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FACTORIAL EXPERIMENTS IN CYCLIC DESIGNS

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The use of cyclic incomplete block designs for factorial experiments is considered. Necessary and sufficient conditions are established under which a cyclic design for an $m \times n$ factorial experiment will give an orthogonal analysis of the main effects and interaction. Various properties, the construction and analysis of such designs are then considered.

1. Introduction. A number of writers have considered the use of factorials in incomplete block designs. In the class of partially balanced incomplete block (PBIB) designs with two associated classes Kramer and Bradley (1957a, b) and Zelen (1958) considered the Group Divisible designs and Bradley, Walpole and Kramer (1960) considered the Latin Square designs with association schemes LS_2 and LS_3. Brenna and Kramer (1961) considered the use of factorials in rectangular lattice designs. Some general results were obtained by Kurkjian and Zelen (1962, 1963). Recently, John and Smith (1972) have developed some general theory for factorials in incomplete block designs. In particular they considered the two factor experiment and obtained some conditions for the designs to satisfy. They showed that all the previous work on factorials satisfied these conditions.

In this paper the use of cyclic designs in factorial experiments will be considered. It will be shown that such designs will have to satisfy certain conditions. These conditions are both necessary and sufficient. Various properties of and the analyses of such designs will then be considered.

Cyclic designs are incomplete block designs consisting in the simplest case of a set of blocks obtained by development of an initial block. More generally, a cyclic design consists of combinations of such sets and will be of size (v, k, r) where v is the number of treatments, k the number of treatments per block and r the number of replications. A cyclic design can, therefore, be specified by its initial block or blocks. The main results on cyclic designs for general k have been given by David and Wolock (1965) and John (1966). A catalogue of cyclic designs is given by John, Wolock and David (1972).

2. Notation. Throughout the paper the following notation will be used:

$\mathbf{1}_m$: a column vector with all m elements unity
$\mathbf{I}_m$: $m \times m$ identity matrix:
$\mathbf{J}_m = \mathbf{1}_m \mathbf{1}_m'$ and $\mathbf{J}_{m\times n} = \mathbf{1}_m \mathbf{1}_n'$.

The suffixes will be omitted if the dimensions are clear from the text.

Let $\mathbf{D} = (d_{ij})$ and $\mathbf{E} = (e_{ij})$ be rectangular matrices of dimensions $m \times n$ and

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$p \times q$ respectively. Then the (right) direct product or Kronecker product of **D** and **E** is written

$$\mathbf{D} \otimes \mathbf{E} = \begin{pmatrix} d_{11}\mathbf{E} & d_{12}\mathbf{E} & \cdots & d_{1n}\mathbf{E} \\ d_{21}\mathbf{E} & d_{22}\mathbf{E} & \cdots & d_{2n}\mathbf{E} \\ \vdots & \vdots & & \vdots \\ d_{m1}\mathbf{E} & d_{m2}\mathbf{E} & \cdots & d_{mn}\mathbf{E} \end{pmatrix}$$

and will have dimensions $mp \times nq$.

3. Two factor experiments in incomplete block designs. This section relies on some of the results given in John and Smith (1972). It is assumed that $v = mn$ and that the intra block model for an incomplete block design is

$$y_{ijs} = \mu + \tau_{ij} + \beta_s + \varepsilon_{ijs}$$
$$(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n; s = 1, 2, \cdots, b)\,.$$

The reduced normal equations for estimating the treatment effects are $\mathbf{A}\hat{\boldsymbol{\tau}} = \mathbf{Q}$, where $\mathbf{A} = r\mathbf{I} - (1/k)\mathbf{NN}'$ and $\mathbf{Q} = \mathbf{T} - (1/k)\mathbf{NB}$. **N** is the $v \times b$ incidence matrix of the design, **T** is the vector of treatment totals and **B** the vector of block totals. A solution to these equations is given by $\hat{\boldsymbol{\tau}} = \mathbf{A}^-\mathbf{Q}$, where $\mathbf{A}^-$ is any generalized inverse of **A**. We shall take $\mathbf{A}^- = \boldsymbol{\Omega}$ where $\boldsymbol{\Omega}^{-1} = \mathbf{A} + a\mathbf{J}$, $a \neq 0$ and **J** a matrix with each element unity. For two factor experiments the treatment effects satisfy the model

$$\tau_{ij} = \alpha_i + \gamma_j + \delta_{ij}$$

that is, $\boldsymbol{\tau} = \mathbf{X}\boldsymbol{\theta}$ where **X** is a $v \times (m + n + v)$ matrix of zeros and ones and of rank v, and $\boldsymbol{\theta}$ is a $(m + n + v)$ parameter vector. Let **X** be partitioned as $(D \,\vdots\, I_v)$ where D is $v \times (m + n)$ and let the contrast matrix $\mathbf{C}^*$ be partitioned similarly as $(G \,\vdots\, C)$. Then the best linear unbiased estimator of $\mathbf{C}^*\boldsymbol{\theta}$ is given by $\mathbf{C}^*\hat{\boldsymbol{\theta}} = \mathbf{C}\hat{\boldsymbol{\tau}}$ and the adjusted sum of squares for testing H_0: $\mathbf{C}^*\boldsymbol{\theta} = \mathbf{0}$ is $(\mathbf{C}\hat{\boldsymbol{\tau}})'(\mathbf{C}\boldsymbol{\Omega}\mathbf{C}')^-\mathbf{C}\hat{\boldsymbol{\tau}}$.

The **C** matrices for estimating and testing the two main effects and the interaction can be defined in a number of ways, and it is well known that the estimators and sums of squares will be invariant to the choice of these matrices. A **C** matrix for testing the α main effect is given by n times the $(m - 1) \times v$ matrix

$$\mathbf{C}_1 = \mathbf{1}_n' \otimes \mathbf{W}_\alpha = (\mathbf{W}_0 \;\; \mathbf{W}_1 \cdots \mathbf{W}_{m-1})$$

where $\mathbf{W}_\alpha = (\mathbf{1}_{m-1} - \mathbf{I}_{m-1})$ is $(m - 1) \times m$, $\mathbf{W}_i$ is $(m - 1) \times n$, $\mathbf{W}_0$ has all elements unity and where $\mathbf{W}_i$ $(i = 1, 2, \cdots, m - 1)$ has all elements in the ith row equal to -1 and all other elements zero. Hence, $\mathbf{W}_0 = -\sum_{i=1}^{m-1} W_i$. A **C** matrix for the γ main effect is m times the $(n - 1) \times v$ matrix

$$\mathbf{C}_2 = \mathbf{W}_\gamma \otimes \mathbf{1}_m' = (\mathbf{W} \;\; \mathbf{W} \cdots \mathbf{W})$$

where $\mathbf{W}_\gamma = (\mathbf{1}_{n-1} - \mathbf{I}_{n-1})$ is $(n - 1) \times n$ and $\mathbf{W} = \mathbf{W}_\gamma$. Finally, for the interaction effect, a suitable C matrix is the $(m - 1)(n - 1) \times v$ matrix

$$\mathbf{C}_3 = \mathbf{W}_\gamma \otimes \mathbf{W}_\alpha\,.$$

The contrast matrices used here are full rank matrices so that it can be shown that $C_i \Omega C_i'$ ($i = 1, 2, 3$) will be nonsingular.

A design will have factorial structure if the adjusted treatment sum of squares is equal to the sum of the adjusted sum of squares for the α, γ and δ effects, which will be the case if and only if $\mathbf{C}_i \Omega \mathbf{C}_j' = \mathbf{0}$ for all $i \neq j$.

The following lemma will prove useful in the next section.

LEMMA 1. *If* $a_1 - a_2 = a_2 - a_3 = \cdots = a_{n-1} - a_n = a_n - a_1$ $(n \geqq 2)$ *then* $a_1 = a_2 = \cdots = a_n$.

PROOF. Let $r = a_i - a_{i+1}$ then $nr = \sum_{i=1}^{n} (a_i - a_{i+1}) = 0$, where $a_1 \equiv a_{n+1}$. Hence $r = 0$.

4. Factorials in cyclic designs. Suppose the $v = mn$ treatments are arranged in a cyclic incomplete block design. Let the matrix Ω be partitioned into a number of $n \times n$ submatrices as follows

$$\Omega = \begin{pmatrix} \Omega_1 & \Omega_2 & \cdots & \Omega_m \\ \Omega_m & \Omega_1 & \cdots & \Omega_{m-1} \\ \cdot & \cdot & \cdot & \cdot \\ \Omega_2 & \Omega_3 & \cdots & \Omega_1 \end{pmatrix}$$

where $\Omega_j = \Omega'_{m-j+2}$ ($j = 2, 3, \cdots, m$).

THEOREM 1. *A necessary and sufficient condition for the cyclic design to have a two factor structure is that, for all j, $\mathbf{\Omega}_j$ is symmetric and circulant and that, apart from the diagonal elements, all submatrices in $\mathbf{\Omega}$ are equal.*

PROOF. If $\mathbf{\Omega}_j$ is symmetric and circulant then the row sums and column sums are all equal to a_j, say. That is, $\mathbf{\Omega}_j \mathbf{1} = a_j \mathbf{1}$. Hence Ω has the same structure as the Ω matrix given in John and Smith (1972). It follows that the condition is necessary.

To prove sufficiency it will be shown that for $\mathbf{C}_1 \Omega \mathbf{C}_3'$ to be the zero matrix the conditions on Ω given in the theorem must hold. Now

$$\Omega \mathbf{C}_3' = \begin{pmatrix} (\Omega_1 - \Omega_2)\mathbf{W}' & (\Omega_1 - \Omega_3)\mathbf{W}' \cdots & (\Omega_1 - \Omega_m)\mathbf{W}' \\ (\Omega_m - \Omega_1)\mathbf{W}' & (\Omega_m - \Omega_2)\mathbf{W}' \cdots & (\Omega_m - \Omega_{m-1})\mathbf{W}' \\ \cdot & \cdot & \cdot \\ (\Omega_2 - \Omega_1)\mathbf{W}' & (\Omega_2 - \Omega_4)\mathbf{W}' \cdots & (\Omega_2 - \Omega_1)\mathbf{W}' \end{pmatrix}$$

and since $\mathbf{C}_1 = (\mathbf{W}_0 \ \mathbf{W}_1 \cdots \mathbf{W}_{m-1})$, where $\mathbf{W}_0 = -\sum_{i=1}^{m-1} \mathbf{W}_i$ it follows that $C_1 \Omega C_3'$ will be made up of terms of the form $\mathbf{W}_i(\Omega_j - \Omega_k)\mathbf{W}'$. For cyclic designs the matrix Ω is a circulant matrix, so that the general structure of Ω_j ($j = 1, 2, \cdots, m$) can be written as

$$\Omega_j = \begin{pmatrix} \omega_1^j & \omega_2^j & \cdots & \omega_{n-1}^j & \omega_n^j \\ \omega_n^{j-1} & \omega_1^j & \cdots & \omega_{n-2}^j & \omega_{n-1}^j \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \omega_3^{j-1} & \omega_4^{j-1} & \cdots & \omega_1^j & \omega_2^j \\ \omega_2^{j-1} & \omega_3^{j-1} & \cdots & \omega_n^{j-1} & \omega_1^j \end{pmatrix}$$

where $\omega_l^0 \equiv \omega_l^m$ $(l = 1, 2, \cdots, n)$. Now let $S_l^{jk} = (\omega_l^{j-1} - \omega_l^j) - (\omega_l^{k-1} - \omega_l^k)$. It can be shown that the ith column of $(\mathbf{\Omega}_j - \mathbf{\Omega}_k)\mathbf{W}'$ is given by $\sum_{l=2}^{i+1} S_l^{jk}$ $(i = 1, 2, \cdots, n-1)$. Hence

$$\mathbf{W}_i(\mathbf{\Omega}_j - \mathbf{\Omega}_k)\mathbf{W}' = -\mathbf{V}_i^{jk} \qquad (i = 0, 1, \cdots, m-1)$$

where $\mathbf{V}_i^{jk}$ has ith row $(i > 0)$

$$(S_2^{jk}, S_2^{jk} + S_3^{jk}, S_2^{jk} + S_3^{jk} + S_4^{jk}, \cdots, \textstyle\sum_{l=2}^{n} S_l^{jk})$$

and zeros elsewhere, and where $\mathbf{V}_0^{jk} = -\sum_{i=1}^{m} \mathbf{V}_i^{jk}$. Let $\mathbf{C}_1 \mathbf{\Omega} \mathbf{C}_3' = (\mathbf{D}_1\ \mathbf{D}_2 \cdots \mathbf{D}_{m-1})$ where $\mathbf{D}_i$ is $(m-1) \times (n-1)$. Then $\mathbf{D}_1 = \mathbf{W}_0(\mathbf{\Omega}_1 - \mathbf{\Omega}_2)\mathbf{W}' + \mathbf{W}_1(\mathbf{\Omega}_m - \mathbf{\Omega}_1)\mathbf{W}' + \cdots + \mathbf{W}_{m-1}(\mathbf{\Omega}_2 - \mathbf{\Omega}_3)\mathbf{W}'$ and, since $\mathbf{W}_0 = -\sum \mathbf{W}_i$, the only nonzero contribution to the first row of $\mathbf{D}_1$ comes from

$$-\mathbf{W}_1(\mathbf{\Omega}_1 - \mathbf{\Omega}_2)\mathbf{W}' + \mathbf{W}_1(\mathbf{\Omega}_m - \mathbf{\Omega}_1)\mathbf{W}' = -(\mathbf{V}_1^{21} + \mathbf{V}_1^{m1})$$

and for each element in this row to be zero we require $S_l^{21} + S_l^{m1} = 0$ for $l = 2, 3, \cdots, n$. By considering the other rows of $\mathbf{D}_1$ we get $S_l^{21} + S_l^{i,i+1} = 0$ or

$$S_l^{i,i+1} = S_l^{i+1,i+2}, \qquad i = 1, 2, \cdots, m;\ i + j \pmod{m},$$

that is,

$$(\omega_l^{i-1} - \omega_l^i) - (\omega_l^i - \omega_l^{i+1}) = (\omega_l^i - \omega_l^{i+1}) - (\omega_l^{i+1} - \omega_l^{i+2})$$

for $l = 2, 3, \cdots, n$. Using Lemma 1 we get

$$\omega_l^1 - \omega_l^2 = \omega_l^2 - \omega_l^3 = \cdots = \omega_l^m - \omega_l^1$$

and using Lemma 1 again

$$\omega_l^1 = \omega_l^2 = \cdots = \omega_l^m \qquad \text{for} \quad l = 2, 3, \cdots, n.$$

With this condition it can also be shown that $\mathbf{D}_i = \mathbf{0}$ $(i = 2, \cdots, m-1)$. This means that $\mathbf{\Omega}_j$ must be a circulant matrix. Now, since $\mathbf{\Omega}$ is symmetric it follows that $\mathbf{\Omega}_j = \mathbf{\Omega}'_{m-j+2}$ and this implies that

$$\omega_l^j = \omega_{n-l+2}^{m-j+1}, \qquad j = 1, 2, \cdots, m;\ l = 2, 3, \cdots, n;\ m - j + 1 \pmod{m}.$$

Hence $\omega_l^1 = \omega_l^2 = \cdots = \omega_l^m = \omega_{n-l+2}^1 = \omega_{n-l+2}^2 = \cdots = \omega_{n-l+2}^m$. Therefore $\mathbf{\Omega}_j$ must be symmetric. It is also clear that, apart from the diagonal elements, the submatrices must be equal.

5. The association scheme. The treatments in cyclic designs are usually labelled $0, 1, 2, \cdots, v-1$. For a two-factor experiment we shall denote the jth treatment in the ith group by V_{ij} $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$. The correspondence between the two methods of labelling is given as follows: V_{ij} is the treatment corresponding to the element in the ith row and jth column of the $m \times n$ table

$$\begin{array}{cccc} 0 & 1 & \cdots & n-1 \\ n & n+1 & \cdots & 2n-1 \\ \cdot & \cdot & \cdot & \cdot \\ (m-1)n & (m-1)n+1 & \cdots & mn-1\,. \end{array}$$

We shall call this array the association scheme of the cyclic design.

All cyclic designs are PBIB designs. The designs considered here have p associate classes where $p \leqq [n/2] + [m/2]$ and where $[j]$ is the integer part of j. From the discussion by Tocher in Pearce (1963) it follows that $\mathbf{\Omega}^{-1}$ will have the same pattern as $\mathbf{\Omega}$, so that it is possible to study the class of cyclic designs having factorial balance by considering either $\mathbf{\Omega}$, $\mathbf{\Omega}^{-1}$ or, equivalently, $\mathbf{NN'}$.

6. Construction of cyclic designs with factorial balance. To obtain cyclic designs with factorial balance we can first check the list of tables given in John, *et al.* (1972). However, these tables list designs with maximum efficiency, and such designs do not necessarily have factorial balance. Otherwise designs can be constructed fairly easily using the method of paired differences given in John (1966). This method enables the first row of $\mathbf{NN'}$ to be obtained from any cyclic design.

For any given cyclic design of size ($v = mn$, k, r) to have factorial structure, the first row of $\mathbf{NN'}$ will be determined by the requirements of Theorem 1. For example, consider four groups of designs for $v = 12$.

m	n	*First row of* $\mathbf{NN'}$
4	3	$r\lambda_1\lambda_1 \quad \lambda_2\lambda_1\lambda_1 \quad \lambda_3\lambda_1\lambda_1 \quad \lambda_2\lambda_1\lambda_1$
3	4	$r\lambda_1\lambda_2\lambda_1 \quad \lambda_3\lambda_1\lambda_2\lambda_1 \quad \lambda_3\lambda_1\lambda_2\lambda_1$
6	2	$r\lambda_1 \quad \lambda_2\lambda_1 \quad \lambda_3\lambda_1 \quad \lambda_4\lambda_1 \quad \lambda_3\lambda_1 \quad \lambda_2\lambda_1$
2	6	$r\lambda_1\lambda_2\lambda_3\lambda_2\lambda_1 \quad \lambda_4\lambda_1\lambda_2\lambda_3\lambda_2\lambda_1$

The λ's given here are the number of times pairs of treatments occur together in a block. If all the λ's are equal to each other then the resulting design will also be a balanced incomplete block (BIB) design. Other designs can be constructed by making some of the λ's equal or by having them all different.

To consider the construction of these cyclic designs further we shall take as an illustration the first of the group of designs considered above, namely $v = 12$, $m = 4$ and $n = 3$. Since $\mathbf{NN'1} = rk\mathbf{1}$, we have that

$$8\lambda_1 + 2\lambda_2 + \lambda_3 = r(k-1)\,.$$

If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, say, the design is BIB. However, since $\lambda(v-1) = r(k-1)$, we must have $r = 11$ or $k = 12$ before a BIB can possibly exist. For other values of r and k we must introduce some partial balance. The most efficient designs here are those with only two different λ values and with the absolute difference of these values being unity. Three examples will be given. For $k = 3$ and $r = 6$ the requirements for factorial balance are satisfied for $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$. If a cyclic design exists here then it can be shown to have an efficiency, as compared with a randomised block design, of 72.3%. Further it will be a PBIB design with two associate classes (PBIB/2). It is not difficult to show, however, that no such design exists. The next most efficient design satisfying the necessary requirements will have $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 0$. One such design is obtained from the initial blocks (0, 1, 4) and (0, 2, 5) with efficiency of 71.5%. It is a PBIB/3 design. Again, for $k = 4$ and $r = 4$ the requirements are satisfied for $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$. Design B36 in John, *et al.* (1972) with initial block (0, 1, 3, 7) satisfies these requirements. It has an efficiency of 81.3% and is a PBIB/2 design. Finally, for $k = 4$ and $r = 8$ the requirements are satisfied for $\lambda_1 = \lambda_3 = 2$ and $\lambda_2 = 3$. Design B39 with initial blocks (0, 1, 2, 5) (0, 2, 6, 9) is appropriate. It has an efficiency of 81.6% and is a PBIB/3 design.

It is interesting to compare these cyclic designs with other incomplete block designs that have been shown to have factorial balance. For $k = 3$ and $r = 6$ a group divisible PBIB/2 design can be used. It is design R17 in the catalogue by Bose, Clatworthy and Shrikhande (1954) and has an efficiency of 72.3%. This design has $m = 6$ and $n = 2$ but can be used for a 3×4 factorial experiment if its association scheme is taken as

$$\begin{matrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 . \end{matrix}$$

For $k = 4$ and $r = 4$ the cyclic design given above is equivalent to R15 in Bose, *et al.* (1954), with association scheme as for R17. For $k = 4$ and $r = 8$ no corresponding PBIB/2 design is available.

7. The analysis. The adjusted sums of squares for the main effects and the interaction have been given by John and Smith (1972). Because of the special structure of $\mathbf{NN}'$ for cyclic designs some further results will be given, which will simplify the analysis. Since $\mathbf{\Omega}^{-1} = r\mathbf{I} - (1/k)\mathbf{NN}' + a\mathbf{J}$, $a \neq 0$, is circulant and has the same pattern as $\mathbf{\Omega}$ it follows that $\mathbf{NN}'$ can be partitioned into a number of $n \times n$ matrices, as follows

$$\mathbf{NN}' = \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 \cdots & \mathbf{B}_m \\ \mathbf{B}_m & \mathbf{B}_1 \cdots & \mathbf{B}_{m-1} \\ \vdots & \vdots & \vdots \\ \mathbf{B}_2 & \mathbf{B}_3 \cdots & \mathbf{B}_1 \end{pmatrix}$$

where $\mathbf{B}_i = \mathbf{B}'_{m-i+2}$ $(i = 2, 3, \cdots, m)$. The elements of the first row of the circulant matrix $\mathbf{B}_i$ $(i = 1, 2, \cdots, m)$ will be denoted by $\{b_i, d_1, d_2, \cdots, d_{n-1}\}$, where $d_j = d_{n-j}$ $(j = 1, 2, \cdots, n-1)$, $b_k = b_{m-k+2}$ $(k = 2, \cdots, m)$ and $b_1 = r$. Since $\mathbf{NN}'\mathbf{1} = rk\mathbf{1}$ we have the following condition on the elements of $\mathbf{NN}'$

$$\textstyle\sum_{i=2}^{m} b_i + m \sum_{j=1}^{n-1} d_j = r(k-1) .$$

The adjusted sum of squares due to the α main effect, based on $m - 1$ degrees of freedom, is $\boldsymbol{\alpha}^{*\prime}\mathbf{V}_1\boldsymbol{\alpha}^*$, where $\boldsymbol{\alpha}^{*\prime} = (\hat{\hat{\tau}}_{1\cdot}, \hat{\hat{\tau}}_{2\cdot}, \cdots, \hat{\hat{\tau}}_{m\cdot})$ and $n\hat{\hat{\tau}}_{i\cdot} = \sum_j \hat{\tau}_{ij}$ $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$. In general, the matrix $\mathbf{V}_1$ will be equal to $\mathbf{P}_1\mathbf{A}\mathbf{P}_1' + c\mathbf{J}$ for any scalar c, where $\mathbf{P}_1 = \mathbf{I}_m \otimes 1_n'$. If $c = (n/k) \sum_{j=i}^{n-1} d_j$, it can be seen that a simple choice of $\mathbf{V}_1$ in this case is the circulant matrix with first row $(n/k)\{r(k-1), -b_2, \cdots, -b_m\}$, that is, a matrix based only on the diagonal elements of $\mathbf{B}_i$ $(i = 1, 2, \cdots, m)$ in $\mathbf{NN}'$. Similarly, the adjusted sum of squares due to the γ main effect, based on $n - 1$ degrees of freedom, is $\boldsymbol{\gamma}^{*\prime}\mathbf{V}_2\boldsymbol{\gamma}^*$ where $\boldsymbol{\gamma}^{*\prime} = (\hat{\hat{\tau}}_{.1}, \hat{\hat{\tau}}_{.2}, \cdots, \hat{\hat{\tau}}_{.n})$ and $m\hat{\hat{\tau}}_{.j} = \sum_i \hat{\tau}_{ij}$ $(i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n)$. The matrix $\mathbf{V}_2$ will be equal to $\mathbf{P}_2\mathbf{A}\mathbf{P}_2' + c\mathbf{J}$ for any scalar c, where $\mathbf{P}_2 = \mathbf{1}_m' \otimes \mathbf{I}_n$. With $c = 0$, a possible choice of $\mathbf{V}_2$ is the circulant matrix with first row $(m^2/k)\{\sum_{j=i}^{n-1} d_j, -d_1, -d_2, \cdots, -d_{n-1}\}$. The adjusted sum of squares for the interaction, based on $(m-1)(n-1)$ degrees of freedom, is $\boldsymbol{\delta}^{*\prime}\mathbf{V}_3\boldsymbol{\delta}^*$ where $\boldsymbol{\delta}^{*\prime} = (\hat{\tau}_{11}^*, \hat{\tau}_{12}^*, \cdots, \hat{\tau}_{mn}^*)$ and $\hat{\tau}_{ij}^* = \hat{\tau}_{ij} - \hat{\hat{\tau}}_{i.} - \hat{\hat{\tau}}_{.j}$. The matrix V_3 will, in general, be equal to $\mathbf{A} + c\mathbf{J}$ for any scalar c. This sum of squares is most easily obtained by subtraction.

The recovery of inter-block information follows in a similar way, using the results given in John and Smith (1972).

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