Relation between differential polynomials and small functions

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Abstract   In this article, we discuss the growth of solutions of the second-order nonhomogeneous linear differential equation

$$f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = F,$$

where a, b are complex constants and $A_j(z) \not\equiv 0$ $(j = 0, 1)$, and $F \not\equiv 0$ are entire functions such that $\max\{\rho(A_j)\ (j = 0, 1), \rho(F)\} < 1$. We also investigate the relationship between small functions and differential polynomials $g_f(z) = d_2 f'' + d_1 f' + d_0 f$, where $d_0(z), d_1(z), d_2(z)$ are entire functions that are not all equal to zero with $\rho(d_j) < 1$ $(j = 0, 1, 2)$ generated by solutions of the above equation.

1. Introduction and statement of results

Throughout this article, we assume that the reader is familiar with the fundamental results and the standard notation of the Nevanlinna value distribution theory (see [14], [20]). In addition, we use $\lambda(f)$ and $\overline{\lambda}(f)$ to denote, respectively, the exponents of convergence of the zero-sequence and the sequence of distinct zeros of f, $\rho(f)$ to denote the order of growth of f. A meromorphic function $\varphi(z)$ is called a small function with respect to $f(z)$ if $T(r, \varphi) = o(T(r, f))$ as $r \to +\infty$, where $T(r, f)$ is the Nevanlinna characteristic function of f.

To give the precise estimate of fixed points, we define the following.

DEFINITION 1.1 ([18, p. 192], [23])
Let f be a meromorphic function, and let $z_1, z_2, \dots (|z_j| = r_j,\ 0 < r_1 \le r_2 \le \cdots)$ be the sequence of the fixed points of f, each point being repeated only once. The exponent of convergence of the sequence of distinct fixed points of f is defined by

$$\overline{\tau}(f) = \inf\Big\{\tau > 0 : \sum_{j=1}^{+\infty} |z_j|^{-\tau} < +\infty\Big\}.$$

Clearly,

$$(1.1) \qquad \overline{\tau}(f) = \varlimsup_{r \to +\infty} \frac{\log \overline{N}(r, 1/(f - z))}{\log r},$$

where $\overline{N}(r, 1/(f-z))$ is the counting function of distinct fixed points of $f(z)$ in $\{|z| < r\}$.

For the second-order linear differential equation

$$(1.2) \qquad f'' + e^{-z}f' + B(z)f = 0,$$

where $B(z)$ is an entire function, it is well known that each solution f of the equation (1.2) is an entire function and that if f_1, f_2 are two linearly independent solutions of (1.2), then by [9, Lemma 3], there is at least one of f_1, f_2 of infinite order. Hence, most solutions of (1.2) have infinite order. But equation (1.2) with $B(z) = -(1 + e^{-z})$ possesses a solution $f(z) = e^z$ of finite order.

A natural question arises: What conditions on $B(z)$ will guarantee that every solution $f \not\equiv 0$ has infinite order? Many authors, Frei [10], Ozawa [21], Amemiya and Ozawa [1], Gundersen [11], and Langley [17], have studied this problem. They proved that when $B(z)$ is a nonconstant polynomial or $B(z)$ is a transcendental entire function with order $\rho(B) \neq 1$, then every solution $f \not\equiv 0$ of (1.2) has infinite order.

In 2002, Z. X. Chen [6] considered the question: What conditions on $B(z)$ when $\rho(B) = 1$ guarantee that every nontrivial solution of (1.2) has infinite order? He proved the following results, which improved results of Frei, Amemiya and Ozawa, Ozawa, Langley, and Gundersen.

THEOREM A ([6, p. 291])
Let $A_j(z) \not\equiv 0$ $(j = 0, 1)$ and $D_j(z)$ $(j = 0, 1)$ be entire functions with $\max\{\rho(A_j)$ $(j = 0, 1), \rho(D_j)$ $(j = 0, 1)\} < 1$, and let a, b be complex constants that satisfy $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$. Then every solution $f \not\equiv 0$ of the equation

$$(1.3) \qquad f'' + \big(D_1(z) + A_1(z)e^{az}\big)f' + \big(D_0(z) + A_0(z)e^{bz}\big)f = 0$$

is of infinite order.

Setting $D_j \equiv 0$ $(j = 0, 1)$ in Theorem A, we obtain the following result.

THEOREM B
Let $A_j(z) \not\equiv 0$ $(j = 0, 1)$ be entire functions with $\max\{\rho(A_j)$ $(j = 0, 1)\} < 1$, and let a, b be complex constants that satisfy $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$. Then every solution $f \not\equiv 0$ of the equation

$$(1.4) \qquad f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = 0$$

is of infinite order.

THEOREM C ([6, p. 291])
Let $A_j(z) \not\equiv 0$ $(j = 0, 1)$ be entire functions with $\rho(A_j) < 1$ $(j = 0, 1)$, and let a, b be complex constants that satisfy $ab \neq 0$ and $a = cb$ $(c > 1)$. Then every solution $f \not\equiv 0$ of the equation (1.4) is of infinite order.

Consider the second-order nonhomogeneous linear differential equation

$$(1.5) \qquad f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = F,$$

where a, b are complex constants and $A_j(z) \not\equiv 0$ $(j = 0, 1)$, $F(z)$ are entire functions with $\max\{\rho(A_j)$ $(j = 0, 1), \rho(F)\} < 1$. In [22], J. Wang and I. Laine have investigated the growth of solutions of (1.5) and have obtained the following.

THEOREM D ([22, p. 40])

Let $A_j(z) \not\equiv 0$ $(j = 0, 1)$ and $F(z)$ be entire functions with $\max\{\rho(A_j)$ $(j = 0, 1), \rho(F)\} < 1$, and let a, b be complex constants that satisfy $ab \neq 0$ and $a \neq b$. Then every nontrivial solution f of equation (1.5) is of infinite order.

The first main purpose of this article is to study the growth and the oscillation of solutions of the second-order linear differential equation (1.5). We prove the following results.

THEOREM 1.1

Let $A_j(z) \not\equiv 0$ $(j = 0, 1)$ and $F \not\equiv 0$ be entire functions with $\max\{\rho(A_j)$ $(j = 0, 1), \rho(F)\} < 1$, and let a, b be complex constants that satisfy $ab \neq 0$ and $a \neq b$. Then every solution f of equation (1.5) is of infinite order and satisfies

$$(1.6) \qquad \overline{\lambda}(f) = \lambda(f) = \rho(f) = \infty.$$

REMARK 1.1

The proof of Theorem 1.1 in which every solution f of the equation (1.5) has infinite order is quite different from that in the proof of Theorem D (see [22]). The main ingredient in the proof is Lemma 2.12.

REMARK 1.2

If $\rho(F) \geq 1$, then equation (1.5) can posses solution of finite order. For instance the equation

$$f'' + e^{-z}f' + e^z f = 1 + e^{2z}$$

satisfies $\rho(F) = \rho(1 + e^{2z}) = 1$ and has a finite order solution $f(z) = e^z - 1$.

THEOREM 1.2

Let $A_j(z) \not\equiv 0$ $(j = 0, 1)$ and $D_j(z)$ $(j = 0, 1)$, $F(z) \not\equiv 0$ be entire functions with $\max\{\rho(A_j)$ $(j = 0, 1), \rho(D_j)$ $(j = 0, 1), \rho(F)\} < 1$, and let a, b be complex constants that satisfy $ab \neq 0$ and $a \neq b$. Then every solution f of the equation

$$(1.7) \qquad f'' + \big(D_1(z) + A_1(z)e^{az}\big)f' + \big(D_0(z) + A_0(z)e^{bz}\big)f = F$$

is of infinite order and satisfies (1.6).

REMARK 1.3

In [22], J. Wang and I. Laine studied equation (1.7) and obtained the same result

as in Theorem 1.2 but under restriction that the complex constants a, b satisfy $ab \neq 0$ and $b/a < 0$.

REMARK 1.4

Setting $D_j \equiv 0$ $(j = 0, 1)$ in Theorem 1.2, we obtain Theorem 1.1.

THEOREM 1.3

Let $A_j(z)$ $(j = 0, 1)$, a, b satisfy the additional hypotheses of Theorem 1.1, and let $F(z)$ be an entire function such that $\rho(F) \geq 1$. Then every solution f of the equation (1.5) satisfies (1.6) with at most one finite-order solution f_0.

Many important results have been obtained on the fixed points of general transcendental meromorphic functions for almost four decades (see [24]). However, there are few studies on the fixed points of solutions of differential equations. It was in the year 2000 that Z. X. Chen first pointed out the relation between the exponent of convergence of distinct fixed points and the rate of growth of solutions of second-order linear differential equations with entire coefficients (see [5]). In [23], Wang and Yi investigated fixed points and hyperorder of differential polynomials generated by solutions of second-order linear differential equations with meromorphic coefficients. In [16], Laine and Rieppo gave an improvement of the results of [23] by considering fixed points and iterated order. In [18], Liu and Zhang have investigated the fixed points and hyperorder of solutions of some higher-order linear differential equations with meromorphic coefficients and their derivatives. Recently, in [2], [3], Belaïdi gave an extension of the results of [18].

We know that a differential equation bears a relation to all derivatives of its solutions. Hence, linear differential polynomials generated by its solutions must have special nature because of the control of differential equations.

The second main purpose of this article is to study the relation between small functions and some differential polynomials generated by solutions of the second-order linear differential equation (1.5). We obtain some estimates of their distinct fixed points.

THEOREM 1.4

Let $A_j(z) \not\equiv 0$ $(j = 0, 1)$ and $F \not\equiv 0$ be entire functions with $\max\{\rho(A_j)$ $(j = 0, 1), \rho(F)\} < 1$, and let a, b be complex constants that satisfy $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$. Let $d_0(z), d_1(z), d_2(z)$ be entire functions that are not all equal to zero with $\rho(d_j) < 1$ $(j = 0, 1, 2)$, and let $\varphi(z)$ be an entire function with finite order. If f is a solution of (1.5), then the differential polynomial $g_f = d_2 f'' + d_1 f' + d_0 f$ satisfies $\overline{\lambda}(g_f - \varphi) = \infty$.

COROLLARY 1.1

Let $A_j(z)$ $(j = 0, 1), F(z), d_j(z)$ $(j = 0, 1, 2)$, a, b satisfy the additional hypotheses of Theorem 1.4. If f is a solution of (1.5), then the differential polynomial $g_f = d_2 f'' + d_1 f' + d_0 f$ has infinitely many fixed points and satisfies $\overline{\tau}(g_f) = \infty$.

Now, let us denote

(1.8) $$\alpha_1 = d_1 - d_2 A_1 e^{az}, \qquad \alpha_0 = d_0 - d_2 A_0 e^{bz},$$

(1.9)
$$\beta_1 = d_2 A_1^2 e^{2az} - \big((d_2 A_1)' + a d_2 A_1 + d_1 A_1\big)e^{az}$$
$$- d_2 A_0 e^{bz} + d_0 + d_1',$$

(1.10) $$\beta_0 = d_2 A_0 A_1 e^{(a+b)z} - \big((d_2 A_0) + b d_2 A_0 + d_1 A_0\big)e^{bz} + d_0',$$

(1.11) $$h = \alpha_1 \beta_0 - \alpha_0 \beta_1,$$

and

(1.12) $$\psi = \frac{\alpha_1(\varphi' - (d_2 F)' - \alpha_1 F) - \beta_1(\varphi - d_2 F)}{h}.$$

THEOREM 1.5

Let $A_j(z)$ $(j = 0, 1)$, $d_j(z)$ $(j = 0, 1, 2)$, a, b satisfy the additional hypotheses of Theorem 1.4, and let $F(z)$ be an entire function such that $\rho(F) \geq 1$. Let $\varphi(z)$ be an entire function with finite order such that $\psi(z)$ is not a solution of equation (1.5). If $f(z)$ is a solution of (1.5), then the differential polynomial $g_f = d_2 f'' + d_1 f' + d_0 f$ satisfies $\overline{\lambda}(g_f - \varphi) = \infty$ with at most one finite-order solution f_0.

Next, we investigate the relation between small functions and differential polynomials of a pair of nonhomogeneous linear differential equations, and we obtain the following result.

THEOREM 1.6

Let $A_j(z)$ $(j = 0, 1)$, $d_j(z)$ $(j = 0, 1, 2)$, a, b satisfy the additional hypotheses of Theorem 1.4. Let $F_1 \not\equiv 0$ and $F_2 \not\equiv 0$ be entire functions such that $\max\{\rho(F_j) : j = 1, 2\} < 1$ and $F_1 - C F_2 \not\equiv 0$ for any constant C, and let $\varphi(z)$ be an entire function with finite order. If f_1 is a solution of the equation

(1.13) $$f'' + A_1(z)e^{az} f' + A_0(z)e^{bz} f = F_1$$

and f_2 is a solution of the equation

(1.14) $$f'' + A_1(z)e^{az} f' + A_0(z)e^{bz} f = F_2,$$

then the differential polynomial $g_{f_1 - C f_2}(z) = d_2(f_1'' - C f_2'') + d_1(f_1' - C f_2') + d_0(f_1 - C f_2)$ satisfies $\overline{\lambda}(g_{f_1 - C f_2} - \varphi) = \infty$ for any constant C.

2. Preliminary lemmas

We need the following lemmas in the proofs of our theorems.

LEMMA 2.1 ([12, p. 90])

Let f be a transcendental meromorphic function of finite order ρ, let $\Gamma = \{(k_1, j_1),$

$(k_2, j_2), \ldots, (k_m, j_m)\}$ *denote a finite set of distinct pairs of integers that satisfy* $k_i > j_i \geq 0$ *for* $i = 1, \ldots, m$, *and let* $\varepsilon > 0$ *be a given constant. Then the following estimations hold.*

(i)   *There exists a set* $E_1 \subset [0, 2\pi)$ *which has linear measure zero, such that if* $\psi \in [0, 2\pi) - E_1$, *then there is a constant* $R_1 = R_1(\psi) > 1$ *such that for all* z *satisfying* $\arg z = \psi$ *and* $|z| \geq R_1$ *and for all* $(k, j) \in \Gamma$, *we have*

$$(2.1) \qquad \left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho - 1 + \varepsilon)}.$$

(ii)   *There exists a set* $E_2 \subset (1, \infty)$ *which has finite logarithmic measure* $lm(E_2) = \int_1^{+\infty} ((\chi_{E_2}(t))/t)\, dt$, *where* χ_{E_2} *is the characteristic function of* E_2, *such that for all* z *satisfying* $|z| \notin E_2 \cup [0, 1]$ *and for all* $(k, j) \in \Gamma$, *we have*

$$(2.2) \qquad \left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho - 1 + \varepsilon)}.$$

LEMMA 2.2 ([8, p. 755])

Let $f(z)$ *be a transcendental meromorphic function of order* $\rho(f) = \rho < +\infty$. *Then for any given* $\varepsilon > 0$, *there exists a set* $E_3 \subset [0, 2\pi)$ *which has linear measure zero, such that if* $\psi_1 \in [0, 2\pi) \backslash E_3$, *then there is a constant* $R_2 = R_2(\psi_1) > 1$ *such that for all* z *satisfying* $\arg z = \psi_1$ *and* $|z| = r \geq R_2$, *we have*

$$(2.3) \qquad \exp\{-r^{\rho + \varepsilon}\} \leq |f(z)| \leq \exp\{r^{\rho + \varepsilon}\}.$$

The next lemma describing the behavior of $e^{P(z)}$, where $P(z)$ is a linear polynomial, is a special case of a more general result in [19, p. 254].

LEMMA 2.3 ([19, p. 254])

Let $P(z) = (\alpha + i\beta)z$, $(\alpha + i\beta \neq 0)$, *and let* $A(z) \not\equiv 0$ *be a meromorphic function with* $\rho(A) < 1$. *Set* $f(z) = A(z)e^{P(z)}$, $z = re^{i\theta}, \delta(P, \theta) = \alpha \cos\theta - \beta \sin\theta$. *Then for any given* $\varepsilon > 0$, *there exists a set* $E_4 \subset [0, 2\pi)$ *which has linear measure zero, such that if* $\theta \in [0, 2\pi) \backslash (E_4 \cup E_5)$, *where* $E_5 = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$ *is a finite set, then for sufficiently large* $|z| = r$, *we have the following.*

(i)   *If* $\delta(P, \theta) > 0$, *then*

$$(2.4) \qquad \exp\{(1 - \varepsilon)\delta(P, \theta)r\} \leq |f(z)| \leq \exp\{(1 + \varepsilon)\delta(P, \theta)r\}.$$

(ii)   *If* $\delta(P, \theta) < 0$, *then*

$$(2.5) \qquad \exp\{(1 + \varepsilon)\delta(P, \theta)r\} \leq |f(z)| \leq \exp\{(1 - \varepsilon)\delta(P, \theta)r\}.$$

LEMMA 2.4 ([4])

Let $A_0, A_1, \ldots, A_{k-1}$, $F \not\equiv 0$, *be finite-order meromorphic functions. If* f *is a meromorphic solution with* $\rho(f) = \infty$ *of the equation*

$$(2.6) \qquad f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = F,$$

then $\overline{\lambda}(f) = \lambda(f) = \rho(f) = \infty$.

LEMMA 2.5

Let a, b be complex numbers such that $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$. We denote index sets by

$$\Lambda_1 = \{0, a\},$$

$$\Lambda_2 = \{0, a, b, 2a, a + b\}.$$

(i) *If $H_j (j \in \Lambda_1)$ and $H_b \not\equiv 0$ are all meromorphic functions of orders that are less than 1, setting $\Psi_1(z) = \sum_{j \in \Lambda_1} H_j(z) e^{jz}$, then $\Psi_1(z) + H_b e^{bz} \not\equiv 0$.*

(ii) *If $H_j (j \in \Lambda_2)$ and $H_{2b} \not\equiv 0$ are all meromorphic functions of orders that are less than 1, setting $\Psi_2(z) = \sum_{j \in \Lambda_2} H_j(z) e^{jz}$, then $\Psi_2(z) + H_{2b} e^{2bz} \not\equiv 0$.*

Proof

We prove only (i) (for the proof of (ii), see [8]). We divide this into two cases.

Case 1. Suppose first that $\arg a \neq \arg b$. Set $\rho(H_0) = \beta < 1$. By Lemma 2.2, for any given ε $(0 < \varepsilon < 1 - \beta)$ there is a set E_3 which has linear measure zero such that if $\arg z = \theta \in [0, 2\pi) \backslash E_3$, then there is $R = R(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| = r > R$, we have

$$(2.7) \qquad |H_0(z)| \leq \exp\{r^{\beta + \varepsilon}\}.$$

By Lemma 2.3, there exists a ray $\arg z = \theta \in [0, 2\pi) \backslash E_3 \cup E_4 \cup E_5$, $E_3 \cup E_4$, $E_5 = \{\theta \in [0, 2\pi) : \delta(az, \theta) = 0 \text{ or } \delta(bz, \theta) = 0\} \subset [0, 2\pi)$ being defined as in Lemma 2.3, $E_3 \cup E_4$ having linear measure zero, E_5 being a finite set, such that

$$\delta(az, \theta) < 0, \qquad \delta(bz, \theta) > 0,$$

and for the above ε, we have for sufficiently large $|z| = r$:

$$(2.8) \qquad |H_b e^{bz}| \geq \exp\{(1 - \varepsilon)\delta(bz, \theta)r\},$$

$$(2.9) \qquad |H_a e^{az}| \leq \exp\{(1 - \varepsilon)\delta(az, \theta)r\} < 1.$$

If $\Psi_1(z) + H_b e^{bz} \equiv 0$, then by (2.7)–(2.9), we have

$$(2.10) \qquad \exp\{(1 - \varepsilon)\delta(bz, \theta)r\} \leq |H_b e^{bz}| \leq \exp\{r^{\beta + \varepsilon}\} + 1.$$

This is a contradiction by $\beta + \varepsilon < 1$. Hence $\Psi_1(z) + H_b e^{bz} \not\equiv 0$.

Case 2. Suppose now $a = cb$ $(0 < c < 1)$. Then for any ray $\arg z = \theta$, we have

$$\delta(az, \theta) = c\delta(bz, \theta).$$

Then by Lemma 2.2 and Lemma 2.3, for any given ε $(0 < \varepsilon < \min((1 - c)/4, 1 - \beta))$ there exist $E_j \subset [0, 2\pi)$ $(j = 3, 4, 5)$ such that E_3, E_4 has linear measure zero and E_5 is a finite set, where E_3, E_4, and E_5 are defined, respectively, as in case 1. We take the ray $\arg z = \theta \in [0, 2\pi) \backslash E_3 \cup E_4 \cup E_5$ such that $\delta(bz, \theta) > 0$, and for sufficiently large $|z| = r$, we have (2.7), (2.8), and

$$(2.11) \qquad |H_a e^{az}| \leq \exp\{(1 + \varepsilon)c\delta(bz, \theta)r\}.$$

If $\Psi_1(z) + H_b e^{bz} \equiv 0$, then by (2.7), (2.8), and (2.11), we have

$$(2.12) \quad \begin{aligned} \exp\{(1-\varepsilon)\delta(bz,\theta)r\} &\leq |H_b e^{bz}| \\ &\leq \exp\{r^{\beta+\varepsilon}\} + \exp\{(1+\varepsilon)c\delta(bz,\theta)r\}. \end{aligned}$$

By $\beta + \varepsilon < 1$ and $4\varepsilon < 1 - c$, we have, as $r \to +\infty$,

$$(2.13) \qquad\qquad \frac{\exp\{r^{\beta+\varepsilon}\}}{\exp\{(1-\varepsilon)\delta(bz,\theta)r\}} \to 0,$$

$$(2.14) \qquad\qquad \frac{\exp\{(1+\varepsilon)c\delta(bz,\theta)r\}}{\exp\{(1-\varepsilon)\delta(bz,\theta)r\}} \to 0.$$

By (2.12)–(2.14), we get $1 \leq 0$. This is a contradiction; hence $\Psi_1(z) + H_b e^{bz} \not\equiv 0$.
□

By interchanging a and b in Lemma 2.5, we easily obtain the following.

LEMMA 2.6

Let a, b be complex numbers such that $ab \neq 0$ and $a = cb$ $(c > 1)$. We denote index sets by

$$\Lambda_1 = \{0, b\}.$$

If $H_j(j \in \Lambda_1)$ and $H_a \not\equiv 0$ are all meromorphic functions of orders that are less than 1, setting $\Psi_1(z) = \sum_{j \in \Lambda_1} H_j(z) e^{jz}$, then $\Psi_1(z) + H_a e^{az} \not\equiv 0$.

LEMMA 2.7 ([15, p. 344])

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function, let $\mu(r)$ be the maximum term, that is, $\mu(r) = \max\{|a_n| r^n; n = 0, 1, \ldots\}$, and let $\nu_f(r)$ be the central index of f, that is, $\nu_f(r) = \max\{m; \mu(r) = |a_m| r^m\}$. Then

$$(2.15) \qquad \nu_f(r) = r \frac{d}{dr} \log \mu(r) < [\log \mu(r)]^2 \leq [\log M(r, f)]^2$$

holds outside a set $E_6 \subset (1, +\infty)$ of r of finite logarithmic measure.

LEMMA 2.8 ([7, p. 55])

Let $f(z)$ be a transcendental entire function. Then there is a set $E_7 \subset (1, +\infty)$ which has finite logarithmic measure, such that for all z with $|z| = r \notin [0, 1] \cup E_7$ at which $|f(z)| = M(r, f)$, we have

$$(2.16) \qquad\qquad \left| \frac{f(z)}{f^{(s)}(z)} \right| \leq 2r^s \quad (s \in \mathbf{N}).$$

To avoid some problems caused by the exceptional set, we recall the following lemma.

LEMMA 2.9 ([13, p. 421])

Let $g : [0, +\infty) \to \mathbb{R}$ and $h : [0, +\infty) \to \mathbb{R}$ be monotone nondecreasing functions

such that $g(r) \leq h(r)$ for all $r \notin [0,1] \cup E_8$, where $E_8 \subset (1, +\infty)$ is a set of finite logarithmic measure. Let $\alpha > 1$ be a given constant. Then there exists an $r_0 = r_0(\alpha) > 0$ such that $g(r) \leq h(\alpha r)$ for all $r \geq r_0$.

LEMMA 2.10

Let $A_j(z) \not\equiv 0$ $(j = 0,1)$ and $D_j(z)$ $(j = 0,1)$ be entire functions with $\max\{\rho(A_j)$ $(j = 0,1), \rho(D_j)$ $(j = 0,1)\} < 1$, and let a, b be complex constants that satisfy $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$. We denote

$$(2.17) \qquad L_f = f'' + \big(D_1(z) + A_1(z)e^{az}\big)f' + \big(D_0(z) + A_0(z)e^{bz}\big)f.$$

If $f \not\equiv 0$ is a finite-order entire function, then $\rho(L_f) \geq 1$.

Proof

We suppose that $\rho(L_f) < 1$, and then we obtain a contradiction.

(i) If $\rho(f) = \rho < 1$, then

$$f'' + \big(D_1(z) + A_1(z)e^{az}\big)f' + \big(D_0(z) + A_0(z)e^{bz}\big)f - L_f = 0$$

has the form

$$\begin{aligned} \Psi_1(z) + H_b e^{bz} = {} & f'' + D_1(z)f' + D_0(z)f - L_f \\ & + A_1(z)f'e^{az} + A_0(z)fe^{bz} = 0, \end{aligned}$$

and this is a contradiction by Lemma 2.5(i).

(ii) If $\rho(f) = \rho \geq 1$, we rewrite

$$(2.18) \qquad \frac{L_f}{f} = \frac{f''}{f} + \big(D_1(z) + A_1(z)e^{az}\big)\frac{f'}{f} + D_0(z) + A_0(z)e^{bz}.$$

Case 1. Suppose first that $\arg a \neq \arg b$. Set

$$\max\big\{\rho(L_f), D_j(z) \ (j = 0,1)\big\} = \beta < 1.$$

Then, for any given ε $(0 < \varepsilon < 1 - \beta)$, we have, for sufficiently large r,

$$(2.19) \qquad |L_f| \leq \exp\{r^{\beta+\varepsilon}\}, \qquad |D_j(z)| \leq \exp\{r^{\beta+\varepsilon}\} \quad (j = 0,1).$$

By Lemma 2.7, we know that there exists a set E_6 with finite logarithmic measure such that for a point z satisfying $|z| = r \notin E_6$ and $|f(z)| = M(r, f)$, we have

$$(2.20) \qquad v_f(r) < [\log M(r, f)]^2.$$

Since f is a transcendental function, we know that $v_f(r) \to \infty$. Then for sufficiently large $|z| = r$ we have $|f(z)| = M(r, f) \geq 1$; then by (2.19),

$$(2.21) \qquad \left|\frac{L_f}{f}\right| \leq |L_f| \leq \exp\{r^{\beta+\varepsilon}\}.$$

Also, by Lemma 2.1, for the above ε there exists a set $E_1 \subset [0, 2\pi)$ which has linear measure zero, such that if $\theta \in [0, 2\pi) - E_1$, then there is a constant $R_1 = R_1(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \geq R_1$, we have

$$(2.22) \qquad \left|\frac{f^{(k)}(z)}{f(z)}\right| \leq |z|^{k(\rho-1+\varepsilon)} \quad (k = 1, 2).$$

By Lemma 2.3, there exists a ray $\arg z = \theta \in [0, 2\pi) \backslash E_1 \cup E_4 \cup E_5$, $E_5 = \{\theta \in [0, 2\pi) : \delta(az, \theta) = 0 \text{ or } \delta(bz, \theta) = 0\} \subset [0, 2\pi)$, $E_1 \cup E_4$ having linear measure zero and E_5 being a finite set, such that

$$\delta(az, \theta) < 0, \qquad \delta(bz, \theta) > 0$$

and for any given ε $(0 < \varepsilon < 1 - \beta)$, by (2.19) and (2.22) we have, for sufficiently large $|z| = r$,

$$(2.23) \qquad |A_0 e^{bz}| \geq \exp\{(1 - \varepsilon)\delta(bz, \theta)r\},$$

$$\left| (D_1(z) + A_1(z)e^{az}) \frac{f'}{f} \right|$$

$$(2.24) \qquad \leq r^{\rho - 1 + \varepsilon} \exp\{r^{\beta + \varepsilon}\} + r^{\rho - 1 + \varepsilon} \exp\{(1 - \varepsilon)\delta(az, \theta)r\}$$

$$\leq 2r^{\rho - 1 + \varepsilon} \exp\{r^{\beta + \varepsilon}\}.$$

By (2.18), (2.19), (2.21), and (2.22)–(2.24), we have

$$(2.25) \qquad \begin{aligned} \exp\{(1 - \varepsilon)\delta(bz, \theta)r\} &\leq |A_0 e^{bz}| \leq 2\exp\{r^{\beta + \varepsilon}\} \\ &\quad + 2r^{\rho - 1 + \varepsilon} \exp\{r^{\beta + \varepsilon}\} + r^{2(\rho - 1 + \varepsilon)}. \end{aligned}$$

This is a contradiction by $\beta + \varepsilon < 1$. Hence $\rho(L_f) \geq 1$.

Case 2. Suppose now that $a = cb$ $(0 < c < 1)$. Then for any ray $\arg z = \theta$, we have

$$\delta(az, \theta) = c\delta(bz, \theta).$$

Then, by Lemma 2.1 and Lemma 2.3, for any given ε $(0 < \varepsilon < \min(2(1 - c)/(1 + c), 1 - \beta))$, there exist $E_j \subset [0, 2\pi)$ $(j = 1, 4, 5)$ such that E_1, E_4 has linear measure zero and E_5 is a finite set, where E_1, E_4, and E_5 are defined, respectively, as in case 1. We take the ray $\arg z = \theta \in [0, 2\pi) \backslash E_1 \cup E_4 \cup E_5$ such that $\delta(bz, \theta) > 0$ and for sufficiently large $|z| = r$, we have (2.23), and by (2.19) and (2.22), we obtain

$$\left| (D_1(z) + A_1(z)e^{az}) \frac{f'}{f} \right| \leq r^{\rho - 1 + \varepsilon} \exp\{r^{\beta + \varepsilon}\}$$

$$(2.26) \qquad\qquad + r^{\rho - 1 + \varepsilon} \exp\{(1 + \varepsilon)c\delta(bz, \theta)r\}.$$

By (2.18), (2.19), (2.21)–(2.23), and (2.26), we have

$$\exp\{(1 - \varepsilon)\delta(bz, \theta)r\}$$

$$\leq |A_0 e^{bz}|$$

$$(2.27) \quad \leq 2\exp\{r^{\beta + \varepsilon}\} + r^{\rho - 1 + \varepsilon} \exp\{r^{\beta + \varepsilon}\}$$

$$\quad + r^{\rho - 1 + \varepsilon} \exp\{(1 + \varepsilon)c\delta(bz, \theta)r\} + r^{2(\rho - 1 + \varepsilon)}$$

$$\leq 3r^{\rho - 1 + \varepsilon} \exp\{r^{\beta + \varepsilon}\} + r^{\rho - 1 + \varepsilon} \exp\{(1 + \varepsilon)c\delta(bz, \theta)r\} + r^{2(\rho - 1 + \varepsilon)}.$$

By ε $(0 < \varepsilon < \min((1-c)/2(1+c), 1-\beta))$, we have, as $r \to +\infty$,

$$(2.28) \qquad \frac{r^{\rho-1+\varepsilon} \exp\{r^{\beta+\varepsilon}\}}{\exp\{(1-\varepsilon)\delta(bz,\theta)r\}} \to 0,$$

$$(2.29) \qquad \frac{r^{\rho-1+\varepsilon} \exp\{(1+\varepsilon)c\delta(bz,\theta)r\}}{\exp\{(1-\varepsilon)\delta(bz,\theta)r\}} \to 0,$$

$$(2.30) \qquad \frac{r^{2(\rho-1+\varepsilon)}}{\exp\{(1-\varepsilon)\delta(bz,\theta)r\}} \to 0.$$

By (2.27)–(2.30), we get $1 \leq 0$. This is a contradiction. Hence $\rho(L_f) \geq 1$. $\qquad \square$

LEMMA 2.11

Let $A_j(z) \not\equiv 0$ $(j = 0, 1)$ and $D_j(z)$ $(j = 0, 1)$ be entire functions with $\max\{\rho(A_j)$ $(j = 0, 1), \rho(D_j)$ $(j = 0, 1)\} < 1$, and let a, b be complex constants that satisfy $ab \neq 0$ and $a = cb$ $(c > 1)$. If $f \not\equiv 0$ is a finite-order entire function, then L_f, which is defined in (2.17), satisfies $\rho(L_f) \geq 1$.

Proof
First, if $f(z) \equiv C \neq 0$, then

$$L_f = \big(D_0(z) + A_0(z)e^{bz}\big)C.$$

Hence $\rho(L_f) = 1$, and Lemma 2.11 holds.

If $f \not\equiv C$, we suppose that $\rho(L_f) < 1$ and then we obtain a contradiction.

(i) If $\rho(f) = \rho < 1$, then

$$f'' + \big(D_1(z) + A_1(z)e^{az}\big)f' + \big(D_0(z) + A_0(z)e^{bz}\big)f - L_f = 0$$

has the form of

$$\Psi_1(z) + H_a e^{az} = f'' + D_1(z)f' + D_0(z)f - L_f$$
$$+ A_0(z)fe^{bz} + A_1(z)f'e^{az} = 0,$$

and this is a contradiction by Lemma 2.6.

(ii) If $\rho(f) = \rho \geq 1$, we rewrite

$$(2.31) \qquad \frac{L_f}{f}\frac{f}{f'} = \frac{f''}{f'} + \big(D_0(z) + A_0(z)e^{bz}\big)\frac{f}{f'} + D_1(z) + A_1(z)e^{az}.$$

By Lemma 2.1, for any given ε $(0 < \varepsilon < \min((c-1)/2(c+1), 1-\beta))$, there exists a set $E_1 \subset [0, 2\pi)$ which has linear measure zero, such that if $\theta \in [0, 2\pi) - E_1$, then there is a constant $R_1 = R_1(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \geq R_1$, we have

$$(2.32) \qquad \left|\frac{f''(z)}{f'(z)}\right| \leq |z|^{\rho-1+\varepsilon}.$$

Also, by Lemma 2.8, there is a set $E_7 \subset (1, +\infty)$ which has finite logarithmic measure such that for all z with $|z| = r \notin [0, 1] \cup E_7$ at which $|f(z)| = M(r, f)$,

we have

$$(2.33) \qquad \left|\frac{f(z)}{f'(z)}\right| \le 2r.$$

For any ray $\arg z = \theta$, we have

$$\delta(az, \theta) = c\delta(bz, \theta).$$

By Lemma 2.3, there exists a ray $\arg z = \theta \in [0, 2\pi) \backslash E_1 \cup E_4 \cup E_5$, $E_5 = \{\theta \in [0, 2\pi) : \delta(az, \theta) = 0 \text{ or } \delta(bz, \theta) = 0\} \subset [0, 2\pi)$, $E_1 \cup E_4$ having linear measure zero and E_5 being a finite set, such that

$$\delta(az, \theta) = c\delta(bz, \theta) > 0,$$

and by (2.19), (2.21), (2.33), and Lemma 2.9, for sufficiently large $|z| = r$ we have

$$(2.34) \qquad |A_1 e^{az}| \ge \exp\{(1 - \varepsilon)c\delta(bz, \theta)r\},$$

$$(2.35) \qquad \left|\frac{L_f}{f} \frac{f}{f'}\right| \le 2r \exp\{r^{\beta + \varepsilon}\},$$

$$(2.36) \qquad \left|\left(D_0(z) + A_0(z)e^{bz}\right)\frac{f}{f'}\right| \le 2r \exp\{r^{\beta + \varepsilon}\} + 2r \exp\{(1 + \varepsilon)\delta(bz, \theta)r\}.$$

By (2.19), (2.31), (2.32), and (2.34)–(2.36), we have

$$\exp\{(1 - \varepsilon)c\delta(bz, \theta)r\} \le |A_1 e^{az}|$$

$$(2.37) \qquad \begin{aligned} &\le 2r \exp\{r^{\beta + \varepsilon}\} + \exp\{r^{\beta + \varepsilon}\} + 2r \exp\{r^{\beta + \varepsilon}\} \\ &\quad + 2r \exp\{(1 + \varepsilon)\delta(bz, \theta)r\} + r^{\rho - 1 + \varepsilon} \\ &\le 5r \exp\{r^{\beta + \varepsilon}\} + 2r \exp\{(1 + \varepsilon)\delta(bz, \theta)r\} + r^{\rho - 1 + \varepsilon}. \end{aligned}$$

By ε $(0 < \varepsilon < \min((c - 1)/2(1 + c), 1 - \beta))$, we have, as $r \to +\infty$,

$$(2.38) \qquad \frac{r \exp\{r^{\beta + \varepsilon}\}}{\exp\{(1 - \varepsilon)c\delta(bz, \theta)r\}} \to 0,$$

$$(2.39) \qquad \frac{r \exp\{(1 + \varepsilon)\delta(bz, \theta)r\}}{\exp\{(1 - \varepsilon)c\delta(bz, \theta)r\}} \to 0,$$

$$(2.40) \qquad \frac{r^{\rho - 1 + \varepsilon}}{\exp\{(1 - \varepsilon)c\delta(bz, \theta)r\}} \to 0.$$

By (2.37)–(2.40), we get $1 \le 0$. This is a contradiction. Hence $\rho(L_f) \ge 1$. $\qquad\square$

Setting $D_j \equiv 0$ $(j = 0, 1)$ in Lemmas 2.10 and 2.11, we obtain the following lemma.

LEMMA 2.12

Let $A_j(z) \not\equiv 0$ $(j = 0, 1)$ be entire functions with $\max\{\rho(A_j) : j = 0, 1\} < 1$, and let a, b be complex constants that satisfy $ab \ne 0$ and $a \ne b$. We denote

$$(2.41) \qquad L_f = f'' + A_1(z)e^{az} f' + A_0(z)e^{bz} f.$$

If $f \not\equiv 0$ is a finite-order entire function, then $\rho(L_f) \geq 1$.

3. Proof of Theorem 1.1

Assume that f is a solution of equation (1.5). We prove that f is of infinite order. We suppose the contrary $\rho(f) < \infty$. By Lemma 2.12, we have $1 \leq \rho(L_f) = \rho(F) < 1$, and this is a contradiction. Hence, every solution of equation (1.5) is of infinite order. By Lemma 2.4, every solution f satisfies (1.6). $\qquad \Box$

4. Proof of Theorem 1.2

By using Lemma 2.10, Lemma 2.11, and a proof similar to that of Theorem 1.1, we obtain Theorem 1.2. $\qquad \Box$

5. Proof of Theorem 1.3

Assume that f_0 is a solution of (1.5) with $\rho(f_0) = \rho < \infty$. If f_1 is another finite-order solution of (1.5), then $\rho(f_1 - f_0) < \infty$, and $f_1 - f_0$ is a solution of the corresponding homogeneous equation (1.4) of (1.5), but $\rho(f_1 - f_0) = \infty$ from Theorems B and C; this is a contradiction. Hence, (1.5) has at most one finite-order solution f_0, and all other solutions f_1 of (1.5) satisfy (1.6) by Lemma 2.4. $\qquad \Box$

6. Proof of Theorem 1.4

Suppose that $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$. We first prove $\rho(g_f) = \rho(d_2 f'' + d_1 f' + d_0 f) = \infty$. Suppose that $f(z)$ is a solution of equation (1.5). Then by Theorem 1.1, we have $\rho(f) = \infty$. First, we suppose that $d_2 \not\equiv 0$. Substituting $f'' = F - A_1 e^{az} f' - A_0 e^{bz} f$ into g_f, we get

$$(3.1) \qquad g_f - d_2 F = (d_1 - d_2 A_1 e^{az}) f' + (d_0 - d_2 A_0 e^{bz}) f.$$

Differentiating both sides of equation (3.1) and replacing f'' with $f'' = F - A_1 e^{az} f' - A_0 e^{bz} f$, we obtain

$$g_f' - (d_2 F)' - (d_1 - d_2 A_1 e^{az}) F$$
$$(3.2) \quad = \left[d_2 A_1^2 e^{2az} - \left((d_2 A_1)' + a d_2 A_1 + d_1 A_1 \right) e^{az} - d_2 A_0 e^{bz} + d_0 + d_1' \right] f'$$
$$+ \left[d_2 A_0 A_1 e^{(a+b)z} - \left((d_2 A_0)' + b d_2 A_0 + d_1 A_0 \right) e^{bz} + d_0' \right] f.$$

Then by (3.1), (3.2), (1.8), (1.9), and (1.10), we have

$$(3.3) \qquad\qquad \alpha_1 f' + \alpha_0 f = g_f - d_2 F,$$

$$(3.4) \qquad\qquad \beta_1 f' + \beta_0 f = g_f' - (d_2 F)' - (d_1 - d_2 A_1 e^{az}) F.$$

Set

$$h = \alpha_1 \beta_0 - \alpha_0 \beta_1$$
$$(3.5) \quad = (d_1 - d_2 A_1 e^{az}) \left[d_2 A_0 A_1 e^{(a+b)z} - \left((d_2 A_0)' + b d_2 A_0 + d_1 A_0 \right) e^{bz} + d_0' \right]$$

$$- (d_0 - d_2 A_0 e^{bz}) \big[d_2 A_1^2 e^{2az} - \big((d_2 A_1)' + a d_2 A_1 + d_1 A_1 \big) e^{az}$$
$$- d_2 A_0 e^{bz} + d_0 + d_1' \big].$$

Now check all the terms of h. Since the term $d_2^2 A_1^2 A_0 e^{(2a+b)z}$ is eliminated, by (3.5) we can write $h = \Psi_2(z) - d_2^2 A_0^2 e^{2bz}$, where $\Psi_2(z)$ is defined as in Lemma 2.5(ii). By $d_2 \not\equiv 0$, $A_0 \not\equiv 0$, and Lemma 2.5(ii), we see that $h \not\equiv 0$. By (3.3), (3.4), and (3.5), we obtain

$$(3.6) \qquad f = \frac{\alpha_1 (g_f' - (d_2 F)' - \alpha_1 F) - \beta_1 (g_f - d_2 F)}{h}.$$

If $\rho(g_f) < \infty$, then by (3.6) we get $\rho(f) < \infty$, and this is a contradiction. Hence, $\rho(g_f) = \infty$.

   Set $w(z) = d_2 f'' + d_1 f' + d_0 f - \varphi$. Since $\rho(\varphi) < \infty$, then $\rho(w) = \rho(g_f) = \rho(f) = \infty$. In order to prove $\overline{\lambda}(g_f - \varphi) = \infty$, we need to prove only $\overline{\lambda}(w) = \infty$. By $g_f = w + \varphi$, we get, from (3.6),

$$(3.7) \qquad f = \frac{\alpha_1 (w' + \varphi' - (d_2 F)' - \alpha_1 F) - \beta_1 (w + \varphi - d_2 F)}{h}.$$

So

$$(3.8) \qquad\qquad\qquad f = \frac{\alpha_1 w' - \beta_1 w}{h} + \psi,$$

where

$$\psi(z) = \frac{\alpha_1 (\varphi' - (d_2 F)' - \alpha_1 F) - \beta_1 (\varphi - d_2 F)}{h}.$$

Substituting (3.8) into equation (1.5), we obtain

$$(3.9) \qquad \begin{aligned} & \frac{\alpha_1}{h} w''' + \phi_2 w'' + \phi_1 w' + \phi_0 w \\ & \qquad = F - \big(\psi'' + A_1(z) e^{az} \psi' + A_0(z) e^{bz} \psi \big) = A, \end{aligned}$$

where ϕ_j $(j = 0, 1, 2)$ are meromorphic functions with $\rho(\phi_j) < \infty$ $(j = 0, 1, 2)$. Since $\rho(\psi) < \infty$, by Theorem 1.1 it follows that $A \not\equiv 0$. By $\alpha_1 \not\equiv 0$, $h \not\equiv 0$, and Lemma 2.4, we obtain $\overline{\lambda}(w) = \lambda(w) = \rho(w) = \infty$, that is, $\overline{\lambda}(g_f - \varphi) = \infty$.

   Now suppose $d_2 \equiv 0$, $d_1 \not\equiv 0$ or $d_2 \equiv 0$, $d_1 \equiv 0$, and $d_0 \not\equiv 0$. Using reasoning similar to that above, we get $\overline{\lambda}(w) = \lambda(w) = \rho(w) = \infty$, that is, $\overline{\lambda}(g_f - \varphi) = \infty$. □

7. Proof of Theorem 1.5

By the hypothesis of Theorem 1.5, $\psi(z)$ is not a solution of equation (1.5). Then

$$(4.1) \qquad\qquad F - \big(\psi'' + A_1(z) e^{az} \psi' + A_0(z) e^{bz} \psi \big) \not\equiv 0.$$

By reasoning similar to that in the proof of Theorem 1.4, we can prove Theorem 1.5. □

REMARK 4.1

The condition "$\psi(z)$ is not a solution of equation (1.5)" in Theorem 1.5 is nec-

essary, because if $\psi(z)$ is a solution of equation (1.5), then we have

$$F - \left(\psi'' + A_1(z)e^{az}\psi' + A_0(z)e^{bz}\psi\right) \equiv 0.$$

8. Proof of Theorem 1.6

Suppose that f_1 is a solution of equation (1.13) and that f_2 is a solution of equation (1.14). Set $w = f_1 - Cf_2$. Then w is a solution of the equation $w'' + A_1(z)e^{az}w' + A_0(z)e^{bz}w = F_1 - CF_2$. By $\rho(F_1 - CF_0) < 1$, $F_1 - CF_2 \not\equiv 0$, and Theorem 1.1, we have $\rho(w) = \infty$. Thus, by Theorem 1.4, we obtain

(5.1) $$\overline{\lambda}(g_{f_1 - Cf_2} - \varphi) = \infty.$$

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