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Comment

Theo Gasser, Christine Jennen-Steinmetz and Joachim Engel

Nonparametric curve estimation is coming of age, and it is thus timely to study the merits of various approaches. Two weighing schemes have been proposed in the kernel estimation literature, called "evaluation weights" and "convolution weights" by Chu and Marron. The goal of their paper is to give a balanced discussion of their merits, based on two complementary philosophies P1 and P2. We feel that the paper falls short of presenting a balanced discussion and often disregards philosophy P1, that is, looking for structure in a set of numbers. For many years the evaluation weights (due to Nadaraya and Watson) have been studied primarily for random design, the convolution weights for fixed design. Random design is defined and treated adequately by the authors, while fixed design is represented by rather peculiar examples (see below). As is common (see, e.g., Silverman, 1984), we define a regular fixed design as $x_i = F^{-1}((i - 0.5)/n)$, $f = F'$, where F is some distribution function with density f. Under standard assumptions, the asymptotic bias and variance for the two weighting schemes are as in Table 1, where $M_2(K) = \int u^2 K(u)\,du$ and $V(K) = \int K(u)^2\,du$.

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VARIANCE

The factor C in the variance of the convolution estimator is 1 for fixed and 1.5 for random design. Thus, we have an increase in variance for convolution weights with respect to the random design only; variances are asymptotically identical for regular fixed design. There is one fixed but not regular design of importance, that is, when we have multiple points, for example, due to rounding. It is easy to modify convolution weights for this design appropriately, and this has been done in our programs.

We are puzzled by the frequent use of the word *efficiency* in Section 3, when in fact only variance is

TABLE 1
Convolution and evaluation weights

	Convolution weights	Evaluation weights
Bias	$\frac{b^2}{2} M_2(K) m''(t)$	$b^2 M_2(K) \cdot \left\{ \frac{m''(t)}{2} + \frac{m'(t) f'(t)}{f(t)} \right\}$
Variance	$\frac{C\sigma^2 V(K)}{nf(t)b}$	$\frac{\sigma^2 V(K)}{nf(t)b}$

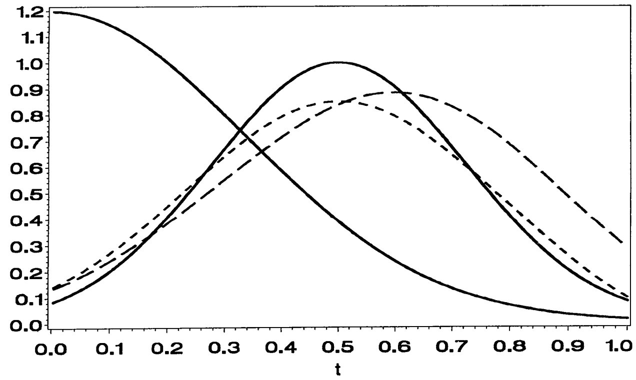

FIG. 1. *True curve: Gauss peak, solid line. Expectation convolution estimation: dotted line. Expectation Nadaraya-Watson estimator: dashed line. Design density: declining solid line.*

at stake. This is unusual for biased estimators and might cause a wrong impression in the hurried reader. A much more common measure of efficiency in nonparametric curve estimation is mean square error. If the authors insisted on cutting down primarily the variance, it would be advisable to use a minimum variance kernel instead of the Gaussian one. The authors admit that their examples showing the "inefficiency" of convolution weights are artificial. Why not provide more realistic ones or limit the conclusions to artificial cases? The design of Figure 2 is okay, but the residual structure is quite peculiar. Leaving apart the very special pattern of observations Y_4 to Y_7, it may be enough to say that their standard deviation is about 10 times the one of the rest of the data. Here, and at other places, the authors freely attribute some phenomena to variance in *one* realization. This seems to us an overinterpretation since for variance we need to take expectation (same for bias).

BIAS

A qualitative and not just a quantitative difference turns up for both types of design with respect to bias. A second term comes up for evaluation weights, depending on m', f' and f in a rather complicated way. Chu and Marron find warm words for this nasty additional bias in Section 4 ("... the bias of $\hat{m}_E$ being the more natural one"): We couldn't disagree more with this point of view. The additional term has awkward consequences of theoretical and practical importance:

1. It is no longer possible to estimate straight lines or pieces of straight lines without bias, and—depending on f'/f—any curvature may arise even for straight lines (see Figure 11 of Chu and Marron).
2. Peaks and other important structure may be shifted around which cannot arise for convolution weights (see our Figure 1). This and the above property of evaluation weights violate philosophy P1 of the authors ("looking for structure").
3. So far there exists no method to correct boundary effects for the evaluation weights, and it might prove to be quite difficult to devise one. This is quite easy to do for the convolution weights and has been implemented by several groups, but not by Chu and Marron. Inappropriately, they thus do not refrain from discussing adverse boundary effects of the convolution weights without performing such a correction. Boundary effects may be a nasty practical problem and usually dominate asymptotic MSE thus preventing us from computing MISE.
4. The important and thorny problem of estimating the optimal bandwidth b_{opt} from the data has experienced a turn by the exploration of "plug-in" estimators. These need estimates of integrated bias squared and integrated variance. While there are reliable algorithms for estimating $\int m''(t)^2\,dt$ (Gasser, Kneip and Köhler, 1991), it is expected to be much harder to estimate $\int (m''(t) + m'(t)f'(t)/f(t))^2\,dt$. In fact, plug-in estimators may be out of reach for evaluation weights.

MINIMAX RESULTS

When applying convolution weights for random design, unfortunately we have to pay a price in variance. Is this worthwhile in terms of MSE? While there is no uniform result, we made a minimax comparison, since a nonparametric method should perform well in a poorly specified, possibly awkward situation (Gasser and Engel, 1990). Chu and Marron find our result unconvincing since we assumed the design density f to be bounded from below on the domain of interest. This is, however, a classical assumption for asymptotics, since it is not meaningful to perform an asymptotic analysis at places where there are hardly any data. It may seem a bit ironical that Chu and Marron make the same assumption "bounded from below" in the same paper (assumption A.4 of Section 3). In an interesting paper, Fan (1990) concludes independently about the Nadaraya-Watson estimator (remark 2, Section 3) "... hence its asymptotic minimax efficiency is arbitrary small."

CONCLUSIONS

Our conclusion is that the convolution weights are clearly superior to evaluation weights for fixed design, since we have the same variance for both methods but a nasty bias for evaluation weights. For random design, the problem seems to us more open: There is a minimax argument, and we would like to repeat a general argument, which is not well quoted by Chu and Marron (Section 3): "The latter authors [Gasser and colleagues] in particular seem to feel that variability is not a major issue, apparently basing their feelings on the premise that it is always easy to gather simply more data." What we said when discussing the structural bias of the evaluation weights was the following (Gasser and Engel, 1990): "These bias problems are particularly accentuated in the scientific process of many empirical sciences: studies are usually replicated by sticking to the design of the previously published study. In this way, qualitatively misleading phenomena as obtained by the Nadaraya-Watson estimator will be attributed even more confidence."

OUTLOOK

One way out of this problem has been opened by Fan (1990), who showed that for random design local polynomials have the same bias as convolution weights and the same variance as evaluation weights (the equivalence of local polynomials to convolution type kernel estimators for fixed design had been shown by Müller, 1987). A further possibility for improving the variance properties of convolution weights has been described by Chu and Marron in Section 6.

Comment

Birgit Grund and Wolfgang Härdle

1. OBJECTIVES OF SMOOTHING

Smoothing has become a standard data analytic tool. A good indicator of this is the increased offer of smoothing procedures in a variety of standard statistical software packages. It is therefore high time to provide background information that enables statisticians and users to critically evaluate the—in the meantime—rich basket of smoothing tools. The paper by Chu and Marron meets this demand for information and compares two different kernel regression estimators on an easy, understandable level. The authors combine successfully careful mathematical discussion with heuristic arguments in a well-done exposition. Cleverly chosen striking examples provide an easy access to not immediately apparent problems in smoothing for data analysis. We congratulate the authors to this valuable contribution.

Among the many objectives of smoothing, there are certainly the two perhaps most discussed. These are P1: to find structure; and P2: to construct estimators from a probability distribution.

We agree that the interplay of these two objectives is vital for an honest parameter-free data

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