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ON THE MINIMAX VALUE IN THE SCALE MODEL WITH TRUNCATED DATA

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Let X be a positive random variable with Lebesgue density $f_\theta(x)$, where θ is the scale parameter, and let Y be a positive random variable independent of X. We consider two models of truncation: the LHS model, where the data consist only of those observations of X for which $X > Y$; and the RHS model, where the data consist of those observations of X for which $X \leq Y$. Consider the problem of estimating θ^s, $s \neq 0$, under a normalized squared error loss function. It is shown that under appropriate assumptions, if $f_1(\cdot)$ varies regularly at 0 (or $+\infty$), then the minimax value in the RHS (LHS) model is equal to 1 for arbitrarily large sample size. This implies the existence of trivial minimax and admissible estimators, which do not depend on the sample at all, in contrast with the scale model without truncation.

1. Introduction. Let X and Y be a pair of independent positive random variables and suppose that the data consist only of those observations of X for which $X > Y$. Such models arise, for example, in situations when measurement of the quantity X is the only way used to identify some physical objects. Then the random variable Y is concentrated on the smallest quantity y_0, which can be observed by a physicist using his measuring instrument; if $X > y_0$, the physicist observes X and identifies some element of the data, but if not, nothing is observed. In Woodroofe (1985) one can find a detailed description of this model in astronomy with a list of references.

The so-called time-truncated sampling in life testing, which relies on putting an unspecified number of items on test until a fixed length of time has elapsed, is another example of models with truncated data. It is usually assumed that each item on test has an exponential distribution with density $(1/\theta)\exp(-t/\theta)$, $t \geq 0, \theta > 0$. Let $[0, y]$ be the duration of experiment, N be the number of failures observed in $[0, y]$ and $T_1 < \cdots < T_N$ be the observed failure times. Then it can easily be shown that $T_1, \dots, T_N$, given $N = n$, have the same joint density as the order statistics of a random sample $X_1, \dots, X_n$ from the density

$$\exp(-x/\theta)/[\theta(1 - \exp(-y/\theta))], \qquad 0 \leq x \leq y.$$

In this paper we investigate the scale parameter model truncated as described previously. Let

$$f_\theta(x) = (1/\theta)f(x/\theta), \qquad \theta \in H \subset (0, \infty), \tag{1.1}$$

be a scale parameter family of probability densities on $(0, \infty)$ relative to Lebesgue measure. Let Y be a positive random variable with cumulative distribution function Q; Y is the (possibly random) truncation time.

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Let X denote the observed truncated random variable. In the left-hand side (LHS) model, X has density

$$g_\theta(x) = C_\theta \int f_\theta(x) 1_{(y,\infty)}(x) Q(dy), \tag{1.2}$$

with

$$C_\theta^{-1} = \int \int f_\theta(x) 1_{(y,\infty)}(x) Q(dy)\, dx.$$

In the right-hand side (RHS) model, X has density

$$g_\theta(x) = C_\theta \int f_\theta(x) 1_{(0,y)}(x) Q(dy), \tag{1.2'}$$

with

$$C_\theta^{-1} = \int \int f_\theta(x) 1_{(0,y)}(x) Q(dy)\, dx.$$

The observed data consist of n independent copies of X, namely, $X_1, \dots, X_n$. The problem treated here is that of estimating θ^s, where $s \neq 0$, under squared error loss with the usual weight $h(\theta) = \theta^{-2s}$.

The main aim of this article is to present a curiosity that concerns the minimax value in both truncation models. The first result is that if there exists

$$\lim_{x \to 0} |xf'(x)/f(x)| < \infty, \tag{1.3}$$

then the minimax value in the RHS model is equal to 1 for every sample size n. This implies the existence of minimax and admissible estimators, which depend neither on the sample $X_1, \dots, X_n$ nor on n. Condition (1.3) is nearly equivalent to regular variation of the density f and its distribution function F (see Remarks 3.3 and 3.5) and is satisfied for distributions belonging to the families of Gamma or F-distributions. The conclusion of practical importance is that minimax estimation for the distributions satisfying (1.3) is very sensitive to the RHS truncation of the data.

In the LHS model the condition $\lim_{x \to \infty} |xf'(x)/f(x)| < \infty$ implies similar effects as (1.3) in the previous case; in particular, for distributions regularly varying at infinity, such as inverse Gamma or F-distributions, trivial estimators may be both minimax and admissible.

Finally, we investigate the minimax value $V_{n,m}$ for estimators based on two i.i.d. samples $X_1, \dots, X_n$ and $X_1', \dots, X_m'$ obtained from the model without truncation and the RHS (LHS) model, respectively. It turns out under conditions like those given previously that $V_{n,m}$ is of order $1/n$ and estimators based only on the first sample still remain minimax after adding the second one.

2. An improper Cramér–Rao lower bound. Initiated by Hodges and Lehmann (1951), the idea of applying the Cramér–Rao inequality in admissibility proofs has received much attention in the literature. See, for example, Blyth and Roberts (1972) or Lehmann (1983) for a modern version of this method being designed to prove both admissibility and minimaxity. In this section we present a new method, which also, explicitly, makes use of the Cramér–Rao inequality.

Let X be the n-dimensional random variable with probability distribution

$$P_\theta(dx) = p_\theta(x) \mu(dx),$$

for $x \in \mathbb{R}^n$, $\theta \in H$, where H is an open interval of the real line and μ is some σ-finite measure on $\mathbb{R}^n$. Suppose that τ is a real-valued estimable function on H and $t(X)$ is an estimator of $\tau(\theta)$. Throughout this article, we shall consider only the estimators for which the Cramér–Rao inequality holds. This restriction is of minor importance because Fabian and Hannan (1977) have proved that, under some regularity conditions, the Cramér–Rao lower bound is valid without any assumptions about the estimator except for the trivial condition

$$\mathrm{Var}_\theta(t) < \infty.$$

More convenient conditions, which imply those of Fabian and Hannan, can be found in Brown (1986). Another set of regularity conditions, which concerns differentiability in quadratic mean of the square root of the density p_θ with respect to θ, is given in Klaassen and van Zwet (1985). Each set of regularity assumptions mentioned implies

$$\frac{\partial}{\partial\theta}\int t(x)p_\theta(x)\,d\mu(x) = \int t(x)\frac{\partial}{\partial\theta}p_\theta(x)\,d\mu(x) \tag{2.1}$$

and

$$E_\theta[t(X) - \tau(\theta)]^2 \geq B^2(\theta) + [\tau'(\theta) + B'(\theta)]^2/I(\theta), \tag{2.2}$$

where

$$I(\theta) = \mathrm{Var}_\theta\left[\frac{\partial}{\partial\theta}\log p_\theta(X)\right] \quad \text{and} \quad B(\theta) = E_\theta[t(X) - \tau(\theta)].$$

Therefore, instead of choosing one of many possible assumptions, we assume simply that (2.1) and (2.2) hold throughout this article, referring the reader to one of the papers qucted previously.

In this section we show that the risk of each estimator that satisfies (2.2) can be bounded from below at some points belonging to the compactification of the parameter space H by a positive quantity that does not depend on the estimator. Since such points may not belong to H, this quantity will be called an improper Cramér–Rao lower bound (ICRB). In the following discussion we present a brief sketch of the proof of ICRB given by Gajek (1987); actually, another proof can be found in Brown (1986).

Assume, for simplicity, that $\tau(\theta) = \theta$. Let h be a positive weight function and observe that (2.2) implies

$$\begin{aligned} E_\theta[t(X) - \theta]^2 h(\theta) &\geq \left[\frac{B^2(\theta)}{\theta^2} + \frac{(1 + B'(\theta))^2}{\theta^2 I(\theta)}\right]\theta^2 h(\theta) \\ &\geq \left[B'(\theta)^2 + \frac{(1 + B'(\theta))^2}{\theta^2 I(\theta)}\right]\theta^2 h(\theta) \\ &\quad + \left[\frac{B^2(\theta)}{\theta^2} - B'(\theta)^2\right]\theta^2 h(\theta) \\ &\geq \frac{\theta^2 h(\theta)}{\theta^2 I(\theta) + 1} + \left[\left(\frac{B(\theta)}{\theta}\right)^2 - B'(\theta)^2\right]\theta^2 h(\theta), \end{aligned} \tag{2.3}$$

where the last inequality follows from $z^2 + a(1+z)^2 \geq a/(a+1)$, which holds for each $a \geq 0$. We shall show that from (2.3),

$$(2.4)\qquad \limsup_{\theta \to 0} E_\theta[t(X) - \theta]^2 h(\theta) \geq \lim_{\theta \to 0} \theta^2 h(\theta)(\theta^2 I(\theta) + 1)^{-1},$$

provided that the limits of $\theta^2 I(\theta)$ and $\theta^2 h(\theta)$ exist as $\theta \to 0$ and that the latter is finite. To begin, observe that the condition $\lim_{\theta \to 0} \theta^2 h(\theta) < \infty$ and the first inequality in (2.3) imply that (2.4) holds if $B(\theta)$ does not tend to 0 as $\theta \to 0$. On the other hand, if $\lim_{\theta \to 0} B(\theta) = 0$, Cauchy's theorem implies that for every $\theta > 0$, there is $\xi \in (0, \theta)$ such that $B(\theta)/\theta = B'(\xi)$. Therefore

$$(2.5)\qquad \liminf_{\theta \to 0} B'(\theta) \leq \liminf_{\theta \to 0} \frac{B(\theta)}{\theta} \leq \limsup_{\theta \to 0} \frac{B(\theta)}{\theta} \leq \limsup_{\theta \to 0} B'(\theta).$$

In order to prove (2.4), it is sufficient to show that

$$(2.6)\qquad \limsup_{\theta \to 0} \left[\left(\frac{B(\theta)}{\theta} \right)^2 - B'(\theta)^2 \right] \geq 0$$

and to apply (2.3). Assume that (2.6) is not fulfilled. Then, for some $\varepsilon > 0$, $(B(\theta)/\theta)^2 - B'(\theta)^2 < 0$ for $\theta \in (0, \varepsilon)$, and by Theorem 5.12 in Rudin (1976), page 108, we obtain that either $B'(\theta) > 0$ for $\theta \in (0, \varepsilon)$ or $B'(\theta) < 0$ for $\theta \in (0, \varepsilon)$. In the first case we have

$$-B'(\theta) < \frac{B(\theta)}{\theta} < B'(\theta),$$

which implies that $B(\theta)/\theta$ is increasing on $(0, \varepsilon)$ and, moreover, that all limits in (2.5), except for the last one, are equal to each other. If they are finite, then (2.6) is satisfied, a contradiction; if not, then (2.4) follows directly from (2.3). The second case can be treated in the same way and this establishes (2.4).

Inequality (2.4) is a special case of the following general ICRB.

(2.7) THEOREM. *Suppose that H is an open interval, τ is a diffeomorphism on H and θ^* is a point (possibly $\pm\infty$) belonging to the compactification of H such that $\lim_{\theta \to \theta^*} \tau(\theta)$ exists and is equal to 0 or $\pm\infty$. Under such regularity conditions that* (2.1) *and* (2.2) *hold, we have*

$$(2.8)\qquad \limsup_{\theta \to \theta^*} E_\theta(t(X) - \tau(\theta))^2 h(\theta) \geq \lim_{\theta \to \theta^*} \frac{\tau^2(\theta) h(\theta)}{(\tau'(\theta)/\tau(\theta))^{-2} I(\theta) + 1},$$

provided that the right-hand side of (2.8) *exists and is finite.*

The right-hand side of (2.8) does not depend on the estimator. It is rather unexpected that, when compared with the corresponding limit of the Cramér–Rao lower bound for unbiased estimators, the denominators differ exactly by 1.

Evidently, (2.8) is immediately applicable in minimaxity proofs. In particular, a number of standard minimaxity results can be quickly derived from it. In the following discussion we present an example of the application of the ICRB in the scale model without truncation; more complex LHS and RHS models are treated in the next sections. A version of the ICRB adapted to admissibility proofs can be found in Gajek (1987).

Let $X_1, \dots, X_n$ be n independent copies of the random variable X in the scale model (1.1). Under such regularity assumptions that (2.1) and (2.2) hold, let V_n be the minimax value for estimating θ^s on the basis of $X_1, \dots, X_n$. Let $I_n(\theta)$ be the Fisher information in the model and observe that

$$
\begin{aligned}
I_n(\theta) &= n\int_0^\infty \left[\frac{\partial}{\partial\theta}\log\left(\frac{1}{\theta}f\left(\frac{x}{\theta}\right)\right)\right]^2 \frac{1}{\theta}f\left(\frac{x}{\theta}\right)dx \\
&= n\int_0^\infty \frac{1}{\theta^2}\left[1+\frac{x}{\theta}f'\left(\frac{x}{\theta}\right)\Big/f\left(\frac{x}{\theta}\right)\right]^2 \frac{1}{\theta}f\left(\frac{x}{\theta}\right)dx \qquad (2.9)\\
&= \frac{n}{\theta^2}E_1[1+Xf'(X)/f(X)]^2.
\end{aligned}
$$

Though the regularity conditions are related to $I_n(\theta) < \infty$, it is convenient to assume explicitly throughout the paper that

$$E_1[1+Xf'(X)/f(X)]^2 < \infty. \qquad (2.10)$$

(2.11) **Proposition.** *If* $H \supset (0, \bar{\theta})$ *for some* $\bar{\theta} > 0$ *or* $H \supset (\underline{\theta}, \infty)$ *for some* $\underline{\theta} < \infty$, *then*

$$V_n \ge \left(1 + ns^{-2}E_1[1+Xf'(X)/f(X)]^2\right)^{-1}.$$

Proof. The result follows from (2.7) and (2.9). □

3. The scale model with truncated data. Let $X_1, \dots, X_n$ be n independent copies of X from the LHS or RHS model, respectively. Assume that (2.1) and (2.2) hold and denote by V_n the minimax value for estimating θ^s, $s \neq 0$, on the basis of the sample $X_1, \dots, X_n$. Let $R(y) = 1 - Q(y)$, $y > 0$.

(3.1) **Theorem.** *Assume that* $H \supset (0, \bar{\theta})$ *for some* $\bar{\theta} > 0$ [*or* $H \supset (\underline{\theta}, \infty)$ *for some* $\underline{\theta} < \infty$]. *If*

(i) *for each* $0 < v < \infty$,

$$\lim_{\theta\to 0^+} Q(v\theta)/E_1[Q(X\theta)] = 0 \qquad \left[\lim_{\theta\to\infty} R(v\theta)/E_1[R(X\theta)] = 0\right],$$

(ii) *the limit*

$$\lim_{x\to\infty} f'(x)x/f(x) \qquad \left[\lim_{x\to 0^+} f'(x)x/f(x)\right]$$

exists and is finite,

then in the LHS (RHS) *model of truncation* $V_n \ge 1$ *for all* n.

The proof of (3.1) is an immediate consequence of Theorem 2.7 and Lemma 4.2, which is given in the next section.

(3.2) **Remark.** Conditions (i) mean that the tails of the truncating distribution Q should be small enough when compared with those of the truncated distribution F. Clearly, (i) is satisfied for any f in the LHS (RHS) model if Q has a support separated from 0 (∞) and $f(x) > 0$, $x \in (0, \infty)$.

(3.3) REMARK. If $\lim_{x \to 0} f'(x)x/f(x) = c$, where $-\infty < c < \infty$, then by l'Hospital's rule

$$\lim_{x \to 0} f(x)x \Big/ \int_0^x f(u)\,du = c + 1. \tag{3.4}$$

Since (3.4) implies $c + 1 \geq 0$, it follows from Feller (1971), Theorem 1(b), page 281, that

(1) $f(x)$ and $F(x)$ both vary regularly at 0 with exponents c and $c+1$, respectively, when $c > -1$;
(2) $F(X)$ varies slowly, when $c = -1$.

(3.5) REMARK. The condition $\lim_{x \to \infty} f'(x)x/f(x) = c$, where $-\infty < c < \infty$, implies

$$\lim_{x \to \infty} f(x)x \Big/ \int_x^\infty f(u)\,du = -c - 1 \geq 0$$

and by Feller (1971), Theorem 1(a), page 281, we have that

(1) $f(x)$ and $1 - F(x) = \int_x^\infty f(u)\,du$ both vary regularly at infinity with exponents c and $c+1$, respectively, when $c < -1$;
(2) $1 - F(x)$ varies slowly, when $c = -1$.

(3.6) REMARK. In a sense, conditions (ii) are symmetrical. Clearly, if (ii) holds for X in one of the models, then it also does in the second model after substitution $X' = 1/X$ and $\theta' = 1/\theta$.

Theorem 3.1 has the following rather unexpected corollary.

(3.7) COROLLARY. *Assume that conditions* (i) *and* (ii) *of* (3.1) *are satisfied. If* $H = [\theta_1, \infty)$, *where* $\infty > \theta_1 > 0$, *then* $t_n = C^s$ *for any* $C \in [\theta_1, 2^{1/s}\theta_1]$ *is a minimax and admissible estimator of* θ^s, $s > 0$, *in the* RHS *model. Similarly, if* $H = (0, \theta_2]$, *where* $\infty > \theta_2 > 0$, *then* $t_n = C^s$ *for any* $C \in [2^{1/s}\theta_2, \theta_2]$ *is a minimax and admissible estimator of* θ^s, $s < 0$, *in the* LHS *model.*

In fact, minimaxity of $t_n = C^s$ follows from (3.1) and the inequality $E_\theta(C^s - \theta^s)^2\theta^{-2s} \leq 1$, which holds for every C such that $\theta_i^s \leq C^s \leq 2\theta_i^s$, $i = 1, 2$, in the RHS or LHS model, respectively. In order to prove admissibility, it is sufficient to notice that $t_n = C^s$ is the unique locally optimal estimator at the point $\theta = C$.

Another anomaly is that the estimator $t_n = 0$, which is rather absurd in the scale model, is minimax for each $s \neq 0$ and every $n > 0$ in both truncation models.

Now, we present some examples of scale parameter families that satisfy conditions (ii).

EXAMPLES. (a) Gamma distributions. Suppose that

$$f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} 1_{(0,\infty)}(x),$$

with $\alpha > 0$. Then $f'(x)x/f(x) \to \alpha - 1$ as $x \to 0$.

(b) Inverse Gamma distributions. Let

$$f(x) = \frac{1}{\Gamma(\alpha) x^{\alpha+1}} e^{-1/x} 1_{(0,\infty)}(x),$$

with $\alpha > 0$. Then $f'(x)x/f(x) \to -\alpha - 1$ as $x \to \infty$.

(c) F-distributions with α and β degrees of freedom. Let

$$f(x) = \frac{\Gamma[(\alpha+\beta)/2]\,\alpha^{\alpha/2}\beta^{\beta/2}}{\Gamma(\alpha/2)\Gamma(\beta/2)} \frac{x^{\alpha/2-1}}{(\beta+\alpha x)^{(\alpha+\beta)/2}} 1_{(0,\infty)}(x),$$

with $\alpha > 0$ and $\beta > 0$. Then

$$\begin{aligned} f'(x)x/f(x) &\to -1 + \alpha/2, \quad \text{as } x \to 0, \\ &\to -1 - \beta/2, \quad \text{as } x \to \infty. \end{aligned}$$

In all the preceding families the truncation anomalies occur provided that assumption (i) of (3.1) is fulfilled. Of course, (i) holds in the case of (3.2); however, many other distributions of Y are possible because the condition is very mild. If, for example, EY is finite, then (i) is satisfied in the RHS model for every f such that $\lim_{x \to 0^+} f(x) > 0$. To see this, observe that G_θ defined by

$$dG_\theta(u) = R(u\theta)\theta\, du (EY)^{-1}$$

converges weakly as $\theta \to \infty$ to the distribution concentrated at 0. Therefore

$$\lim_{\theta\to\infty} \frac{R(u\theta)}{E_1[R(X\theta)]} = \lim_{\theta\to\infty} \frac{\theta R(u\theta)(EY)^{-1}}{\int f(u)\, dG_\theta(u)} = 0.$$

Though the Gamma distributions are very sensitive only to the RHS truncation of the data, in the LHS case some modification of standard procedures are also needed. As an example, consider the data $X_1, \dots, X_n$ from the exponential distribution truncated by Y, which is concentrated at some $y_0 > 0$. The usual estimator of θ in the model without truncation is $t_n = (n+1)^{-1}\sum_{i=1}^n X_i$, which is minimax and admissible if $\theta \in H = (0, \infty)$. One can verify directly that in the LHS model this estimator produces unbounded risk for each $y_0 > 0$. It is well known that

$$\hat{t}_n = (n+1)^{-1} \sum_{i=1}^{n} (X_i - y_0)$$

is a minimax and admissible modification of t_n, provided that y_0 is known. If y_0 is not known, a natural modification of t_n would be

$$t_n^* = c_n \sum_{i=1}^{n} (X_i - \min(X_1, \dots, X_n)),$$

which, however, could not be proved to be minimax by a straightforward use of the Cramér–Rao inequality (2.2) for the simple reason that the bound is not sharp then. In fact, y_0 (if unknown) is a kind of nuisance parameter and the Cramér–Rao inequality can be improved as is shown by Klaassen and van Zwet (1985) and Klaassen, van der Vaart and van Zwet (1987).

4. Two-sample problem. Now we shall investigate the minimax value $V_{n,m}$ for estimators of θ^s based on two independent samples $X_1, \dots, X_n$ and $X_1', \dots, X_m'$, which are fully observable and RHS (LHS) truncated, respectively.

(4.1) PROPOSITION. *Under the assumptions of* (3.1),

$$V_{n,m} \geq \left(1 + ns^{-2}E_1[1 + Xf'(X)/f(X)]^2\right)^{-1},$$

in both truncation models.

Proofs of (4.1) and (3.1) are based on the following lemma.

(4.2) LEMMA. *Let $I(\theta)$ denote the Fisher information of X truncated by Y in the LHS (or RHS) model. If conditions* (i) *and* (ii) *of* (3.1) *are satisfied, then*

$$\theta^2 I(\theta) \to 0,$$

as $\theta \to 0$ (or $\theta \to \infty$).

PROOF. We shall prove the lemma only for the LHS model because the second model can be treated by analogy. From (1.2), it follows that $g_\theta(x) = C_\theta f_\theta(x)Q(x)$, where $C_\theta^{-1} = \int f_\theta(x)Q(x)\,dx$. Therefore

$$I(\theta) = E_\theta\left[\frac{\partial/\partial\theta\, f_\theta(X)}{f_\theta(X)} - \frac{\partial/\partial\theta \int f_\theta(x)Q(x)\,dx}{\int f_\theta(x)Q(x)\,dx}\right]^2$$

$$= E_\theta\left[\frac{\partial/\partial\theta\, f_\theta(X)}{f_\theta(X)}\right]^2 - 2\frac{\int \partial/\partial\theta\, f_\theta(x)Q(x)\,dx}{\int f_\theta(x)Q(x)\,dx}E_\theta\left[\frac{\partial/\partial\theta\, f_\theta(X)}{f_\theta(X)}\right] + \left[\frac{\int \partial/\partial\theta\, f_\theta(x)Q(x)\,dx}{\int f_\theta(x)Q(x)\,dx}\right]^2$$

$$(4.3)\qquad = \int\left[\frac{\partial/\partial\theta\, f_\theta(x)}{f_\theta(x)}\right]^2 g_\theta(x)\,dx - \left[\frac{\int \partial/\partial\theta\, f_\theta(x)Q(x)\,dx}{\int f_\theta(x)Q(x)\,dx}\right]^2$$

$$= \theta^{-2}\left\{\int\left(1 + \frac{f'(x/\theta)x/\theta}{f(x/\theta)}\right)^2 \frac{f(x/\theta)Q(x)\,dx}{\int f(x/\theta)Q(x)\,dx} - \left[\int\left(1 + \frac{f'(x/\theta)x/\theta}{f(x/\theta)}\right)\frac{f(x/\theta)Q(x)\,dx}{\int f(x/\theta)Q(x)\,dx}\right]^2\right\}$$

$$= \theta^{-2}\left\{\int\left(1 + \frac{f'(u)u}{f(u)}\right)^2 d\psi_\theta(u) - \left[\int\left(1 + \frac{f'(u)u}{f(u)}\right)d\psi_\theta(u)\right]^2\right\},$$

where

$$d\psi_\theta(u) = f(u)Q(u\theta)\,du/\left(\int f(u)Q(u\theta)\,du\right).$$

Since for any $0 < v < \infty$,

$$\psi_\theta(v) = \int_0^v f(u)Q(u\theta)\,du \Big/ \int f(u)Q(u\theta)\,du$$
$$\leq Q(v\theta) \Big/ \int f(u)Q(u\theta)\,du,$$

therefore from (i) it follows that ψ_θ converges weakly (as $\theta \to 0^+$) to the distribution concentrated at infinity. Hence, by (i), (ii) and (2.10), we have

$$\int \left(1 + \frac{f'(u)u}{f(u)}\right)^2 d\psi_\theta(u) \underset{\theta \to 0^+}{\to} \lim_{x \to \infty} \left(1 + \frac{f'(x)x}{f(x)}\right)^2. \tag{4.4}$$

In a similar way, we can obtain

$$\int \left(1 + \frac{f'(u)u}{f(u)}\right) d\psi_\theta(u) \underset{\theta \to 0^+}{\to} \lim_{x \to \infty} \left(1 + \frac{f'(x)x}{f(x)}\right). \tag{4.5}$$

Eventually, from (4.3)–(4.5), the result follows. □

Theorem 3.1 follows directly from (2.7) and (4.2), whereas (4.1) can be established by compiling proofs of (2.11) and (3.1).

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