The Annals of Statistics
1988, Vol. 16, No. 3, 1327–1329

ON THE STOCHASTIC ORDERING OF ABSOLUTE UNIVARIATE GAUSSIAN RANDOM VARIABLES[1]

BY PAUL S. HORN

University of Cincinnati

This paper presents necessary and sufficient conditions for the stochastic ordering of absolute univariate Gaussian random variables. This result is based on the behavior of the particular Gaussian density at zero, which is equivalent to restrictions on the mean and variance.

1. Introduction. In this paper we are concerned with the stochastic ordering of the absolute value of Gaussian random variables. Specifically, we say that the random variable X is stochastically smaller than the random variable Z, or $X <^{\text{st}} Z$, if and only if $P(X \leq t) \geq P(Z \leq t)$ for all t [Marshall and Olkin (1979), page 481]. It will be shown that the value of the density function at zero is necessary and sufficient for the stochastic ordering of the absolute value of an arbitrary Gaussian random variable with respect to the standard Gaussian random variable.

2. Main result. Before proving the main result, we need the following

LEMMA. *Suppose that* $0 < a \leq 1$ *and* $b > 0$ *and define*

$$f(x) = \cosh(x) - a\exp\{bx^2\}.$$

Then either $f(\cdot)$ *is negative on* $(0, \infty)$ *or changes sign exactly once from a positive to a negative sign.*

PROOF. The sign of $f(x)$ is the same as that of

$$g(x) = \log\cosh(x) - \log a - bx^2.$$

Thus we have

$$g'(x) = \tanh(x) - 2bx$$

and

$$g''(x) = \cosh^{-2}(x) - 2b.$$

Since $g''(0) = 1 - 2b$, $g''(\infty) = -2b$ and $g''(\cdot)$ is decreasing on $(0, \infty)$, either $g''(\cdot)$ is negative on $(0, \infty)$ or changes sign once from a positive to a negative sign. Further, as a result of the behavior of $g''(\cdot)$ and the fact that $g'(0) = 0$ and $g'(\infty) = -\infty$, we see that $g'(\cdot)$ is also either negative on $(0, \infty)$ or changes sign once from a positive to a negative sign. Finally, since $g'(\cdot)$ behaves as previously

Received February 1987; revised September 1987.
[1]Research supported in part by NSF Grant DMS-87-06044.
AMS 1980 *subject classifications*. Primary 62E99; secondary 60E05.
Key words and phrases. Folded normal, majorization, noncentral chi-square, stochastic ordering.

stated and $g(0) = -\log a \geq 0$ and $g(\infty) = -\infty$, it follows that $g(\cdot)$ is either negative on $(0, \infty)$ or changes sign once from a positive to a negative sign. □

We now state the main result.

THEOREM. *Let $X \sim N(\mu, \sigma^2)$ and $Z \sim N(0,1)$. Then*

$$|X| <^{\mathrm{st}} |Z| \text{ if and only if } \phi_x(0) \geq \phi_z(0),$$

where $\phi_x(\cdot)$ is the density function of X and $\phi_z(\cdot)$ is the density function of Z.

PROOF. We first note that the condition on the values of the density functions at zero, $\phi_x(0) \geq \phi_z(0)$, is equivalent to a restriction on μ and σ, namely, $\sigma^2 \leq 1$ and $\mu^2 \leq -\sigma^2 \ln \sigma^2$. Next we note that

$$\begin{aligned} &|X| <^{\mathrm{st}} |Z| \\ &\Leftrightarrow P(|X| \leq t) \geq P(|Z| \leq t) \quad \text{for all } t \\ &\Leftrightarrow P(-t \leq X \leq t) \geq P(-t \leq Z \leq t) \end{aligned} \tag{1}$$

and so the stochastic ordering condition is equivalent to the property that X is more peaked about zero than Z [Birnbaum (1945)].

Now (1) can be rewritten as

$$P((-t-\mu)/\sigma \leq Z \leq (t-\mu)/\sigma) \geq P(-t \leq Z \leq t) \quad \text{for } t > 0. \tag{2}$$

Note that $\sigma^2 \leq 1$ is necessary for (2) to hold. If not, then the interval $[(-t-\mu)/\sigma, (t-\mu)/\sigma]$ is at most as large as $[-t, t]$, but the latter is symmetric about zero. Define

$$B(t) = (\Phi((t+\mu)/\sigma) - \Phi(t)) - (\Phi(t) - \Phi((t-\mu)/\sigma)),$$

where $\Phi(\cdot)$ is the standard Gaussian distribution function and note that (2) is equivalent to $B(t) \geq 0$ for all $t \geq 0$. Since $B(0) = 0$, $B'(0) \geq 0$ is necessary, i.e.,

$$B'(0) = \left(2/(\sigma\sqrt{2\pi})\right)\exp\left(-\mu^2/(2\sigma^2)\right) - \sqrt{2/\pi} \geq 0$$

or

$$\mu^2 \leq -\sigma^2 \ln \sigma^2.$$

Thus necessity is proved.

For sufficiency, assume that $\sigma^2 \leq 1$ and $\mu^2 \leq -\sigma^2 \ln \sigma^2$. If $\sigma^2 = 1$, then $\mu = 0$ and $B \equiv 0$. Suppose therefore that $\sigma^2 < 1$. Note that the following properties imply $B(t) \geq 0$ for all $t \geq 0$:

(i) $B(0) = 0$.
(ii) $B(t) > 0$ for all $t \geq |\mu|/(1-\sigma)$.
(iii) $B'(t)$ is either negative on $(0, \infty)$ or changes sign exactly once from a positive to a negative sign.

Property (i) follows from previous arguments. Property (ii) is true since

$$t \geq |\mu|/(1-\sigma) \text{ implies } [-t, t] \subseteq [(-t-\mu)/\sigma, (t-\mu)/\sigma]$$

and (2) holds.

To show that property (iii) holds, note

$$
\begin{aligned}
B'(t) &= \left(1/\sigma\sqrt{2\pi}\right)\exp\left(-(t+\mu)^2/2\sigma^2\right) - \sqrt{2/\pi}\,\exp(-t^2/2) \\
&\quad + \left(1/\sigma\sqrt{2\pi}\right)\exp\left(-(t-\mu)^2/2\sigma^2\right) \qquad (3)\\
&= (1/\sigma)\sqrt{2/\pi}\,\cosh(\mu t/\sigma^2)\exp\left(-(t^2+\mu^2)/2\sigma^2\right) - \sqrt{2/\pi}\,\exp(-t^2/2).
\end{aligned}
$$

Thus $B'(t)$ has the same sign as

$$\cosh(x) - \sigma\exp(\mu^2/2\sigma^2)\exp\left(\sigma^2(1-\sigma^2)x^2/2\mu^2\right), \qquad (4)$$

where $x = |\mu|t/\sigma^2$. Since $\mu^2 \le -\sigma^2 \ln\sigma^2$ and $\sigma^2 < 1$,

$$\sigma\exp\left(\mu^2/2\sigma^2\right) \le 1 \quad \text{and} \quad \sigma^2(1-\sigma^2)/2\mu^2 > 0.$$

Thus property (iii) holds from the lemma and sufficiency is proved. □

3. Discussion. It should be noted that (3) is the difference of two folded normal densities, where a folded normal random variable is the absolute value of a normal random variable [Leone, Nelson and Nottingham (1961)]. It is also clear that the results proved here apply more generally to even convex functions of Gaussian random variables. If $\psi(\cdot)$ is an even convex function, then

$$\psi(X) <^{\mathrm{st}} \psi(Z) \Leftrightarrow P(\psi(X) \le u) \ge P(\psi(Z) \le u) \quad \text{for all } u,$$

which is equivalent to (1) when $t = \inf\{x: \psi(x) \ge u\}$.

Last, we consider the preceding situation when $\psi(X) = X^2$. The theorem thus provides necessary and sufficient conditions such that $X^2 <^{\mathrm{st}} \chi_1^2$, where X is $N(\mu, \sigma^2)$ and χ_1^2 is a chi-squared random variable with one degree of freedom. Taking this a step further, it is easily shown that

$$\sigma^2\chi_1^2\left(\mu^2/\sigma^2\right) <^{\mathrm{st}} \chi_1^2 <^{\mathrm{st}} \chi_1^2\left(\mu^2/\sigma^2\right)$$

if and only if the conditions of the theorem hold, where $\chi_1^2(\lambda)$ is a noncentral χ_1^2 with noncentrality parameter λ. Thus χ_1^2 is stochastically ordered between a noncentral chi-square and a constant times a noncentral chi-square under certain conditions.

REFERENCES

BIRNBAUM, Z. W. (1945). On random variables with comparable peakedness. *Ann. Math. Statist.* **16** 76–81.

LEONE, F. C., NELSON, L. S. and NOTTINGHAM, R. B. (1961). The folded normal distribution. *Technometrics* **3** 543–551.

MARSHALL, A. W. and OLKIN, I. (1979). *Inequalities: Theory of Majorization and Its Applications.* Academic, New York.

DEPARTMENT OF MATHEMATICAL SCIENCES
OLD CHEMISTRY BUILDING (ML 25)
UNIVERSITY OF CINCINNATI
CINCINNATI, OHIO 45221-0025