{(t,z)\in B\times D:\psi(t,z)<0\}$. Then the arc $l''=\{t\in B:\phi(t)=0\}$, where $\phi(t)=-|t|^2-2\Re t$, divides B into two domains $B'=\{t\in B:\phi(t)<0\}$ and $B''=\{t\in B:\phi(t)>0\}$ such that $\frac{\partial\psi}{\partial z}(t,t)=0$ for $t\in l''$, $R(t)=\emptyset$ for $t\in B'\cup l''$ and $R(t)=\{|z-t|^2<\phi(t)\}$ for $t\in B''$. The mapping $\mathfrak{z}:t\in l''\to z(t)=t$ so that $(t,t)\in\partial\mathcal{R}$ but $t\notin\partial R(t)$, and each $R(t),t\in B''$ is a disk $\{|z-t|<\phi(t)\}$ that shrinkingly approaches the singular point $z=t^0$ as $t\to t^0\in l''$.

Since the Stein manifold carries a C^ω strictly plurisubharmonic exhaustion function, we immediately have the following.

LEMMA 5.2. *Let $\mathcal{R}:t\in\Delta\to R(t)$ satisfy (a)–(d). Let $\xi,\eta\in\Gamma(\Delta,\mathcal{R})$ such that $\xi\cap\eta=\emptyset$. Assume that*

($\star$) *$R(t),t\in\Delta$ is homeomorphic to a domain in $\mathbb{C}_w$ bounded by a finite number, say, ν, of contours, where ν is independent of $t\in\Delta$.*

Then, for $t_0\in\Delta$, there exists a disk $B\Subset\Delta$ of center t_0 such that we find an increasing sequence $\{\mathcal{R}'_n\}_n$ of case (c1) with $\lim_{n\to\infty}\mathcal{R}'_n=\mathcal{R}|_B$.

Let $\mathcal{R}:t\in\Delta\to R(t)$ satisfy (a)–(d), and let $\xi,\eta\in\Gamma(\Delta,\mathcal{R})$ such that $\xi\cap\eta=\emptyset$. We fix a small disk $B\Subset\Delta$ so that we can fix local parameters (t,z) of $\xi|_B$ and $\eta|_B$ in $\mathcal{R}|_B$ and so that $\{\mathcal{R}_n\}_n$ satisfies conditions in "Preparation" to these Δ and B. Precisely, we define

(5.2) $\mathcal{R}_n:=$ the connected component of $\mathcal{R}(a_n)|_B$ that contains $g|_B$,

which satisfies cases (1) and (2) for (5.1). We put $\mathcal{R}_n=\bigcup_{t\in B}(t,R_n(t))$, and for $t\in B$ we denote by $R'_n(t)$ the connected component of $R_n(t)$ that contains $g(t)$ (connecting $\xi(t)$ and $\eta(t)$) and put $\mathcal{R}'_n=\bigcup_{t\in B}(t,R'_n(t))$. Though $\partial R'(t)$ may not be smooth, each $R'_n(t)$ admits the L_1-function $p_n(t,z)$ and the L_1-constant α_n for $(R'_n(t),0,\eta(t))$, where $B\times\{|z|<r_1\}$ and $\bigcup_{t\in B}(t,\{|z-\eta(t)|<r_2)$ are π-local coordinates for ξ and η, and similarly for $q_n(t,z)$ and $\beta_n(t)$. In one complex variable it is known (see [1, Chapter III, Section 8]) that $p_n(t,z)$ uniformly converges to a certain function $p(t,z)$ on any compact set in $R(t)\setminus\{\xi(t),\eta(t)\}$. Thus, $p(t,z)$ is harmonic on $R(t)\setminus\{\xi(t),\eta(t)\}$ with the same pole as $p_n(t,z)$ at $\xi(t)$ and $\eta(t)$. Putting $\alpha(t)=$

$\lim_{z\to\eta(t)}(p(t,z)-\log|z-\eta(t)|)$, we have $\lim_{n\to\infty}\alpha_n(t)=\alpha(t)$. We also call $p(t,z)$ and $\alpha(t)$ the L_1-*function* and the L_1-*constant* for $(R(t),0,\eta(t))$. Similarly, we define the L_0-function $q(t,z)$ and the L_0-constant $\beta(t)$, and call $s(t):=\alpha(t)-\beta(t)$ the *harmonic span* for $(R(t),\xi(t),\eta(t))$. Since $R(t)$ is planar, we have $s_n(t)\searrow s(t)$ as $n\to\infty$. Their proofs in [1] imply that, for $K\Subset\mathcal{R}|_B\setminus\{\xi|_B,\eta|_B\}$,

$$p_n(t,z), q_n(t,z), p(t,z), q(t,z) \text{ are uniformly bounded on } K. \tag{5.3}$$

Though $p(t,z),q(t,z),\alpha(t),\beta(t)$ depend on the choice of local coordinates about $\xi(t)$ and $\eta(t)$, $s(t)$ does not depend on it, so that $s(t)(\geq 0)$ is a function on B and on Δ.

Using this notation, we have the following approximation condition.

THEOREM 5.1. *Let $\mathcal{R}:t\in\Delta\to R(t)$ satisfy (b)–(d), where Δ is an open Riemann surface. Let $\xi,\eta\in\Gamma(\Delta,\mathcal{R})$ such that $\xi\cap\eta=\emptyset$, and let $s(t)$ denote the harmonic span for $(R(t),\xi(t),\eta(t))$. Assume that*

(∗) *for any $t_0\in\Delta$, there exists a small disk $B\Subset\Delta$ of center t_0 such that we find an increasing sequence $\{\mathcal{R}'_n\}_n$ of case (c1) such that* $\lim_{n\to\infty}\mathcal{R}'_n=\mathcal{R}|_B$.

Then

(1) *$s(t)$ is subharmonic on Δ;*

(2) *(simultaneous uniformization) if $s(t)$ is harmonic on Δ, then $\mathcal{R}$ is π-biholomorphic to a univalent domain in $\Delta\times\mathbb{P}$.*

Proof. To show (1), let $t_0\in\Delta$. Then we have a disk $B\subset\Delta$ with condition (∗). By Lemma 5.1(2)(ii), $s_n(t)$ is C^1 subharmonic on B; hence, $s(t)$ is subharmonic on B and on Δ. To prove (2), we cover Δ by small disks $\{B_i\}_{i=1,2,\dots}$ with condition (∗); that is, for fixed B_i, we find an increasing sequence $\{\mathcal{R}'_n\}_n$ (depending on B_i) such that each $\mathcal{R}'_n$ is of case (c1) and $\lim_{n\to\infty}\mathcal{R}'_n=\mathcal{R}|_{B_i}$. We divide the proof into two steps.

STEP 1. Each $\mathcal{R}|_{B_i}$, $i=1,2,\dots$, is π-biholomorphic to a univalent domain $\mathcal{D}_i$ in $B\times\mathbb{P}$.

In fact, we simply write $B=B_i$. We put $\mathcal{R}'_n=\bigcup_{t\in B}(t,R'_n(t)), n=1,2,\dots$, and consider $p_n(t,z),q_n(t,z)$ and $s_n(t)$ for each $(R'_n(t),0,\eta(t))$, $t\in B$ as above. We put

$$\begin{aligned} P_n(t,z)&=e^{p_n(t,z)+ip_n(t,z)^*}, & P(t,z)&=e^{p(t,z)+ip(t,z)^*},\\ Q_n(t,z)&=e^{q_n(t,z)+iq_n(t,z)^*}, & Q(t,z)&=e^{q(t,z)+iq(t,z)^*}, \end{aligned}\tag{5.4}$$

which are all 0 at $z=\eta(t)$ and normalized

$$\frac{1}{z}+(\text{holomorphic function}) \quad \text{near } z=0. \tag{5.5}$$

For $t\in B$, $P_n(t,z)$ and $Q_n(t,z)$ uniformly converge to $P(t,z)$ and $Q(t,z)$ on any compact set in $R(t)$; $w=P_n(t,z)$ is a circular slit mapping on $R'_n(t)$, and similarly $w=Q_n(t,z)$ is a radial slit one. Hence, $P(t,z)$ and $Q(t,z)$ are univalent functions on $R(t)$. We also call $P(t,z)$ the *circular slit mapping* for $(R(t),0,\eta(t))$, and similarly, we call $Q(t,z)$ the *radial slit mapping*. For Step 1 it suffices to show that

(a) the harmonicity of $s(t)$ on B implies that $P(t,z)$ is holomorphic for two complex variables (t,z) in $\mathcal{R}|_B\setminus\{\xi|_B\}$.

In fact, fix a point (t_0,z_0) in $\mathcal{R}|_B\setminus\{\xi|_B,\eta|_B\}$, and let $B_0\times V\Subset\mathcal{R}|_B\setminus\{\xi|_B,\eta|_B\}$ be a bidisk centered at (t_0,z_0), a local coordinate of a neighborhood of (t_0,z_0). We put $f(t,z):=\frac{\partial p(t,z)}{\partial z}$ for $(t,z)\in B_0\times V$. From (5.5) it suffices for (a) to prove that $f(t,z)$ is holomorphic for (t,z) in $B_0\times V$. Since each $f(t,z), t\in B_0$ is holomorphic for $z\in V$ and since $f(t,z)$ is uniformly bounded in $B_0\times V$ by (5.3), it thus suffices for (a) to show that, for any fixed $z'\in V$, it holds $\frac{\partial f(t,z')}{\partial\bar t}=0$ on B_0 in the sense of distribution; that is, it holds, for any $\varphi(t)=\varphi(t_1+it_2)\in C_0^\infty(B_0)$,

$$I:=\int_{B_0} f(t,z')\frac{\partial\varphi(t)}{\partial\bar t}\,dt_1\,dt_2=0. \tag{5.6}$$

To prove this by contradiction, assume that $I\neq 0$. We fix a small disk $V_0=\{|z-z'|<r_0\}\Subset V$ of center z', so that we have $R'_n(t)\Supset V_0$ for any $t\in B_0$ and $n\geq\exists n_0$. We see from the mean-value theorem for holomorphic functions for z that

$$I=\frac{1}{\pi r_0^2}\iint_{B_0\times V_0} f(t,z)\frac{\partial\varphi(t)}{\partial\bar t}\,dt_1\,dt_2\,dx\,dy.$$

We put $f_n(t,z)=\frac{\partial p_n(t,z)}{\partial z}$ in $B_0\times V$. Since $\lim_{n\to\infty} f_n(t,z)=f(t,z)$ uniformly on V_0 for a fixed $t\in B_0$ and since $f_n(t,z), f(t,z)$ are uniformly bounded in $B_0\times V_0$ by (5.3), the Lebesgue bounded theorem implies that

$$I=\frac{1}{\pi r_0^2}\lim_{n\to\infty}\iint_{B_0\times V_0} f_n(t,z)\frac{\partial\varphi(t)}{\partial\bar t}\,dt_1\,dt_2\,dx\,dy.$$

Therefore,

$$\Big|\frac{1}{\pi r_0^2}\iint_{B_0\times V_0} f_n(t,z)\frac{\partial\varphi(t)}{\partial\bar t}\,dt_1\,dt_2\,dx\,dy\Big| \ge \frac{|I|}{2} > 0 \quad \text{for } n \ge \exists N.$$

On the other hand, using Lemma 5.1(2)(ii) under Theorem 5.1($*$), we see that, for a fixed $z\in V_0$, $p_n(t,z)$, and hence $f_n(t,z)$ is of class C^1 for $t\in B_0$. It follows that

$$\int_{B_0} f_n(t,z)\frac{\partial\varphi(t)}{\partial\bar t}\,dt_1\,dt_2 = -\int_{B_0}\varphi(t)\frac{\partial f_n(t,z)}{\partial\bar t}\,dt_1\,dt_2.$$

Hence, putting $I_0=(\pi r_0^2|I|)/2>0$, we have from the Schwarz inequality that

$$\begin{aligned} I_0^2 &\le \Big(\iint_{B_0\times V_0}|\varphi(t)|^2\,dt_1\,dt_2\,dx\,dy\Big)\\ &\quad\times\Big(\iint_{B_0\times V_0}\Big|\frac{\partial f_n(t,z)}{\partial\bar t}\Big|^2\,dt_1\,dt_2\,dx\,dy\Big)\\ &=: C\iint_{B_0\times V_0}\Big|\frac{\partial f_n(t,z)}{\partial\bar t}\Big|^2\,dt_1\,dt_2\,dx\,dy, \end{aligned}$$

where $C>0$ is independent of n. Lemma 4.1 and $\mathcal{L}'$(i) in “Preparation” for the pseudoconvex domain $\mathcal{R}_n'$ imply that

$$0 \le \frac{4}{\pi}\int_{R_n'(t)}\Big|\frac{\partial f_n(t,z)}{\partial\bar t}\Big|^2\,dx\,dy \le \frac{\partial^2 s_n(t)}{\partial t\partial\bar t} \quad \text{for any } t\in B\setminus\mathcal{L}'.$$

Since $\mathcal{L}'$ (depending on n) consists of a finite number of C^ω arcs in B, $R_n'(t)\supset V_0$ for $n\ge n_0$, and $f_n\in C^1(B_0\times V_0)$, it follows that

$$\begin{aligned} I_0^2 &\le C\iint_{(B_0\setminus\mathcal{L}')\times V_0}\Big|\frac{\partial f_n(t,z)}{\partial\bar t}\Big|^2\,dt_1\,dt_2\,dx\,dy\\ &\le \frac{C\pi}{4}\int_{B_0\setminus\mathcal{L}'}\frac{\partial^2 s_n(t)}{\partial t\partial\bar t}\,dt_1\,dt_2. \end{aligned}$$

We fix a disk $B_1 : B_0 \Subset B_1 \Subset B$ and a C_0^∞ function $\varphi_1(t)\ge 0$ on B_1 such that $\varphi_1(t)\equiv 1$ on B_0. Since $\frac{\partial^2 s_n(t)}{\partial t\partial\bar t}\ge 0$ on $B_1\setminus\mathcal{L}'$, we have that

$$\int_{B_0\setminus\mathcal{L}'}\frac{\partial^2 s_n(t)}{\partial t\partial\bar t}\,dt_1\,dt_2 \le \int_{B_1\setminus\mathcal{L}'}\varphi_1(t)\frac{\partial^2 s_n(t)}{\partial t\partial\bar t}\,dt_1\,dt_2.$$

Since $s_n(t)$ is of class C^1 on B and $\varphi_1(t)\equiv 0$ on ∂B_1, we have that

$$\int_{B_1\setminus\mathcal{L}'} \varphi_1(t)\frac{\partial^2 s_n(t)}{\partial t\partial\bar{t}}\,dt_1\,dt_2=\int_{B_1} s_n(t)\frac{\partial^2\varphi_1(t)}{\partial t\partial\bar{t}}\,dt_1\,dt_2,$$

both being equal to $-(1/4)\int_{B_1}(\frac{\partial\varphi_1}{\partial t_1}\frac{\partial s_n}{\partial t_1}+\frac{\partial\varphi_1}{\partial t_2}\frac{\partial s_n}{\partial t_2})\,dt_1\,dt_2$. Therefore,

$$\begin{aligned}0<I_0^2 &\le \frac{C\pi}{4}\int_{B_1} s_n(t)\frac{\partial^2\varphi_1(t)}{\partial t\partial\bar{t}}\,dt_1\,dt_2\\ &\to \frac{C\pi}{4}\int_{B_1} s(t)\frac{\partial^2\varphi_1(t)}{\partial t\partial\bar{t}}\,dt_1\,dt_2 \quad \text{as } n\to\infty\\ &=0 \quad \text{by the harmonicity of } s(t) \text{ on } B,\end{aligned}$$

which is a contradiction, and Step 1 is proved.

STEP 2. Assertion (2) is true.

In fact, fix $B_i, i=1,2,\dots$, and let $P_i(t,z)$ denote the circular slit mapping for $(R(t),0,\eta(t))$ used in (a) in Step 1 for $\mathcal{R}|_{B_i}$. From the theory of one complex variable, for a fixed $t\in B_i\cap B_j$, there exists $a_{ij}(t)\neq 0$ such that $P_i(t,z)=a_{ij}(t)P_j(t,z)$ on $R(t)$. Since $a_{ij}(t)$ is holomorphic on $B_i\cap B_j$ and since Δ is an open Riemann surface, we have a nonvanishing holomorphic function $a_i(t)$ on B_i such that $a_{ij}(t)=a_j(t)/a_i(t)$ on $B_i\cap B_j$. Thus, $a_i(t)P_i(t,z)$ on $B_i, i=1,2,\dots$ defines a holomorphic function $\mathcal{P}(t,z)$ on $\mathcal{R}$, so that $T:(t,z)\in\mathcal{R}\to(t,w)=(t,\mathcal{P}(t,z))\in B\times\mathbb{P}_w$ proves Step 2. □

COROLLARY 5.1 (Rigidity). *Let $\mathcal{R}: t\in\Delta\to R(t)$ satisfy (a)–(d). Assume that*

- (i) *$R(t), t\in B$ satisfies Lemma 5.2 $(\star)$, so that $R(t)$ has ν (ideal) boundary components;*
- (ii) *there exists at least one (ideal) boundary component $C(t)$ of $R(t), t\in\Delta$ such that $C(t)$ moves homotopically with $t\in\Delta$ in $\mathcal{R}$ and $C(t)$ is of positive harmonic measure on $R(t)$.*

Let $\xi,\eta\in\Gamma(\Delta,\mathcal{R})$ such that $\xi\neq\eta$, and let $s(t), t\in\Delta$ denote the harmonic span for $(R(t),\xi(t),\eta(t))$. Then we have the following.

- (1) *In the case where Δ is an open Riemann surface, $s(t)$ is harmonic on Δ if and only if $\mathcal{R}$ is π-biholomorphic to a domain $(\Delta\times D)\setminus\Xi$ where D is a circular μ slit domain in $\mathbb{P}_w$ and $\Xi:=t\in\Delta\to\{\xi_k(t)\}_{k=1,\dots,\mu'}\subset D$ is a multivalent holomorphic section of $\Delta\times D$ over Δ, where $\mu\ge 1$ and $\mu+\mu'=\nu$. Thus $s(t)$ is constant on Δ.*

(2) *In the case where Δ is a compact Riemann surface, then $\mathcal{R}$ is π-biholomorphic to the product $\Delta\times(D\setminus\{a_k\}_{=1,\dots,\mu'})$, where D is a circular μ slit domain in $\mathbb{P}_w$ and $a_k\in D$.*

Proof. Since the proofs are similar, we prove (2). By (i), we cover Δ with disks $\{B_i\}_{i=1,\dots,m}$ which satisfies Theorem 5.1(∗), so that $s(t)$ is subharmonic on B_i and on Δ; hence, $s(t)=$ constant on Δ. We fix B_i. Then by the proof of Theorem 5.1(2), the circular slit mapping $P_i(t,z)$ for $(R(t),0,\eta(t))$ is holomorphic for $t\in B_i$. Since $D_i(t):=P_i(t,R(t))$ is a circular slit domain in $\mathbb{P}_w$ with ν circular arcs $\{A_j^{(1)}(t),A_j^{(2)}(t)\}$ (depending on B_i), some of which may be a point $A_j^{(1)}(t)=A_j^{(2)}(t)=:\xi_j(t)$, Behnke [3, p. 352] implies that each $A_j^{(k)}(t)$ is holomorphic on B_i. We rename j such that arc $\{A_1^{(1)}(t),A_1^{(2)}(t)\}=P_i(t,C(t))$ for $C(t)$ in (ii); $\{A_j^{(1)}(t),A_j^{(2)}(t)\}, j=2,\dots,\mu(\le\nu)$, are arcs and the rest are points, say, $\xi_k(t), k=1,\dots,\mu'$. Under the homotopy condition for $C(t)$, we see by the same argument as in Theorem 4.1(3)(ii) that, if we put $\widetilde{P}_i(t,z):=P_i(t,z)/A_1^{(1)}(t)$ on $\mathcal{R}_{B_i}$ and $\widetilde{\xi}_k(t):=\xi_k(t)/A_1^{(1)}(t)$ on B_i, then $\widetilde{P}_i(t,z)=\widetilde{P}_j(t,z)$ on $\mathcal{R}|_{B_i\cap B_j}$ for all i,j. We thus have a holomorphic function $\widetilde{P}(t,z)$ for $(t,z)\in\mathcal{R}$ such that $T:(t,z)\in\mathcal{R}\to(t,w)=(t,\widetilde{P}(t,z))\in\Delta\times\mathbb{P}_w$ is a π-biholomorphism from $\mathcal{R}$ onto $(\Delta\times D)\setminus\widetilde{\Xi}$, where D is a circular μ slit domain in $\mathbb{P}_w$ and where $\widetilde{\Xi}=\{\widetilde{\xi}_k\}_{k=1,\dots,\mu'}$ is a μ'-valent holomorphic section of $\Delta\times D$ over Δ. Taking the fundamental polynomials of $\{\widetilde{\xi}_k(t)\}_{k=1}^{\mu'}$ on Δ, we see that each $\widetilde{\xi}_k(t)$ is a constant a_k on Δ, which proves (2). □

Applying Corollary 5.1 to the special case (c′): each $R(t), t\in\Delta$ conformally equivalent to a disk D, we have the following.

Corollary 5.2. *We have the following.*

(1) *Corollary 4.1 holds under the weaker condition for $\mathcal{R}:t\in\Delta\to R(t)$, which satisfies (a), (b), (c′), and (d).*
(2) *Let $\mathcal{R}:t\in\Delta\to R(t)$ satisfy (b), (c′), and (d), where Δ is a compact Riemann surface. Then, if there exist two distinct $\xi_i\in\Gamma(\Delta,\mathcal{R}), i=1,2$, then $\mathcal{R}$ is equivalent to the trivial $\Delta\times D$.*

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