Differentiability of spectral functions for nonsymmetric diffusion processes

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Abstract Let $L = (1/2)\nabla \cdot a\nabla + b \cdot \nabla + W$ be a critical elliptic operator on $\mathbb{R}^d$. For a certain class of potential functions, we consider the generalized principal eigenvalues $\lambda(t)$ of $L_t = L + tV$. We show that it is differentiable if and only if L_0 is null critical.

1. Introduction

We study the behavior of the generalized principal eigenvalues $\lambda(t)$ of operators $L_t = L + tV$ $(t \in \mathbb{R})$. Here $L := (1/2)\nabla \cdot a\nabla + b \cdot \nabla + W$ is an elliptic operator on $\mathbb{R}^d$ and V is a potential function on $\mathbb{R}^d$. We call the function $\lambda(t)$ the *spectral function*, and we are interested in the behavior of $\lambda(t)$, in particular, its differentiability. A precise definition of the generalized principal eigenvalue is given later. We note that $\lambda(t)$ is the bottom of the real part of the spectrum of L_t. So it is an important characteristic for the large time behavior of the semigroup associated with L_t. We suppose that V is nonnegative. We may also suppose that $\lambda(0) = 0$ by replacing L with $L - \lambda(0)$ if necessary. Since $\lambda(t)$ is convex, there are two possibilities:

(i) $\lambda(t)$ is flat to zero for a small positive t;
(ii) $\lambda(t)$ rises immediately as t increases.

Therefore one question arises. Is the initial rising steep or gradual? We show by using the method in [5] that if the operator L is positive critical, then $\lambda'(+0) > 0$, and if L is null critical, then $\lambda'(+0) = 0$. In the latter case, $\lambda(t)$ is differentiable at all $t \in \mathbb{R}$. Heuristically if the operator L is positive critical, then $\lambda(t)$ responds sensitively.

2. Preliminary

Let us consider an elliptic partial differential operator $L := (1/2)\nabla \cdot a\nabla + b \cdot \nabla + W$ on $\mathbb{R}^d$. Let V be a potential function with compact support. We assume that the coefficients a, b, W, and V satisfy the following conditions.

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CONDITION A.1

There exists an $\alpha \in (0,1)$ such that for every relatively compact domain D' with $\overline{D'} \subset \mathbb{R}^d$,

(i) $a, b \in C^{1,\alpha}(D'), W, V \in C^{\alpha}(D')$, and $\nabla \cdot b \geq -c, -W \geq -c$ for some positive constant c;

(ii) $a = \{a_{ij}\}$ is uniformly elliptic; that is, there exists a constant $\mu > 0$ such that for any $x \in D'$,

$$\xi \cdot a(x)\xi \geq \mu|\xi|^2, \quad \xi \in \mathbb{R}^d;$$

(iii) $x \cdot b(x) \leq K(|x|^2+1), \xi \cdot a(x)\xi \leq K(|x|^2+1)|\xi|^2$ for some positive constant K.

We note that condition (iii) is used only in Lemma 4.5.

For the operator L, we denote the set of *positive harmonic functions* by $C_L(\mathbb{R}^d)$:

$$C_L(\mathbb{R}^d) = \big\{u > 0 : u \in C^2(\mathbb{R}^d), Lu = 0\big\}.$$

Following [3], we define the criticality as follows:

(1) If L has 0 Green function, then L is called subcritical.
(2) If L is not subcritical and $C_L(\mathbb{R}^d) \neq \emptyset$, then L is called critical.
(3) If $C_L(\mathbb{R}^d) = \emptyset$, then L is called supercritical.

In the critical case, $C_L(\mathbb{R}^d)$ is one-dimensional, that is, its positive harmonic function is unique up to positive constant multiplication. In that case, we denote the positive harmonic function by ϕ_c and call it the *ground state* of L. For an operator $L = (1/2)\nabla \cdot a\nabla + b \cdot \nabla + W$, we define its dual operator $\widetilde{L}$ by $\widetilde{L} = (1/2)\nabla \cdot a\nabla - b \cdot \nabla - \nabla \cdot b + W$.

The criticality properties of both L and $\widetilde{L}$ coincide; that is, L is critical (subcritical, supercritical) if and only if $\widetilde{L}$ is critical (subcritical, supercritical). In the critical case, both L and $\widetilde{L}$ have ground states ϕ_c and $\tilde{\phi}_c$. Moreover, we classify its critical property as follows

(2-i) If L is critical and $\phi_c\tilde{\phi}_c \in L^1(\mathbb{R}^d, dx)$, then L is positive (or product L^1) critical.

(2-ii) If L is critical and $\phi_c\tilde{\phi}_c \notin L^1(\mathbb{R}^d, dx)$, then L is null (or product not L^1) critical.

In the subcritical case, $C_L(\mathbb{R}^d)$ is not empty. Hence if L is not supercritical, then $C_L(\mathbb{R}^d)$ is not empty. We take a $\phi \in C_L(\mathbb{R}^d)$ and consider an h-transformation of L with respect to ϕ:

$$L^{\phi} := \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla + a\frac{\nabla\phi}{\phi} \cdot \nabla.$$

L^{ϕ} is a diffusion operator. It is known that

(1) L is subcritical $\iff$ L^{ϕ} is transient;
(2-i) L is positive critical $\iff$ L^{ϕ} is positive recurrent;
(2-ii) L is null critical $\iff$ L^{ϕ} is null recurrent.

Therefore the criticality property is regarded as a generalization of the recurrence property.

Using the criticality property, we define λ_c by

$$\lambda_c := \sup\{\lambda \in \mathbb{R} : L - \lambda \text{ is supercritical}\}.$$

The real number λ_c is called the *generalized principal eigenvalue* of L. We now assume that $L = L_0$ is subcritical. Let $\hat{L} = (1/2)(L + \widetilde{L})$ be the symmetric part of L. Define $\hat{G}_\alpha(x,y) = \int_0^\infty e^{-\alpha t}\hat{p}_t(x,y)\,dx$.

We assume the following conditions.

(A.2) We have $\lim_{\alpha\to\infty} \|\hat{G}_\alpha V\|_\infty = 0$ (Kato class).

(A.3) We have $\lim_{n\to\infty} \|\hat{G}_\alpha(V\mathbb{1}_{D_n^c})\|_\infty = 0$ (tightness). Here a sequence of relatively compact sets $\{D_n\}$ is an approximation of $\mathbb{R}^d$; $\overline{D_n} \subset D_{n+1}, \bigcup_{n=1}^\infty D_n = \mathbb{R}^d$.

In the case $a = I,\ b = 0,\ d = 1, 2$, B. Simon [4] and M. Klaus [2] obtained its perturbation series around $t = 0$. For example, Klaus proved that $\lambda(t)^{1/2} = (t/\sqrt{2})\int_{\mathbb{R}} V(x)\,dx - (t^2/\sqrt{2})\int_{\mathbb{R}}\int_{\mathbb{R}} V(x)|x-y|V(y)\,dx\,dy + O(t^3)$ as $t \searrow 0$, where $L = (1/2)\frac{d^2}{dx^2}$ and V satisfies $\int_{\mathbb{R}} |V(x)|(1+|x|)\,dx$. Further, M. Takeda and K. Tsuchida [5] showed that $\lambda(t)$ is differentiable when $a = I,\ b = 0,\ d \le 4$ and showed the necessary and sufficient condition for differentiability of $\lambda(t)$ for symmetric stable process. In this article we consider differentiability of $\lambda(t)$ of L_t by using the method of [5].

Finally, we note that the differentiability of $\lambda(t)$ is crucial when we prove the large deviation principle of the additive functional $\int_0^t V(X_s)\,ds$ by employing the Gärtner-Ellis theorem.

3. Main theorem and examples

We suppose that $L(= L_0)$ is subcritical. We also assume that $\lambda(0) = 0$. Conditions (A.2) and (A.3) imply that L_t is a compact perturbation of L (see [5]), and thus there exists some constant $t_0 > 0$ such that L_{t_0} is critical (see [3, p. 267] for details).

THEOREM 3.1

Assume that L_{t_0} is null critical; then $\lambda'(t_0+) = 0$. Moreover, $\lambda(t)$ is a C^1-function on $\mathbb{R}$.

Figure 1 shows examples of $\lambda(t)$ in the case $L = (1/2)\triangle$ on $\mathbb{R}^d$ and $V = \mathbb{1}_{\{|x|<1\}}$. The left is three-dimensional (L_{t_0} is null critical); the right is five-dimensional (L_{t_0} is positive critical).

4. Proof of the main theorem

Let $(\hat{\mathcal{E}}, C_0^\infty(\mathbb{R}^d))$ be a symmetric bilinear form defined by $\hat{\mathcal{E}}(u,v) = (1/2)\int_{\mathbb{R}^d} \nabla u \cdot a\nabla v\,dx + (1/2)\int_{\mathbb{R}^d}(\nabla\cdot b)uv\,dx - \int_{\mathbb{R}^d} Wuv\,dx$. Since $\nabla\cdot b$ and $-W$ are bounded below, we can take some constant α such that $\hat{\mathcal{E}}_\alpha$ is positive and closable on

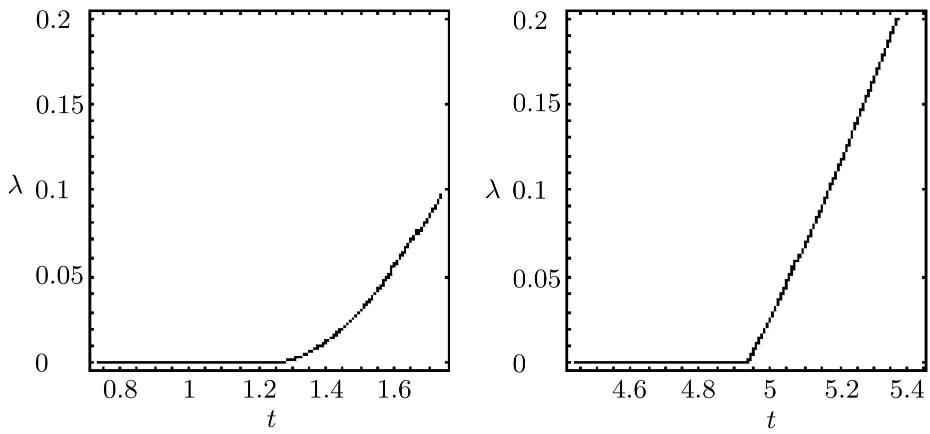

Figure 1

$L^2(\mathbb{R}^d, dx)$. The closure of $(\hat{\mathcal{E}_\alpha}, C_0^\infty(\mathbb{R}^d))$ is a symmetric Dirichlet form $(\hat{\mathcal{E}}_\alpha, \mathcal{F})$. Moreover, if $u \in D(L) \cap \mathcal{F}$, then $\hat{\mathcal{E}}_\alpha(u,u) = (-Lu, u) + \alpha(u,u)$.

A positive Radon measure μ is said to be smooth if $\mu(A) = 0$ for any measurable set A of zero capacity and there exists an increasing sequence of compact sets $\{F_n\}$ such that $\mu(F_n) < \infty$ and $\mathrm{Cap}(K \setminus F_n) \to 0$ $(n \to \infty)$ for every compact set $K \subset \mathbb{R}^d$. We need the following lemma (see [6]).

LEMMA 4.1

Let $u \in \tilde{\mathcal{F}}$ be a quasi-continuous version of $u \in \mathcal{F}$, and let $\hat{G}_\alpha\mu(x) = \int_{\mathbb{R}^d} \hat{G}_\alpha(x, y)\mu(dy)$; then

$$\int_{\mathbb{R}^d} u^2(x)\mu(dx) \le \|\hat{G}_\alpha \mu\|_\infty \hat{\mathcal{E}}_\alpha(u,u).$$

If the operator L is positive critical, then $\phi_c \in C_L(\mathbb{R}^d), \widetilde{\phi}_c \in C_{\widetilde{L}}(\mathbb{R}^d)$ are the ground states of L and $\widetilde{L}$. We set $g_c := \sqrt{\phi_c \widetilde{\phi}_c}$. Since $g_c \in L^2(\mathbb{R}^d, dx)$ by definition of the positive criticality, we normalize g_c so that $\|g_c\|_2 = 1$. Then we have the following lemma.

LEMMA 4.2

Let $H(u) := \int_{\mathbb{R}^d} (Lu/u) g_c^2 \, dx$; then $\inf_{u>0, u \in C^2(\mathbb{R}^d)} H(u)$ can be attained at $u = \phi_c$ and its infimum coincides with λ_c.

Proof

Noting that $H(e^w) = \int_{\mathbb{R}^d} ((1/2)\nabla \cdot a\nabla w + b \cdot \nabla w + (1/2)\nabla w \cdot a \nabla w + Ww) g_c^2 \, dx$, we see that $H(e^{tv_1 + (1-t)v_2}) = tH(e^{v_1}) + (1-t)H(e^{v_2}) - t(1-t)\int_{\mathbb{R}^d} (1/2)\nabla(v_1 - v_2) \cdot a \nabla(v_1 - v_2) \, dx \le tH(e^{v_1}) + (1-t)H(e^{v_2})$. Let $\psi(\varepsilon) := H(\phi_c e^{\varepsilon v})$. Then we can show that $\psi'(+0) = \int_{\mathbb{R}^d} ((1/2)\nabla \cdot a \nabla v + b \cdot \nabla v + (1/2)\frac{\nabla \phi_c}{\phi_c} \cdot a \nabla v + Wv) g_c^2 \, dx = \int_{\mathbb{R}^d} (L^{\phi_c} v)\phi_c \widetilde{\phi}_c \, dx = \int_{\mathbb{R}^d} L(\phi_c v)\widetilde{\phi}_c \, dx = \int_{\mathbb{R}^d} \phi_c v (\widetilde{L}\widetilde{\phi}_c) \, dx = 0.$ $\square$

We have immediately $(-Lg_c, g_c) \le (-L\phi_c, \widetilde{\phi}_c)$ from $H(g_c) = \int_{\mathbb{R}^d} g_c L g_c\, dx \ge \int_{\mathbb{R}^d} \widetilde{\phi}_c L\phi_c\, dx$. Suppose that $g_c \in \mathcal{F}$. Then, by combining Lemmas 4.1 and 4.2, we have the following lemma. One of the sufficient conditions for $g_c \in \mathcal{F}$ is given later.

LEMMA 4.3

Assume that μ is a smooth measure and that $g_c \in \mathcal{F}$. Then

$$\int_{\mathbb{R}^d} \phi_c(x)\widetilde{\phi}_c(x)\mu(dx) \le \|\hat{G}_\alpha \mu\|_\infty \big((-L\phi_c, \widetilde{\phi}_c) + \alpha\big).$$

We consider a one-parameter family of operators $L_t = L + tV$ $(t \in \mathbb{R})$. Assume that L_{t_0} is critical. For $t > t_0$, $\lambda(t) > 0$ and the ground state ϕ_t is in $L^2(\mathbb{R}^d, dx)$. Assume that L_{t_0} is critical. For $t > t_0$, $\lambda(t) > 0$ and the ground state ϕ_t is in $L^2(\mathbb{R}^d, dx)$. Since L_t is a holomorphic family of closed operators (see [1]), we can get that $\lambda(t)$ is analytic in variable t $(t > t_0)$ by the analytic perturbation theory, and we have the following.

LEMMA 4.4

Let $t > t_0$. Then $\lambda(t+\varepsilon) = \lambda(t) + \varepsilon \int_{\mathbb{R}^d} \phi_t \widetilde{\phi}_t V\, dx + o(\varepsilon)$.

Let $\{D_n\}$ be an approximation of $\mathbb{R}^d$ given in (A.3). We take $x_0 \in D_1$ and set $C_t = 1/\phi_t(x_0)$. Let $t_n \searrow t_0$. We see from Harnack inequality that $\{C_{t_n}\phi_{t_n}\}$ is uniformly bounded and equicontinuous on D_1, so we can choose a subsequence of $\{C_{t_n}\phi_{t_n}\}$ which converges uniformly on D_1. We denote the subsequence by $\{C_{t_n^{(1)}}\phi_{t_n^{(1)}}\}$. Next take a subsequence $\{C_{t_n^{(2)}}\phi_{t_n^{(2)}}\}$ of $\{C_{t_n^{(1)}}\phi_{t_n^{(1)}}\}$ so that it converges uniformly on D_2. By the same procedure, we take a subsequence $\{C_{t_n^{(m+1)}}\phi_{t_n^{(m+1)}}\}$ of $\{C_{t_n^{(m)}}\phi_{t_n^{(m)}}\}$ so that it converges uniformly on D_{m+1}. Then $C_{t_0}\phi_{t_0}(x) = \lim_{n\to\infty} C_{t_n^{(n)}}\phi_{t_n^{(n)}}(x)$. Since the limit is unique, we can get that $C_t\phi_t \to C_{t_0}\phi_{t_0}$ locally uniformly as $t \searrow t_0$. Now we are ready to give a proof of the main theorem.

Proof of Theorem 3.1

Applying Lemma 4.4, we have $\lambda'(t) = \int_{\mathbb{R}^d} \phi_t\widetilde{\phi}_t V\, dx$ for $t > t_0$. Therefore it is enough to show that if L_{t_0} is null critical, then $\lim_{t\to t_0} \int_{\mathbb{R}^d} \phi_t\widetilde{\phi}_t V\, dx = 0$.

We first note that

$$\limsup_{t\to t_0} \int_{\mathbb{R}^d} \phi_t\widetilde{\phi}_t V\, dx \le \limsup_{t\to t_0} \int_{D_n} \phi_t\widetilde{\phi}_t V\, dx + \limsup_{t\to t_0} \int_{D_n^c} \phi_t\widetilde{\phi}_t V\, dx.$$

On the other hand, by Fatou's lemma, we have

$$1 = \liminf_{t\to t_0} \int_{\mathbb{R}^d} \phi_t\widetilde{\phi}_t\, dx \ge \int_{\mathbb{R}^d} \liminf_{t\to t_0} \phi_t\widetilde{\phi}_t\, dx = C_{t_0}\widetilde{C}_{t_0} \int_{\mathbb{R}^d} \phi_{t_0}\widetilde{\phi}_{t_0}\, dx.$$

Since L_{t_0} is null critical, we have $C_{t_0}\widetilde{C}_{t_0} = 0$. Hence $\phi_t\widetilde{\phi}_t$ tends to zero locally uniformly as $t \to t_0$. Therefore for fixed n the first term converges to zero.

For the second term, applying Lemma 4.3, we have

$$\int_{D_n^c} \phi_t \widetilde{\phi}_t V\, dx \le \|\hat{G}_\alpha(V \mathbb{1}_{D_n^c})\|_\infty \big((-L\phi_t, \widetilde{\phi}_t) + \alpha\big).$$

We note that from the condition of tightness with respect to V, $\|\hat{G}_\alpha(V \mathbb{1}_{D_n^c})\|_\infty \to 0$ $(n \to \infty)$. Again applying Lemma 4.3,

$$\int_{\mathbb{R}^d} \phi_t \widetilde{\phi}_t V\, dx \le \|\hat{G}_\alpha V\|_\infty \Big(\alpha - \lambda(t) + t \int_{\mathbb{R}^d} \phi_t \widetilde{\phi}_t V\, dx\Big).$$

We have immediately

$$\int_{\mathbb{R}^d} \phi_t \widetilde{\phi}_t V\, dx \le \frac{\|\hat{G}_\alpha V\|_\infty (\alpha - \lambda(t))}{1 - t\|\hat{G}_\alpha V\|_\infty}.$$

Since V is in Kato class, $\lim_{\alpha\to\infty} \|\hat{G}_\alpha V\|_\infty = 0$. Hence we take α such that $t_0\|\hat{G}_\alpha V\|_\infty < 1$, and then we have

$$\limsup_{t\to t_0} \int_{\mathbb{R}^d} \phi_t \widetilde{\phi}_t V\, dx < \infty.$$

Therefore the second term converges to zero. □

The next lemma gives one of the sufficient conditions for $g_c \in \mathcal{F}$.

LEMMA 4.5

For a positive critical operator $L = (1/2)\nabla \cdot a\nabla + b\cdot\nabla + W$, assume that

$$x \cdot b(x) \le K(|x|^2 + 1),$$
$$\xi \cdot a(x)\xi \le K(|x|^2 + 1)|\xi|^2$$

for some constant K. Then the the geometric mean $g_c = \sqrt{\phi_c \widetilde{\phi}_c} \in \mathcal{F}$.

Proof

Let $\varphi : \mathbb{R} \to [0, 1]$ be a smooth function such that $\varphi(t) = 1$ for $t \in [0, 1]$, $\varphi(t) = 0$ for $t \ge 2$, and $-2 \le \varphi'(t) \le 0$. For an $n \in \mathbb{N}$, we define $\chi_n(x) = \varphi(|x|/n)$. We can easily have $\nabla\chi_n(x) = (1/n)\varphi'(|x|/n)x/|x|$. Using this, we have the following equality by direct calculation and integration by parts:

$$\begin{aligned}
&\int_{\mathbb{R}^d} L(\chi_n g_c)\chi_n g_c\, dx \\
&= \int_{\mathbb{R}^d} \Big\{(\nabla g_c \cdot a\nabla\chi_n)\chi_n g_c + \frac{1}{2}(\nabla\cdot a\nabla\chi_n)\chi_n g_c^2 \\
&\qquad + (b\cdot\nabla\chi_n)\chi_n g_c^2 + (Lg_c)g_c\chi_n^2\Big\}\, dx \\
&= \int_{\mathbb{R}^d} \Big\{-\frac{1}{2}g_c^2 \nabla\chi_n \cdot a\nabla\chi_n + (b\cdot\nabla\chi_n)\chi_n g_c^2 + (Lg_c)g_c\chi_n^2\Big\}\, dx.
\end{aligned}$$

For the first term, we have

$$\begin{aligned}\int_{\mathbb{R}^d} -\frac{1}{2} g_c^2 \nabla\chi_n \cdot a\nabla\chi_n \,dx &= -\frac{1}{2}\int_{\mathbb{R}^d} g_c^2 \frac{1}{n^2}\varphi'\Big(\frac{|x|}{n}\Big)^2 \frac{x}{|x|}\cdot a\frac{x}{|x|}\,dx\\ &\ge -\frac{1}{2}\int_{\mathbb{R}^d} g_c^2 \frac{1}{n^2}\varphi'\Big(\frac{|x|}{n}\Big)^2 K(|x|^2+1)\,dx\\ &\ge -\frac{1}{2}\int_{\mathbb{R}^d} g_c^2 \frac{1}{n^2}\varphi'\Big(\frac{|x|}{n}\Big)^2 K(4n^2+1)\,dx\\ &\ge -9K\int_{\mathbb{R}^d} g_c^2\,dx.\end{aligned}$$

For the second term, we have

$$\begin{aligned}\int_{\mathbb{R}^d} (b\cdot\nabla\chi_n)\chi_n g_c^2\,dx &= \int_{\mathbb{R}^d} \chi_n g_c^2 b\cdot \frac{1}{n}\varphi'\Big(\frac{|x|}{n}\Big)\frac{x}{|x|}\,dx\\ &\ge \int_{\mathbb{R}^d} \chi_n g_c^2 \frac{1}{n}\varphi'\Big(\frac{|x|}{n}\Big)\frac{K(|x|^2+1)}{|x|}\,dx\\ &\ge \int_{\mathbb{R}^d} \chi_n g_c^2 \frac{1}{n}\varphi'\Big(\frac{|x|}{n}\Big)K(|x|+1)\,dx\\ &\ge \int_{\mathbb{R}^d} \chi_n g_c^2 \frac{1}{n}\varphi'\Big(\frac{|x|}{n}\Big)K(2n+1)\,dx\\ &\ge -5K\int_{\mathbb{R}^d} g_c^2\,dx.\end{aligned}$$

Noting that $\int_{\mathbb{R}^d} L(\chi_n g_c)\chi_n g_c\,dx = -\hat{\mathcal{E}}(\chi_n g_c, \chi_n g_c)$, we can get

$$\hat{\mathcal{E}}(\chi_n g_c,\chi_n g_c) \le 14K\int_{\mathbb{R}^d} g_c^2\,dx - \int_{\mathbb{R}^d} (Lg_c)g_c\chi_n^2\,dx.$$

Therefore

$$\limsup_{n\to\infty} \hat{\mathcal{E}}(\chi_n g_c,\chi_n g_c) \le 14K\int_{\mathbb{R}^d} g_c^2\,dx - \int_{\mathbb{R}^d} (Lg_c)g_c\,dx.$$

Since $-\int_{\mathbb{R}^d}(Lg_c)g_c\,dx<\infty$ by Lemma 4.2, we have shown $g_c\in D(\mathcal{E})$.

This concludes the proofs of Lemma 4.5 and Theorem 3.1. □

5. One-dimensional case

In the one-dimensional case, there is a necessary and sufficient criterion for a diffusion being either recurrent or transient. Indeed, let $L=(1/2)a(x)\frac{d^2}{dx^2}+b(x)\frac{d}{dx}$ on (α,β), where $-\infty\le\alpha<\beta\le\infty$. Then the corresponding diffusion to L is recurrent if and only if, for any $x_0\in(\alpha,\beta)$,

$$\int_\alpha^{x_0} \exp\Big(-\int_{x_0}^x \frac{2b}{a}(s)\,ds\Big)\,dx = \int_{x_0}^\beta \exp\Big(-\int_{x_0}^x \frac{2b}{a}(s)\,ds\Big)dx = \infty.$$

In general, for a $\phi\in C_L(\mathbb{R}^d)$, the critical properties of L and L^ϕ are the same. In the sequel we determine the criticality of L^ϕ from which the corresponding diffusion is recurrent.

Let $L=(1/2)a(x)\frac{d^2}{dx^2}+b(x)\frac{d}{dx}+W(x)$ on $(0,\infty)$ be critical, where $a(x)>0$ and $a(x),b(x),W(x)\in C^1((0,\infty))$ and $V\in C^1((0,\infty))$ is compactly supported. Let $\lambda(t)$ be the generalized principal eigenvalue of $L+tV$.

In this assumption, we can get the following theorem.

THEOREM 5.1
We have $\lambda'(+0)>0$ if and only if L is product L^1 critical.

Proof
For a fixed $t>0$, if $\lambda=\lambda(t)$, then $L_{t,\lambda}=(1/2)a(x)\frac{d^2}{dx^2}+b(x)\frac{d}{dx}+W(x)+tV(x)-\lambda$ on $(0,\infty)$ is also critical (see [3]). From now on we assume that $\lambda>\lambda(t)$. We denote by $u(x,t,\lambda)$ the ground state of $L_{t,\lambda}$. If $x\notin\operatorname{supp}V$, then u is the solution to the equation

$$\Big(\frac{1}{2}a(x)\,\frac{d^2}{dx^2}+b(x)\,\frac{d}{dx}+W(x)-\lambda\Big)w(x)=0.$$

Since $L-\lambda$ is subcritical, let $I(x,\lambda)$ be its increasing solution of $(L-\lambda)w=0$, and let $K(x,\lambda)$ be its decreasing solution of $(L-\lambda)w=0$. We note that $I(0,\lambda)=K(\infty,\lambda)=0$ and assume that $I(x_0,\lambda)=K(x_0,\lambda)=1$. Then a general solution is $c_1I(x,\lambda)+c_2K(x,\lambda)$. From the boundary condition at x_0, c_1,c_2 are satisfied:

$$\begin{cases}u(x_0,t,\lambda)=c_1I(x_0,\lambda)+c_2K(x_0,\lambda),\\ u'(x_0,t,\lambda)=c_1I'(x_0,\lambda)+c_2K'(x_0,\lambda).\end{cases}$$

Here $'$ denotes the derivative w.r.t x variable. If $\lambda=\lambda(t)$, then $L_{t,\lambda}$ is critical, so that $c_1=0$. Therefore t and $\lambda=\lambda(t)$ satisfy

$$(*)\qquad K'(x_0,\lambda)u(x_0,t,\lambda)-K(x_0,\lambda)u'(x_0,t,\lambda)=0.$$

We set

$$G(x_0,t,\lambda)=u'(x_0,t,\lambda)-u(x_0,t,\lambda)\frac{K'(x_0,\lambda)}{K(x_0,\lambda)};$$

then $(*)$ can be rewritten as $G(x_0,t,\lambda)=0$. Differentiating $G(x_0,t,\lambda)$ in t, we find that

$$\frac{\partial}{\partial t}G(x_0,t,\lambda)t'(\lambda)+\frac{\partial}{\partial\lambda}G(x_0,t,\lambda)=0.$$

We now regard t and λ as independent variables. Let

$$\begin{aligned}W(x,t,\lambda)&:=\begin{vmatrix}u(x,t,\lambda) & \partial_tu(x,t,\lambda)\\ G(x,t,\lambda) & \partial_tG(x,t,\lambda)\end{vmatrix}\\ &=\begin{vmatrix}u(x,t,\lambda) & \partial_tu(x,t,\lambda)\\ u'(x,t,\lambda)-u(x,t,\lambda)k(x,\lambda) & \partial_tu'(x,t,\lambda)-\partial_tu(x,t,\lambda)k(x,\lambda)\end{vmatrix}\\ &=\begin{vmatrix}u(x,t,\lambda) & \partial_tu(x,t,\lambda)\\ u'(x,t,\lambda) & \partial_tu'(x,t,\lambda)\end{vmatrix}.\end{aligned}$$

The function u satisfies

$$\frac{1}{2}a(x)u''(x)+b(x)u'(x)+W(x)u(x)+tV(x)u(x)-\lambda u(x)=0.$$

Differentiating the left side in t, we have

$$\frac{1}{2}a\partial_t u''+b\partial_t u'+(W+tV-\lambda)\partial_t u+Vu=0.$$

Therefore

$$\begin{aligned}
W'(x,t,\lambda)&=\begin{vmatrix} u(x,t,\lambda) & \partial_t u(x,t,\lambda)\\ u''(x,t,\lambda) & \partial_t u''(x,t,\lambda)\end{vmatrix}\\
&=\begin{vmatrix} u & \partial_t u\\ -\frac{2b}{a}u'-\frac{2}{a}(W+tV-\lambda)u & -\frac{2b}{a}\partial_t u'-\frac{2}{a}(W+tV-\lambda)\partial_t u-\frac{2}{a}Vu\end{vmatrix}\\
&=\begin{vmatrix} u & \partial_t u\\ -\frac{2b}{a}u' & -\frac{2b}{a}\partial_t u'-\frac{2}{a}Vu\end{vmatrix}\\
&=-\frac{2b}{a}W(x,t,\lambda)-\frac{2}{a}V(x)u^2(x,t,\lambda).
\end{aligned}$$

Noting that $W(0,t,\lambda)=0$, we can get

$$W(x_0,t,\lambda)=-\frac{1}{a}\int_0^{x_0}\frac{2V(x)}{a}u^2(x,t,\lambda)\exp\Big(\int_{x_0}^{x}\frac{2b}{a}(s)\,ds\Big)\,dx<0.$$

Since $G(x_0,t_0,0)=0$, we have proved that $\partial_t G(x_0,t_0,0)<0$.

We have $t'(+0)=\infty$ if and only if $\lim_{\lambda\to 0}\frac{\partial}{\partial\lambda}G(x_0,t,\lambda)=\infty$. Set $k(x_0,\lambda):=(K'(x_0,\lambda))/(K(x_0,\lambda))$. The function $k(x_0,\lambda)$ diverges as $\lambda\to 0$ if and only if $t'(+0)=\infty$. Differentiating $k(x_0,\lambda)$ in λ, we have

$$\frac{\partial}{\partial\lambda}k(x_0,\lambda)=\frac{1}{K^2(x_0,\lambda)}\begin{vmatrix} K(x_0,\lambda) & \partial_\lambda K(x_0,\lambda)\\ K'(x_0,\lambda) & \partial_\lambda K'(x_0,\lambda)\end{vmatrix}.$$

Recall that $K(x,\lambda)$ satisfies

$$\frac{1}{2}a(x)K''(x,\lambda)+b(x)K'(x)+\big(W(x)-\lambda\big)K(x,\lambda)=0.$$

Differentiating in λ, we have,

$$\frac{1}{2}a(x)\partial_\lambda K''(x,\lambda)+b(x)\partial_\lambda K'(x)+\big(W(x)-\lambda\big)\partial_\lambda K(x,\lambda)-K(x,\lambda)=0.$$

Thus

$$\begin{vmatrix} K(x,\lambda) & \partial_\lambda K(x,\lambda)\\ K'(x,\lambda) & \partial_\lambda K'(x,\lambda)\end{vmatrix}'=-\frac{2b}{a}(x)\begin{vmatrix} K(x,\lambda) & \partial_\lambda K(x,\lambda)\\ K'(x,\lambda) & \partial_\lambda K'(x,\lambda)\end{vmatrix}+\frac{2K^2(x,\lambda)}{a(x)}.$$

$K(\infty,\lambda)=0$ and $(K^2(y,\lambda))/(a(y))\exp(\int_{x_0}^{y}\frac{2b}{a}(s)\,ds)$ is integrable on (x_0,∞). From this,

$$\partial_\lambda k(x_0,\lambda)=\frac{2}{K^2(x_0,\lambda)}\exp\Big(-\int_{x_0}^{x_0}\frac{2b}{a}(s)\,ds\Big)\int_{\infty}^{x_0}\frac{K^2(y,\lambda)}{a(y)}\exp\Big(\int_{x_0}^{y}\frac{2b}{a}(s)\,ds\Big)\,dy.$$

Letting $\lambda \to 0$, $K^2(x,\lambda) \to u^2(x)$ on each point $x \in [x_0, \infty)$. Therefore if the integral

$$\int_{\infty}^{x_0} \frac{u^2(y)}{a(y)} \exp\Big(\int_{x_0}^{y} \frac{2b}{a}(s)\,ds\Big)\,dy$$

converges, then $\partial_\lambda k(x_0,\lambda)$ also converges. Then L is product L^1-critical if and only if the above integral is finite, and we have shown the theorem. $\square$

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