TESTING FOR LACK OF FIT IN NONLINEAR REGRESSION

BY JAMES W. NEILL

Kansas State University

The problem of testing the correctness of a nonlinear response function against unspecified general alternatives is considered. The proposed test statistic is a modification of a nonlinear analogue to the well-known linear regression lack-of-fit test and can be used with or without replication. Asymptotically valid critical points can be obtained from a central $F$-distribution. Also, when the null model is the orthogonal projection of the true model, the test statistic is asymptotically comparable to a random variable with a noncentral $F$-distribution.

1. Introduction. A nonlinear regression model with replication can be represented by

$$Y_{ik} = G(x_i, \theta) + \epsilon_{ik},$$

where $i = 1, 2, \ldots, M$, $k = 1, 2, \ldots, n_i$, and $n_i > 1$ for at least one $i$. The random errors $\epsilon_{ik}$ are assumed to be independent and identically distributed with zero mean and unknown finite variance $\sigma^2 > 0$. The $x_i$ are fixed input regression vectors contained in a compact subset $X$ of $\mathbb{R}^p$ and $\theta$ is an unknown $p$-dimensional parameter vector contained in a compact subset $\Theta$ of $\mathbb{R}^p$. In addition, the real-valued response function $G$ is assumed to be defined and continuous on $X \times \Theta$.

When the response function is linear in the unknown parameter vector, Graybill (1976) discussed some reasons for taking repeated observations at fixed values of the input vectors. These same reasons are noted to be applicable when the response function is nonlinear in the parameter vector. In particular, one important aspect of such a nonlinear model is that it allows one to test the correctness of the response function $G$ against unspecified general alternatives.

Let the total number of observations be denoted by $N = \sum n_i$ and let $\bar{G}(\theta)$ be defined by

$$\bar{G}(\theta) = \left( G(x_1, \theta)j_{n_1}', \ldots, G(x_M, \theta)j_{n_M}' \right)' ,$$

where $j_{n_i}$ is an $n_i \times 1$ vector of ones, $i = 1, 2, \ldots, M$. Also, $Y = (Y_{11}, \ldots, Y_{1n_1}, \ldots, Y_{Mn_M})'$ and $\bar{Y} = (\bar{Y}_1, j_{n_1}', \ldots, \bar{Y}_M, j_{n_M}')'$. To test

$$H_0: E(Y) = \bar{G}(\theta)$$

vs.

$$H_1: E(Y) \neq \bar{G}(\theta),$$

let the statistic $F$ be defined by

$$F = \frac{\left[(N - M)/(M - p)\right] \left(\|\bar{Y} - \bar{G}(\theta)\|^2/\|Y - \bar{Y}\|^2\right)}{\left(\|\bar{Y} - \bar{G}(\theta)\|^2/\|Y - \bar{Y}\|^2\right)} ,$$

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where $\hat{\theta}$ is the least-squares estimator of the parameter vector $\theta$ under model (1) as defined by Jennrich (1969), and $\|v\|^2$ represents $\sum v_i^2$, the squared norm of $v$ in $\mathbb{R}^d$. By Theorem 1 of Section 3, an asymptotic size $\alpha$ test of $H_0$ vs. $H_i$, is: Reject $H_0$ if observed $F > F_{M-p,N-M}^\alpha$, where $F_{M-p,N-M}^\alpha$ is the $(1 - \alpha)$th quantile of the central $F$-distribution $F(M - p, N - M)$. The preceding test is a nonlinear analogue to the well-known lack-of-fit--pure-error test for linear regression model adequacy with replication. However, as in the case of linear regression models, replication is not always possible or available in many nonlinear experimental applications.

Wald (1943), Gallant (1975a–c) and Milliken and DeBruin (1978) discussed test procedures to test general hypotheses about the parameters of a nonlinear model. In addition, Milliken and Graybill (1970) considered a model with linear terms in an unknown parameter vector and known nonlinear functions of estimable functions of the same parameter vector. However, in each of the preceding papers, a known correct functional form for the response function was assumed. Test statistics were then developed to test hypotheses about the parameters appearing in the assumed response function. Also, White (1981) proposed a test for nonlinear model misspecification based on the difference between the least-squares estimator and a weighted least-squares estimator of the unknown parameter vector in a specified nonlinear model. Assuming that the specified model is correct up to an independent additive error, a heuristically determined choice of the weight function was given by White.

The purpose of this paper is to present a statistical test to determine the correctness of a nonlinear response function, with or without replication, against unspecified general alternatives. Section 2 defines a nonlinear regression model without replication and develops a test statistic for model correctness. Null and nonnull asymptotic results are derived in Sections 3 and 4, respectively. Examples are discussed in Section 5.

2. Nonreplicated nonlinear models. An extension of the model given by (1) to a nonlinear regression model without replication can be represented by

$$Y_{ik} = G(x_{ik}, \theta) + \epsilon_{ik},$$

where $i = 1, 2, \ldots, M$, $k = 1, 2, \ldots, n_i$ and $n_i > 1$ for at least one $i$. The regression vectors $x_{ik}$ are considered to be of the form $x_i + \delta_{ik}$, where $\delta_{ik}$ characterizes the perturbation in the regression vector for the $k$th observation in the $i$th group. Except for the additional subscript $k$ on the regression vectors that allows for nonreplication, the models given by (1) and (2) are identical. A statistical test to determine the correctness of the response function $G$ appearing in model (2) against unspecified general alternatives will now be discussed.

Let $G(\theta)$ be defined by

$$G(\theta) = \left(G(x_{11}, \theta), \ldots, G(x_{1n_1}, \theta), \ldots, G(x_{Mn_M}, \theta)\right)^\prime.$$

Analogous to Neill and Johnson (1985), let $Y^*_{ik} = Y_{ik} - [G(x_{ik}, \theta) - G(x_i, \theta)]$ for $i = 1, 2, \ldots, M$, $k = 1, 2, \ldots, n_i$. Thus $Y^* = Y - [G(\theta) - \overline{G(\theta)}]$ so that $Y^* = \overline{G(\theta)} + \epsilon$ under model (2). Accordingly, the correctness of the response function
could be determined by the F-statistic given in Section 1 with Y replaced by \(Y^*\), provided \(Y^*\) were observable. As an alternative to \(Y^*\), define \(\hat{Y}^* = Y - [G(\hat{\theta}) - G(\hat{\theta})]\), where \(\hat{\theta}\) is the least-squares estimator of \(\theta\) under model (2). Then, using the asymptotic scheme given later and the uniform continuity of \(G, Y_{ik}^* - \hat{Y}_{ik}^* \to 0\) with probability 1 (w.p.1) as \(N \to \infty\). Let \(H_0(\Delta)\) denote the hypothesis that \(Y\) is correctly modeled by (2). Similarly, \(H_a(\Delta)\) represents unspecified general alternatives. To test

\[ H_0(\Delta): E(Y) = G(\theta) \]

vs.

\[ H_a(\Delta): E(Y) \neq G(\theta), \]

let the statistic \(\hat{F}^*\) be defined by

\[ \hat{F}^* = \left[ \left( \frac{N - M}{M - p} \right) \right] \left( \frac{\|\hat{Y}^* - \bar{G}(\tilde{\theta})\|^2}{\|\hat{Y}^* - \tilde{Y}^*\|^2} \right), \]

where \(\hat{Y}_{ik}^* = \sum_{i=1}^{n_i} \hat{Y}_{iik}/n_i\). By Theorem 1 of Section 3, an asymptotic size \(\alpha\) test of \(H_0(\Delta)\) vs. \(H_a(\Delta)\) is: Reject \(H_0(\Delta)\) if observed \(\hat{F}^* > F_{\alpha-p,N-M}^\alpha\).

In order to obtain asymptotic properties of \(\hat{F}^*\), suppose that \(X\) is partitioned into \(M\) cells. The partition is then refined in such a way that the volume of the largest cell converges to zero as \(M \to \infty\), where the number of observations per cell \(\{n_i: i = 1, 2, \ldots, M\}\), is uniformly bounded for all \(N\) and \(N/M\) is bounded away from 1. To facilitate the proofs of certain results, it will be assumed that the partition sequence is regular. In other words, a partition is a set of \(M\) \(s\)-dimensional rectangles and the maximum diameter over all \(M\) rectangles converges to zero as \(M \to \infty\). The data points for a common cell would then comprise a group of near replicate observations.

3. Asymptotic results for the null case. In addition to the assumption that the response function \(G\) in model (2) is continuous on the compact set \(X \times \Theta\), the following assumption is sufficient to determine the asymptotic distribution of \(\hat{F}^*\) under \(H_0(\Delta)\). The notation \(\langle u, v \rangle\) represents \(\Sigma u_i v_i\), the inner product of \(u\) and \(v\) in \(R^d\).

**Assumption (a).** \(\langle G(\tilde{\theta}), G(\tilde{\theta}) \rangle/N\) converges uniformly for all \(\bar{\theta}\) and \(\tilde{\theta}\) in \(\Theta\) as \(N \to \infty\) and \(\|G(\tilde{\theta}) - G(\theta)\|^2/N \to Q(\hat{\theta})\), where \(Q\) has a unique minimum at \(\tilde{\theta} = \theta\).

Any vector \(\tilde{\theta}_N\) contained in \(\Theta\), which minimizes \(Q_N(\hat{\theta}) = \|Y - G(\hat{\theta})\|^2/N\), is called a least-squares estimator of \(\theta\) based on the first \(N\) values of the response under model (2). By Lemma 2 of Jennrich (1969), there exists a least-squares estimator of \(\theta\) under \(H_0(\Delta)\). The following lemma corresponds to Theorem 6 of Jennrich.

**Lemma 1.** Under \(H_0(\Delta)\) and Assumption (a), \(\hat{\theta}_N \to \theta\) w.p. 1 as \(N \to \infty\).

* In Lemma 2 it will be assumed that the random errors have finite fourth moments. However, with regard to the response function, only the assumption that \(G\) is continuous on \(X \times \Theta\) will be used.
LEMMA 2. \( \|\hat{\mathbf{Y}}^* - \tilde{\mathbf{Y}}^*\|^2/(N - M) \to_p \sigma^2 \) as \( N \to \infty \) under \( H_0(\Delta) \).

PROOF. Under \( H_0(\Delta) \),
\[
\mathbf{Y}^* - \hat{\mathbf{Y}}^* = (\mathbf{e} - \tilde{\mathbf{e}}) + (G(\mathbf{\theta}) - \overline{G}(\mathbf{\theta})) - (G(\hat{\mathbf{\theta}}) - \overline{G}(\hat{\mathbf{\theta}})),
\]
where
\[
\tilde{\mathbf{e}}_{ik} = \sum_{i=1}^{n_i} \mathbf{e}_{ii}/n_i \quad \text{and} \quad \overline{G}(\mathbf{\theta}) = (\overline{G}_{1,\mathbf{\theta}}(\mathbf{\theta}) j'_{n_1}, \ldots, \overline{G}_{M,\mathbf{\theta}}(\mathbf{\theta}) j'_{n_M})',
\]
with
\[
\overline{G}_{i,\mathbf{\theta}} = \sum_k G(x_{ik}, \mathbf{\theta})/n_i, \quad i = 1, 2, \ldots, M, \quad k = 1, 2, \ldots, n_i.
\]
Since \( \|\mathbf{e} - \tilde{\mathbf{e}}\|^2/(N - M) \to_p \sigma^2 \) as \( N \to \infty \), it suffices to establish that \( \|G(\mathbf{\theta}) - \overline{G}(\mathbf{\theta})\|^2/(N - M) \to 0 \) and \( \|G(\hat{\mathbf{\theta}}) - \overline{G}(\hat{\mathbf{\theta}})\|^2/(N - M) \to_p 0 \) as \( N \to \infty \).

Let \( b_{iM}(\mathbf{\theta}) = \sum_k (G_{ik}(\mathbf{\theta}) - \overline{G}_{i,\mathbf{\theta}}(\mathbf{\theta}))^2/n_i \) and note that
\[
\|G(\mathbf{\theta}) - \overline{G}(\mathbf{\theta})\|^2/(N - M) = \sum_i \left[ \frac{\text{Mn}_i/(N - M)}{b_{iM}(\mathbf{\theta})/M} \right] \frac{b_{iM}(\mathbf{\theta})/M}{M} \to 0, \quad \text{as} \; N \to \infty.
\]

The proof that \( \|G(\hat{\mathbf{\theta}}) - \overline{G}(\hat{\mathbf{\theta}})\|^2/(N - M) \to_p 0 \) as \( N \to \infty \) is similar. Finally, \( \langle \mathbf{e} - \tilde{\mathbf{e}}, G(\mathbf{\theta}) - \overline{G}(\mathbf{\theta}) \rangle/(N - M) \to_p 0 \) as \( N \to \infty \) by the Schwarz inequality and the preceding convergence results. The convergence of the remaining inner product terms to zero is similar. \( \square \)

The asymptotic null distribution of \( \hat{\mathbf{Y}}^* \) will be derived next. Since the proof of Theorem 1 makes use of Lemma 2, it will be assumed that the random errors have finite fourth moments. Let \( F_0 = [(N - M)/(M - p)](\|\tilde{\mathbf{e}}\|^2/\|\mathbf{e} - \tilde{\mathbf{e}}\|^2). \)

THEOREM 1. Under \( H_0(\Delta) \) and Assumption (a), \( \hat{\mathbf{Y}}^* - F_0 \to_p 0 \) as \( N \to \infty \).

PROOF. Note that
\[
\hat{\mathbf{Y}}^* - F_0 = \left( \frac{\|\hat{\mathbf{Y}}^* - \overline{G}(\hat{\mathbf{\theta}})\|^2 - \|\tilde{\mathbf{e}}\|^2}{(M - p)} \right) + \left( \frac{\|\tilde{\mathbf{e}}\|^2/(M - p)}{(1 - \|\hat{\mathbf{Y}}^* - \tilde{\mathbf{Y}}^*\|^2/\|\mathbf{e} - \tilde{\mathbf{e}}\|^2)}/(\|\hat{\mathbf{Y}}^* - \tilde{\mathbf{Y}}^*\|^2/(N - M)) \right) \cdot \frac{\|\hat{\mathbf{Y}}^* - \tilde{\mathbf{Y}}^*\|^2/(N - M)}{M}.
\]

From Lemma 2, \( \|\hat{\mathbf{Y}}^* - \tilde{\mathbf{Y}}^*\|^2/(N - M) \to_p \sigma^2 \) and \( \|\mathbf{e} - \tilde{\mathbf{e}}\|^2/(N - M) \to_p \sigma^2 \) as \( N \to \infty \). Thus, since \( \|\tilde{\mathbf{e}}\|^2/(M - p) \to \sigma^2 \) as \( N \to \infty \), it suffices to show that \( \|\hat{\mathbf{Y}}^* - \overline{G}(\mathbf{\theta})\|^2 - \|\tilde{\mathbf{e}}\|^2)/M \to_p 0 \) as \( N \to \infty \). Since \( \hat{\mathbf{Y}}^* = \mathbf{Y} - [\overline{G}(\mathbf{\theta}) - \overline{G}(\hat{\mathbf{\theta}})], \) it
follows that $\hat{\tilde{Y}}^* - \tilde{G}(\hat{\theta}) = [\tilde{G}(\theta) - \tilde{G}(\hat{\theta})] + \tilde{\varepsilon}$ under $H_0(\Delta)$. Thus it suffices to show that $\|\tilde{G}(\theta) - \tilde{G}(\hat{\theta})\|^2/M \to_0$ as $N \to \infty$. Let $a_{iM}(\theta, \hat{\theta}) = (\tilde{G}_i(\theta) - \tilde{G}_i(\hat{\theta}))^2$ and note that

$$\|\tilde{G}(\theta) - \tilde{G}(\hat{\theta})\|^2/M = \sum_i n_i a_{iM}(\theta, \hat{\theta})/M.$$ 

Since $\hat{\theta} \to \theta$ w.p.1 as $N \to \infty$ by Lemma 1 and by the uniform continuity of $G$ on $X \times \Theta$, it follows that for $\theta$ in $\Theta$ and every $\varepsilon^* > 0$, there exists an $M_0$ such that $a_{iM}(\theta, \hat{\theta}) < \varepsilon^*$ for all $i = 1, 2, \ldots, M$ and $M \geq M_0$ w.p.1. Thus, since the sets \{ $n_i$: $i = 1, 2, \ldots, M$ \} are uniformly bounded for all $N$, $\Sigma_i n_i a_{iM}(\theta, \hat{\theta})/M \to 0$ w.p.1 as $N \to \infty$. \hfill \Box

Since $[(M - p)/M]F_0$ is distributed according to the central $F$-distribution $F(M, N - M)$ with normality, Theorem 1 implies the test procedure based on $\hat{F}^*$ is an asymptotic size $\alpha$ test of $H_0(\Delta)$ vs. $H_0(\Delta)$.

4. Asymptotic results for the nonnull case. Under $H_{\alpha}(\Delta)$, suppose that the true mean response differs from the specified response function by a real-valued function $H$. More precisely, suppose

$$Y_{ik} = G(x_{ik}, \theta) + H(z_{ik}, \gamma) + \varepsilon_{ik},$$

for $i = 1, 2, \ldots, M$, $k = 1, 2, \ldots, n_i$, where the regression vectors $z_{ik}$ are contained in a compact subset $Z$ of $R^l$ and vary continuously with the input vectors contained in $X$. Also, $\gamma$ is an unknown parameter vector contained in a compact subset $\Gamma$ of $R^q$. Lastly, $H$ is assumed to be continuous on $Z \times \Gamma$.

By Lemma 2 of Jennrich, there exists a least-squares estimator $\hat{\theta}_N$ of $\theta$ under (3) when the response is modeled by (2). In addition to Assumption (a), the following assumption will be used to show nonnull asymptotic results. Let $H(\gamma) = (H(z_{11}, \gamma), \ldots, H(z_{1n_1}, \gamma), \ldots, H(z_{Mn_M}, \gamma))'$. Note that the inner product condition for $G$ and $H$ given in Assumption (b) is a nonlinear analogue to the definition of orthogonal designs for linear regression models. This orthogonality requirement will be used to show the consistency of $\hat{\theta}_N$ for $\theta$ under alternative (3), and consequently the nonnull behavior of $\hat{F}^*$.

**Assumption (b)**. $\|H(\gamma)\|^2/N \to R(\gamma) < \infty$ for $\gamma$ in $\Gamma$ as $N \to \infty$ and $\langle G(\hat{\theta}), H(\gamma) \rangle/N \to 0$ uniformly for all $\hat{\theta}$ in $\Theta$ as $N \to \infty$.

**Lemma 3.** Under $H_{\alpha}(\Delta)$ defined by (3) and Assumptions (a) and (b), $\hat{\theta}_N \to \theta$ w.p.1 as $N \to \infty$.

**Proof.** Using Theorem 4 of Jennrich and Assumptions (a) and (b), it can be shown that $Q_N(\hat{\theta}) \to Q(\theta) + R(\gamma) + \sigma^2$ uniformly for all $\hat{\theta}$ in $\Theta$ and w.p.1 as $N \to \infty$. The proof is completed in a manner analogous to Theorem 6 of Jennrich. \hfill \Box

The following lemma shows that the pseudo pure error mean square is consistent for $\sigma^2$ regardless of whether or not the specified model is correct. In
Lemma 4 and Theorem 2, it will be assumed that the errors have finite fourth moments.

**LEMMA 4.** \[ \| \hat{Y}^* - \hat{Y}^* \|^2 / (N - M) \to_p \sigma^2 \text{ as } N \to \infty \text{ under } H_0(\Delta) \text{ defined by (3)}. \]

The proof of Lemma 4 is similar to the proof of Lemma 2 and is thus omitted. The asymptotic nonnull behavior of \( \hat{F}^* \) under (3) is considered in Theorem 2. Let \( F_0 = [(N - M)/(M - p)][\| \hat{H}(\gamma) + \hat{e} \|^2 / ||e - \hat{e}||^2] \) with

\[ \hat{H}(\gamma) = \left( H(z_1, \gamma) j_{n_1}', \ldots, H(z_M, \gamma) j_{n_M}' \right)', \]

where the \( z_i \) are representative group vectors analogous to the \( x_i, i = 1, 2, \ldots, M. \) Also, let \( \hat{H}(\gamma) = (\hat{H}_1(\gamma) j_{n_1}', \ldots, \hat{H}_M(\gamma) j_{n_M}')', \) where \( \hat{H}_i(\gamma) = \sum_k H(z_{ik}, \gamma)/n_i, \)

\( i = 1, 2, \ldots, M. \)

**THEOREM 2.** Under \( H_0(\Delta) \text{ defined by (3)} \) and Assumptions (a) and (b), \( \hat{F}^* - F_0 \to_p 0 \text{ as } N \to \infty. \)

**PROOF.** Similar to the proof of Theorem 1 and by Lemma 4, it suffices to show that \( \| \hat{F}^* - \hat{G}(\theta) \|^2 - \| \hat{H}(\gamma) + \hat{e} \|^2 \)/\( M \to_p 0 \text{ as } N \to \infty. \) Hence it suffices to note that \( \| \hat{G}(\theta) - \hat{G}(\hat{\theta}) \|^2 / M \to 0 \text{ as } N \to \infty \) by Lemma 3, and that \( \| \hat{H}(\gamma) - \hat{H}(\gamma) \|^2 / M \to 0 \text{ as } N \to \infty \) by the uniform continuity of \( H \) on \( Z \times \Gamma. \)

By Theorem 2, \( \hat{F}^* \) is asymptotically comparable to a random variable with the noncentral \( F \)-distribution \( F(M, N - M, \lambda), \) where the noncentrality parameter \( \lambda = \| \hat{H}(\gamma) \|^2 / \sigma^2. \)

5. **Examples.** Consider a regression model in which the response function of model (2) is given by \( G(x_{ik}, \theta) = \langle a(\theta), x_{ik} \rangle, \) where \( a \) is a continuous function mapping \( \Theta \) into \( R^s. \) The preceding model was discussed by Malinvaud (1970) who noted that this model covers many cases of nonlinear regression, often after a redefinition of explanatory variables and parameters.

To test the correctness of this response function, assume that the specified design space \( X \) has been partitioned into \( M \) cells so that data points for a common cell comprise a group of near replicate observations. In addition, the test procedure based on \( \hat{F}^* \) is naturally implemented by choosing \( x_i = \bar{x}_i, \) in \( \hat{G}(\theta), \) where \( \bar{x}_i, \) is the average input vector for the \( i \)th group of near replicates. The statistic \( \hat{F}^* \) can be computed as

\[ \hat{F}^* = [(N - M)/(M - p)] \left( \| \hat{r} \|^2 / ||r - \hat{r}||^2 \right), \]

where \( r = (r_{ik}) \) denotes the residual vector for model (2) and \( r_{ik} = \sum_{j=1}^{n_i} r_{ij}/n_i \) for \( i = 1, 2, \ldots, M, \) \( k = 1, 2, \ldots, n_i. \) For the case \( s = 1, \) let \( x = (x_{ik}) \) and note that Assumption (a) is satisfied by requiring that \( \| x \|^2 / N \) converges as \( N \to \infty \) and \( (a(\hat{\theta}) - a(\theta))^2 \) has a unique minimum at \( \hat{\theta} = \theta. \) By Theorem 1, \( H_0(\Delta) \) is rejected with asymptotic size \( \alpha \) if \( \hat{F}^* > \chi^2_{\alpha, M-p, N-M}. \)
With regard to power, consider alternatives of the form (3) with \( H(z_{ik}, \gamma) = \langle b(\gamma), z_{ik} \rangle \), where \( b(\gamma) \) is a continuous function mapping \( \Gamma \) into \( R^t \). For the case \( t = 1 \), let \( z = (z_{ik}) \) and note that Assumption (b) is satisfied by requiring that \( \|z\|^2/N \) converges as \( N \to \infty \) and \( \langle x, z \rangle /N \to 0 \) as \( N \to \infty \). By Theorem 2, \( \tilde{F}^a \) is asymptotically comparable to a random variable with the noncentral \( F \)-distribution \( F(M - p, N - M, \lambda) \), where \( \lambda = [b(\gamma)]^2\|\tilde{z}\|^2/\sigma^2 \) and \( \tilde{z}_{ik} = \sum_{j=1}^p z_{ij}/n_i \).

As a second example, consider a model in which \( G(x_{ik}, \theta) = \exp(\langle \theta, x_{ik} \rangle) \), where \( X \) and \( \Theta \) have the same dimension \( p \). Assuming \( X \) has been partitioned and \( x_i \) defined as in the previous example, the correctness of this exponential function can be determined by comparing the computed value of (4) to \( F_{M-p, N-M}^a \). For the case \( p = 1 \), suppose \( (x_{ik}) \) is a sequence of input values in \( X \) whose sample distribution function \( F_N \) approaches a distribution function \( F \) completely. Then, by Theorem 1 of Jennrich (1969),

\[
\langle G(\tilde{\theta}), G(\tilde{\theta}) \rangle /N \to \int \exp(\tilde{\theta}x)\exp(\tilde{\theta}x) \, dF(x),
\]
uniformly for all \( \tilde{\theta} \) and \( \tilde{\theta} \) in \( \Theta \) as \( N \to \infty \). In addition,

\[
||G(\tilde{\theta}) - G(\theta)||^2/N \to \int (\exp(\tilde{\theta}x) - \exp(\theta x))^2 \, dF(x),
\]
for which a unique minimum occurs at \( \tilde{\theta} = \theta \). Thus Assumption (a) is satisfied. By Theorem 1, the correctness of this exponential response function is rejected with asymptotic size \( \alpha \) if \( \tilde{F}^a > F_{M-p, N-M}^a \). For alternatives of the form (3) with \( Z = X \), Theorem 1 of Jennrich implies that

\[
\langle \exp(\tilde{\theta}), H(\gamma) \rangle /N \to \int \exp(\tilde{\theta}x)H(x, \gamma) \, dF(x),
\]
uniformly for all \( \tilde{\theta} \) in \( \Theta \) as \( N \to \infty \) and that

\[
||H(\gamma)||^2/N \to \int (H(x, \gamma))^2 \, dF(x) < \infty, \quad \text{for } \gamma \in \Gamma,
\]
as \( N \to \infty \). Assumption (b) is satisfied by requiring that

\[
\int \exp(\tilde{\theta}x)H(x, \gamma) \, dF(x) = 0
\]
for all \( \tilde{\theta} \) in \( \Theta \) in which case Theorem 2 is applicable. Although Assumption (b) may be difficult to satisfy exactly, it may be possible to choose the sequence \( (x_{ik}) \) so that the integral of \( \exp(\tilde{\theta}x)H(x, \gamma) \) with respect to \( dF \) measure approximates zero for all \( \tilde{\theta} \) in \( \Theta \). Thus an approximate evaluation of power for a particular alternative would be obtained by Theorem 2. The case for which \( Z \neq X \) is obtained similarly by Theorem 1 of Jennrich.

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Department of Statistics
Kansas State University
Manhattan, Kansas 66506