# Comment

**Peter R. Freeman**

Professor Ferguson has written an entertaining paper that for me had the unputdownable qualities of a good detective novel as I followed all the twists and turns of the plot on the way to discovering who solved it. There can't be many other stories, surely, in which the author takes over and performs the deed himself on the final page.

I must begin by taking the opportunity of recording my personal debt to the two papers by Lindley and by Gilbert and Mosteller. It was their elegance and beautiful clarity that first kindled by own interest in the secretary problem and in other sequential decision problems, especially those relating to statistical inference. They still vividly convey the excitement and sheer fun of proposing and solving a whole series of increasingly complex problems, giving new graduate students a better idea of what it's like to do successful research in applied probability than anything else I know.

The secretary problem also serves as an excellent case-study of the evolution of research. It starts as an intriguing, not exactly practical but at least realistic, problem. It quickly leads into a branching process at each node of which one of the simple assumptions can be replaced by something more general. Before we know where we are (well, within 25 years, anyway), one problem has turned into hundreds and an army of academics is solving them while simultaneously inventing thousands more. This is all understandable and to some extent admirable, but as one whose interests have turned ever more towards practicalities over the years, I have to ask the embarrassing question: Has anyone *ever* used a secretary-type approach to solve *any* real problem of practical importance? Is anybody willing to admit, for example, to having actually chosen a secretary or a wife using the "optimal" policy?

More seriously, the most common defect in papers that I get to referee in this area is that they assume rather than prove that the optimal policy will be of the form "reject the first $r - 1$ applicants, then accept the first that . . .". This is, of course, not necessarily true, as Presman and Sonin (1972) were the first to show. I'd be very interested to know what is currently the strongest statement that can be made about conditions under which it *is* true. I suspect that there is still fame, if not fortune, awaiting the first person to make progress with the deep issues underlying sequential optimality.

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# Rejoinder

**Thomas S. Ferguson**

One aspect brought out by the comments of the discussants of this paper is the vitality and robustness of the class of problems engendered by the secretary problem. Some new results and some open problems are mentioned in passing. There's life in the old secretary yet!

So much so that my title has come in for criticism. Steve Samuels has punctured my balloon by pointing out that there is still a point to be resolved in what I call *the* secretary problem. According to him, it is not enough to show that the problem can be solved to within $\varepsilon$ for every $\varepsilon > 0$; one must also determine whether or not it can be solved for $\varepsilon = 0$. I admit that as I have worked on this problem (since 1961) I was under the impression that no optimal strategy existed for the player who chooses the set of numbers, and that elementary methods should suffice to show this. Now that he points it out, I can see that the proof of this conjecture, if indeed it is true, is by no means easy, requiring, as it does, a strengthening of Hill's result. Yet, I would be very surprised if one could find an optimal strategy for the numbers chooser, and so I am willing to conjecture that no such strategy exists. I admit I am on shaky ground as I can see no reason for the validity of Samuels' condition (a) to imply the invalidity of his condition (b)—and this is just for the case $n = 3$.

This mathematically interesting open problem prompts me to ask if the secretary problem will ever be "solved." Maybe it's like the central limit theorem—every now and then one hears something new about it, even in the independent, identically distributed case.

I expected the discussants to disagree with my "definition" of a secretary problem; my main objective was to make a firm distinction between it and the offshoots of the Cayley problem. However, I did not expect disagreement as to whether a given problem satisfied the definition. Professor Robbins and Professor Sakaguchi both give me an opportunity to clarify the definition. Robbins and I seem to confuse each other. He does not call the game of googol a secretary problem. Since the payoff depends only on the ranks of the numbers and not upon their actual values, it is a secretary problem by my definition. Also, observe the transition from the simplest secretary problem to googol. First, move to the full-information case, solved by Gilbert and Mosteller (1966). Then move to the no-information case, in which the solution to the simplest secretary problem is minimax, as shown in Samuels (1981). Now, wonder if the value exists and if there is a least favorable distribution, as Samuels does above, or an $\varepsilon$-least favorable distribution, as found in Section 8. It's clear that these are all variations of the same basic underlying problem. The latter is googol, and they are all secretary problems.

In his last paragraph, Robbins states a very pretty extension of the already beautiful result of Chow, Moriguti, Robbins and Samuels (1964). When he says, "Down with googol and up with problems like these!", I agree with the last five words.

In his discussion, Sakaguchi gives a different variation of the Cayley problem. The observations, $X_1, \cdots, X_k$, are chosen without replacement from the set $\{1, \cdots, n\}$, as in the Cayley problem, but the payoff is one or zero depending on whether or not you stop at the largest of the $X_j$. Since the payoff depends only on the relative ranks of the observations and not otherwise on their actual values, this is a secretary problem by my definition. Had Cayley stated the problem in this way, I would have credited him with initiating the class of secretary problems; as it is, he deserves credit for initiating an equally large and perhaps more realistic and important class of problems. The problem treated by Chen and Starr (1980), on the other hand, is not a secretary problem by my definition. It belongs to Cayley's class.

The problem posed by Sakaguchi may be classified as a problem with partial information. The joint distribution of the $X_1, \cdots, X_k$ is exchangeable. It is interesting to note that one has slightly more knowledge about the observations than one does in the so-called full-information case in which the observations are iid. Sakaguchi has set up the problem in a form used by Dynkin (1963), and it also turns out that the one-stage look-ahead rule is optimal. This rule is simply: stop at the first state $(m, y)$ such that $\Phi(m, y) \le 1$, where

$$\Phi(m,y) = \sum_{m<i\le k} \sum_{y<j\le n} \frac{(y-k)!\,(j-i)!}{(j-k)!\,(y-i+1)!}.$$

It is easily seen that $\Phi(m, y)$ is decreasing in both $m$ and $y$, so that, applying the theory of Chow and Robbins (1961), the problem is monotone, and the 1-sla is optimal.

The second problem of Sakaguchi, showing that Rose's solution to the problem of choosing the second-best candidate is minimax, is much more difficult than the problems treated by Stewart and Samuels. Nor do the methods of Section 8 help. Even more difficult would be the minimax version of the minimum expected rank problem of Chow, Moriguti, Robbins and Samuels. I believe it would be comparable in difficulty to the problem of showing that the sample distribution function is minimax as an estimate of the true distribution function under the Cramér-von Mises loss function,

$$L(F, \hat{F}) = \int \frac{(F(t) - \hat{F}(t))^2}{F(t)(1 - F(t)}\, dF(t).$$

However, progress on this difficult problem has been made by Larry Brown and Qiqing Yu (personal communications), using two completely different approaches. Possibly one of these methods will be useful in the secretary problem.

The third problem of Sakaguchi, as I understand it, is googol with the restriction that the number chooser be restricted to choosing non-negative integers. The number chooser can find $\varepsilon$-optimal strategies that choose integer values by modifying the procedure of Section 8 to replacing the random $X_j$ by their closest integer values. By choosing $\alpha$ somewhat larger, he may ensure that all of the resulting integers are distinct with probability $1 - \varepsilon$, and concede the game to his opponent if two of the integers happen to be equal. Part of the difficulty with this problem is in the statement; one must decide what happens in the case of ties. For a treatment of such problems, see Campbell (1982, 1984). Once that question is settled, the simplest thing to do would be to use the discrete analog of (8.1); namely, $\theta$ is inverse power with density proportional to $\theta^{\alpha+1}$, $\theta = m_0, m_0 + 1, \cdots,$ and $X_1, \cdots, X_n$ given $\theta$ is id uniform on the set $\{1, \cdots, \theta\}$. It is doubtful that the negative binomial/beta model or the Poisson/gamma model are useful for this particular study.

Professor Freeman models the evolution of secretary problems as a branching process and asks an embarrassing question: Has any of this been of use in solving a real problem of practical importance? This raises in my mind another related question: Why has the secretary problem been so susceptible to this explosion of ideas? Certainly, it is a picturesque problem with a neat solution that can be explained to someone not in the field, but it is a little mysterious why this problem seems to attract people more than the Cayley type problems, especially since the latter are more realistic. Perhaps it is because the branching factor (the number of branches at a node) in Freeman's model is especially high for secretary problems. Maybe it's fame that researchers seek since it's certainly not fortune.

Since I have been challenged to come up with applied or practical aspects of the secretary problem, let me do the best I can. Morris DeGroot suggested that I mention Dennis Lindley's statement that nobody should get married before the age of 26. I certainly agree with that statement, especially in view of the burgeoning world population and the difficulties which various countries, notably China and India, have encountered in their efforts to control the problem. I applaud such efforts, but it seems clear that success will be more easily achieved if we can convince individual citizens that it is in their own interests to postpone marriage until a more mature and knowledgeable age is reached. Therefore, I have been searching for models of the marriage problem that indicate a later age for marriage than just avoiding the first 36.8% = $1/e$ of the opportunities presented. Most of the models have been disappointing. For example, in the model of Frank and Samuels (1980), you win if you select a spouse who is one of the $r$ best of the $n$ opportunities you will have in your lifetime; yet for large $n$ and moderately large $r$, you should start accepting a relatively best candidate after 28.3% of the opportunities have passed.

Thus, it was a great pleasure to see the model of Sakaguchi (1984), both because of its realism and because of its suggestion of when one should start considering marriage. The model is so simple and pertinent that it is surprising that it has not been suggested earlier. In this model, you win if you marry the best of the candidates, you lose if you marry one of the candidates who is not the best, and you draw if you stay single. Replacing win, lose, and draw by +1, −1, and 0, respectively, Sakaguchi shows that for a large number of prospective candidates, one should not start accepting a relatively best candidate until about 60.7% = $1/\sqrt{e}$ of the opportunities have passed. In this way, it is hoped that the secretary problem has made a modest contribution to world population control!

## ADDITIONAL REFERENCES


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