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DIFFERENTIAL EQUATIONS AND OPTIMAL CHOICE PROBLEMS

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Asymptotic forms for the optimal payoff and optimal stop rule for a generalized class of "secretary" problems are obtained by the analysis of a related family of ordinary differential equations.

1. Introduction. The stop rule problems to be considered in this paper are generalizations of the "secretary problem" for bounded payoffs. A subclass of such problems is considered in Gusein-Zade [6].

Let q map the positive integers into $[0, 1]$ with $q(k) \downarrow 0$. Let a probability $(N!)^{-1}$ be attached to each permutation σ of the first N integers, and let $\{X_k\}$ $k = 1, \cdots, N$ be the sequence of independent random variables where X_k is the rank of $\sigma(k)$ among $\sigma(1), \sigma(2), \cdots, \sigma(k)$. The optimal choice problem consists in determining v_N, t_N where

$$v_N = \max_t Eq(\sigma(t)) = Eq(\sigma(t_N)) \tag{1.1}$$

as t runs through stop rules on the sequence $X_1, X_2, \cdots, X_N$. This paper is concerned with asymptotic forms for v_N and t_N. Our principal result is expressed in the following theorem.

THEOREM 1.1. *Let*

$$R_k(\alpha) = \sum_k^\infty q(l)\binom{l-1}{k-1}\alpha^k(1-\alpha)^{l-k}, \qquad \alpha \in (0, 1].$$

Then the differential equation

$$g'(\alpha) = -\alpha^{-1} \sum_1^\infty (R_k(\alpha) - g(\alpha))^+; \qquad g(1) = 0$$

has a unique solution on $[0, 1]$ *and*

$$|v_N - g(0)| \leqq C_1[\ln N/N^{\frac{1}{2}} + q(C_2 \ln N)]$$

where C_1 *and* C_2 *are determined constants.*

Analogous results for optimal choice problems involving unbounded functions, q, are treated in [8], and will be the subject of a forthcoming paper. The latter results generalize Chow, Moriguti, Robbins and Samuels [2].

2. Basic recursions and difference equations. Let q be a function on the positive integers, I, such that

(a) $q(1) = 1$,
(b) $q(l) \downarrow 0$.

We call q a payoff function. Let σ be any permutation of the first N integers; we assume all $N!$ permutations to be equally likely. We define X_r to be the

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relative rank of $\sigma(r)$ among $\sigma(1), \sigma(2), \cdots, \sigma(r)$. The random variables, X_r, $r = 1, 2, \cdots, N$, called observations, are independent with distributions

$$P(X_r = k) = \frac{1}{r} \quad \text{if} \quad 1 \leqq k \leqq r$$
$$= 0 \quad \text{otherwise.}$$

Let us set

$$Q(r, k) = E(q(\sigma(r)) \mid X_r = k) = \sum_k^{N-(r-k)} q(l) \frac{\binom{l-1}{k-1}\binom{N-l}{r-k}}{\binom{N}{r}} .$$

Our objective is to find asymptotic forms and estimates for the values v_N, t_N defined by

$$v_N = \max_t Eq(\sigma(t)) = \max_t EQ(t, X_t) = EQ(t_N, X_{t_N}) \tag{2.1}$$

where t runs through all stop rules on the sequence X_r, $r = 1, \cdots, N$. We call v_N the optimal payoff and t_N the optimal stop rule. We sometimes call v_N a utility. A standard recursive technique for generating v_N and t_N is the following.

Set

$$v_N(N) = Eq(\sigma(N))$$
$$v_N(N, k) = q(k) , \qquad k = 1, 2, \cdots, N$$
$$v_N(r, k) = \max(Q(r, k), Ev_N(r + 1, X_{r+1}))$$
$$v_N(r) = Ev_N(r, X_r) = \frac{1}{r} \sum_1^r v_N(r, k) .$$

We then have

$$v_N(r) = E \max(Q(r, X_r), v_N(r + 1)) = \frac{1}{r} \sum_1^r \max(Q(r, k), v_N(r + 1)) . \tag{2.2}$$

Since $v_N(r) = \sup_{t \geqq r} Eq(\sigma(t))$, we see that $v_N(r) \geqq v_N(r + 1)$ and $v_N = v_N(1)$. This recursive technique, called a "backwards recursion," defines the optimal stop rule, t_N, by the prescription that one stop with the observation $X_r = k$ unless $v_N(r, k) > Q(r, k)$, i.e., unless it pays more to continue.

We will presently prove that

$$Q(r, k) \geqq Q(r, k + 1)$$
$$Q(r + 1, k) \geqq Q(r, k) .$$

We can therefore characterize our optimal stop rule t_N, (which is the collection of pairs (r, k) such that $Q((r, k) \geqq v_N(r + 1))$ as a tuple $(r_1, r_2, \cdots, r_N)$ where $r_1 \leqq r_2 \leqq \cdots \leqq r_N$ and where one stops with observation $X_r = k$ provided $r \geqq r_k$.

The procedure outlined above is a rewording of the procedures found in [4], [6], and [2].

We now state

PROPOSITION 2.1. (*Basic Recursion.*)

$$Q(r, k) = \frac{k}{r+1} Q(r+1, k+1) + \left(1 - \frac{k}{r+1}\right) Q(r+1, k) .$$

PROOF. Straightforward calculation.

This recursion will be used often in this paper. A first application is:

COROLLARY 2.1. (a) $Q(r, k) \geqq Q(r, k+1)$
(b) $Q(r+1, k) \geqq Q(r, k)$.

PROOF. (a) Note first: $Q(N, k+1) = q(k+1) \leqq q(k) = Q(N, k)$. Then backwards, assuming the result for $r+1$, we have

$$\begin{aligned} Q(r, k+1) &= \frac{k+1}{r+1} Q(r+1, k+2) + \left(1 - \frac{k+1}{r+1}\right) Q(r+1, k+1) \\ &= \frac{k}{r+1} Q(r+1, k+2) + \left(1 - \frac{k}{r+1}\right) Q(r+1, k+1) \\ &\quad + \frac{1}{r+1} [Q(r+1, k+2) - Q(r+1, k+1)] \\ &\leqq \frac{k}{r+1} Q(r+1, k+2) + \left(1 - \frac{k}{r+1}\right) Q(r+1, k+1) \\ &\leqq \frac{k}{r+1} Q(r+1, k+1) + \left(1 - \frac{k}{r+1}\right) Q(r+1, k) = Q(r, k) . \end{aligned}$$

(b) Use the same procedure as for (a). □

Let us now normalize $v_N(\cdot)$ by setting $v_N(r) = f_N(r/N)$. We have then:

$$f_N(1) = \frac{1}{N} \sum_1^N g(l) = \frac{s(N)}{N}$$

$$f_N\left(\frac{r}{N}\right) = \frac{1}{r} \sum_1^r \max\left(Q(r, k), f_N\left(\frac{r+1}{N}\right)\right) .$$

We extend f_N by linear interpolation to the entire unit interval and we rewrite this recursion in the form

$$f_N\left(\frac{r}{N}\right) - f_N\left(\frac{r+1}{N}\right) = \frac{1}{r} \sum_1^r \left(Q(r, k) - f_N\left(\frac{r+1}{N}\right)\right)^+ .$$

$$f_N(1) = \frac{s(N)}{N} .$$

We note that $v_N = f_N(0)$.

These last equations resemble

$$g(1) = 0$$

$$g_N\left(\frac{r}{N}\right) - g_N\left(\frac{r+1}{N}\right) = \frac{1}{r+1} \sum_1^{r+1} \left(R_k\left(\frac{r+1}{N}\right) - g_N\left(\frac{r+1}{N}\right)\right)^+$$

where

$$R_k(\alpha) = \sum_k^\infty q(l) \binom{l-1}{k-1} \alpha^k (1-\alpha)^{l-k} \qquad \alpha \in [0, 1] .$$

Now g_N, extended by linear interpolation to [0, 1], can be shown to be a piecewise differentiable function which in the limit tends to the function g which uniquely satisfies

$$g(1) = 0$$

$$g'(\alpha) = -\frac{1}{\alpha} \sum_1^\infty (R_k(\alpha) - g(\alpha))^+ \qquad \alpha \in [0, 1).$$

One employs the standard Picard method for ordinary differential equations to establish the existence of g and the fact that $\lim g_N = g$. The details, lengthy but straightforward, can be found in [8]. Using estimates for the difference $R_k(r/N) - Q(r, k)$, one deduces

THEOREM 2.1.

$$||f_N - g|| \leqq 10 \frac{\log N}{N^{\frac{1}{2}}} + 30q\left(\left[\frac{\alpha_*}{2} \log N\right]\right)$$

provided $\log N \geqq \max(4, 12\alpha_*)$ *where* $R_1(\alpha_*) \leqq .25$.

REMARK 2.1.

(i) Since $|v_N - g(0)| = |f_N(0) - g(0)| \leqq \sup_{0 \leqq \alpha \leqq 1} |f_N(\alpha) - g(\alpha)| = ||f_N - g||$, we see that $v_N \to g(0)$.

(ii) If q is truncated, i.e., if there exists m such that $q(l) = 0$ all $l > m$, then $||f_N - g|| < 10 \log N/N^{\frac{1}{2}}$. Actually, much more can be said. We have the stronger inequality:

$$||f_N - g|| \leqq C_m/N \qquad \text{for large } N$$

where C_m is a constant that depends only on m. Again, the details are spelled out in [8].

(iii) Note that g is non-increasing.

(iv) The functions R_k play an important role in this paper. We list some properties:

(a) $R_k'(\alpha) = k\alpha^{-1}(R_k(\alpha) - R_{k+1}(\alpha)) = k \sum_k^\infty (q(l) - q(l+1))\binom{l}{k}\alpha^{k-1} \times (1-\alpha)^{l-k}$.

(b) If $q(k) > 0$, then R_k is *strictly* increasing and $R_k(\alpha) > R_{k+1}(\alpha)$, all $\alpha \in (0, 1]$.

(c) $\sum_1^m R_l(\alpha) = \alpha s(m) + m\alpha \int_\alpha^1 t^{-2} R_{m+1}(t)\, dt$.

(d) $R_k(0) = 0$; $R_k(1) = q(k)$.

(v) The properties listed above imply that $g' \equiv 0$ on $[0, \alpha_1]$ where α_1 is the unique solution for the equation $R_1(\alpha_1) = g(\alpha_1)$. Thus, $\lim_{N\to\infty} v_N = R_1(\alpha_1)$.

We described t_N, the optimal stop rule, as an N-tuple of increasing integers $(r_1, r_2, \cdots, r_N)$ where one stops on observation $x_r = k$ provided $r \geqq r_k$. The value r_k was the minimal integer for which $Q(r, k) > v_N(r+1)$. This condition, in the limit as $N \to \infty$, leads to

THEOREM 2.2. *The optimal stop rule* $t_N = (r_1, r_2, \cdots, r_N)$ *satisfies*

$$\lim_{N\to\infty} r_k/N = \alpha_k$$

where $0 < \alpha_1 < \alpha_2 < \cdots$ *is a sequence uniquely determined by the equations*:

$$R_k(\alpha_k) = g(\alpha_k) .$$

There is a converse of sorts to the last theorem, namely, that if $\bar{v}_N$ is the utility associated with the stop rule $\bar{t}_N = ([\alpha_1 N], [\alpha_2 N], \cdots, N)$, then $\lim_{N\to\infty} \bar{v}_N = \lim_{N\to\infty} v_N = g(0)$. We will sketch a proof of this result in the last section of this paper.

3. Applications. Our objective in this section is to determine the sequence $\{\alpha_n\}$ and the utility $v = R_1(\alpha_1)$. Our differential equation

$$g' = -\frac{1}{\alpha} \textstyle\sum_1^\infty (R_k - g)^+ ; \qquad g(1) = 0 \tag{3.1}$$

reduces on the interval $[\alpha_n, \alpha_{n+1}]$ to the equation

$$g' = -\frac{1}{\alpha} \textstyle\sum_1^n (R_k - g) \tag{3.2}$$

with boundary conditions

$$\begin{aligned} g(\alpha_n) &= R_n(\alpha_n) \\ g(\alpha_{n+1}) &= R_{n+1}(\alpha_{n+1}) . \end{aligned} \tag{3.3}$$

Equivalently, we can write (3.2) in the form $(g(\alpha)/\alpha^n)' = -\alpha^{-(n+1)} \sum_1^n R_k(\alpha)$ on $[\alpha_n, \alpha_{n+1}]$. Now for $n \geqq 2$, $[\alpha^{-n} \sum_1^{n-1} R_k(\alpha)]' = -\alpha^{-(n+1)}(n-1) \sum_1^n R_k(\alpha)$ follows immediately from the properties of R_k'. Thus, on $[\alpha_n, \alpha_{n+1}]$ we have $(g(\alpha)/\alpha^n)' = (n-1)^{-1}[\alpha^{-n} \sum_1^{n-1} R_k]'$ for $n \geqq 2$. Using our boundary conditions (3.3) we conclude

$$\frac{1}{\alpha_n{}^n} \textstyle\sum_1^{n-1} (R_k(\alpha_n) - R_n(\alpha_n)) = \frac{1}{\alpha_{n+1}^n} \sum_1^{n-1} (R_k(\alpha_{n+1}) - R_{n+1}(\alpha_{n+1})) . \tag{3.4}$$

When $n = 1$, we see that (3.2) becomes $(g(\alpha)/\alpha)' = -R_1(\alpha)/\alpha^2$ or, using (3.3),

$$\frac{R_2(\alpha_2)}{\alpha_2} - \frac{R_1(\alpha_1)}{\alpha_1} = -\int_{\alpha_1}^{\alpha_2} \frac{R_1(\alpha)}{\alpha^2} d\alpha . \tag{3.5}$$

Now

$$\frac{d}{d\alpha} \textstyle\sum_1^\infty \frac{s(k)}{k} (1-\alpha)^k = -\sum_1^\infty s(k)(1-\alpha)^{k-1} = -\frac{R_1(\alpha)}{\alpha^2} ,$$

the integrand in (3.5). So we can rewrite (3.5) in the form:

$$\textstyle\sum_1^\infty \frac{s(k)}{k} (1-\alpha_1)^k - \frac{R_1(\alpha_1)}{\alpha_1} = \sum_1^\infty \frac{s(k)}{k} (1-\alpha_2)^k - \frac{R_2(\alpha_2)}{\alpha_2} . \tag{3.6}$$

DEFINITION 3.1. Set

$$H_1(\alpha) = \textstyle\sum_1^\infty \frac{s(k)}{k} (1-\alpha)^k - \frac{R_1(\alpha)}{\alpha}$$

$$G_1(\alpha) = H_1(\alpha) + \frac{1}{\alpha} (R_1(\alpha) - R_2(\alpha))$$

and for $n \geqq 2$

$$H_n(\alpha) = \frac{1}{\alpha^n} \sum_1^{n-1} (R_k(\alpha) - R_n(\alpha))$$

$$G_n(\alpha) = \frac{1}{\alpha^n} \sum_1^{n-1} (R_k(\alpha) - R_{n+1}(\alpha)) .$$

From (3.4) and (3.6) we see that our sequence $\{\alpha_n\}$ must satisfy

$$H_n(\alpha_n) = G_n(\alpha_{n+1}) . \tag{3.7}$$

Let us now list some properties for H_n and G_n which will be useful for examining (3.7).

REMARK 3.1.

(A) $0 \leqq H_n(\alpha) \leqq G_n(\alpha) \leqq H_{n+1}(\alpha)$ for $n \geqq 2$.

(B) $H_n'(\alpha) = -(n-1)H_{n+1}(\alpha)$ for $n \geqq 2$.

Thus, H_n is non-increasing.

(C) $G_n(\alpha) = H_n(\alpha) + \alpha^{-n}(n-1)(R_n - R_{n+1})$, $n \geqq 2$.

(D) $G_n'(\alpha) \leqq -(n-1)H_{n+1}(\alpha)$, $n \geqq 2$.

Thus, G_n is non-increasing.

(E) $H_n(\alpha) = \sum_{k=0}^{\infty} \binom{n-2+k}{k}[s(n+k-1) - (n+k-1)q(n+k)](1-\alpha)^k$, $n \geqq 2$. For by repeated application of (B), we get $H_n^{(k)} = (-1)^k k! \binom{n-2k+1}{k} H_{n+k}$ and further $H_{n+k}(1) = s(n+k-1) - (n+k-1)q(n+k)$, so we simply take the Taylor expansion of H_n around $\alpha = 1$.

(F) $\lim_{\alpha\to 0+} H_n(\alpha) = \infty$. $n \geqq 2$. For we can write

$$H_2(\alpha) = \alpha^{-1}R_1'(\alpha) = \alpha^{-1} \sum_1^{\infty} l(q(l) - q(l+1))(1-\alpha)^{l+1} , \tag{3.8}$$

and clearly $\lim_{\alpha\to 0+} H_2(\alpha) = \infty$. Now use the fact that $H_n \geqq H_2$ for $n \geqq 2$. *We clearly have also that* $\lim_{\alpha\to 0+} G_n(\alpha) = \infty$, $n \geqq 2$.

(G) $H_n(1) = s(n-1) - (n-1)q(n)$, $n \geqq 2$.
$G_n(1) = s(n-1) - (n-1)q(n+1)$, $n \geqq 2$.

(H) If $q(m+1) = 0$, then $k \geqq 1$ implies $H_{m+k}(\alpha) = H_{m+1}(\alpha) = s(m)/\alpha^m$, $G_{m+k}(\alpha) = G_{m+1}(\alpha) = s(m)/\alpha^m$, since $R_{m+k} \equiv 0$ and $\sum_1^m R_k(\alpha) = \alpha s(m)$.

(I) H_n is strictly decreasing on $(0, 1]$ for $n \geqq 2$. G_n is strictly decreasing on $(0, 1]$ for $n \geqq 2$.

(J) $H_1'(\alpha) = -H_2(\alpha)$, $G_1'(\alpha) \leqq -H_2(\alpha)$. *In particular, H_1 and G_1 are strictly decreasing on* $(0, 1]$.

(K) $\lim_{\alpha\to 0+} H_1(\alpha) = \infty$.

(L) Given any $\alpha \in (0, 1]$, there exists a unique $\tilde{\alpha} \in (0, 1]$, $\tilde{\alpha} \leqq \alpha$ such that $H_n(\tilde{\alpha}) = G_n(\alpha)$. This result follows from the facts that $G_n \geqq H_n$, H_n is strictly decreasing, and $H_n(0^+) = \infty$.

PROPOSITION 3.1. *Let q be truncated at m. Then the sequence $\{\alpha_n\}$, $n = 1, \cdots, m$ which defined the limiting optimal stop rule is generated recursively by the formulas*:

$$\alpha_{m+1} = 1 , \qquad H_n(\alpha_n) = G_n(\alpha_{n+1}) \qquad n = 1, \cdots, m .$$

EXAMPLE 3.1. (a) Let

$$q(l) = 1 \quad \text{if } l = 1$$
$$= 0 \quad \text{otherwise.}$$

Then $R_1(\alpha) = \alpha$, $R_k(\alpha) = 0$ for $k > 1$. $H_1(\alpha) = \sum_1^\infty k^{-1}(1-\alpha)^k - 1 = -\log\alpha - 1$, $G_1(\alpha) = -\log\alpha$. Now $\alpha_2 = 1$, and α_1 is the solution to $-\log\alpha_1 - 1 = -\log\alpha_2 = -\log 1 = 0$, i.e., $\log\alpha_1 = -1$ so $\alpha_1 = 1/e$.

(b) Let

$$q(l) = 1 \quad \text{if } l = 1, 2$$
$$= 0 \quad \text{otherwise.}$$

Then $R_1(\alpha) = 2\alpha - \alpha^2$, $R_2(\alpha) = \alpha^2$. So $H_1(\alpha) = 2\alpha - 2\log\alpha - 2$, $H_2(\alpha) = 2\alpha^{-1} - 2$, $G_1(\alpha) = -2\log\alpha$, $G_2(\alpha) = 2\alpha^{-1} - 1$. Thus, starting with $\alpha_3 = 1$, we have $2\alpha_2^{-1} - 2 = \frac{2}{1} - 1 = 1$, i.e., $\alpha_2 = \frac{2}{3}$ and $2\alpha_1 - 2\log\alpha_1 - 2 = -\log\frac{2}{3}$, i.e., $\alpha_1 - \log\alpha_1 \sim 1.40$ so $\alpha_1 \sim .35$. Let us note that in case (a), our utility v satisfies $v = R_1(1/e) = 1/e$, and in case (b) we have $v = R_1(.35) = 2(.35) - (.35)^2 \sim .58$.

We also note that for any q truncated at m we have: $H_m(\alpha) = s(m)\alpha^{-(m-1)} - mq(m)$. Since $G_m(1) = s(m-1)$, we have $\alpha_m = [s(m)/\{s(m-1) + mq(m)\}]^{(m-1)^{-1}}$; a generalization of a formula found in [6].

REMARK 3.2. It may happen that, although q is not truncated, we can still find $(\alpha_1, \alpha_2, \cdots)$. The idea is to set up the recursion in a form which expresses α_m as a limit depending in an essential manner on the fact that $\lim_{n\to\infty}\alpha_n = 1$, which is proven in [8]. Let us illustrate this.

A special class of payoffs. Let $\beta \in (0, 1)$. We set

$$q_\beta(k) = 1 \quad \text{if } k = 1$$
$$= \frac{\beta(\beta+1), \cdots, (\beta+k-2)}{(k-1)!} \quad \text{if } k \geqq 2\,. \tag{3.9}$$

We can write this payoff in the form $q_\beta(k) = (-1)^{k-1}\binom{-\beta}{k-1}$. We note that

(a) $q_\beta(k+1)/q_\beta(k) = 1 - (1-\beta)/k \leqq 1$,

(b) $q_\beta(k) = \prod_1^{k-1}(1 - (1-\beta)/l)$.

Thus, for large k, we have $\log q_\beta(k) \leqq -(1-\beta)\sum_1^{k-1} l^{-1} \to -\infty$, i.e., $q_\beta(k) \downarrow 0$. In fact, let us note that $\log q_\beta(k) \leqq -(1-\beta)\int_1^k s^{-1}\,ds = -(1-\beta)\log k$, so $q_\beta(k) \leqq k^{-(1-\beta)}$ while $\log q_\beta(k) \geqq -(1-\beta)\{1 + \int_1^k s^{-1}\,ds\}$, from which $q_\beta(k) \geqq e^{-(1-\beta)}k^{-(1-\beta)} \geqq e^{-2}\cdot k^{-(1-\beta)}$. This last inequality shows that $\sum_{k=1}^\infty q_\beta(k) = \infty$ for all $\beta \in (0, 1]$.

(c) $\beta \geqq \bar\beta$, $q_\beta \geqq q_{\bar\beta}$.

(d) $R_1(\alpha) = \alpha^{1-\beta}$.

(e) $R_n(\alpha) = q_\beta(n)\alpha^{1-\beta}$. This follows immediately from the recursion $R_{n+1} = R_n - \alpha n^{-1}R_n'$. *In what follows,* $\beta < 1$.

(f) $H_2(\alpha) = (1-\beta)\alpha^{-1-\beta}$.

So $H_1(\alpha) = (1-\beta)\beta^{-1}\alpha^{-\beta} - \beta^{-1}$ *since* $H_1' = -H_2$ *and* $H_1(1) = -1$.

(g) $H_n(\alpha) = (1 - \beta)\beta^{-1}(n-1)q_\beta(n)\alpha^{-\beta-(n-1)}$, $n \geqq 2$. Here we simply use the recursion: $H_n' = -(n-1)H_{n+1}$.

(h) $G_1(\alpha) = (\beta^{-1} - \beta)\alpha^{-\beta} - \beta^{-1}$.

(i) $G_n(\alpha) = (1 - \beta)\beta^{-1}(n+\beta)n^{-1}(n-1)q_\beta(n)\alpha^{-\beta-(n-1)}$, $n \geqq 2$.

These formulas allow a straighforward calculation for the sequence $\{\alpha_n\}$.

We have the recursion $H_n(\alpha_n) = G_n(\alpha_{n+1})$ reducing to $\alpha_n{}^{-\beta-(n-1)} = (n + \beta)n^{-1}\alpha_{n+1}^{-\beta-(n-1)}$, i.e.,

$$\frac{\alpha_n}{\alpha_{n+1}} = \left(1 + \frac{\beta}{n}\right)^{1/(-\beta-(n-1))}$$

for $n \geqq 2$. Now since $\alpha_n \to 1$, we have

$$\alpha_m = \prod_m^\infty \left(1 + \frac{\beta}{n}\right)^{-1/(\beta+(n-1))}, \qquad m \geqq 2 .$$

Further, the recursion $H_1(\alpha_1) = G_1(\alpha_2)$ reduces to

$$\alpha_1/\alpha_2 = (1+\beta)^{-1/\beta} .$$

So

$$\alpha_1 = \prod_1^\infty (1 + \beta/l)^{-1/(\beta+(l-1))} .$$

Finally

$$v = g(\alpha_1) = R_1(\alpha_1) = \prod_1^\infty \left(1 + \frac{\beta}{l}\right)^{\frac{1}{1-(l/(1-\beta))}} \tag{3.10}$$

$$g(\alpha_n) = R_n(\alpha_n) = q_\beta(n) \prod_n^\infty \left(1 + \frac{\beta}{l}\right)^{\frac{1}{1-(l/(1-\beta))}} .$$

REMARK 3.3. Note that

$$\begin{aligned} \ln \frac{1}{\alpha_1} &= \sum_1^\infty \frac{1}{\beta + (l-1)} \log\left(1 + \frac{\beta}{l}\right) \\ &\leqq \sum_1^\infty \frac{1}{\beta + (l-1)} \frac{\beta}{l} \\ &\leqq 1 + \beta \sum_2^\infty \frac{1}{l(l-1)} = 1 + \beta . \end{aligned}$$

Thus $\alpha_1{}^{-1} \leqq e^{1+\beta}$ or $\alpha_1 \geqq e^{-1-\beta} \geqq e^{-2}$. *So α_1 is bounded away from zero uniformly in β.*

On the other hand,

$$\begin{aligned} \log \frac{1}{\alpha_1} &= \sum_1^\infty \frac{1}{\beta + (l-1)} \log\left(1 + \frac{\beta}{l}\right) \\ &\geqq \sum_1^\infty \frac{1}{\beta + (l-1)} \frac{1}{1 + \beta/l} \frac{\beta}{l} \\ &= \beta \sum_1^\infty \frac{1}{(\beta + l)(\beta + l - 1)} = \beta \sum_1^\infty \left\{\frac{1}{\beta + l - 1} - \frac{1}{\beta + l}\right\} \\ &= \beta(1/\beta) = 1 ; \end{aligned}$$

therefore $\alpha_1 \leqq 1/e$.

PROPOSITION 3.2. *Let* $\beta \in (0, 1)$, q_β *as defined above. Then* $1/e^2 \leqq \alpha_1 \leqq 1/e$.

4. Asymptotically optimal stop rules. Our objective in this section is to sketch a proof of the following:

THEOREM 4.1. *Let* $\bar{t}_N = ([\alpha_1 N], [\alpha_2 N], \cdots, N)$, $\bar{v}_N = Eq(\sigma \bar{t}_N))$. *Then*

$$\lim_{N\to\infty} \bar{v}_N = g(0) .$$

We begin by defining:

$$p_1 = 1$$

$$p_{l,l+1} = p(x_t > l, [\alpha_l N] \leqq t < [\alpha_{l+1} N])$$

$$p_k = \prod_1^{k-1} p_{l,l+1} \qquad k > 1$$

$$u_k = \sum_{[\alpha_k N]}^{[\alpha_{k+1} N]-1} \sum_{l=1}^k p(x_t = l;\, x_s > k \text{ for } [\alpha_l N] \leqq s < t) \cdot Q(t, l) .$$

Then, for any fixed m we have

$$\bar{v}_N \geqq \sum_1^m p_k m_k \qquad \text{for } N \geqq m .$$

Using the equation

$$p_{l,l+1} = \prod_{[\alpha_l N]}^{[\alpha_{l+1} N]-1} (1 - l/t)$$

and letting $N \to \infty$, we get

$$p_{l,l+1} \to \left(\frac{\alpha_l}{\alpha_{l+1}}\right)^l$$

and

$$u_k \to \alpha_k{}^k \int_{\alpha_k}^{\alpha_{k+1}} \frac{1}{\lambda^{k+1}} \sum_1^k R_l(\lambda)\, d\lambda$$

so that

$$\liminf v_N \geqq \sum_{k=1}^m \left(\prod_1^k \alpha_l\right) \int_{\alpha_k}^{\alpha_{k+1}} \frac{1}{\lambda^{k+1}} \sum_1^k R_l(\lambda)\, d\lambda .$$

Now for $k \geqq 2$ we have

$$\frac{1}{\lambda^{k+1}} \sum_1^k R_l(\lambda) = -\frac{1}{k-1}\left(\frac{1}{\lambda^k} \sum_1^{k-1} R_l(\lambda)\right)' .$$

So we can write

$$\liminf \bar{v}_N \geqq \alpha_1 \int_{\alpha_1}^{\alpha_2} \frac{R_1(\lambda)}{\lambda^2}\, d\lambda - \sum_2^m \frac{1}{k-1} (G_k(\alpha_{k+1}) - H_k(\alpha_k)) \prod_1^k \alpha_l$$

$$- \sum_2^m \left(\frac{R_{k+1}(\alpha_{k+1})}{\alpha_{k+1}^k} - \frac{R_k(\alpha_k)}{\alpha_k^k}\right) \prod_1^k \alpha_l .$$

The second term is zero while the third term reduces to

$$\frac{\alpha_1}{\alpha_2} R_2(\alpha_2) - \frac{\prod_1^m \alpha_l}{\alpha_{m+1}^m} R_{m+1}(\alpha_{m+1}) \geqq \frac{\alpha_1}{\alpha_2} R_2(\alpha_2) - R_{m+1}(\alpha_{m+1})$$

$$\geqq \frac{\alpha_1}{\alpha_2} R_2(\alpha_2) - q(m+1) .$$

Now

$$\int_{\alpha_1}^{\alpha_2} \frac{R_1(\lambda)}{\lambda^2}\, d\lambda = \frac{R_1(\alpha_1)}{\alpha_1} - \frac{R_2(\alpha_2)}{\alpha_2},$$

so we have

$$\liminf \bar{v}_N \geqq R_1(\alpha_1) - q(m+1)\,.$$

Letting $m \to \infty$ we have $\liminf \bar{v}_N \geqq R_1(\alpha_1)$. Since $v_N \geqq \bar{v}_N$ and $v_N \to R_1(\alpha_1) = g(0)$, we have

$$\lim \bar{v}_N = g(0)\,.$$

Note. To be completely rigorous,

$$\lim u_k = \alpha_k{}^k \int_{\alpha_k}^{\alpha_{k+1}} \frac{1}{\lambda^{k+1}} \sum_1^k R_l(\lambda)\, d\lambda$$

is true only if q is truncated. One can get around this restriction quite easily, however, and the end result is the same. Again, details can be found in [8].

REFERENCES

[1] Breiman, L. (1964). Stopping rule problems, in *Applied Combinatorial Mathematics*, ed. E. Beckenbach. Wiley, New York.

[2] Chow, Y. S., Moriguti, S., Robbins, H. and Samuels, E. (1964). Optimum selection based on relative rank (The 'Secretary Problem'). *Israel J. Math.* **2** 81–90.

[3] Coddington, E. and Levison, N. (1955). *Theory of Ordinary Differential Equations*. McGraw-Hill, New York, Chapter 1.

[4] Dynkin, E. B. and Yushkevich, A. A. (1969). *Markov Processes; Theorems and Problems*. Plenum Press, New York, Chapter 3.

[5] Gilbert, J. P. and Mosteller, F. (1966). Recognizing the maximum of a sequence. *J. Amer. Statist. Assoc.* **61** 35–73.

[6] Gusein-Zade, S. M. (1966). The problem of choice and the optimum stopping rule for a sequence of independent trials. *Theor. Probability Appl.* **11** 472–476.

[7] Lindley, D. V. (1961). Dynamic programming and decision theory. *Appl. Statist.* **10** 39–51.

[8] Mucci, A. G. (0000). Differential equations related to optimal selection problems. Ph. D. dissertation.

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