The Annals of Statistics
1987, Vol. 15, No. 2, 800–804

A COMPLETE CLASS THEOREM FOR ESTIMATING A NONCENTRALITY PARAMETER[1]

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In statistical decision theory, an important question is to characterize the admissible rules. In this paper, we establish complete class theorems for estimating the noncentrality parameter of noncentral chi-square and noncentral F distributions under squared error loss. Under a minor assumption, any admissible estimator must be a generalized Bayes rule. Using this result, we prove that the positive part of the UMVUE is inadmissible.

1. Introduction. In statistical decision theory, an important question is to characterize the admissible rules. One very useful result would be that any admissible estimator must be a generalized Bayes rule. This result is true for estimating the natural parameter of a p-dimensional exponential family under squared error loss. See Sacks (1963), Brown (1971) and Berger and Srinivasan (1978). All three papers used the fact that any generalized Bayes rule for an exponential family can be written as a Laplace transform of some measure. Hence, it is very difficult to extend their proof to distributions not in the exponential family.

In this paper, we establish complete class theorems for estimating the noncentrality parameter of noncentral chi-square and noncentral F distributions under squared error loss. Our approach is quite different from previous papers since neither distribution belongs to the exponential family.

Noncentral chi-square and F distributions occur frequently in statistical testing procedures such as the analysis of variance for tests of homogeneity and most large sample tests. Estimation of the corresponding noncentrality parameter provides useful information for the power of different tests and has received considerable attention. Perlman and Rasmussen (1975), Neff and Strawderman (1976), Alam and Saxena (1982) and Chow and Hwang (1982) have dealt with problems related to the performance of the UMVUE and how to improve upon it. It is obvious that the UMVUE is inadmissible because it is not always positive. The positive part of the UMVUE dominates the UMVUE and hence the question of its admissibility is of interest. By the complete class theorem we establish later on, we prove that the positive part of the UMVUE is inadmissible. Below we establish the uniqueness property of generalized Bayes rules for a general class of distributions including noncentral chi-square and noncentral F distributions. This result is also needed to prove our complete class theorems in Section 3.

Received February 1985; revised September 1986.

[1]This paper was part of the author's Ph.D. dissertation, prepared at Cornell University. This research was supported by an RSDF grant from Northeastern University.

AMS 1980 *subject classifications*. Primary 62C07; secondary 62C15.

Key words and phrases. Complete class theorem, noncentrality parameter, generalized Bayes rules.

2. Uniqueness theorem. Let X be an observation with sample space $\mathscr{X} \subset (0, \infty)$ and parameter space $\Theta = [0, \infty)$. The distributions on $\mathscr{X}$ are assumed to have density with respect to a Lebesgue measure of the form

$$f_\theta(x) = h(x) \sum_{j=0}^{\infty} a_j \beta(\theta) \theta^j x^j, \tag{2.1}$$

where $h(x) > 0$ on $\mathscr{X}$, $a_j > 0$ for all j and $\beta(\theta)$ is a continuous positive function on Θ. Let $S \subset \mathscr{X}$ and G be a σ-finite measure on Θ. An estimator δ is called a generalized Bayes rule on S relative to G if it minimizes the posterior expected loss. Throughout this paper we assume that squared error loss is considered.

THEOREM 2.1. *Let $f_\theta(x)$ be of the form* (2.1). *Let δ_G (δ_H, respectively) be the generalized Bayes rule relative to G (H, respectively). Assume that G, H are renormalized so that $\int \beta(\theta) G(d\theta) = \int \beta(\theta) H(d\theta)$. Suppose that for some nondegenerate interval (ξ_0, ξ_1) in $\mathscr{X}$,*

$$\delta_G(x) = \delta_H(x), \quad \text{for } x \in (\xi_0, \xi_1) \text{ a.e.} \tag{2.2}$$

If $\sum_{j=1}^{\infty} a_j^{1/2j}$ diverges, then $G = H$.

PROOF. Since (2.2) holds on (ξ_0, ξ_1) a.e., we get that, for $x \in (\xi_0, \xi_1)$ a.e.,

$$\frac{\int_\Theta \theta \sum_{j=0}^{\infty} a_j \beta(\theta) \theta^j x^j G(d\theta)}{\int_\Theta \sum_{j=0}^{\infty} a_j \beta(\theta) \theta^j x^j G(d\theta)} = \frac{\int_\Theta \theta \sum_{j=0}^{\infty} a_j \beta(\theta) \theta^j x^j H(d\theta)}{\int_\Theta \sum_{j=0}^{\infty} a_j \beta(\theta) \theta^j x^j H(d\theta)}.$$

Fubini's theorem and rearrangement of terms yield

$$\sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} a_i \int_\Theta \theta^{i+1} \beta(\theta) G(d\theta) a_{k-i} \int_\Theta \theta^{k-i} \beta(\theta) H(d\theta) \right) x^k$$
$$= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} a_i \int_\Theta \theta^{i+1} \beta(\theta) H(d\theta) a_{k-i} \int_\Theta \theta^{k-i} \beta(\theta) G(d\theta) \right) x^k,$$
$$\text{for } x \in (\xi_0, \xi_1) \text{ a.e.}$$

The coefficients corresponding to x^k must be identical. For $k = 0$, this yields

$$\int_\Theta \theta \beta(\theta) G(d\theta) = \int_\Theta \theta \beta(\theta) H(d\theta).$$

It then follows by induction on k that

$$\mu_n = \int_\Theta \theta^n \beta(\theta) G(d\theta) = \int_\Theta \theta^n \beta(\theta) H(d\theta), \quad \text{for } n = 0, 1, 2, \dots.$$

Since $\sum_{j=0}^{\infty} a_j \mu_j x^j$ is finite, the power series converges on a nondegenerate interval and there exists a constant $R > 0$ such that

$$(a_j \mu_j)^{1/j} < R, \quad \text{for } j = 1, 2, \dots.$$

This implies that

$$\mu_j^{-1/j} > \frac{a_j^{1/j}}{R}.$$

Since $\sum_{j=1}^{\infty} a_j^{1/2j}$ diverges, $\sum_{j=1}^{\infty} \mu_j^{-1/2j}$ also diverges. By Shohat and Tamarkin (1943), $\beta(\theta)G(d\theta) = \beta(\theta)H(d\theta)$. Since $\beta(\theta) > 0$, $G = H$. □

REMARK 1. For the case $\mathscr{X} \subset (-\infty, \infty)$ and $\Theta = (-\infty, \infty)$, the uniqueness theorem holds if $\sum_{j=1}^{\infty} \mu_{2j}^{-1/2j} = \infty$. For details see Chow (1983).

3. Complete class theorems for the noncentrality parameter. Let $R(\theta, \delta_0)$ denote the risk of δ_0 under squared error loss.

THEOREM 3.1. *Let $X \sim \chi_n^2(\theta)$, the noncentral chi-square distribution with n degrees of freedom and noncentrality parameter θ with parameter space $\Theta = [0, \infty)$. Let $\delta_0(X)$ be an admissible estimator of θ. If $\delta_0(X)$ also satisfies the condition*

$$(*) \qquad \text{for some } k, \quad R(\theta, \delta_0) = O(\theta^k) \text{ as } \theta \to \infty,$$

then $\delta_0(X)$ must be a generalized Bayes rule (a.e. with respect to Lebesgue measure). Furthermore, the set of generalized Bayes procedures satisfying $(*)$ forms a complete class relative to all procedures satisfying $(*)$.

PROOF. The estimator δ_0, being admissible, is nonnegative and nondecreasing by virtue of the MLR property of the noncentral chi-square distribution [see Brown, Cohen and Strawderman (1976)]. The rest of the proof involves verifying the conditions of the theorem of Brown (1980). There are two cases:

CASE (i): *δ_0 is unbounded.* Given $x_1 > 0$ and $k > 0$, there exists $x_2 > x_1$ such that $\delta_0(x_1) < \delta_0(x_2) - k$. Define $\delta'(x) = \max(\delta_0(x), \delta_0(x_2))$ and set $B(\theta) = R(\theta, \delta_0) - R(\theta, \delta')$.

It follows from the asymptotic behavior of $f_\theta(x)$ for large θ and the monotonicity of δ_0 that

$$B(\theta) \geq k\theta \int_0^{x_1} f_\theta(x)\,dx \tag{3.1}$$

and

$$B(\theta) \leq 2\delta_0(x_2)\theta \int_0^{x_2} f_\theta(x)\,dx, \tag{3.2}$$

for all large θ ($\gg \delta_0(x_2)$). Let $h(\theta) = \max(1, 1/B(\theta))$. Plainly,

$$\liminf_{\theta \to \infty} h(\theta)\left[R(\theta, \delta_0) - R(\theta, \delta')\right] = 1. \tag{3.3}$$

Next, we want to show that for all $x < x_1$,

$$\begin{aligned} &\lim_{\theta \to \infty} h(\theta)(L(\theta, a) + 1) f_\theta(x) = 0, \quad \text{for all } a \in [0, \infty), \\ &\sup_\theta h(\theta) R(\theta, \delta_0) f_\theta(x) < \infty. \end{aligned} \tag{3.4}$$

In view of $(*)$ and the continuity of $R(\theta, \delta_0)$, (3.4) would follow if, for all $t < x_1$ and $m \geq 1$,

$$\lim_{\theta \to \infty} \theta^m h(\theta) f_\theta(t) = 0 \tag{3.5}$$

and this is immediate from (3.2) and the fact that for $y < z$, $f_\theta(y)/f_\theta(z) = O(e^{\sqrt{\theta(y-z)}})$.

Thus, the theorem of Brown (1980) is applicable and it implies that $\delta_0(x)$ is generalized Bayes for almost all $x \in (0, x_1)$. Since x_1 is arbitrary, $\delta_0(x)$ is generalized Bayes on every interval of the form $(0, t)$. Now appeal to the uniqueness theorem to conclude that δ_0 is a generalized Bayes rule a.e.

CASE (ii): *δ_0 is bounded.* Let K be such that $\delta_0(x) < K, \forall x$. Define $\delta'(x) = K_1 > K$. Arguments similar to those in case (i) imply δ_0 is a generalized Bayes rule a.e.

For both cases we have proved that every admissible rule satisfying $(*)$ must be a generalized Bayes rule. Since the admissible rules form a complete class [Brown (1987), Corollary 4A.6], the set of generalized Bayes rules forms a complete class relative to all rules satisfying $(*)$. □

The same complete class theorem holds for the noncentral F distribution with similar proof. For details see Chow (1983).

4. Applications of the complete class theorems. For estimating the noncentrality parameter θ of the noncentral chi-square distribution, de Waal (1974) has shown that $x + n$ is the generalized Bayes estimator for θ with respect to a noninformative prior distribution. We know that $x + n$ is inadmissible since it is dominated by $x - n$, the UMVUE. Note that $E_\theta(x - n) = \theta$. Furthermore, since $\theta = [0, \infty)$, $(x - n)^+ = \max(0, x - n)$ dominates $x - n$. It is interesting to consider the admissibility of $\delta_0(x) = (x - n)^+$. Note that δ_0 was conjectured to be inadmissible by many statisticians including Alam and Saxena (1982). This is first established below.

THEOREM 4.1. *For estimation of the noncentrality parameter θ of the noncentral chi-square distribution under squared error loss, $\delta_0(x) = (x - n)^+$ is inadmissible.*

PROOF. Calculation yields $R(x - n, \theta) = 2n + 4\theta$. Since we know that $(x - n)^+$ dominates $x - n$, we get

$$R(\delta_0, \theta) \le 2n + 4\theta \le O(\theta). \tag{4.1}$$

By (4.1) the $(*)$ condition in Theorem 3.1 is satisfied. If δ_0 is admissible, then by Theorem 3.1 it must be a generalized Bayes rule a.e. However, Theorem 2.1 implies that the only generalized Bayes rule which is equal to δ_0 a.e. must be the zero estimator since δ_0 is zero on the nondegenerate interval $(0, n)$. This implies that δ_0 is zero a.e., which is not true. δ_0 is then inadmissible. □

Let Z be a noncentral F distributed random variable with n, m degrees of freedom and noncentrality parameter θ. Assume that $m \ge 5$. Perlman and Rasmussen (1975) have shown that the UMVUE of θ is $(m - 2)z - n$. They also showed that the linear estimator $a\{(m - 2)z - n\}$ dominates the UMVUE for all $\theta \ge 0$ provided that

$$\max\left(0, \frac{m-6}{m-2}\right) \le a < 1.$$

Note that the estimator $a\{(m-2)z-n\}$ is negative for $z<n/(m-2)$. Since $\Theta=[0,\infty)$, this estimator is dominated by its positive part, $\delta_a(z)=a\{(m-2)z-n\}^+$. Therefore, it is interesting to consider the admissibility of $\delta_a(z)$.

THEOREM 4.2. *For estimation of the noncentrality parameter θ of the noncentral F distribution under squared error loss, $\delta_a(z)=a\{(m-2)z-n\}^+$ is inadmissible for any $a\neq 0$.*

PROOF. Clearly if $a<0$, $\delta_a(z)$ is inadmissible since it is uniformly dominated by the zero estimator. For $a>0$ the risk of δ_a can be shown to be $O(\theta^2)$, and arguments similar to those in Theorem 4.1 complete the proof. □

Acknowledgments. My deepest thanks go to Professor Larry Brown for bringing this problem to my attention, his continued encouragement and numerous valuable suggestions. The author also wants to thank Professor Jiunn Hwang, the Associate Editor and the referees for their helpful comments and careful reading of the paper.

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