Multivariate FMRAs and FMRA frame wavelets for reducing subspaces of $L^2(\mathbb{R}^d)$

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Abstract This article addresses frame multiresolution analyses (FMRAs) and FMRA frame wavelets in the setting of reducing subspaces of $L^2(\mathbb{R}^d)$. For a general expansive matrix, we obtain a characterization and some conditions for a frame-scaling function to generate an FMRA, and we prove that an arbitrary reducing subspace must admit an FMRA. For an expansive matrix M with $|\det M| = 2$, we establish a sufficient and necessary condition for FMRAs to admit a single FMRA frame wavelet, give an explicit construction of FMRA frame wavelets, and study the relation between s-frame wavelets and FMRA frame wavelets. These results are also new in the setting of $L^2(\mathbb{R}^d)$.

1. Introduction

An at most countable set $\{f_i : i \in I\}$ in a separable Hilbert space $\mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2 \tag{1}$$

for $f \in \mathcal{H}$, where A, B are called the lower frame bound and the upper frame bound, respectively. In particular, $\{f_i : i \in I\}$ is called a *Parseval frame* for $\mathcal{H}$ if $A = B = 1$ in (1). The fundamentals of frames can be found in [8], [14] and [6]. $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{Z}$ denotes the set of integers. Given $d \in \mathbb{N}$, we denote by $L^2(\mathbb{R}^d)$ the Hilbert space of square-integrable functions on $\mathbb{R}^d$, by x_k the kth component of x, by $|x|$ the Euclidean norm of x for $x \in \mathbb{R}^d$, and by $\mathbb{T}^d = [-1/2, 1/2)^d$ the d-dimensional torus, respectively. For a Lebesgue measurable set E in $\mathbb{R}^d$, we denote by $|E|$ its measure, denote by χ_E the characteristic function of E, and define $(E)\tilde{} := E + \mathbb{Z}^d$. A $(d \times d)$-matrix M is called an *expansive matrix* if it is an integer matrix with all its eigenvalues greater than 1 in the module. We denote by M^* the transpose of M. Throughout this article, relations between two measurable sets in $\mathbb{R}^d$ such as equality, disjointness, or inclusion are always understood up to a set of measure zero, and similarly, equality or inequality

Kyoto Journal of Mathematics, Vol. 50, No. 1 (2010), 83–99
DOI 10.1215/0023608X-2009-006,
Received July 6, 2009. Revised September 2, 2009. Accepted October 1, 2009.
Mathematics Subject Classification: 42C40.
Authors' work supported by National Natural Science Foundation of China grant no. 10671008, Beijing Natural Science Foundation grant no. 1092001, PHR (IHLB), and the project sponsored by SRF for ROCS, SEM of China.

between measurable functions is always understood in the "almost-everywhere" sense. For a measurable function f on $\mathbb{R}^d$, we define its support by

$$\operatorname{supp}(f) = \{x \in \mathbb{R}^d : f(x) \neq 0\}.$$

It is well defined up to a set of measure zero. The Fourier transform of an arbitrary $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is defined by

$$\hat{f}(\cdot) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \cdot \rangle}\, dx$$

on $\mathbb{R}^d$, and the Fourier transforms of the functions in $L^2(\mathbb{R}^d)$ are understood as its unitary extension. Let Ω be a measurable set in $\mathbb{R}^d$ with nonzero measure. We denote by $FL^2(\Omega)$ the closed subspace of $L^2(\mathbb{R}^d)$ of the form

$$FL^2(\Omega) := \{f \in L^2(\mathbb{R}^d) : \hat{f}(\cdot) = 0 \text{ on } \mathbb{R}^d \setminus \Omega\}.$$

Given a $d \times d$ expansive matrix M, define the *dilation operator* D and the *shift operator* T_k on $L^2(\mathbb{R}^d)$ for each $k \in \mathbb{Z}^d$ by

$$Df(\cdot) := |\det M|^{1/2} f(M\cdot), \qquad T_k f(\cdot) := f(\cdot - k)$$

for $f \in L^2(\mathbb{R}^d)$. Obviously, they are both unitary operators on $L^2(\mathbb{R}^d)$. Given f, $g \in L^2(\mathbb{R}^d)$, define the *bracket product* of f and g by

$$[f, g](\cdot) := \sum_{k \in \mathbb{Z}^d} \hat{f}(\cdot + k)\overline{\hat{g}(\cdot + k)}$$

on $\mathbb{R}^d$. In particular, write

$$\Phi_f := [f, f].$$

We denote by $V_0(f)$ the closed linear span of $\{T_k f : k \in \mathbb{Z}^d\}$, that is,

$$V_0(f) := \overline{\operatorname{span}}\{T_k f : k \in \mathbb{Z}^d\}.$$

A function f is called a *frame function* with frame bounds A and B if $\{T_k f : k \in \mathbb{Z}^d\}$ is a frame for $V_0(f)$ with frame bounds A and B and is called a *Parseval frame function* if $\{T_k f : k \in \mathbb{Z}^d\}$ is a Parseval frame for $V_0(f)$. A function f is said to be *M-refinable* if there exists a $\mathbb{Z}^d$-periodic measurable function m_f such that

$$\hat{f}(M^* \cdot) = m_f(\cdot)\hat{f}(\cdot)$$

on $\mathbb{R}^d$, where m_f is called the *symbol* of f. We always denote by m_f its symbol for an arbitrary M-refinable function f. A function f is called an *M-refinable frame function* (*M-refinable Parseval frame function*) if it is M-refinable and a frame function (a Parseval frame function). A closed subspace X of $L^2(\mathbb{R}^d)$ is called a *reducing subspace* if $DX = X$ and $T_k X = X$ for each $k \in \mathbb{Z}^d$. The following proposition provides us with a characterization of reducing subspaces.

PROPOSITION 1 ([7, THEOREM 1])

Given a $d \times d$ expansive matrix M, a closed subspace X of $L^2(\mathbb{R}^d)$ is a reducing subspace if and only if $X = FL^2(\Omega)$ for some measurable set Ω in $\mathbb{R}^d$ satisfying $\Omega = M^*\Omega$.

By Proposition 1, a nonzero reducing subspace X always corresponds to a set $\Omega \subset \mathbb{R}^d$ with nonzero measure for which

$$M^*\Omega = \Omega \qquad \text{and} \qquad X = FL^2(\Omega).$$

So, to be specific, we denote a reducing subspace by $FL^2(\Omega)$ instead of X. In particular, $L^2(\mathbb{R}^d)$ is a reducing subspace of $L^2(\mathbb{R}^d)$, and $FL^2([0,\infty))$ (Hardy space) is also a reducing subspace of $L^2(\mathbb{R})$.

Given $\psi \in FL^2(\Omega)$, ψ is called a *frame wavelet* for $FL^2(\Omega)$ if $\{D^j T_k \psi : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ is a frame for $FL^2(\Omega)$; ψ is called an *s-frame wavelet* for $FL^2(\Omega)$ if $\hat{\psi} = \chi_E$ for some measurable set E in $\mathbb{R}^d$, and $\{D^j T_k \psi : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ is a Parseval frame for $FL^2(\Omega)$.

DEFINITION 1

Given a $d \times d$ expansive matrix M and a reducing subspace $FL^2(\Omega)$ of $L^2(\mathbb{R}^d)$, a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $FL^2(\Omega)$ is called a frame multiresolution analysis (FMRA) associated with M for $FL^2(\Omega)$ if the following conditions are satisfied:

(i) $V_j \subset V_{j+1}$ for $j \in \mathbb{Z}$;
(ii) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = FL^2(\Omega)$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
(iii) $V_j = D^j V_0$ for $j \in \mathbb{Z}$;
(iv) there exists $\phi \in FL^2(\Omega)$ such that $\{T_k \phi : k \in \mathbb{Z}^d\}$ is a frame for V_0.

Herein, we call ϕ a *frame-scaling function* of the FMRA. From the definition, we know that ϕ is an M-refinable frame function satisfying

$$V_0 = V_0(\phi), \qquad V_j = D^j V_0 \quad \text{for } j \in \mathbb{Z}. \tag{2}$$

So we also say that ϕ generates the FMRA. Given an FMRA $\{V_j\}_{j\in\mathbb{Z}}$ for $FL^2(\Omega)$, we always denote by W_j the orthogonal complement of V_j in V_{j+1} for $j \in \mathbb{Z}$. Then $W_j = D^j W_0$ for $j \in \mathbb{Z}$. A function ψ in $FL^2(\Omega)$ is called an FMRA *frame wavelet* for $FL^2(\Omega)$ if there exists an FMRA $\{V_j\}_{j\in\mathbb{Z}}$ for $FL^2(\Omega)$ such that $\psi \in W_0$ and that ψ is a frame wavelet for $FL^2(\Omega)$, which is equivalent to $\{T_k \psi : k \in \mathbb{Z}^d\}$ being a frame for W_0 since $FL^2(\Omega) = \bigoplus_{j\in\mathbb{Z}} W_j$.

Ron and Shen in [12] investigated the construction of frame wavelets for $L^2(\mathbb{R}^d)$ in a general multiresolutional analysis (MRA) setting. Benedetto and Li in [1] obtained an explicit expression of FMRA frame wavelets for $L^2(\mathbb{R})$ when $M = 2$. In [9], an explicit expression of FMRA frame wavelets for $L^2(\mathbb{R}^2)$ was established for four classes of 2×2 expansive matrices with determinant ± 2. Dai, Diao, Gu, and Han in [7] characterized s-frame wavelets for a general $FL^2(\Omega)$. Lian and Li in [11] discussed one-dimensional FMRA frame wavelets for $FL^2(\Omega)$ when $M = 2$. This article addresses multivariate FMRAs and FMRA frame wavelets associated with a general expansive matrix in the setting of reducing subspaces.

In Section 2, we investigate frame-scaling functions generating an FMRA for a reducing subspace $FL^2(\Omega)$ related to a general expansive matrix M. A characterization and some sufficient conditions for such frame-scaling functions are established. It is also proved that an arbitrary reducing subspace must admit an FMRA. Section 3 is devoted to the case $|\det M| = 2$. We characterize FMRAs admitting a single FMRA frame wavelet for a reducing subspace and obtain an explicit expression of FMRA frame wavelet. Furthermore, we discuss the relation between s-frame wavelets and FMRA frame wavelets. These results are also new in the setting of $L^2(\mathbb{R}^d)$.

2. The characterization of frame-scaling functions generating an FMRA for $FL^2(\Omega)$

Given a $d \times d$ expansive matrix M and a reducing subspace $FL^2(\Omega)$ of $L^2(\mathbb{R}^d)$. In this section, we characterize frame-scaling functions generating FMRAs, give a sufficient condition for such frame-scaling functions in the setting of $FL^2(\Omega)$, and discuss how to obtain an FMRA for a reducing subspace from an FMRA for $L^2(\mathbb{R}^d)$. Furthermore, we prove that an arbitrary reducing subspace of $L^2(\mathbb{R}^d)$ always admits an FMRA. For this purpose, we first introduce the notion of quasi norm associated with M.

DEFINITION 2

Given a $d \times d$ expansive matrix M, a nonnegative function ρ defined on $\mathbb{R}^d$ is called a quasi norm associated with M if the following conditions hold:

(i) $\rho(x) = 0$ if and only if $x = 0$;

(ii) there exists a constant c_1 such that $\rho(x+y) \le c_1(\rho(x) + \rho(y))$ for x, $y \in \mathbb{R}^d$;

(iii) ρ is continuous on $\mathbb{R}^d$ and smooth on $\mathbb{R}^d \setminus \{0\}$;

(iv) there exists a constant $\delta > 1$ such that $\rho(M^*\cdot) = \delta\rho(\cdot)$;

(v) there exist positive constants c, α_1, α_2, β_1, β_2 such that

$$c^{-1}|x|^{\alpha_1} \le \rho(x) \le c|x|^{\beta_1} \quad \text{when } |x| \le 1,$$
$$c^{-1}|x|^{\alpha_2} \le \rho(x) \le c|x|^{\beta_2} \quad \text{when } |x| \ge 1.$$

The existence of such a quasi norm can be found in [13]. It is easy to check that, for $\alpha > 0$, ρ^α is also a quasi norm if ρ is a quasi norm.

For an arbitrary $\phi \in L^2(\mathbb{R}^d)$, define V_j as in (2) for $j \in \mathbb{Z}$. Then $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ by [3, Theorem 1.1]. So, we have the following proposition by Definition 1.

PROPOSITION 2

Given a $d \times d$ expansive matrix M and a reducing subspace $FL^2(\Omega)$ of $L^2(\mathbb{R}^d)$, let $\phi \in FL^2(\Omega)$, and let V_j be defined as in (2). Then $\{V_j\}_{j\in\mathbb{Z}}$ is an FMRA *associated with M for $FL^2(\Omega)$ if and only if*

(i) *ϕ is an M-refinable frame function;*

(ii) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = FL^2(\Omega)$.

By [5, Theorem 2.1], we have the following.

PROPOSITION 3

A function ϕ in $L^2(\mathbb{R}^d)$ is a frame function with frame bounds A and B if and only if $A\chi_{\mathrm{supp}(\Phi_\phi)}(\cdot) \le \Phi_\phi(\cdot) \le B\chi_{\mathrm{supp}(\Phi_\phi)}(\cdot)$ on $\mathbb{R}^d$. In particular, ϕ is a Parseval frame function if and only if $\Phi_\phi(\cdot) = 1$ on $\mathrm{supp}(\Phi_\phi)$.

PROPOSITION 4

Given a $d\times d$ expansive matrix M and a set $\{\varepsilon_i : 0 \le i \le |\det M| - 1\}$ of representatives of distinct cosets in $\mathbb{Z}^d/M^*\mathbb{Z}^d$, let $f,\ g \in L^2(\mathbb{R}^d)$ be two M-refinable functions. Then

$$\begin{aligned}[f,g](M^*\cdot) = \sum_{i=0}^{|\det M|-1} & m_f\big(\cdot + (M^*)^{-1}\varepsilon_i\big) \\ & \times \overline{m_g\big(\cdot + (M^*)^{-1}\varepsilon_i\big)}[f,g]\big(\cdot + (M^*)^{-1}\varepsilon_i\big)\end{aligned} \tag{3}$$

on $\mathbb{R}^d$. In particular,

$$\Phi_f(M^*\cdot) = \sum_{i=0}^{|\det M|-1} \big|m_f\big(\cdot + (M^*)^{-1}\varepsilon_i\big)\big|^2 \Phi_f\big(\cdot + (M^*)^{-1}\varepsilon_i\big) \tag{4}$$

on $\mathbb{R}^d$.

Proof

Note that

$$\mathbb{Z}^d = \bigcup_{i=0}^{|\det M|-1} (M^*\mathbb{Z}^d + \varepsilon_i),$$

where the union is a disjoint union. By the refinable properties of f and g, we can obtain the proposition, and we omit the details here. □

PROPOSITION 5

Given a $d\times d$ expansive matrix M, for an arbitrary M-refinable frame function f, the symbol m_f can be chosen such that $m_f \in L^\infty(\mathbb{T}^d)$.

Proof

Suppose that f is an M-refinable frame function with frame bounds A and B. By Propositions 4 and 3, we have

$$\begin{aligned}\Phi_f(M^*\cdot) &= |m_f(\xi)|^2\Phi_f(\cdot) + \sum_{i=1}^{|\det M|-1} \big|m_f\big(\cdot + (M^*)^{-1}\varepsilon_i\big)\big|^2 \Phi_f\big(\cdot + (M^*)^{-1}\varepsilon_i\big) \\ &\ge |m_f(\cdot)|^2\Phi_f(\cdot),\end{aligned}$$

which implies that

$$A|m_f(\cdot)|^2 \le B$$

on $\mathrm{supp}(\Phi_f)$. Choose $m_f(\cdot) = 0$ on $\mathbb{R}^d \setminus \mathrm{supp}(\Phi_f)$. Then

$$\hat{f}(M^*\cdot) = m_f(\cdot)\hat{f}(\cdot)$$

on $\mathbb{R}^d$ with $m_f \in L^\infty(\mathbb{T}^d)$. The proof is completed. $\square$

By an argument similar to that in [11, Theorem 1], we have the following proposition.

PROPOSITION 6

Given a $d \times d$ expansive matrix M and an M-refinable function ϕ, define V_j as in (2) for $j \in \mathbb{Z}$. Then

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = FL^2(\Omega)$$

with $\Omega = \bigcup_{j\in\mathbb{Z}} (M^*)^j \,\mathrm{supp}(\hat{\phi})$.

By Propositions 2, 5, and 6, we have the following theorem.

THEOREM 1

Given a $d \times d$ expansive matrix M and a reducing subspace $FL^2(\Omega)$ of $L^2(\mathbb{R}^d)$, let $\phi \in FL^2(\Omega)$, and let V_j be defined as in (2). Then $\{V_j\}_{j\in\mathbb{Z}}$ is an FMRA *associated with M for $FL^2(\Omega)$ if and only if*

(i) *ϕ is an M-refinable frame function with $m_\phi \in L^\infty(\mathbb{T}^d)$;*
(ii) *$\bigcup_{j\in\mathbb{Z}} (M^*)^j \,\mathrm{supp}(\hat{\phi}) = \Omega$.*

REMARK 1

In particular, if $\Omega = \mathbb{R}^d$ in the theorem, we obtain a characterization of $\{V_j\}_{j\in\mathbb{Z}}$ as an FMRA associated with M for $L^2(\mathbb{R}^d)$.

THEOREM 2

Given a $d \times d$ expansive matrix M, let E be a bounded set with nonzero measure satisfying $E \subset M^*E$, let $\Omega = \bigcup_{j\in\mathbb{Z}} (M^*)^j E$, and let ϕ be a function in $L^2(\mathbb{R}^d)$ such that, for some $0 < A \le B < \infty$,

$$A\chi_E(\cdot) \le |\hat{\phi}(\cdot)| \le B\chi_E(\cdot) \tag{5}$$

on $\mathbb{R}^d$. Then ϕ generates an FMRA *associated with M for $FL^2(\Omega)$ if and only if*

$$\big((M^*)^{-1}E\big)^{\sim} \cap E = (M^*)^{-1}E, \tag{6}$$

$$\begin{gathered} T_k\Big(\frac{\hat{\phi}(M^*\cdot)}{\hat{\phi}(\cdot)}\Big) = \frac{\hat{\phi}(M^*\cdot)}{\hat{\phi}(\cdot)} \\ \text{on } \big((M^*)^{-1}E\big) \cap \big((M^*)^{-1}E + k\big) \text{ for } k \in \mathbb{Z}^d. \end{gathered} \tag{7}$$

Proof

Since E is bounded, there exists $0 < A' \le B' < \infty$ such that $A'\chi_{(E)^{\sim}}(\cdot) \le \Phi_\phi(\cdot) \le B'\chi_{(E)^{\sim}}(\cdot)$ by (5), and thus ϕ is a frame function by Proposition 3. So, by Theorem 1, ϕ generates an FMRA for $FL^2(\Omega)$ if and only if ϕ is M-refinable. Note that $E \subset M^*E$. By (5), it is equivalent to the fact that there exists $m_\phi \in L^\infty(\mathbb{T}^d)$ such that

$$m_\phi(\cdot) = \begin{cases} \frac{\hat\phi(M^*\cdot)}{\hat\phi(\cdot)} & \text{on } (M^*)^{-1}E, \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

on E. Therefore, we need to prove only that (8) holds for some $m_\phi \in L^\infty(\mathbb{T}^d)$ if and only if (6) and (7) hold.

Necessity. Since m_ϕ is $\mathbb{Z}^d$-periodic, from (8) we have

$$\big(E \setminus (M^*)^{-1}E\big) \cap \big((M^*)^{-1}E\big)^{\sim} = \emptyset.$$

So, $((M^*)^{-1}E)^{\sim} \cap E \subset (M^*)^{-1}E$. Also, observing that $(M^*)^{-1}E \subset E$ leads to (6). For any $\xi \in ((M^*)^{-1}E) \cap ((M^*)^{-1}E + k)$, we have

$$T_k\Big(\frac{\hat\phi(M^*\xi)}{\hat\phi(\xi)}\Big) = T_k\big(m_\phi(\xi)\big) = m_\phi(\xi) = \frac{\hat\phi(M^*\xi)}{\hat\phi(\xi)},$$

which gives (7).

Sufficiency. Define

$$m_\phi(\cdot) = \begin{cases} T_k\big(\frac{\hat\phi(M^*\cdot)}{\hat\phi(\cdot)}\big) & \text{on } (M^*)^{-1}E + k \text{ with } k \in \mathbb{Z}^d, \\ 0 & \text{otherwise,} \end{cases}$$

which is well defined by (7) and the fact $E \subset M^*E$. Indeed, since $E \subset M^*E$, we have $\xi - k \in (M^*)^{-1}E \subset E$ for $\xi \in (M^*)^{-1}E + k$. So $\hat\phi(\xi - k) \neq 0$; therefore $T_k(\frac{\hat\phi(M^*\cdot)}{\hat\phi(\cdot)})$ makes sense. For $\xi \in ((M^*)^{-1}E + k_1) \cap ((M^*)^{-1}E + k_2)$ with $k_1 \neq k_2$, let $\xi - k_1 = \eta$; then $\eta \in (M^*)^{-1}E$ and $\eta - (k_1 - k_2) \in (M^*)^{-1}E$. So $\eta \in ((M^*)^{-1}E) \cap ((M^*)^{-1}E + (k_1 - k_2))$, which implies that

$$\frac{\hat\phi(M^*\eta)}{\hat\phi(\eta)} = T_{k_2-k_1}\Big(\frac{\hat\phi(M^*\eta)}{\hat\phi(\eta)}\Big) = \frac{\hat\phi\big(M^*(\eta - (k_2 - k_1))\big)}{\hat\phi(\eta - (k_2 - k_1))}$$

by (7). It follows that

$$\frac{\hat\phi(M^*(\xi - k_1))}{\hat\phi(\xi - k_1)} = \frac{\hat\phi(M^*(\xi - k_2))}{\hat\phi(\xi - k_2)},$$

namely,

$$T_{k_1}\Big(\frac{\hat\phi(M^*\xi)}{\hat\phi(\xi)}\Big) = T_{k_2}\Big(\frac{\hat\phi(M^*\xi)}{\hat\phi(\xi)}\Big).$$

It is easy to check that m_ϕ is $\mathbb{Z}^d$-periodic, so $m_\phi \in L^\infty(\mathbb{T}^d)$ due to (5). From (6), we can obtain (8). Indeed, on E,

$$m_\phi(\cdot) = \begin{cases} T_k\big(\frac{\hat{\phi}(M^*\cdot)}{\hat{\phi}(\cdot)}\big) & \text{on } (M^*)^{-1}E + k \text{ with } k \in \mathbb{Z}^d, \\ 0 & \text{otherwise,} \end{cases}$$

$$(9) \qquad = \begin{cases} T_k\big(\frac{\hat{\phi}(M^*\cdot)}{\hat{\phi}(\cdot)}\big) & \text{on } E \cap ((M^*)^{-1}E + k) \text{ with } k \in \mathbb{Z}^d, \\ 0 & \text{otherwise.} \end{cases}$$

By (6), we have

$$E \setminus \bigcup_{k\in\mathbb{Z}^d} \big(((M^*)^{-1}E + k) \cap E\big) = E \setminus (M^*)^{-1}E,$$

from which it follows that $m_\phi(\cdot) = 0$ on $E \setminus (M^*)^{-1}E$ by (9). However, $m_\phi(\cdot) = \frac{\hat{\phi}(M^*\cdot)}{\hat{\phi}(\cdot)}$ on $(M^*)^{-1}E$. Equation (8) therefore holds. □

THEOREM 3

Given a $d \times d$ expansive matrix M. An arbitrary reducing subspace $FL^2(\Omega)$ of $L^2(\mathbb{R}^d)$ admits an FMRA *associated with M.*

Proof

Suppose that ρ is a quasi norm associated with M as in Definition 2. Define

$$G_0 = \{\xi \in \mathbb{R}^d : \rho(\xi) \le 1\};$$

then

$$M^*G_0 = \{\eta \in \mathbb{R}^d : \rho(\eta) \le \delta\}.$$

However, by (iv) and (v) in Definition 2, we have $G_0 \subset M^*G_0$, and G_0 is bounded, which implies that $(M^*)^{j_0}G_0 \subset \mathbb{T}^d$ for some $j_0 \in \mathbb{Z}$. Again by the continuity of ρ and the fact that $\rho(0) = 0$, G_0 contains a neighborhood of the origin zero. Take $G = (M^*)^{j_0}G_0$. Then

$$G \subset M^*G \qquad \text{and} \qquad \mathbb{R}^d = \bigcup_{j\in\mathbb{Z}} (M^*)^j G.$$

Define ϕ_0 via its Fourier transform by $\hat{\phi}_0 = \chi_G$. It is easy to check that (6) and (7) hold for G. Then ϕ_0 generates an FMRA associated with M for $L^2(\mathbb{R}^d)$ by Theorem 2. Define ϕ via its Fourier transform by $\hat{\phi} = \chi_E$ with $E = G \cap \Omega$. Then

$$E \subset M^*E \qquad \text{and} \qquad \Omega = \bigcup_{j\in\mathbb{Z}} (M^*)^j E,$$

and (6) and (7) hold for E. So ϕ generates an FMRA associated with M for $FL^2(\Omega)$ by Theorem 2. The proof is completed. □

THEOREM 4

Given a $d\times d$ expansive matrix M and a measurable set E in $\mathbb{R}^d$ satisfying $E\subset M^*E$, let $\Omega=\bigcup_{j\in\mathbb{Z}}(M^*)^jE$. Assume that $f\in L^2(\mathbb{R}^d)$ generates an FMRA *associated with M for $L^2(\mathbb{R}^d)$ and that $|E\cap G|>0$, where $G=\operatorname{supp}(\hat{f})$. Define ϕ via its Fourier transform by*

$$\hat{\phi}(\cdot)=\hat{f}(\cdot)\chi_E(\cdot) \tag{10}$$

on $\mathbb{R}^d$. Then ϕ generates an FMRA *associated with M for $FL^2(\Omega)$ if and only if*

(i) $(M^*)^{-1}(E\cap G)=((M^*)^{-1}(E\cap G))^{\sim}\cap(E\cap G)$;
(ii) *there exists a positive constant C such that $\Phi_\phi(\cdot)\geq C$ on $(E\cap G)^{\sim}$.*

Proof

Since f generates an FMRA for $L^2(\mathbb{R}^d)$, in view of Remark 1, we have $\bigcup_{j\in\mathbb{Z}}(M^*)^jG=\mathbb{R}^d$. It follows that

$$\begin{aligned}\bigcup_{j\in\mathbb{Z}}(M^*)^j(E\cap G)&=\Big(\bigcup_{j\in\mathbb{Z}}(M^*)^jE\Big)\cap\Big(\bigcup_{j\in\mathbb{Z}}(M^*)^jG\Big)\\&=\bigcup_{j\in\mathbb{Z}}(M^*)^jE=\Omega,\end{aligned}$$

where we have used the fact that $E\subset M^*E$ and $G\subset M^*G$ due to the refinable property of f. Note that $\operatorname{supp}(\hat{\phi})=E\cap G$. By Theorem 1, to finish the proof, we need to prove only that ϕ is M-refinable with $m_\phi\in L^\infty(\mathbb{T}^d)$ if and only if

$$(M^*)^{-1}(E\cap G)=\big((M^*)^{-1}(E\cap G)\big)^{\sim}\cap(E\cap G). \tag{11}$$

Suppose that there exists $m_\phi\in L^\infty(\mathbb{T}^d)$ such that

$$\hat{\phi}(M^*\cdot)=m_\phi(\cdot)\hat{\phi}(\cdot) \tag{12}$$

on $\mathbb{R}^d$. It follows that $((M^*)^{-1}(E\cap G))^{\sim}\subset\operatorname{supp}(m_\phi)$ by $\mathbb{Z}^d$-periodicity of m_ϕ, which together with (12) implies that

$$\begin{aligned}&\big((M^*)^{-1}(E\cap G)\big)^{\sim}\cap(E\cap G)\subset\operatorname{supp}(m_\phi)\cap(E\cap G)\\&\quad=\operatorname{supp}\big(\hat{\phi}(M^*\cdot)\big)\\&\quad=(M^*)^{-1}(E\cap G).\end{aligned}$$

However, we also have $(M^*)^{-1}(E\cap G)\subset E\cap G$ by (12). Equation (11) therefore follows.

Conversely, suppose that (11) holds. Since $\operatorname{supp}(\hat{\phi})=E\cap G$, we have

$$\hat{\phi}(M^*\cdot)=\hat{f}(M^*\cdot)\chi_E(M^*\cdot)=\hat{f}(M^*\cdot)\chi_{(M^*)^{-1}(E\cap G)}(\cdot)$$

on $\mathbb{R}^d$ by (10), which together with (11) leads to

$$\hat{\phi}(M^*\cdot)=\hat{f}(M^*\cdot)\chi_{((M^*)^{-1}(E\cap G))^{\sim}}(\cdot)\chi_{E\cap G}(\cdot)\quad\text{on }\mathbb{R}^d.$$

Since $f \in L^2(\mathbb{R}^d)$ generates an FMRA for $L^2(\mathbb{R}^d)$, by Remark 1 there exists $m_f \in L^\infty(\mathbb{T}^d)$ such that $\hat{f}(M^*\cdot) = m_f(\cdot)\hat{f}(\cdot)$ on $\mathbb{R}^d$. So

$$\hat{\phi}(M^*\cdot) = m_f(\cdot)\chi_{((M^*)^{-1}(E\cap G))\tilde{}}(\cdot)\hat{f}(\cdot)\chi_E(\cdot) = m_f(\cdot)\chi_{((M^*)^{-1}(E\cap G))\tilde{}}(\cdot)\hat{\phi}(\cdot)$$

on $\mathbb{R}^d$, and consequently, ϕ is M-refinable with $m_\phi \in L^\infty(\mathbb{T}^d)$. The proof is completed. □

3. The construction of FMRA frame wavelets

This section is devoted to the construction of FMRA frame wavelets. In the setting of reducing subspace $FL^2(\Omega)$ with $|\det M| = 2$, we obtain an explicit expression of FMRA frame wavelets. The relation between s-frame wavelets and FMRA-frame wavelets is also studied.

PROPOSITION 7

Given a $d \times d$ expansive matrix M, let $f \in L^2(\mathbb{R}^d)$ be an M-refinable frame function. Define $f^\sharp$ via its Fourier transform by

$$\hat{f^\sharp}(\xi) := \begin{cases} \frac{\hat{f}(\xi)}{\sqrt{\Phi_f(\xi)}} & \xi \in \mathrm{supp}(\hat{f}), \\ 0 & \textit{otherwise}. \end{cases}$$

Then $f^\sharp$ is an M-refinable Parseval frame function, and $V_0(f) = V_0(f^\sharp)$.

Proof

By the definition of $f^\sharp$, we have $f^\sharp \in V_0(f)$ and $\mathrm{supp}(\hat{f^\sharp}) = \mathrm{supp}(\hat{f})$, which implies that $V_0(f) = V_0(f^\sharp)$ by [2, Corollary 2.2]. However, $f^\sharp$ is a Parseval frame function by Proposition 3. So, to finish the proof, we only need to prove that $f^\sharp$ is M-refinable. Since f is M-refinable, we have $(M^*)^{-1}\mathrm{supp}(\hat{f}) \subset \mathrm{supp}(\hat{f})$, and $m_f(\cdot) = 0$ on $\mathrm{supp}(\hat{f}) \setminus ((M^*)^{-1}\mathrm{supp}(\hat{f}))$. It follows that $f^\sharp$ is M-refinable with

$$m_{f^\sharp}(\cdot) = \sqrt{\frac{\Phi_f(\cdot)}{\Phi_f(M^*\cdot)}}\, m_f(\cdot)\chi_{((M^*)^{-1}\mathrm{supp}(\hat{f}))\tilde{}}(\cdot).$$

The proof is completed. □

PROPOSITION 8

Given a $d \times d$ expansive matrix M and an M-refinable frame function f, define $V_0(f)$ and $V_1(f)$ as in (2) and $W_0(f)$ as the orthogonal complement of $V_0(f)$ in $V_1(f)$. Let $\{\varepsilon_i : 0 \le i \le |\det M| - 1\}$ be a set of representatives of distinct cosets in $\mathbb{Z}^d/M^*\mathbb{Z}^d$. Then, for an arbitrary $g \in V_1(f)$, $g \in W_0(f)$ if and only if

$$\sum_{i=0}^{|\det M|-1} m_g\big(\cdot + (M^*)^{-1}\varepsilon_i\big)\overline{m_f\big(\cdot + (M^*)^{-1}\varepsilon_i\big)}\Phi_f\big(\cdot + (M^*)^{-1}\varepsilon_i\big) = 0$$

on $\mathbb{R}^d$, where m_g is a $\mathbb{Z}^d$-periodic measurable function satisfying $\hat{g}(\cdot) = m_g((M^*)^{-1}\cdot)\hat{f}((M^*)^{-1}\cdot)$.

Proof
$g \in W_0(f)$ if and only if $g \perp V_0(f)$, which, by a simple argument, is equivalent to $[g, f] = 0$. However, by a simple computation, we have

$$[g,f](\cdot) = \sum_{i=0}^{|\det M|-1} m_g\big((M^*)^{-1}\cdot + (M^*)^{-1}\varepsilon_i\big) \times \overline{m_f\big((M^*)^{-1}\cdot + (M^*)^{-1}\varepsilon_i\big)}\,\Phi_f\big((M^*)^{-1}\cdot + (M^*)^{-1}\varepsilon_i\big),$$

which leads to the proposition. The proof is completed. □

Now we turn to FMRA frame wavelet construction in the setting of reducing subspaces when $|\det M| = 2$. For this purpose, we quote the following proposition.

PROPOSITION 9 ([10, LEMMA 4.2])
Given a $d \times d$ expansive matrix M with $|\det M| = 2$, let $\{0, \delta\}$ be a set of representatives of distinct cosets in $\mathbb{Z}^d/M^*\mathbb{Z}^d$. Then there exists $1 \le k_0 \le d$ such that $(2(M^*)^{-1}\delta)_{k_0}$ is an odd number.

THEOREM 5
Given a $d \times d$ expansive matrix M with $|\det M| = 2$ and an M-refinable frame function ϕ, let $\{0, \delta\}$ be a set of representatives of distinct cosets in $\mathbb{Z}^d/M^*\mathbb{Z}^d$. Write $F = \operatorname{supp}(\Phi_\phi)$, and write $F_\delta = F - (M^*)^{-1}\delta$. Define $V_0(\phi)$ and $V_1(\phi)$ as in (2), and define $W_0(\phi)$ as the orthogonal complement of $V_0(\phi)$ in $V_1(\phi)$. Then there exists $\psi \in W_0(\phi)$ such that $\{T_k\psi : k \in \mathbb{Z}^d\}$ is a frame for $W_0(\phi)$ if and only if $F \cap F_\delta \subset (M^*)^{-1}F$.

Proof
By Proposition 7 and its proof, without loss of generality, we assume that ϕ is an M-refinable Parseval frame function, and we only need to prove that, under such assumption, there exists $\psi \in W_0(\phi)$ such that $\{T_k\psi : k \in \mathbb{Z}^d\}$ forms a Parseval frame for $W_0(\phi)$ if and only if $F \cap F_\delta \subset (M^*)^{-1}F$.

For necessity, suppose that $\{T_k\psi : k \in \mathbb{Z}^d\}$ is a Parseval frame for $W_0(\phi)$. Write $\operatorname{supp}(\Phi_\psi) = E$. Then $\Phi_\psi = \chi_E$ by Proposition 3. Let $\{0, \sigma\}$ be a set of representatives of distinct cosets in $\mathbb{Z}^d/M\mathbb{Z}^d$, and define

$$\phi_1(\cdot) = \sqrt{2}\phi(M\cdot), \qquad \phi_2(\cdot) = \sqrt{2}\phi(M\cdot - \sigma). \tag{13}$$

Then $\{T_k\phi_i(\cdot) : i = 1, 2, k \in \mathbb{Z}^d\}$ is a Parseval frame for $V_1(\phi)$. Note that $\{T_k\phi, T_k\psi : k \in \mathbb{Z}^d\}$ is also a Parseval frame for $V_1(\phi)$. So, by [4, Lemma 2.3], we have

$$\Phi_{\phi_1}(\cdot) + \Phi_{\phi_2}(\cdot) = \Phi_\phi(\cdot) + \Phi_\psi(\cdot). \tag{14}$$

By a simple computation, (14) can be rewritten as

$$\Phi_\phi\big((M^*)^{-1}\cdot\big) + \Phi_\phi\big((M^*)^{-1}(\cdot + \delta)\big) = \Phi_\phi(\cdot) + \Phi_\psi(\cdot),$$

which implies that

$$\chi_{M^*F}(\cdot)+\chi_{M^*F_\delta}(\cdot)=\chi_F+\chi_E$$

by Proposition 3. It follows that $F\cap F_\delta\subset (M^*)^{-1}F$.

For sufficiency, suppose $F\cap F_\delta\subset (M^*)^{-1}F$. Define ψ via its Fourier transform by

$$\hat{\psi}(\cdot)=m_\psi\big((M^*)^{-1}\cdot\big)\hat{\phi}\big((M^*)^{-1}\cdot\big)$$

with

$$m_\psi(\xi)=\begin{cases} e^{2\pi i\xi_{k_0}}\overline{m_\phi(\xi+(M^*)^{-1}\delta)}, & \xi\in F\cap F_\delta,\\ 1, & \xi\in (F\setminus F_\delta)\setminus (M^*)^{-1}F,\\ 0 & \text{otherwise,}\end{cases}$$

where k_0 is as in Proposition 9. Next, we divide the proof into three steps to prove that ψ is as desired.

Step 1: $\psi\in W_0(\phi)$. By Propositions 3 and 8, we only need to prove that

$$\overline{m_\phi(\cdot)}m_\psi(\cdot)\chi_F(\cdot)+\overline{m_\phi\big(\cdot+(M^*)^{-1}\delta\big)}m_\psi\big(\cdot+(M^*)^{-1}\delta\big)\chi_{F_\delta}(\cdot)=0. \tag{15}$$

It clearly holds on $\mathbb{R}^d\setminus(F\cup F_\delta)$. So we only need to prove that (15) holds on $F\cup F_\delta$.

Applying Propositions 3 and 4 to ϕ here, we have

$$\chi_{(M^*)^{-1}F}(\cdot)=|m_\phi(\cdot)|^2\chi_F(\cdot)+\big|m_\phi\big(\cdot+(M^*)^{-1}\delta\big)\big|^2\chi_{F_\delta}(\cdot), \tag{16}$$

which implies that

$$(M^*)^{-1}F\subset F\cup F_\delta, \tag{17}$$

$$m_\phi(\cdot)=0\quad \text{on } (F\setminus F_\delta)\setminus (M^*)^{-1}F, \tag{18}$$

$$m_\phi\big(\cdot+(M^*)^{-1}\delta\big)=0\quad \text{on } (F_\delta\setminus F)\setminus (M^*)^{-1}F. \tag{19}$$

By (17),

$$\begin{aligned} F\cup F_\delta=(F\cap F_\delta)&\cup\big(((M^*)^{-1}F)\setminus(F\cap F_\delta)\big)\\ &\cup\big((F\setminus F_\delta)\setminus (M^*)^{-1}F\big)\cup\big((F_\delta\setminus F)\setminus (M^*)^{-1}F\big).\end{aligned}$$

By (18) and (19), (15) holds on $((F\setminus F_\delta)\setminus (M^*)^{-1}F)\cup((F_\delta\setminus F)\setminus (M^*)^{-1}F)$. Also, observing that $((M^*)^{-1}F)\setminus(F\cap F_\delta)$ is $(M^*)^{-1}\delta$-shift invariant, we have

$$m_\psi(\cdot)=m_\psi\big(\cdot+(M^*)^{-1}\delta\big)=0$$

on $((M^*)^{-1}F)\setminus(F\cap F_\delta)$, which gives (15) on this set. On $F\cap F_\delta$, a simple computation leads to (15) due to the fact that $e^{2\pi i(\xi+(M^*)^{-1}\delta)_{k_0}}=-e^{2\pi i\xi_{k_0}}$. (15) therefore follows.

Step 2: *ψ is a Parseval frame function.* By a simple computation and Proposition 3, we have

$$\Phi_\psi(\cdot)=\big|m_\psi\big((M^*)^{-1}\cdot\big)\big|^2\chi_{M^*F}(\cdot)+\big|m_\psi\big((M^*)^{-1}\cdot+(M^*)^{-1}\delta\big)\big|^2\chi_{M^*F_\delta}(\cdot), \tag{20}$$

which together with the definition of m_ψ implies that

$$\Phi_\psi = \chi_E,$$

where $E = M^*((F \cap F_\delta) \cup [(F \cup F_\delta) \setminus (M^*)^{-1}F])$. So ψ is a Parseval frame function by Proposition 3.

Step 3: *$\{T_k\psi : k \in \mathbb{Z}^d\}$ is complete in $W_0(\phi)$.* Suppose $f \in W_0(\phi)$, $f \perp V_0(\psi)$. We only need to prove that $f = 0$. Since $f \in W_0(\phi)$, we have $f \perp V_0(\phi)$ and

$$\hat{f}(M^*\cdot) = m_f(\cdot)\hat{\phi}(\cdot) \tag{21}$$

for some $\mathbb{Z}^d$-periodic measurable function m_f. By an argument similar to that in Proposition 8, we have the equation system

$$\overline{m_\phi(\cdot)}m_f(\cdot)\chi_F(\cdot) + \overline{m_\phi\big(\cdot + (M^*)^{-1}\delta\big)}m_f\big(\cdot + (M^*)^{-1}\delta\big)\chi_{F_\delta}(\cdot) = 0, \tag{22}$$

$$\overline{m_\psi(\cdot)}m_f(\cdot)\chi_F(\cdot) + \overline{m_\psi\big(\cdot + (M^*)^{-1}\delta\big)}m_f\big(\cdot + (M^*)^{-1}\delta\big)\chi_{F_\delta}(\cdot) = 0. \tag{23}$$

By (21), it suffices to prove that $m_f = 0$ on F. On $F \cap F_\delta$, the coefficient determinant of the equation system

$$\det\begin{pmatrix} \overline{m_\phi(\xi)} & \overline{m_\phi(\xi + (M^*)^{-1}\delta)} \\ \overline{m_\psi(\xi)} & \overline{m_\psi(\xi + (M^*)^{-1}\delta)} \end{pmatrix} = -e^{-2\pi i \xi_{k_0}} \neq 0,$$

which implies that $m_f(\xi) = 0$. On $(F \setminus F_\delta) \cap (M^*)^{-1}F$, we have $|m_\phi(\cdot)| = 1$. Combined with (22), it follows that $m_f(\cdot) = 0$. On $(F \setminus F_\delta) \setminus (M^*)^{-1}F$, $m_\psi(\cdot) = 1$, which leads to $m_f(\cdot) = 0$ by (23). Therefore, $m_f(\cdot) = 0$ on F. The proof is completed. □

THEOREM 6

Given a $d \times d$ expansive matrix M with $|\det M| = 2$ and a reducing subspace $FL^2(\Omega)$ of $L^2(\mathbb{R}^d)$, let $\{0, \delta\}$ be a set of representatives of distinct cosets in $\mathbb{Z}^d/M^*\mathbb{Z}^d$ with $(2(M^*)^{-1}\delta)_{k_0}$ being odd for some $1 \le k_0 \le d$, let $\{V_j\}_{j\in\mathbb{Z}}$ be an FMRA associated with M for $FL^2(\Omega)$, let ϕ be a frame-scaling function of the FMRA, and let W_0 be the orthogonal complement of V_0 in V_1. Write $F = \operatorname{supp}(\Phi_\phi)$ and $F_\delta = F - (M^*)^{-1}\delta$. Then there exists $\psi \in W_0$ such that $\{T_k\psi : k \in \mathbb{Z}^d\}$ is a frame for W_0 if and only if $F \cap F_\delta \subset (M^*)^{-1}F$. In this case, define ψ via its Fourier transform by $\hat{\psi}(\cdot) = m_\psi((M^*)^{-1}\cdot)\hat{\phi}((M^*)^{-1}\cdot)$ with

$$m_\psi(\xi) = \begin{cases} \lambda e^{2\pi i\mu\xi_{k_0}}\overline{m_\phi(\xi + (M^*)^{-1}\delta)}\Phi_\phi(\xi + (M^*)^{-1}\delta), & \xi \in F \cap F_\delta, \\ a, & \xi \in (F \setminus F_\delta) \setminus (M^*)^{-1}F, \\ b, & \xi \in (F_\delta \setminus F) \setminus (M^*)^{-1}F, \\ 0 & \textit{otherwise}, \end{cases}$$

where a, b, $\lambda \in \mathbb{C}$, $\mu \in \{1, -1\}$, $a\lambda \neq 0$. Then $\{T_k\psi : k \in \mathbb{Z}^d\}$ is a frame for W_0, and thus $\{D^jT_k\psi : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ is a frame for $FL^2(\Omega)$.

Proof

The first part is an immediate consequence of Theorem 5. For the rest, replacing, respectively, (15), (16), (20), (22), and (23) by the following equations, then, by the same procedure as in Theorem 5, we can finish the proof:

$$\begin{aligned}&\overline{m_\phi(\cdot)}m_\psi(\cdot)\Phi_\phi(\cdot)\\&\quad+\overline{m_\phi\big(\cdot+(M^*)^{-1}\delta\big)}m_\psi\big(\cdot+(M^*)^{-1}\delta\big)\Phi_\phi\big(\cdot+(M^*)^{-1}\delta\big)=0,\end{aligned}\tag{24}$$

$$\Phi_\phi(M^*\cdot)=|m_\phi(\cdot)|^2\Phi_\phi(\cdot)+\big|m_\phi\big(\cdot+(M^*)^{-1}\delta\big)\big|^2\Phi_\phi\big(\cdot+(M^*)^{-1}\delta\big),\tag{25}$$

$$\begin{aligned}\Phi_\psi(\cdot)&=\big|m_\psi\big((M^*)^{-1}\cdot\big)\big|^2\Phi_\phi\big((M^*)^{-1}\cdot\big)\\&\quad+\big|m_\psi\big((M^*)^{-1}\cdot+(M^*)^{-1}\delta\big)\big|^2\Phi_\phi\big((M^*)^{-1}\cdot+(M^*)^{-1}\delta\big),\end{aligned}\tag{26}$$

$$\begin{aligned}&m_f(\xi)\overline{m_\phi(\xi)}\Phi_\phi(\xi)\\&\quad+m_f\big(\xi+(M^*)^{-1}\delta\big)\overline{m_\phi\big(\xi+(M^*)^{-1}\delta\big)}\Phi_\phi\big(\xi+(M^*)^{-1}\delta\big)=0,\end{aligned}\tag{27}$$

$$\begin{aligned}&m_f(\xi)\overline{m_\psi(\xi)}\Phi_\phi(\xi)\\&\quad+m_f\big(\xi+(M^*)^{-1}\delta\big)\overline{m_\psi\big(\xi+(M^*)^{-1}\delta\big)}\Phi_\phi\big(\xi+(M^*)^{-1}\delta\big)=0.\end{aligned}\tag{28}$$

The proof is completed. □

In what follows, we focus on the relation between s-frame wavelets and FMRA frame wavelets. For this purpose, we quote the following proposition, which is another statement of [7, Theorem 2].

PROPOSITION 10

Given a $d\times d$ expansive matrix M and a reducing subspace $FL^2(\Omega)$ of $L^2(\mathbb{R}^d)$, a function ψ defined by $\hat{\psi}=\chi_E$ for some measurable set E in $\mathbb{R}^d$ is an s-frame wavelet for $FL^2(\Omega)$ if and only if

(i) $E\cap(E+k)=\emptyset$ *for* $k\in\mathbb{Z}^d\setminus\{0\}$;
(ii) $E\cap(M^*)^jE=\emptyset$ *for* $j\in\mathbb{Z}\setminus\{0\}$;
(iii) $\bigcup_{j\in\mathbb{Z}}(M^*)^jE=\Omega$.

THEOREM 7

Given a $d\times d$ expansive matrix M (not necessarily $|\det M|=2$) and a reducing subspace $FL^2(\Omega)$ of $L^2(\mathbb{R}^d)$. Let E be a measurable set in $\mathbb{R}^d$ satisfying

$$\bigcup_{j\in\mathbb{Z}}(M^*)^jE=\Omega,\tag{29}$$

$$E\cap(M^*)^jE=\emptyset\quad \textit{for } j\in\mathbb{Z}\setminus\{0\},\tag{30}$$

$$(M^*)^{-j}E\subset\mathbb{T}^d\quad \textit{when } j\geq j_0 \textit{ for some } j_0\in\mathbb{Z}.\tag{31}$$

Define ψ via its Fourier transform by $\hat{\psi}(\cdot)=\chi_{(M^*)^{-j_0}E}(\cdot)$ on $\mathbb{R}^d$. Then ψ is an FMRA frame wavelet.

Proof
We use the notation in Theorem 6. Write $E_0 = \bigcup_{j=-\infty}^{-j_0-1} (M^*)^j E$, and define ϕ via its Fourier transform by $\hat{\phi} = \chi_{E_0}$. Then, by (29) and (31),

$$E_0 \subset M^* E_0 \subset \mathbb{T}^d, \tag{32}$$

$$\bigcup_{j\in\mathbb{Z}} (M^*)^j E_0 = \bigcup_{j\in\mathbb{Z}} (M^*)^j E = \Omega. \tag{33}$$

By (32), we have

$$\begin{aligned} (E_0)^{\sim} &\subset (M^*)^{-1}(\mathbb{T}^d + M^*\mathbb{Z}^d), \\ (E_0)^{\sim} - (M^*)^{-1}\delta &\subset (M^*)^{-1}(\mathbb{T}^d + M^*\mathbb{Z}^d - \delta), \end{aligned} \tag{34}$$

which implies that

$$(E_0)^{\sim} \cap \big((E_0)^{\sim} - (M^*)^{-1}\delta\big) = \emptyset. \tag{35}$$

From (32), we also have $\Phi_\phi = \chi_{(E_0)^{\sim}}$, ϕ is M-refinable, and thus ϕ is an M-refinable frame function by Proposition 3. Also in view of (33), ϕ generates an FMRA $\{V_j\}_{j\in\mathbb{Z}}$ for $FL^2(\Omega)$ by Theorem 1, where $V_j = \overline{\operatorname{span}}\{D^j T_k \phi : k \in \mathbb{Z}^d\}$. Note that $F = (E_0)^{\sim}$ and $F_\delta = (E_0)^{\sim} - (M^*)^{-1}\delta$. Choose $m_\psi = \chi_{(F\setminus F_\delta)\setminus (M^*)^{-1}F}$, and define ψ via its Fourier transform by

$$\hat{\psi}(\cdot) = m_\psi\big((M^*)^{-1}\cdot\big)\hat{\phi}\big((M^*)^{-1}\cdot\big). \tag{36}$$

Then, by (35) and Theorem 6, ψ is a frame wavelet associated with the FMRA $\{V_j\}_{j\in\mathbb{Z}}$. From (36), it follows that

$$\hat{\psi} = \chi_{(M^*E_0)\cap((M^*(F\setminus F_\delta))\setminus F)}.$$

Observing that $F = (E_0)^{\sim}$ and $E_0 \subset \mathbb{T}^d$, we have

$$\begin{aligned} (M^*E_0) \cap \big((M^*(F\setminus F_\delta))\setminus F\big) &= (M^*E_0) \setminus \big((M^*(E_0)^{\sim} - \delta) \cup (E_0)^{\sim}\big) \\ &= (M^*E_0) \setminus (E_0)^{\sim} \end{aligned}$$

by (35). However, by (32) and (30), we have

$$(M^*E_0) \setminus (E_0)^{\sim} = (M^*E_0) \setminus E_0 = (M^*)^{-j_0} E.$$

Therefore,

$$\hat{\psi} = \chi_{(M^*)^{-j_0} E}.$$

The proof is completed. □

Given a $d \times d$ expansive matrix M and a reducing subspace $FL^2(\Omega)$ of $L^2(\mathbb{R}^d)$. Suppose ψ defined by $\hat{\psi} = \chi_E$ is an s-frame wavelet for $FL^2(\Omega)$, and E is bounded. Then, by Proposition 10 and Theorem 7, ψ_0 defined by $\hat{\psi}_0 = \chi_{(M^*)^{-j_0} E}$ for some $j_0 \in \mathbb{Z}$ is an FMRA frame wavelet. However, ψ is not necessarily an FMRA frame wavelet. Next, we give such an example.

EXAMPLE 1

Given $1/2 \leq \alpha, \beta \leq 1$, an expansive matrix $M = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$, and a reducing subspace $FL^2(\Omega)$ of $L^2(\mathbb{R}^2)$, where $\Omega = \bigcup_{j\in\mathbb{Z}} (M^*)^j E$ with

$$E = \Big\{\xi \in [0,1)^2 : 0 \leq \xi_1 \leq \alpha, -\frac{\beta}{\alpha}(\xi_1 - \alpha) \leq \xi_2 \leq 1\Big\} \\ \cup \big\{\xi \in [0,1)^2 : \alpha \leq \xi_1 \leq 1, 0 \leq \xi_2 \leq 1\big\},$$

then ψ defined by $\hat{\psi} = \chi_E$ is an s-frame wavelet for $FL^2(\Omega)$, while it cannot be derived from an FMRA for a reducing subspace.

Proof

It is easy to check that

$$E \cap (E + k) = \emptyset \quad \text{for } 0 \neq k \in \mathbb{Z}^2, \tag{37}$$

$$E \cap (M^*)^j E = \emptyset \quad \text{for } 0 \neq j \in \mathbb{Z}. \tag{38}$$

So ψ is an s-frame wavelet for $FL^2(\Omega)$ by Proposition 10. Next, we prove by contradiction that ψ cannot be derived from an FMRA for a reducing subspace. Suppose that ψ is derived from an FMRA $\{V_j\}_{j\in\mathbb{Z}}$ for some reducing subspace with ϕ being its frame-scaling function. Then the reducing subspace must be $FL^2(\Omega)$ since ψ is an s-frame wavelet for $FL^2(\Omega)$. Denote by W_j the orthogonal complement of V_j in V_{j+1}. Then

$$V_0 = \bigoplus_{j=-\infty}^{-1} W_j.$$

Also, we observe that $\mathrm{supp}(\hat{f}_j) \subset (M^*)^j E$ for $f_j \in W_j$ leads to $\mathrm{supp}(\hat{\phi}) \subset \bigcup_{j=-\infty}^{-1} (M^*)^j E$. Since $D^j\psi \in V_0$ for $j < 0$, we have $(M^*)^j E = \mathrm{supp}((D^j\psi)\hat{}\,) \subset \mathrm{supp}(\hat{\phi})$ for $j < 0$, and consequently,

$$\mathrm{supp}(\hat{\phi}) = \bigcup_{j=-\infty}^{-1} (M^*)^j E. \tag{39}$$

By the refinable property of ϕ, we have $(M^*)^{-1}\mathrm{supp}(\hat{\phi}) = \mathrm{supp}(\hat{\phi}) \cap \mathrm{supp}(m_\phi)$, which implies that

$$\bigcup_{j=-\infty}^{-2} (M^*)^j E = \Big(\bigcup_{j=-\infty}^{-1} (M^*)^j E\Big) \cap \mathrm{supp}(m_\phi) \tag{40}$$

by (39). Take $D = \{\xi \in (M^*)^{-2}E : -1/2 \leq \xi_1 \leq -1/4, -1/2 \leq \xi_2 \leq -1/4\}$. Then $|D| > 0$, and $m_\phi(\cdot) \neq 0$ on D. From (38) and (40), it follows that $m_\phi(\cdot) = 0$ on $(M^*)^{-1}E$, and thus $m_\phi(\cdot) = 0$ on $D + \binom{1}{0}$ due to the fact that $D + \binom{1}{0} \subset (M^*)^{-1}E$. It contradicts the fact that $m_\phi(\cdot) \neq 0$ on D by $\mathbb{Z}^2$-periodicity of m_ϕ. The proof is completed. $\square$

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