*The Annals of Statistics*
1988, Vol. 16, No. 1, 323–341

# COMBINATION OF REPRODUCTIVE MODELS

BY O. E. BARNDORFF-NIELSEN AND P. BLÆSILD

*University of Aarhus*

Suppose $s$ is a random variate that follows a statistical model with parameter $\omega$, and let $s_1, s_2, \ldots, s_n, \ldots$ be independent and identically distributed observations of $s$. The model is *reproductive* in $s$ and $\omega$ if for any $n$ the mean $\bar{s} = (s_1 + \cdots + s_n)/n$ follows the same model as $s$ but with parameter $n\omega$ instead of $\omega$. Suitable combinations of reproductive models yield reproductive models for higher-dimensional variates. This combination technique is discussed and illustrated by examples. It is possible, in particular, to construct reproductive combinations of gamma, inverse-Gaussian and Gaussian distributions, determined by a regression structure, which may conveniently be described in graph-theoretic terms. The graph-theoretical interpretation makes it feasible to draw conclusions about conditional independencies in the models concerned, by means of a very general result for Markovian-type probability laws on graphs due to Kiiveri, Speed and Carlin (1984). Most of the models discussed are exponential, of a form, which in conjunction with the reproductivity, implies various useful distributional properties, derivable from the general theory of reproductive exponential models.

**1. Introduction.** Let $\mathscr{D} = \{D(\omega) \colon \omega \in \Omega\}$, where $\Omega \subset R^k$, be a parametric family of probability distributions on a sample space $\mathscr{X}$ and let $s$ be a statistic on $\mathscr{X}$. We write $s \sim D(\omega)$ if $s$ is distributed according to $D(\omega)$, and we let $s_1, s_2, \ldots, s_n$ denote independent and identically distributed copies of $s$.

The pair $(\mathscr{D}, s)$ is said to be *reproductive* in $\omega$ if for all $\omega \in \Omega$ and all $n = 1, 2, \ldots,$ we have

(i) $n\Omega \subset \Omega$

and

(ii) if $s \sim D(\omega)$, then $\bar{s} = n^{-1}(s_1 + \cdots + s_n) \sim D(n\omega)$.

When it appears from the context what $\mathscr{D}$ or $s$ is at issue, we simply speak of $s$ or $\mathscr{D}$ as being reproductive.

Our definition of reproductivity of $(\mathscr{D}, s)$ refers to a specific parametrization $\omega$. To make the definition intrinsic to the pair $(\mathscr{D}, s)$, one could simply call $(\mathscr{D}, s)$ reproductive if it is reproductive in the above sense for some parametrization of $\mathscr{D}$.

For any statistic $s$ of dimension $k_s$, say, we denote the Fourier–Laplace transform of $s$ by $\bar{\varphi}_s$. That is,

$$\bar{\varphi}_s(z; \omega) = E_\omega\{e^{z \cdot s}\}, \qquad z \in Z_{s,\omega},$$

Received November 1984; revised March 1987.

*AMS* 1980 *subject classifications*. 62E10, 62E15, 62F99, 62J99.

*Key words and phrases*. Chi-squared distribution, conditional independence, exponential models, gamma distribution, Gaussian distribution, inverse-Gaussian distribution, Markovian probability laws, maximum-likelihood estimation, oriented graphs, stable distributions.

where $Z_{s,\omega}$ is the set of complex vectors $z$ for which the mean value exists. Writing elements $z$ of $\mathbb{C}^{k_s}$ in the form $z = \operatorname{Re} z + i \operatorname{Im} z$, where $\operatorname{Re} z$ and $\operatorname{Im} z$ belong to $R^{k_s}$, the set $Z_{s,\omega}$ is of the form $Z^*_{s,\omega} + iR^{k_s}$, where $Z^*_{s,\omega}$ is a convex subset of $R^{k_s}$ containing 0. The restrictions of $\bar{\varphi}_s$ to $iR^{k_s}$ and $Z^*_{s,\omega}$ are, respectively, the characteristic function and the Laplace transform, which will be denoted by $\varphi_s$ and $\varphi^*_s$, i.e.,

$$\varphi_s(\zeta;\omega) = \bar{\varphi}_s(i\zeta;\omega), \qquad \zeta \in R^{k_s},$$

and

$$\varphi^*_s(\zeta;\omega) = \bar{\varphi}_s(\zeta;\omega), \qquad \zeta \in Z^*_{s,\omega}.$$

With this notation condition (ii) can be reexpressed as

$$\bar{\varphi}_{\bar{s}}(z; n\omega) = \bar{\varphi}_s(n^{-1}z;\omega)^n, \qquad z \in (nZ_{s,\omega}) \cap Z_{\bar{s},n\omega},$$

or, equivalently, as

$$\varphi_{\bar{s}}(\zeta; n\omega) = \varphi_s(n^{-1}\zeta;\omega)^n, \qquad \zeta \in R^{k_s}.$$

Finally, if the interior of $(nZ^*_{s,\omega}) \cap Z^*_{\bar{s},n\omega}$ is not the empty set, condition (ii) is equivalent to

$$\varphi^*_{\bar{s}}(\zeta; n\omega) = \varphi^*_s(n^{-1}\zeta;\omega)^n, \qquad \zeta \in \operatorname{int}((nZ^*_{s,\omega}) \cap Z^*_{\bar{s},n\omega}).$$

A survey of examples and properties of reproductive models, with particular emphasis on reproductive exponential models for $s$ having minimal exponential representation of the form

$$a(\omega)b(s)\,e^{\omega_1 \cdot H(s) + \omega_2 \cdot s}, \tag{1.1}$$

has been given by Barndorff-Nielsen and Blæsild (1984a). [See also Barndorff-Nielsen and Blæsild (1983a, b) and Blæsild and Jensen (1985).] In (1.1), $(\omega_1, \omega_2)$ is a partition of the $k$-dimensional canonical parameter $\omega$ into components of dimension $k_1$ and $k_2$, respectively, and $(H(s), s)$ is a similar partition of the minimal sufficient statistic. If $\tau = (\tau_1, \tau_2)$ denotes the mean value of $(H(s), s)$, the condition (ii) is in this case equivalent to

$$\omega_2(\omega_1, \tau_2) = -\omega_1 h(\tau_2),$$

where $h(\tau_2) = \partial H^{\mathrm{T}}(\tau_2)/\partial\tau_2$, T indicating transposition. One of the properties of a reproductive exponential model of the form (1.1) is that the components $\hat{\omega}_1$ and $\hat{\tau}_2$ of the maximum-likelihood estimator of the mixed parameter $(\omega_1, \tau_2)$ are independent, i.e.,

$$\hat{\omega}_1 \perp \hat{\tau}_2.$$

A number of other useful properties of such models are reviewed in connection with a particular example in Section 3 of the present paper.

It is possible to combine reproductive models in a nontrivial manner so as to obtain further reproductive models, for higher-dimensional variates. Some first examples of this were discussed in Barndorff-Nielsen and Blæsild (1983b). Here a more systematic investigation will be presented.

Let $\tilde{\mathscr{D}} = \{\tilde{D}(\tilde{\omega}): \tilde{\omega} \in \tilde{\Omega}\}$ and $\mathscr{D}^\dagger = \{D^\dagger(\omega^\dagger): \omega^\dagger \in \Omega^\dagger\}$ be two parametric families of distributions and let $u$ and $z$ be two statistics such that $(\tilde{\mathscr{D}}, u)$ and

$(\mathscr{D}^\dagger, z)$ are reproductive in $\tilde{\omega}$ and $\omega^\dagger$, respectively. Furthermore, let $v$ be a statistic such that for all $u$ the conditional distribution of $v$ given $u$ is $D^\dagger(\omega^\dagger(\omega_c, u))$, where $\omega_c \in \Omega_c$ and $\omega^\dagger(\omega_c, u)$ indicates a function of $\omega_c$ and $u$ with values in $\Omega^\dagger$. The resulting distribution for $(u, v)$ is denoted $[\tilde{D}, D^\dagger](\tilde{\omega}, \omega_c)$ and is called a *combination* from $\tilde{\mathscr{D}}$ and $\mathscr{D}^\dagger$. Let $\mathscr{D}$ denote the class of combinations, i.e., $\mathscr{D} = \{[\tilde{D}, D^\dagger](\tilde{\omega}, \omega_c)\colon (\tilde{\omega}, \tilde{\omega}_c) \in \tilde{\Omega} \times \Omega_c\}$. In Section 2 we give sufficient conditions for $(\mathscr{D}, (u, v))$ to be reproductive in $(\tilde{\omega}, \omega_c)$ with or without the additional property that the three classes of distributions $\tilde{\mathscr{D}}$, $\mathscr{D}^\dagger$ and $\mathscr{D}$ are all exponential and of the type (1.1). We also present some simple examples of such combinations. Combination of several reproductive models in a somewhat more intricate manner can also result in reproductive models. This is discussed in Section 3 for the case where the elements of the combination are gamma, inverse-Gaussian or Gaussian distributions. Graph-theoretic considerations are helpful here. In particular, for certain types of graphs $G$, we define what we call inverse-Gaussian models with graph $G$ and inverse-Gaussian–Gaussian models with graph $G$. These models are exponential and are shown to have $c|\hat{j}|^{1/2}\bar{L}$ exact [cf. Barndorff-Nielsen (1980, 1983) and Blæsild and Jensen (1985)]. The graph-theoretic formulations makes it possible to apply directly a very general theorem concerning Markovian-type probability laws on graphs, due to Kiiveri, Speed and Carlin (1984), to certain of the models considered in Section 3. The theorem yields conditional independence properties of those models and this is briefly discussed at the end of Section 3. Several examples are given to illustrate the general results.

Parametric statistical models, whose structure can to a large extent be described and interpreted by means of certain types of graphs, have recently been delineated and studied in a number of important papers; see Darroch, Lauritzen and Speed (1980), Wermuth and Lauritzen (1983), Kiiveri, Speed and Carlin (1984) and Lauritzen and Wermuth (1984). The employment of graph-theoretic concepts and results has been demonstrated to be a very powerful approach in the investigation and applications of the models concerned. The discussion given in Section 3 is partly inspired by those works.

**2. Sufficient conditions for combinations to be reproductive.** Throughout this section we use the notation introduced in Section 1. Furthermore, we let $\mathscr{D}_{v|u}$ denote the class of conditional distributions of $v$ given $u$, i.e., $\mathscr{D}_{v|u} = \{D^\dagger(\omega^\dagger(\omega_c, u))\colon \omega_c \in \Omega_c\}$, and we denote the dimensions of $u$ and $v$ by $k_u$ and $k_v$, respectively. Note that $\mathscr{D}_{v|u}$ is reproductive in $\omega_c$ if and only if for all $n = 1, 2, \ldots,$ $n\Omega_c \subseteq \Omega_c$ and $\omega^\dagger(n\omega_c, u) = n\omega^\dagger(\omega_c, u)$. With this notation we have

THEOREM 2.1. *Let $U_{\tilde{\omega}}$ denote the support of $u$ under $\tilde{D}(\tilde{\omega})$. If*

(i) $n\Omega_c \subseteq \Omega_c$ *for* $n = 1, 2, \ldots,$

*and*

(ii) *for $\tilde{\omega} \in \tilde{\Omega}$ and $u \in U_{\tilde{\omega}}$, the characteristic function for the conditional distribution of $v$ given $u$ is of the form*

$$\varphi_{v|u}(\eta; \omega_c) = e^{c_1(\eta, \omega_c) + c_2(\eta, \omega_c)\cdot u}, \qquad \eta \in R^{k_v}, \tag{2.1}$$

*for some functions* $c_1$ *and* $c_2$ *such that for* $i = 1, 2$ *and* $n = 1, 2, \ldots,$

$$c_i(\eta, n\omega_c) = nc_i(n^{-1}\eta, \omega_c), \qquad \eta \in R^{k_v},\ \omega_c \in \Omega_c, \tag{2.2}$$

*then* $\mathscr{Q}_{v|u}$ *is reproductive with respect to* $\omega_c$ *for* $u \in U_{\tilde{\omega}}$, $\tilde{\omega} \in \tilde{\Omega}$.

*If, in addition,*

(iii) *the class* $\mathscr{Q}$ *of distributions for* $u$ *is reproductive with respect to* $\tilde{\omega}$,

*then the class of combinations from* $\tilde{\mathscr{Q}}$ *and* $\mathscr{Q}^{\dagger}$ *is reproductive in* $\omega = (\tilde{\omega}, \omega_c) \in \Omega = \tilde{\Omega} \times \Omega_c$.

PROOF. For $\eta \in R^{k_v}$ one has, using (ii), that

$$\begin{aligned}\varphi_{v|u}(n^{-1}\eta; \omega_c)^n &= e^{nc_1(n^{-1}\eta, \omega_c) + nc_2(n^{-1}\eta, \omega_c)\cdot u}\\ &= e^{c_1(\eta, n\omega_c) + c_2(\eta, n\omega_c)\cdot u}\\ &= \varphi_{v|u}(\eta; n\omega_c).\end{aligned}$$

Together with (i) this implies that the first part of the theorem is true.

From conditions (i) and (iii) it follows that $n\Omega \subseteq \Omega$ for $n = 1, 2, \ldots,$ and since (ii) implies that $\operatorname{Re} c_2(\eta, \omega_c) \in Z^*_{u, \tilde{\omega}}$ for $\omega_c \in \Omega_c$, $\tilde{\omega} \in \tilde{\Omega}$ and $\eta \in R^{k_v}$, we have for $\zeta \in R^{k_u}$ and $\eta \in R^{k_v}$ that

$$\begin{aligned}\varphi_{u,v}(\zeta, \eta; \tilde{\omega}, \omega_c) &= E_{\tilde{\omega}}\big(e^{i\zeta\cdot u}E_{\omega_c}(e^{i\eta\cdot v}|u)\big)\\ &= E_{\tilde{\omega}}\big(e^{i\zeta\cdot u + c_1(\eta, \omega_c) + c_2(\eta, \omega_c)\cdot u}\big)\\ &= e^{c_1(\eta, \omega_c)}\overline{\varphi}_u(i\zeta + c_2(\eta, \omega_c); \tilde{\omega}),\end{aligned}$$

and we obtain, using (iii), that

$$\begin{aligned}\varphi_{u,v}(n^{-1}\zeta, n^{-1}\eta; \tilde{\omega}, \omega_c)^n &= e^{nc_1(n^{-1}\eta, \omega_c)}\overline{\varphi}_u(in^{-1}\zeta + c_2(n^{-1}\eta, \omega_c); \tilde{\omega})^n\\ &= e^{c_1(\eta, n\omega_c)}\overline{\varphi}_u(i\zeta + nc_2(n^{-1}\eta, \omega_c); n\tilde{\omega})\\ &= e^{c_1(\eta, n\omega_c)}\overline{\varphi}_u(i\zeta + c_2(\eta, n\omega_c); n\tilde{\omega})\\ &= \varphi_{u,v}(\zeta, \eta; n\tilde{\omega}, n\omega_c).\end{aligned}$$

The proof is now complete. □

In applications of Theorem 2.1 the family $\mathscr{Q}$ will often be defined by specifying the class of marginal distributions of $u$ and the family of conditional characteristic functions (2.1). It is then necessary, of course, to make sure that the joint distribution of $u$ and $v$ is well defined, i.e., it must be checked that

$$\operatorname{Re} c_2(\eta, \omega_c) \in Z^*_{u, \tilde{\omega}},$$

for $\omega_c \in \Omega_c$, $\tilde{\omega} \in \tilde{\Omega}$ and $\eta \in R^{k_v}$.

EXAMPLE 2.1. For fixed $(\alpha, \beta)$ let $\mathscr{S}_{\alpha,\beta}$ be the scale-parameter family generated by the one-dimensional stable law with characteristic function

$$\varphi(\zeta) = \exp(-|\zeta|^{\alpha}\{1 + i\beta\zeta/|\zeta|\tan(\pi\alpha/2)\}),$$

where $\alpha \in (0,2)$, $\alpha \neq 1$ and $|\beta| \leq 1$, and let $S_{\alpha,\beta}(\omega)$ be the element in $\mathscr{S}_{\alpha,\beta}$ having scale parameter $\omega^{(\alpha-1)/\alpha}$, i.e., $S_{\alpha,\beta}(\omega)$ is the distribution with characteristic function

$$\varphi(\zeta;\omega) = \exp\left(-\omega^{1-\alpha}|\zeta|^{\alpha}\{1 + i\beta\zeta/|\zeta|\tan(\pi\alpha/2)\}\right).$$

Clearly, $\mathscr{S}_{\alpha,\beta}$ is reproductive in $\omega$.

Let $0 < \tilde{\alpha} < 1$ and let $\mathscr{S}_{\tilde{\alpha},1}$ be the class of marginal distributions for $u$, i.e., $\mathscr{S}_{\tilde{\alpha},1} = \tilde{\mathscr{Q}}$. In this case $\tilde{\Omega} = R_+$ and $U_{\tilde{\omega}} = R_+$ for all $\tilde{\omega} \in \tilde{\Omega}$. Furthermore, let $\mathscr{Q}^{\dagger} = \mathscr{S}_{\alpha^{\dagger},\beta^{\dagger}}$, where $\alpha^{\dagger} \neq 1$. Specifying the conditional distribution of $v$ given $u$ by

$$v|u \sim S_{\alpha^{\dagger},\beta^{\dagger}}\left(\omega_c u^{(1-\alpha^{\dagger})^{-1}}\right),$$

we have that the conditional characteristic function is of the form (2.1) with

$$c_1(\eta;\omega_c) = 0, \qquad c_2(\eta;\omega_c) = -\omega_c^{1-\alpha^{\dagger}}|\eta|^{\alpha^{\dagger}}\{1 + i\beta^{\dagger}\eta/|\eta|\tan(\pi\alpha^{\dagger}/2)\}.$$

Theorem 2.1 now implies that the class of combinations $[S_{\tilde{\alpha},1}, S_{\alpha^{\dagger},\beta^{\dagger}}](\tilde{\omega},\omega_c)$, $(\tilde{\omega},\omega_c) \in R_+^2$, is reproductive in $(\tilde{\omega},\omega_c)$.

Under the assumption (2.1) and the additional requirement that the distribution of $u$ is not concentrated on an affine subspace of $R^{k_u}$, it is often possible to show that reproductivity of $\mathscr{Q}_{v|u}$ is equivalent to (2.2) at least for $\eta$ in a neighbourhood of 0. A similar remark applies if the characteristic function in (2.1) is replaced by the Laplace transform. This appears from Theorem 2.2, which constitutes an analogue of Theorem 2.1 based on Laplace transforms instead of characteristic functions.

It is useful to have both Theorems 2.1 and 2.2 available. In particular, whereas Theorem 2.1 covers Example 2.1, it is not suited for the study of reproductivity in exponential families, which is our prime interest in this paper. On the other hand, Theorem 2.2 is geared to the latter purpose but does not encompass a number of other interesting cases, among which is Example 2.1.

THEOREM 2.2. *Suppose that*

(i) *for $\tilde{\omega} \in \tilde{\Omega}$ the smallest affine subspace of $R^{k_u}$ containing $U_{\tilde{\omega}}$ is $R^{k_u}$, i.e.,*

$$\dim(\text{aff } U_{\tilde{\omega}}) = k_u, \qquad \tilde{\omega} \in \tilde{\Omega},$$

*and*

(ii) *for every $\tilde{\omega} \in \tilde{\Omega}$, $\omega_c \in \Omega_c$ and $u \in U_{\tilde{\omega}}$ the cumulant transform for the conditional distribution of $v$ given $u$ exists in a neighbourhood of 0 and is there of the form*

$$\kappa_{v|u}(\eta;\omega_c) = c_1(\eta,\omega_c) + c_2(\eta,\omega_c)\cdot u, \tag{2.1'}$$

*for some functions $c_1$ and $c_2$ such that $c_1(0,\omega_c) = 0$,*

*then $\mathscr{Q}_{v|u}$ is reproductive with respect to $\omega_c$ if and only if for $n = 1,2,\ldots,$ we* *have* $n\Omega_c \subseteq \Omega_c$ *and*

$$(2.2') \qquad c_i(\eta, n\omega_c) = nc_i(n^{-1}\eta, \omega_c), \qquad i = 1,2,\ \eta \in N_{n,\omega_c},\ \omega_c \in \Omega_c,$$

*for* $N_{n,\omega_c}$ *some neighbourhood of* 0.

*If* $\mathscr{Q}_{v|u}$ *is reproductive with respect to* $\omega_c$, *if the conditions* (i) *and* (ii) *are fulfilled and if, in addition,*

(iii) *the class* $\tilde{\mathscr{Q}}$ *of distributions for u is reproductive with respect to* $\tilde{\omega}$, *and*
(iv) *for every* $\tilde{\omega} \in \tilde{\Omega}$ *the cumulant transform* $\kappa_u(\cdot\,; \tilde{\omega})$ *of u exists in a neighbourhood of* 0,

*then the class of combinations from* $\tilde{\mathscr{Q}}$ *and* $\mathscr{Q}^\dagger$ *is reproductive with respect to* $\omega = (\tilde{\omega}, \omega_c) \in \Omega = \tilde{\Omega} \times \Omega_c$.

PROOF. Suppose $\mathscr{Q}_{v|u}$ is reproductive with respect to $\omega_c$ for $\tilde{\omega} \in \tilde{\Omega}$ and $u \in U_{\tilde{\omega}}$. It follows that for $u \in U_{\tilde{\omega}}$ and $\eta$ in a neighbourhood of 0, one has

$$n\kappa_{v|u}(n^{-1}\eta; \omega_c) = \kappa_{v|u}(\eta; n\omega_c),$$

or, using (2.1′), that

$$nc_1(n^{-1}\eta, \omega_c) - c_1(\eta, n\omega_c) = \big(c_2(\eta, n\omega_c) - nc_2(n^{-1}\eta, \omega_c)\big) \cdot u.$$

Using (i), one now obtains (2.2′). Conversely, (2.1′) and (2.2′) imply that in a neighbourhood of 0,

$$n\kappa_{v|u}(n^{-1}\eta; \omega_c) = \kappa_{v|u}(\eta; n\omega_c).$$

Consequently, $\mathscr{Q}_{v|u}$ is reproductive with respect to $\omega_c$ and the first part of the theorem is proved.

As the next step we note that conditions (i) and (ii) imply that the functions $c_1$ and $c_2$ are continuous in $\eta$ in a neighbourhood of 0 and that $c_2(0, \omega_c) = 0$. Now, let $\kappa_{u,v}$ denote the cumulant transform for $(u, v)$. For $(\zeta, \eta)$ in a neighbourhood of $(0, 0)$ one has, using (2.1′), (2.2′), (iii) and (iv),

$$\begin{aligned}\kappa_{u,v}(\zeta, \eta; n\tilde{\omega}, n\omega_c) &= c_1(\eta, n\omega_c) + \kappa_u(\zeta + c_2(\eta, n\omega_c); n\tilde{\omega}) \\ &= nc_1(n^{-1}\eta, \omega_c) + \kappa_u\big(\zeta + nc_2(n^{-1}\eta, \omega_c); n\tilde{\omega}\big) \\ &= nc_1(n^{-1}\eta, \omega_c) + n\kappa_u\big(n^{-1}\zeta + c_2(n^{-1}\eta; \omega_c); \tilde{\omega}\big) \\ &= n\kappa_{u,v}(n^{-1}\zeta, n^{-1}\eta; \tilde{\omega}, \omega_c)\end{aligned}$$

and the proof of the theorem is complete. □

In the next theorem we consider a situation where the two classes of distributions $\tilde{\mathscr{Q}}$ and $\mathscr{Q}_{v|u}$ are exponential, of the form (1.1), as well as reproductive and where this implies the same two properties of the combination $\mathscr{Q}$. Furthermore,

$\tilde{\mathcal{Q}}$ and $\mathcal{Q}$ will be noncurved exponential families. To be specific, we assume that

> $\tilde{\mathcal{Q}}$, the family of marginal distributions for $u$, is an exponential family of order $\tilde{k}$ and with minimal representation

$$\tilde{a}(\tilde{\theta})\tilde{b}(u)\, e^{\tilde{\theta}_1 \cdot \tilde{H}(u) + \tilde{\theta}_2 \cdot u}, \qquad \tilde{\theta} \in \tilde{\Theta}, \tag{2.3}$$

> where $(\tilde{\theta}_1, \tilde{\theta}_2)$ is a partition of the minimal canonical parameter $\tilde{\theta}$ into components of dimension $\tilde{k} - k_u$ and $k_u$, respectively, and where $\tilde{H}$ is a $(\tilde{k} - k_u)$-dimensional statistic; furthermore, the parameter domain $\tilde{\Theta}$ is an open subset of $R^{\tilde{k}}$,

and

> for every $u \in U$, the common support of the distributions in $\tilde{\mathcal{Q}}$, the family $\mathcal{Q}_{v|u}$ of conditional distributions of $v$ given $u$ has an exponential representation of the form

$$a^{\dagger}(\theta^{\dagger})b^{\dagger}(v; u)\, e^{\theta_1^{\dagger} \cdot H^{\dagger}(v) + \theta_2^{\dagger} \cdot v}, \tag{2.4}$$

> where $(\theta_1^{\dagger}, \theta_2^{\dagger})$ is a partition of the canonical parameter $\theta^{\dagger} = \theta^{\dagger}(u)$, a vector of dimension $k^{\dagger}$, into components of dimension $k^{\dagger} - k_v$ and $k_v$, respectively, and where $(H^{\dagger}(v), v)$ is a similar partition of the canonical statistic $t^{\dagger}(v)$.

The following theorem gives a set of sufficient conditions for the resulting class of combinations to be a noncurved exponential model of the form (1.1).

THEOREM 2.3. *Suppose that conditions* (2.3) *and* (2.4) *are fulfilled and that, in addition,*

(i) *$\tilde{\mathcal{Q}}$ is reproductive with respect to $\tilde{\theta}$;*

(ii) *for every $u \in U$ the family $\mathcal{Q}_{v|u}$ is reproductive with respect to $\theta_c$ varying in an open subset $\Theta_c$ of $R^{k_c}$, $k_c \geq k_v$; and*

(iii) *there exists a $(k_c - k_v) \times k^{\dagger}$ matrix function $Y(u)$ such that the following three conditions are fulfilled for every $u$ in $U$:*

$$\theta^{\dagger}(u) = \theta_{c1}Y(u) + (0, \theta_{c2}), \tag{2.5}$$

*where $(\theta_{c1}, \theta_{c2})$ is a partition of $\theta_c$ into components of dimension $k_c - k_v$ and $k_v$, respectively,*

(2.6) *the components in the $k_c$-dimensional vector $(t^{\dagger}(v)Y(u)^{\mathrm{T}}, v)$ are affinely independent,*

*and*

$$a^{\dagger}(\theta^{\dagger}) = a_1(\theta_c)b_1(u)\, e^{-S_1(\theta_{c1}) \cdot \tilde{H}(u) - R_2(\theta_c) \cdot u}, \tag{2.7}$$

*where $S_1$ and $R_2$ are vector functions of dimension $\tilde{k} - k_u$ and $k_u$, respectively,* *such that* $S_1(n\theta_{c1}) = nS_1(\theta_{c1})$ *and* $R_2(n\theta_c) = nR_2(\theta_c)$ *for all* $\theta_c \in \Theta_c$ *and all* $n = 1, 2, \dots$ .

*Then the class of combinations* $\mathscr{Q} = \{[\tilde{D}, D^\dagger](\tilde{\theta}, \theta_c): (\tilde{\theta}, \theta_c) \in \tilde{\Theta} \times \Theta_c\}$ *is a* $(\tilde{k} + k_c, \tilde{k} + k_c)$ *exponential family with minimal representation*

$$(2.8) \qquad a(\theta)b(u, v)\, e^{\theta_{11}\cdot \tilde{H}(u) + \theta_{12}\cdot t^\dagger(v)Y(u)^T + \theta_{21}\cdot u + \theta_{22}\cdot v},$$

*where* $\theta = (\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}) = (\tilde{\theta}_1 - S_1(\theta_{c1}), \theta_{c1}, \tilde{\theta}_2 - R_2(\theta_c), \theta_{c2})$, $a(\theta) = \tilde{a}(\tilde{\theta})a_1(\theta_c)$ *and* $b(u, v) = \tilde{b}(u)b_1(u)b^\dagger(v; u)$. *Furthermore,* $\mathscr{Q}$ *is reproductive in* $\theta$ [*or in* $(\tilde{\theta}, \theta_c)$].

PROOF. From (2.3), (2.4) and (2.7) it follows that the distribution for $(u, v)$ can be represented as in (2.8). This representation is minimal according to (2.6) and (ii), and (2.3) and (ii) imply that the parameter set $\Theta$ for $\mathscr{Q}$ has a nonempty interior. From formula (2.7) it follows that for $\eta$ in a neighbourhood of $0 \in R^{k_v}$,

$$E_\theta(e^{\eta\cdot v}|u) = \frac{a^\dagger(\theta^\dagger(u))}{a^\dagger(\theta^\dagger(u) + (0, \eta))}$$
$$= \frac{a_1(\theta_c)}{a_1(\theta_c + (0, \eta))}\, e^{(R_2(\theta_c + (0, \eta)) - R_2(\theta_c))\cdot u},$$

and an application of Theorem 2.2 completes the proof. □

According to Blæsild and Jensen (1985), the only exponential families of order 2 and with minimal representation of the form (1.1), which are reproductive in $s$ with $\omega = (\omega_1, \omega_2)$ as reproductivity parameter, are those corresponding to the normal, the inverse-Gaussian and the gamma distributions, respectively. Denoting these distributions by $N(\xi, \sigma^2)$, $N^-(\chi, \psi)$ and $\Gamma(\alpha, \beta)$, their probability density functions are, respectively,

$$(2.9) \qquad \varphi(x; \xi, \sigma^2) = (2\pi\sigma^2)^{-1/2}\, e^{-\xi^2/(2\sigma^2)}\, e^{-x^2/(2\sigma^2) + x\xi/\sigma^2},$$
$$x \in R,\ \xi \in R, \text{ and } \sigma > 0,$$

$$(2.10) \qquad \varphi^-(x; \chi, \psi) = (\chi/2\pi)^{1/2}\, e^{\sqrt{\chi\psi}} x^{-3/2}\, e^{-\chi x^{-1/2} - \psi x/2},$$
$$x > 0,\ \chi > 0, \text{ and } \psi \ge 0,$$

and

$$(2.11) \qquad \gamma(x; \alpha, \beta) = \beta^\alpha/\Gamma(\alpha)x^{-1}\, e^{\alpha \log x - \beta x}, \qquad x > 0,\ \alpha > 0,\ \beta > 0.$$

In Table 1 we present essentially all possible reproductive combinations of two of these three models omitting, however, the trivial combinations, i.e., the combinations for which $u$ and $v$ are independent, and the combinations resulting in a curved exponential model. The combinations in Table 1 resulting in exponential families may be shown to be reproductive using Theorem 2.3, whereas the reproductivity for the rest of the combinations in Table 1 follow from Theorem 2.2. Note, that in the cases where $v|u \sim N^-(\chi u, \psi)$, a direct

TABLE 1
*Reproductive models for $(u, v)$ obtained by suitable combination of two of the models $N$, $N^-$ and $\Gamma$*

| $u$ | $v \mid u$ | |
|---|---|---|
| $N(\tilde{\xi}, \tilde{\sigma}^2)$ | $N(\xi u + \alpha, \sigma^2)$ | +5 |
| $N^-(\tilde{\chi}, \tilde{\psi})$ | $N(\xi u + \alpha, \sigma^2 u)$ | +5 |
| $N^-(\tilde{\chi}, \tilde{\psi})$ | $N(\xi u, \sigma^2 u)$ | +4 |
| $N^-(\tilde{\chi}, \tilde{\psi})$ | $N^-(\chi u^2, \psi)$ | +4 |
| $N^-(\tilde{\chi}, \tilde{\psi})$ | $\Gamma(\alpha u, \beta)$ | — |
| $\Gamma(\tilde{\alpha}, \tilde{\beta})$ | $N(\xi u, \sigma^2 u)$ | +4 |
| $\Gamma(\tilde{\alpha}, \tilde{\beta})$ | $N^-(\chi u^2, \psi)$ | +4 |
| $\Gamma(\tilde{\alpha}, \tilde{\beta})$ | $\Gamma(\alpha u, \beta)$ | — |

Trivial combinations and combinations giving a curved exponential model for $(u, v)$ are omitted. In the last column it is indicated whether or not the model for $(u, v)$ is exponential and if so the order of the model is given.

argument shows that one may, in fact, take $\Theta_c = \{(\chi, \psi): \chi > 0, \psi \geq 0\}$, whereas Theorem 2.3 only applies for $(\chi, \psi) \in \text{int}\, \Theta_c$, the interior of $\Theta_c$. Similar remarks apply to the situations in Section 3 where the inverse-Gaussian distribution occurs.

The class $\mathscr{N}_r$ of $r$-dimensional normal distributions $N_r(\xi, \Sigma)$ with mean value $\xi$ and positive definite variance $\Sigma$, provides another example of an exponential family of the form (1.1), which is reproductive in the canonical parameter. Furthermore, the class $\mathscr{N}^-$ of inverse-Gaussian distributions may be combined with $\mathscr{N}_r$ to yield a new reproductive and exponential model. Details of this are available in Barndorff-Nielsen and Blæsild (1984b).

**3. Reproductive combinations of gamma, inverse-Gaussian and Gaussian models.** Suitable combinations of several gamma, inverse-Gaussian and Gaussian models yield further examples of reproductive models. A general scheme for such combinations may conveniently be described by means of graph-theoretic terms in the following way.

Suppose $G$ is a finite, *directed* and *acyclic* graph, i.e., every edge of $G$ is equipped with a direction and it is not possible to travel along edges from any vertex and back to that vertex by a route that respects the orientation of the edges. A vertex $i$ is said to *precede* a vertex $j$, and we write $i \prec j$, if $j$ can be reached from $i$ via edges while respecting the orientation of the edges. If $i$ precedes $j$ and if there is an edge from $i$ to $j$, we indicate this by $i \to j$. For any vertex $j$ of $G$ let $q(j)$ be the number of edges with $j$ as an endpoint and directed away from $j$, and let $r(j)$ be the number of edges with $j$ as an endpoint and directed toward $j$. If $q(j) = 0$, then $j$ is called a *terminal* vertex, and if $r(j) = 0$, then $j$ is said to be an *initial* vertex.

With each vertex $j$ of $G$ we now associate a random variable $u_j$ and we let $u = u_G = \{u_j\colon j \in G\}$. Furthermore, for each $j$ that is not an initial vertex, let $c^j = \{c_{ij}\colon i \prec j\}$ be a set of known nonnegative constants of which one at least is positive. We let $u^j = \{u_i\colon i \prec j\}$ and

$$c^j \cdot u^j = \sum_{i \prec j} c_{ij} u_i,$$

with the convention that if $j$ is an initial vertex, then $c^j \cdot u^j$ is interpreted as 1.

A variety of reproductive models for $u_G$ can be obtained, extending the kind of constructions exhibited in Table 1, by suitable choice of the marginal distributions for $u_j$, where $j \in G$ is an initial vertex, and of the conditional distributions of $u_{j'}$ given $u^{j'}$, where $j' \in G$ is not an initial vertex. Some of these reproductive models for $u_G$ are exponential models of the type specified by formula (1.1), curved or noncurved, whereas others are not exponential. To distinguish between the models, we say that for a particular model for $u_G$ a vertex $j \in G$ is of inverse-Gaussian, Gaussian or gamma *type*, according to the type of the conditional distribution of $u_j$ given $u^j$. In graphical representations the three types of vertices will be symbolized by ●, ○ and □, respectively. Here we confine ourselves to presenting three classes of noncurved exponential models for $u_G$ of the form (1.1). In Section 3.1 we consider combinations involving only inverse-Gaussian distributions. Combinations for which some of the terminal vertices are of Gaussian type, all other vertices being of inverse-Gaussian type are treated in Section 3.2. Finally, in Section 3.3 we consider the case where the initial vertices of $G$ are all of gamma type and the remaining vertices are of inverse-Gaussian type. At the end of each subsection the corresponding class of combinations is illustrated by an example.

3.1. *Combinations of inverse-Gaussian models.* Let $\chi = \{\chi_j\colon j \in G\}$ and $\psi = \{\psi_j\colon j \in G\}$ be two sets of parameters with $\chi_j > 0$ and $\psi_j \geq 0$.

DEFINITION 3.1. *The inverse-Gaussian distribution $N_G^-(\chi, \psi)$ with graph G and associated constants $\{c_{ij}\colon i \prec j\}$* is the probability distribution of the random variate $u$ given by the probability density function

$$\varphi_G^-(u; \chi, \psi) = \prod_{j \in G} \varphi^-\left(u_j; (c^j \cdot u^j)^2 \chi_j, \psi_j\right). \tag{3.1}$$

The *inverse-Gaussian model $\mathcal{N}_G^-$ with graph G and associated constants* $\{c_{ij}\colon i \prec j\}$ is the collection of such distributions as $\chi$ and $\psi$ vary freely (i.e., $\chi_j > 0$, $\psi_j \geq 0$ and $j \in G$). In the special case where $c_{ij} = 1$ if $i \to j$ and $c_{ij} = 0$ otherwise, this model is referred to simply as the *inverse-Gaussian model with graph G*.

Note that, as may easily be shown, the conditional distribution of $u_j$ given $u^j$ is $\varphi^-(u_j; (c^j \cdot u^j)^2 \chi_j, \psi_j)$. The mean value of this distribution is $\sqrt{\chi_j/\psi_j}\, c^j \cdot u^j$ so that the coefficients have an interpretation as regression coefficients.

We need to prove that the right-hand side of (3.1) is, in fact, a probability density function. This may be done by induction relative to $|G|$, the number of vertices of $G$. Suppose the result has been established for $|G| \leq n$. Let $G$ be any directed acyclic graph with $|G| = n + 1$. There exists at least one terminal vertex $k$ of $G$ for otherwise $G$ could not be acyclic. The graph $G^0$ obtained by deleting from $G$ the vertex $k$ and all edges with $k$ as endpoint is again a directed and acyclic graph, and if we let $u^0 = \{u_j: j \in G^0\}$, $\chi^0 = \{\chi_j: j \in G^0\}$ and $\psi^0 = \{\psi_j: j \in G^0\}$, then

$$\varphi_G^-(u; \chi, \psi) = \varphi^-\big(u_k; (c^k \cdot u^k)^2 \chi_k, \psi_k\big)\varphi_{G^0}^-(u^0; \chi^0, \psi^0). \tag{3.2}$$

It follows that $\varphi_G^-(u; \chi, \psi)$ integrates to 1 as was to be shown.

The model $\mathscr{N}_G^-$ is a full exponential model of order $k = 2|G|$ and with exponential representation

$$\varphi_G^-(u; \chi, \psi) = a(\theta)b(u)\, e^{\theta_1 \cdot H(u) + \theta_2 \cdot u}, \tag{3.3}$$

where $\theta = (\theta_1, \theta_2)$,

$$\theta_1 = -\tfrac{1}{2}\chi, \qquad \theta_2 = \Big\{-\tfrac{1}{2}\psi_j + \sum_{j \prec i} c_{ji}\sqrt{\chi_i \psi_i} : j \in G\Big\}, \tag{3.4}$$

$$a(\theta) = (2\pi)^{-|G|/2}\Big(\prod_{j \in G} \chi_j\Big)^{1/2} e^{\Sigma_* \sqrt{\chi_j \psi_j}}, \tag{3.5}$$

$$b(u) = \prod_{j \in G} \big\{u_j^{-3/2} c^j \cdot u^j\big\}, \tag{3.6}$$

and

$$H(u) = \{H_j(u): j \in G\} = \big\{u_j^{-1}(c^j \cdot u^j)^2: j \in G\big\}. \tag{3.7}$$

Here $\Sigma_*$ indicates summation over all initial vertices and in an obvious notation $\theta_1 \cdot H(u) = \Sigma_{j \in G}\theta_{1j}H_j(u)$, $\theta_2 \cdot u = \Sigma_{j \in G}\theta_{2j}u_j$.

Furthermore, $\mathscr{N}_G^-$ is reproductive in $u$ with $\theta = (\theta_1, \theta_2)$ as reproductivity parameter. This may be proved by induction, as was done previously, using Theorem 2.3 in connection with (3.2).

As mentioned in the Introduction, the structure of reproductive exponential models of the general form (3.3) [i.e., without the particular properties (3.4)–(3.7)] has been studied in Barndorff-Nielsen and Blæsild (1983a, b). Note, however, that in the latter of these papers a reproductive exponential model of this type was called strongly reproductive. We now apply some of the results from those papers to the $\mathscr{N}_G^-$ models.

If $\tau = (\tau_1, \tau_2)$ denotes the mean-value parameter, i.e., $\tau = (E_\theta H(u), E_\theta u)$, Theorem 3.2 in Barndorff-Nielsen and Blæsild (1983b) implies that $\theta_2$, considered as a function of the mixed parameter $(\theta_1, \tau_2)$, is

$$\theta_2 = -\theta_1 h(\tau_2). \tag{3.8}$$

Here $h$ is the $|G| \times |G|$ matrix-valued function

$$h(\tau_2) = \frac{\partial H^{\mathrm{T}}}{\partial \tau_2}(\tau_2) = \left\{\frac{\partial H_j}{\partial \tau_{2i}}(\tau_2)\right\}_{i \in G,\, j \in G},$$

where T indicates transposition. Thus

$$\theta_{2i} = -\sum_{j \in G} \theta_{1j} \frac{\partial H_j}{\partial \tau_{2i}}(\tau_2), \qquad i \in G.$$

The components of the Legendre transform $\check{H}$ of $H$ evaluated at $\tau_2$, i.e., $\check{H}(\tau_2) = \{\check{H}_j(\tau_2)\colon j \in G\} = \tau_2 h^{\mathrm{T}}(\tau_2) - H(\tau_2)$, are easily found to be

$$\check{H}_j(\tau_2) = \begin{cases} -2\tau_{2j} = -2\sqrt{\chi_j/\psi_j}, & \text{if } j \text{ is an initial vertex,} \\ 0, & \text{otherwise.} \end{cases}$$

According to Theorem 5.1 in Barndorff-Nielsen and Blæsild (1983a), the norming constant for the reproductive exponential model (3.3) is of the form

$$a(\theta) = e^{\theta_1 \cdot \check{H}(\tau_2) - M(\theta_1)},$$

and, using (3.5), one obtains for the $\mathscr{N}_G^-$ model (3.3) that

$$e^{-M(\theta_1)} = (2\pi)^{-|G|/2}\Big(\prod_{j \in G} \chi_j\Big)^{1/2} = (2\pi)^{-|G|/2}\Big(\prod_{j \in G} -2\theta_{1j}\Big)^{1/2}. \tag{3.9}$$

Consider the quantity $p(u) = \{p_j(u)\colon j \in G\}$ given by

$$p(u) = p(u;\tau_2) = H(u) - uh^{\mathrm{T}}(\tau_2) + \check{H}(\tau_2).$$

With the convention that $c^j \cdot u^j = 1$ and $c^j \cdot \tau_2^j = 1$ for an initial vertex $j$, the components of $p(u)$ are

$$p_j(u) = \big(c^j \cdot \big(u_j \tau_2^j - \tau_{2j} u^j\big)\big)^2 / \big(u_j \tau_{2j}^2\big), \qquad j \in G.$$

Since the Laplace transform for $p(u)$ is [cf. Barndorff-Nielsen and Blæsild (1983a), Corollary 5.1],

$$E_{\theta_1}\{e^{\lambda \cdot p(u)}\} = e^{M(\theta_1+\lambda) - M(\theta_1)} = \prod_{j \in G} \big(1 - 2\lambda_j/\chi_j\big)^{-1/2},$$

where $\lambda = \{\lambda_j\colon j \in G\}$, it follows that the components of $p(u)$ are mutually independent and that

$$p_j(u) \sim \Gamma\big(\tfrac{1}{2}, \chi_j/2\big), \qquad j \in G.$$

For a sample $u_1, \dots, u_n$ from the $N_G^-(\chi, \psi)$ distribution let $\bar{u} = n^{-1}(u_1 + \cdots + u_n)$, $\overline{H} = n^{-1}(H(u_1) + \cdots + H(u_n))$ and $\bar{p} = n^{-1}(p(u_1) + \cdots + p(u_n))$. It follows from Corollary 5.4 in Barndorff-Nielsen and Blæsild (1983a) that $\overline{H} - H(\bar{u})$ and $\bar{u}$ are independent, i.e.,

$$\overline{H} - H(\bar{u}) \perp \bar{u},$$

and that this statement is equivalent to

$$\hat{\theta}_1 \perp \hat{\tau}_2,$$

where $(\hat{\theta}_1, \hat{\tau}_2)$ is the maximum-likelihood estimator of the mixed parameter

$(\theta_1, \tau_2)$. Furthermore, the Laplace transform of $\overline{H} - H(\bar{u})$ is

$$E_{\theta_1}\{e^{\lambda\cdot(\overline{H}-H(\bar{u}))}\} = e^{-\{M(n\theta_1+\lambda)-M(n\theta_1)\}+n\{M(\theta_1+n^{-1}\lambda)-M(\theta_1)\}}$$
$$= \prod_{j\in G}\left(1 - 2\lambda_j/(n\chi_j)\right)^{-(n-1)/2},$$

implying that the components of $\overline{H} - H(\bar{u}) = \{\overline{H}_j - H_j(\bar{u})\colon\ j \in G\}$ are mutually independent and that

$$\overline{H}_j - H_j(\bar{u}) \sim \Gamma\left((n-1)/2, n\chi_j/2\right), \qquad j \in G. \tag{3.10}$$

Combining (3.10) with Theorem 2 of Blæsild and Jensen (1985), one sees that the general formula $c|\hat{j}|^{1/2}\overline{L}$ for the distribution of the maximum-likelihood estimator [Barndorff-Nielsen (1980, 1983)] is exact when applied to the maximum-likelihood estimator $(\hat{\chi}, \hat{\psi})$ from the $\mathcal{N}_G^-$ model (3.3).

Finally, let

$$\acute{q} = p(\bar{u}) = H(\bar{u}) - \bar{u}h^{\mathrm{T}}(\tau_2) + \check{H}(\tau_2),$$

and

$$\acute{w} = \overline{H} - H(\bar{u}) = \bar{p} - \acute{q}.$$

Theorem 3.1 in Barndorff-Nielsen and Blæsild (1983b) then states that the components in the decomposition of $\bar{p}$,

$$\bar{p} = \acute{q} + \acute{w},$$

are independent, i.e.,

$$\acute{q} \perp \acute{w},$$

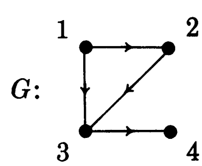
and since the Laplace transform of $\acute{q}$ is

$$E_{\theta_1}\{e^{\lambda\cdot\acute{q}}\} = e^{M(n\theta_1+\lambda)-M(n\theta_1)}$$
$$= \prod_{j\in G}\left(1 - 2\lambda_j/(n_{\chi_j})\right)^{-1/2},$$

the components $\{\acute{q}_j\colon\ j \in G\}$ of $\acute{q}$ are mutually independent and

$$\acute{q}_j \sim \Gamma\left(\tfrac{1}{2}, n\chi_j/2\right), \qquad j \in G.$$

EXAMPLE 3.1. The inverse-Gaussian distribution with the graph




has probability density function

$$\varphi_G^-(u;\chi,\psi) = \varphi^-(u_1;\chi_1,\psi_1)\varphi^-\left(u_2; u_1^2\chi_2,\psi_2\right)$$
$$\times\varphi^-\left(u_3;(u_1+u_2)^2\chi_3,\psi_3\right)\varphi^-\left(u_4;u_3^2\chi_4,\psi_4\right),$$

which is of the form (3.3) with

$$\theta_1 = -\tfrac{1}{2}\chi,$$
$$\theta_2 = \left(-\tfrac{1}{2}\psi_1 + \sqrt{\chi_2\psi_2} + \sqrt{\chi_3\psi_3}, -\tfrac{1}{2}\psi_2 + \sqrt{\chi_3\psi_3}, -\tfrac{1}{2}\psi_3 + \sqrt{\chi_4\psi_4}, -\tfrac{1}{2}\psi_4\right),$$
$$a(\theta) = (2\pi)^{-2}(\chi_1\chi_2\chi_3\chi_4)^{1/2} e^{\sqrt{\chi_1\psi_1}},$$
$$b(u) = u_1^{-1/2}u_2^{-3/2}u_3^{-1/2}u_4^{-3/2}(u_1 + u_2),$$

and

$$H(u) = \left(u_1^{-1}, u_2^{-1}u_1^2, u_3^{-1}(u_1 + u_2)^2, u_4^{-1}u_3^2\right).$$

To illustrate how the previous general results may be applied in problems of inference, suppose we have a sample $(u_{i1}, \ldots, u_{i4})$, $i = 1, \ldots, n$, from the present inverse-Gaussian model with graph $G$. Let $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_4)$. Then the four quantities

$$z_1 = \overline{H}_1 - H_1(\bar{u}) = n^{-1}\sum_i u_{i1}^{-1} - \bar{u}_1^{-1},$$
$$z_2 = \overline{H}_2 - H_2(\bar{u}) = n^{-1}\sum_i u_{i2}^{-1}u_{i1}^2 - \bar{u}_2^{-1}\bar{u}_1^2,$$
$$z_3 = \overline{H}_3 - H_3(\bar{u}) = n^{-1}\sum_i u_{i3}^{-1}(u_{i1} + u_{i2})^2 - \bar{u}_3^{-1}(\bar{u}_1 + \bar{u}_2)^2,$$
$$z_4 = \overline{H}_4 - H_4(\bar{u}) = n^{-1}\sum_i u_{i4}^{-1}u_{i3}^2 - \bar{u}_4^{-1}\bar{u}_3^2$$

are independent and $\chi^2$-distributed on $n - 1$ degrees of freedom and with respective scale parameters $(n\chi_1)^{-1}$, $(n\chi_2)^{-1}$, $(n\chi_3)^{-1}$ and $(n\chi_4)^{-1}$. Based on these distributional properties, testing of the identity of two or more of the parameters $\chi_1$, $\chi_2$, $\chi_3$ and $\chi_4$ may be carried out by an $F$-test or a Bartlett-type test. Suppose next that $\chi_1$, $\chi_2$, $\chi_3$ and $\chi_4$ are taken as identical, the common value being denoted by $\chi$. The likelihood ratio for testing identity of the parameters $\psi_1$, $\psi_2$, $\psi_3$ and $\psi_4$ is then simply

$$Q = \hat{L}/\acute{L} = (\acute{\chi}/\hat{\chi})^{-2n},$$

where $\hat{}$ and $\acute{}$ indicate maximum-likelihood estimation with and without the null hypothesis, respectively. The ratio $\acute{\chi}/\hat{\chi}$ may be rewritten as

$$\acute{\chi}/\hat{\chi} = 1 + y/z_{\cdot},$$

where $nz_{\cdot} = n(z_1 + z_2 + z_3 + z_4)$ is $\chi^2$-distributed on $4(n - 1)$ degrees of freedom and with scale parameter $\chi$, whereas

$$y = \sum_{j=1}^{4} H_j(\bar{u}) - \frac{(2\bar{u}_1 + \bar{u}_2 + \bar{u}_3 + 1)^2}{\bar{u}_1 + \bar{u}_2 + \bar{u}_3 + \bar{u}_4}.$$

Furthermore, since $z_{\cdot}$ is a function of $\{\overline{H}_j - H_j(\bar{u});\ j = 1, \ldots, 4\}$ and $y$ is a function of $\bar{u}$, Corollary 5.4 in Barndorff-Nielsen and Blæsild (1983a) implies that $z_{\cdot}$ and $y$ are independent. It follows that $y/z_{\cdot}$ is a natural alternative test statistic and that, under the null hypothesis, $y/z.$ is asymptotically $F$-distributed on $(3, 4(n-1))$ degrees of freedom.

It may be noted that the base model for $(u_1, u_2, u_3, u_4)$ can be interpreted in terms of four independent Brownian motions and a suitable observational scheme, in analogy with the interpretation of the $[N^-, N^-]$ or ●→—● model discussed in Barndorff-Nielsen and Blæsild (1983b). Under such an interpretation the two hypotheses $\chi_1 = \chi_2 = \chi_3 = \chi_4$ and $\psi_1 = \psi_2 = \psi_3 = \psi_4$ given $\chi_1 = \chi_2 = \chi_3 = \chi_4$ correspond, respectively, to $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2$ and to $\mu_1 = \mu_2 = \mu_3 = \mu_4$ given $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2$, where $\sigma_i^2$ is the diffusion coefficient and $\mu_i$ is the drift coefficient of the $i$th Brownian motion, $i = 1, 2, 3, 4$.

3.2. *Combinations of inverse-Gaussian and Gaussian models.* Let $G_0 \subseteq G$ be the set of terminal vertices of Gaussian type and assume for simplicity that none of these vertices are initial, and let $G^0$ be the graph obtained from $G$ by deleting the vertices in $G_0$ together with all the edges leading to those vertices. Moreover, let $u^0 = \{u_j\colon j \in G^0\}$ and let $\chi = \{\chi_j\colon j \in G^0\}$, $\psi = \{\psi_j\colon j \in G^0\}$, $\xi = \{\xi_j\colon j \in G_0\}$ and $\sigma^2 = \{\sigma_j^2\colon j \in G_0\}$ be sets of parameters with $\chi_j > 0$, $\psi_j \geq 0$, $-\infty < \xi_j < \infty$ and $\sigma_j > 0$.

DEFINITION 3.2. *The inverse-Gaussian–Gaussian model $\mathscr{N}_G^{\mp}$ with graph $G$ and associated constants* $\{c_{ij}\colon i \prec j\}$ is given by the model function

$$(3.11)\qquad \varphi_G^{\mp}(u; \chi, \psi, \xi, \sigma^2) = \varphi_{G^0}^{-}(u^0; \chi, \psi) \prod_{j \in G_0} \varphi\big(u_j; c^j \cdot u^j(\xi_j, \sigma_j^2)\big).$$

In the special case where $c_{ij} = 1$ if $i \to j$ and $c_{ij} = 0$ otherwise, this model is referred to simply as the *inverse-Gaussian–Gaussian model* with graph $G$.

Note that with the notation used in Sections 1 and 2 the distribution (3.11) may also be denoted by $[N_{G^0}^-, N_{G_0}](\chi, \psi, \xi, \sigma^2)$.

The model $\mathscr{N}_G^{\mp}$ with associated constants $\{c_{ij}\colon i \prec j\}$ is a full exponential model of order $k = 2|G|$ and with minimal representation

$$(3.12)\qquad \varphi_G^{\mp}(u; \chi, \psi, \xi, \sigma^2) = a(\theta) b(u)\, e^{\theta_1 \cdot H(u) + \theta_2 \cdot u},$$

where $\theta = (\theta_1, \theta_2)$,

$$\theta_{1j} = \begin{cases} -\frac{1}{2}\chi_j, & j \in G^0, \\ -\frac{1}{2}\sigma_j^{-2}, & j \in G_0, \end{cases}$$

$$\theta_{2j} = \begin{cases} -\frac{1}{2}\psi_j + \sum\limits_{\substack{j \prec i \\ i \in G^0}} c_{ji}\sqrt{\chi_i \psi_i} - \frac{1}{2} \sum\limits_{\substack{j \prec i \\ i \in G_0}} c_{ji}\xi_i^2 \sigma_i^{-2}, & j \in G^0, \\ \xi_j \sigma_j^{-2}, & j \in G_0, \end{cases}$$

$$a(\theta) = (2\pi)^{-|G|/2} \Big(\prod_{j \in G^0} \chi_j\Big)^{1/2} \Big(\prod_{j \in G_0} \sigma_j^2\Big)^{-1/2} e^{\Sigma_* \sqrt{\chi_j \psi_j}},$$

$$b(u) = \prod_{j \in G^0} \big(u_j^{-3/2} c^j \cdot u^j\big) \prod_{j \in G_0} (c^j \cdot u^j)^{-1/2},$$

and

$$H_j(u) = \begin{cases} u_j^{-1}(c^j \cdot u^j)^2, & j \in G^0, \\ u_j^2(c^j \cdot u^j)^{-1}, & j \in G_0. \end{cases}$$

Using Theorem 2.3 and (3.11), it may be established, as for $\mathscr{N}_G^-$, that $\mathscr{N}_G^{\mp}$ is reproductive in $u$ with $\theta = (\theta_1, \theta_2)$ as reproductivity parameter. Moreover, since (3.12) is of the same form as (3.3) further results analogous to those for the $\mathscr{N}_G^-$ model hold for the $\mathscr{N}_G^{\mp}$ model. Here we just mention that

$$(3.13) \quad p_j(u) = \begin{cases} \left(c^j \cdot \left(u_j \tau_2^j - \tau_{2j} u^j\right)\right)^2 / (u_j \tau_{2j})^2, & j \in G^0, \\ \left(c^j \cdot \left(u_j \tau_2^j - \tau_{2j} u^j\right)\right)^2 / \left((c^j \cdot u^j)(c^j \cdot \tau_2^j)^2\right), & j \in G_0, \end{cases}$$

and that the $p_j(u)$, $j \in G$, are mutually independent and

$$(3.14) \qquad p_j(u) \sim \begin{cases} \Gamma\left(\frac{1}{2}, \chi_j/2\right), & j \in G^0, \\ \Gamma\left(\frac{1}{2}, \sigma_j^{-2}/2\right), & j \in G_0. \end{cases}$$

Also, the components of $\acute{q} = p(\bar{u})$ and $\acute{w} = \overline{H} - H(\bar{u}) = \bar{p} - \acute{q}$ are mutually independent and

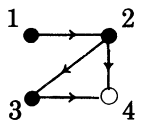
$$(3.15) \qquad \acute{q}_j \sim \begin{cases} \Gamma\left(\frac{1}{2}, n\chi_j/2\right), & j \in G^0, \\ \Gamma\left(\frac{1}{2}, n\sigma_j^{-2}/2\right), & j \in G_0, \end{cases}$$

and

$$(3.16) \qquad \overline{H}_j - H_j(\bar{u}) \sim \begin{cases} \Gamma\left((n-1)/2, n\chi_j/2\right), & j \in G^0, \\ \Gamma\left((n-1)/2, n\sigma_j^{-2}/2\right), & j \in G_0. \end{cases}$$

Finally, $c|\hat{j}|^{1/2}\overline{L}$ is exact for $(\hat{\chi}, \hat{\psi}, \hat{\xi}, \hat{\sigma}^2)$.

EXAMPLE 3.2. The inverse-Gaussian–Gaussian distribution $N_G^{\mp}$ with the graph

G: 


has probability density function

$$\varphi_G^{\mp}(u; \chi, \psi, \xi, \sigma^2) = \varphi^-(u_1; \chi_1, \psi_1)\varphi^-(u_2; u_1^2\chi_2, \psi_2)\varphi^-(u_3; u_2^2\chi_3, \psi_3) \times \varphi\left(u_4; (u_2 + u_3)(\xi_4, \sigma_4^2)\right),$$

which is of the form (3.12) with

$$\theta_1 = \left(-\tfrac{1}{2}\chi_1, -\tfrac{1}{2}\chi_2, -\tfrac{1}{2}\chi_3, -\tfrac{1}{2}\sigma_4^{-2}\right),$$

$$\theta_2 = \left(-\tfrac{1}{2}\psi_1 + \sqrt{\chi_2\psi_2}, -\tfrac{1}{2}\psi_2 + \sqrt{\chi_3\psi_3} - \tfrac{1}{2}\xi_4^2\sigma_4^{-2}, -\tfrac{1}{2}\psi_3 - \tfrac{1}{2}\xi_4^2\sigma_4^{-2}, \xi_4\sigma_4^{-2}\right),$$

$$a(\theta) = (2\pi)^{-2}\left(\chi_1\chi_2\chi_3\sigma_4^{-2}\right)^{1/2} e^{-\sqrt{\chi_1\psi_1}/2},$$

$$b(u) = u_1^{-1/2}u_2^{-1/2}u_3^{-3/2}(u_1 + u_2)^{-1/2},$$

and

$$H(u) = \left(u_1^{-1}, u_2^{-1}u_1^2, u_3^{-1}u_2^2, (u_2 + u_3)^{-1}u_4^2\right).$$

3.3. *Combinations of gamma and inverse-Gaussian models.* Let $G^0$ denote the initial vertices of $G$ and let $G_0$ be the graph obtained from $G$ by deleting all vertices in $G^0$ and all edges leading from those vertices. Furthermore, let $\alpha = \{\alpha_j: j \in G^0\}$, $\beta = \{\beta_j: j \in G^0\}$, $\chi = \{\chi_j: j \in G_0\}$ and $\psi = \{\psi_j: j \in G_0\}$ be sets of parameters such that $\alpha_j > 0$, $\beta_j > 0$, $\chi_j > 0$ and $\psi_j \geq 0$.

DEFINITION 3.3. *The gamma–inverse-Gaussian distribution* $[\Gamma_{G^0}, N^-_{G_0}]$ *with graph* $G$ *and associated constants* $\{c_{ij}: i \prec j\}$ has probability density function

$$[\gamma_{G^0}, \varphi^-_{G_0}](u; \alpha, \beta, \chi, \psi) = \prod_{j\in G^0} \gamma(u_j; \alpha_j, \beta_j) \prod_{j\in G_0} \varphi^-\left(u_j; (c^j \cdot u^j)^2\chi_j; \psi_j\right). \tag{3.17}$$

In the special case where $c_{ij} = 1$ if $i \to j$ and $c_{ij} = 0$ otherwise, this model is referred to simply as the *gamma–inverse-Gaussian distribution* with graph $G$.

The corresponding family of distributions with $\alpha$, $\beta$, $\chi$ and $\psi$ varying freely is a full exponential model of order $k = 2|G|$ and with minimal representation

$$[\gamma_{G^0}, \varphi^-_{G_0}](u; \alpha, \beta, \chi, \psi) = a(\theta)b(u)\, e^{\theta_1\cdot H(u) + \theta_2\cdot u}, \tag{3.18}$$

where

$$\theta_{1j} = \begin{cases} \alpha_j, & j \in G^0, \\ -\tfrac{1}{2}\chi_j, & j \in G_0, \end{cases}$$

$$\theta_{2j} = \begin{cases} -\beta_j + \sum\limits_{\substack{j\prec i \\ i\in G_0}} c_{ji}\sqrt{\chi_i\psi_i}, & j \in G^0, \\ -\tfrac{1}{2}\psi_j + \sum\limits_{\substack{j\prec i \\ i\in G^0}} c_{ji}\sqrt{\chi_i\psi_i}, & j \in G_0, \end{cases}$$

$$a(\theta) = (2\pi)^{-|G_0|/2} \prod_{j\in G^0} \left(\beta_j^{\alpha_j}/\Gamma(\alpha_j)\right)\left(\prod_{j\in G_0} \chi_j\right)^{1/2},$$

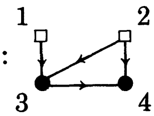
$$b(u) = \prod_{j\in G^0} u_j^{-1} \prod_{j\in G_0} \left(u_j^{-3/2} c^j \cdot u^j\right),$$

and

$$H_j(u) = \begin{cases} \log u_j, & j \in G^0, \\ u_j^{-1}(c^j \cdot u^j)^2, & j \in G_0. \end{cases}$$

Furthermore, because of (3.18), results analogous to those for the $\mathscr{N}_G^-$ and $\mathscr{N}_G^{\mp}$ models hold for the class of gamma–inverse-Gaussian distribution with graph $G$. In this case, however, the formula $c|\hat{j}|^{1/2}\bar{L}$ is not exact for $(\hat{\alpha}, \hat{\beta}, \hat{\chi}, \hat{\psi})$.

EXAMPLE 3.3. The gamma–inverse-Gaussian distribution with the graph

G: 


has probability density function

$$\begin{aligned} &[\gamma_{\{1,2\}}, \varphi^-_{\{3,4\}}](u; \alpha, \beta, \chi, \psi) \\ &\quad = \gamma(u_1; \alpha_1, \beta_1)\gamma(u_2; \alpha_2, \beta_2)\varphi^-\big(u_3; (u_1 + u_2)^2\chi_3, \psi_3\big) \\ &\quad \times \varphi^-\big(u_4; (u_2 + u_3)^2\chi_4, \psi_4\big), \end{aligned}$$

which is of the form (3.18) with

$$\begin{aligned} \theta_1 &= \big(\alpha_1, \alpha_2, -\tfrac{1}{2}\chi_3, -\tfrac{1}{2}\chi_4\big), \\ \theta_2 &= \big(-\beta_1 + \sqrt{\chi_3\psi_3}, -\beta_2 + \sqrt{\chi_3\psi_3} + \sqrt{\chi_4\psi_4}, -\tfrac{1}{2}\psi_3 + \sqrt{\chi_4\psi_4}, -\tfrac{1}{2}\psi_4\big), \\ a(\theta) &= (2\pi)^{-1}\beta_1^{\alpha_1}\beta_2^{\alpha_2}/(\Gamma(\alpha_1)\Gamma(\alpha_2))(\chi_3\chi_4)^{1/2}, \\ b(u) &= u_1^{-1}u_2^{-1}u_3^{-3/2}u_4^{-3/2}(u_1 + u_2)(u_2 + u_3), \end{aligned}$$

and

$$H(u) = \big(\log u_1, \log u_2, u_3^{-1}(u_1 + u_2)^2, u_4^{-1}(u_2 + u_3)^2\big).$$

As indicated in the beginning of this section, it is possible to have all three types of vertices in the graph $G$ and still preserve reproductivity and the exponential form (1.1) for the model with graph $G$. The simplest instance of such a model is indicated by the graph □→●→○.

Various independence properties of the reproductive combinations considered in this section have already been established by means of the general theory of reproductive exponential models. Certain results on conditional independencies in such combinations follow easily from a very general theorem concerning Markovian-type models on graphs established by Kiiveri, Speed and Carlin (1984). For a detailed discussion of this the reader is referred to Barndorff-Nielsen and Blæsild (1984b). Here we restrict ourselves to noting that the "main theorem" of Kiiveri, Speed and Carlin (1984) can be applied to those reproductive exponential models discussed in the present section for which the associated constants satisfy

$$c_{ij} = \begin{cases} 1, & i \to j, \\ 0, & \text{otherwise,} \end{cases}$$

cf. definitions (3.1), (3.11) and (3.17). For instance, for the $N_G^-$ distribution in Example 3.1, the theorem shows that $(u_1, u_2)$ and $u_4$ are conditionally independent given $u_3$.

**Acknowledgments.** We are grateful to an Associate Editor and a referee for constructive criticisms.

## REFERENCES

BARNDORFF-NIELSEN, O. E. (1980). Conditionality resolutions. *Biometrika* **67** 293–310.

BARNDORFF-NIELSEN, O. E. (1983). On a formula for the distribution of the maximum likelihood estimator. *Biometrika* **70** 343–365.

BARNDORFF-NIELSEN, O. E. and BLÆSILD, P. (1983a). Exponential models with affine dual foliations. *Ann. Statist.* **11** 753–769.

BARNDORFF-NIELSEN, O. E. and BLÆSILD, P. (1983b). Reproductive exponential models. *Ann. Statist.* **11** 770–782.

BARNDORFF-NIELSEN, O. E. and BLÆSILD, P. (1984a). Reproductive models. *Encyclopedia of Statistical Sciences*. To appear.

BARNDORFF-NIELSEN, O. E. and BLÆSILD, P. (1984b). Combination of reproductive models. Research Report 107, Dept. Theoretical Statistics, Aarhus Univ.

BLÆSILD, P. and JENSEN, J. L. (1985). Saddlepoint formulas for reproductive exponential models. *Scand. J. Statist.* **12** 193–202.

DARROCH, J. N., LAURITZEN, S. L. and SPEED, T. P. (1980). Markov fields and log-linear interaction models for contingency tables. *Ann. Statist.* **8** 522–539.

KIIVERI, H., SPEED, T. P. and CARLIN, J. B. (1984). Recursive causal models. *J. Austral. Math. Soc. Ser. A* **36** 30–52.

LAURITZEN, S. L. and WERMUTH, N. (1984). Mixed interaction models. Research report, Inst. Elec. Syst., Aalborg Univ.

WERMUTH, N. and LAURITZEN, S. L. (1983). Graphical and recursive models for contingency tables. *Biometrika* **70** 537–552.

DEPARTMENT OF THEORETICAL STATISTICS
INSTITUTE OF MATHEMATICS
UNIVERSITY OF AARHUS
NY MUNKEGADE
DK-8000 AARHUS C
DENMARK