The Annals of Statistics
1987, Vol. 15, No. 2, 552–567

ESTIMATING TRAJECTORIES[1]

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Let f be a continuously differentiable function from $[0,1]$ to the complex plane. Suppose that "at time n" we are given the random set $\{f(k/n) + e_{n,k}: 1 \le k \le n\}$, where the random errors $e_{n,k}$ are i.i.d. and $(\operatorname{Re} e_{n,1}, \operatorname{Im} e_{n,1})$ is $N\left((0,0), \sigma^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$ with σ^2 known. *We do not know* which datum belongs to which position $\theta = k/n$, $k = 1, 2, \ldots, n$. In general, f cannot be determined. In this paper it is shown that a random set T_n can be constructed such that with probability one, T_n converges in the Hausdorff sense to the trajectory $f([0,1])$.

0. Introduction. Let f be a continuously differentiable function from $[0,1]$ to the complex plane $\mathbb{C}$. Since regression problems have been extensively studied in the literature of statistics, it is fairly easy to construct a consistent estimator of f from the data

$$Z_{n,k} = f(k/n) + e_{n,k}, \qquad k = 1, 2, \ldots, n, \tag{0.1}$$

where the complex-valued random errors $e_{n,k}$ are i.i.d. and $(\operatorname{Re} e_{n,1}, \operatorname{Im} e_{n,1})$ is Gaussian $N\left((0,0), \sigma^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$ with σ^2 a known constant. In this paper we shall consider the same estimation problem but with the data (0.1) unlabelled: We are only given the random set

$$\{Z_{n,k}: k = 1, 2, \ldots, n\} \tag{0.2}$$

and do not know which datum belongs to which position $\theta = k/n$, $k = 1, 2, \ldots, n$. To our knowledge this problem has not been previously studied.

Even asymptotically, f will not, in general, be uniquely determined by the random set (0.2). Consider, for example, the special case $f(\theta) = \exp(i2\pi\theta)$ (a circle!). It will be impossible to distinguish f from one of its rotations f_α defined by $f_\alpha(\theta) = \exp[i2\pi(\theta + \alpha)]$, $0 \le \theta \le 1$. Instead, we shall estimate the *trajectory* of f

$$L \equiv f([0,1])$$

from the observed random set (0.2).

The main purpose of this paper is to construct a consistent estimator for L. By using Bessel functions and the Hankel transform, we shall construct from the data set (0.2) a random set T_n that converges, as $n \to \infty$, with probability one to the trajectory L in the Hausdorff sense. Here, as usual, the Hausdorff distance

Received June 1984; revised August 1986.

[1]This work was partially supported by NSF Grant MCS 76-07203 and the National Science Council of the Republic of China.

AMS 1980 *subject classifications*. Primary 62G05; secondary 62J99.

Key words and phrases. Nonparametric estimation, unlabelled data, random set, trajectory estimation.

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between two sets S and T is defined by

$$d_H(S, T) = \max\Big(\sup_{x\in S} \operatorname{dist}(x, T),\ \sup_{y\in T} \operatorname{dist}(S, y)\Big),$$

and S_n converges to T in the Hausdorff sense ($\lim S_n = T$) if $\lim_n d_H(S_n, T) = 0$.

1. Preliminary results on the Bessel functions. As mentioned in the introduction, we shall need some results from the theory of Bessel functions. For convenience, these results are collected in this section.

(a) The Bessel function J_n of order n, $n = 0, 1, 2, \ldots$, can be represented [Watson (1944), page 15] as

$$J_n(z) = \sum_{m=0}^{\infty} (-1)^m (z/2)^{n+2m}/(m!(n+m)!), \qquad z \in \mathbb{C}. \tag{1.1}$$

Hence, $J_n(z)$ has a zero of order n at $z = 0$ and is real when z is real.

(b) Term-by-term differentiation of (1.1) shows [Watson (1944), page 18]

$$d(zJ_1(z))/dz = zJ_0(z), \qquad d(J_0(z))/dz = -J_1(z). \tag{1.2}$$

Integrating the first equation from 0 to δr, we obtain, after a simple calculation,

$$J_1(\delta r) = \delta^{-1} r \int_0^{\delta} u J_0(ru)\, du, \qquad \delta, r > 0. \tag{1.3}$$

(c) [Watson (1944), pages 20 and 25]

$$\begin{aligned} J_0(z) &= (2\pi)^{-1} \int_0^{2\pi} \exp(-iz \sin\theta)\, d\theta \\ &= (2\pi)^{-1} \int_0^{2\pi} \exp(iz\cos\theta)\, d\theta. \end{aligned} \tag{1.4}$$

Hence $|J_0(z)| \le 1$ when z is real.

$$J_1(z) = \pi^{-1} z \int_{-1}^{1} \sqrt{1-t^2} \exp(izt)\, dt = \pi^{-1} z \int_{-1}^{1} \sqrt{1-t^2} \cos(zt)\, dt. \tag{1.5}$$

(d) Since $2^{-1} \le \cos u \le 1$ on $[-1, 1]$, it follows from (1.5) that

$$4^{-1} u \le J_1(u) \le 2^{-1} u, \quad \text{for } 0 \le u \le 1.$$

(e) The second equation in (1.2) and (d) imply that $d(J_0(z))/dz < 0$ on $(0, 1]$. Hence $J_0(u)$ is strictly decreasing on $[0, 1]$. Then

$$2^{-1} \le J_0(1) \le J_0(u) \le J_0(0) = 1, \qquad 0 \le u \le 1,$$

can be proved by using (1.4).

(f) [Watson (1944), page 195]. When u is positive and large,

$$J_n(u) \approx \sqrt{\frac{2}{\pi u}} \cos\left(u - \frac{n\pi}{2} - \frac{\pi}{4}\right), \qquad n = 0, 1, 2, \ldots.$$

In particular, $\sup_{u\ge 0} \sqrt{u}\, |J_n(u)| < \infty$.

(g) *Hankel inversion theorem* [Watson (1944), page 456]. Let h be any function satisfying $\int_0^\infty |h(u)| \sqrt{u}\, du < \infty$. Then for any $\nu \ge -\frac{1}{2}$

$$\int_0^\infty r J_\nu(tr)\, dr \int_0^\infty u J_\nu(ru) h(u)\, du = 2^{-1}(h(t^+) + h(t^-)),$$

provided that the positive number t lies inside an interval in which h has a bounded total variation.

(h) For any $c > 0$, (1.5) implies

$$\int_0^c J_1(u)/u\,du = 2\pi^{-1}\int_0^1 \sqrt{1-t^2}\,dt \int_0^c \cos(ut)\,du = 2\pi^{-1}\int_0^1 \sqrt{1-t^2}\,t^{-1}\sin(ct)\,dt.$$

From this we see two things: (i) $\int_0^c J_1(u)/u\,du > 0$, because $\sqrt{1-t^2}\,t^{-1}$ is decreasing on $(0,1)$ and $\sin(ct)$ is alternating periodically; and (ii) for $c \geq 2\pi$,

$$\int_0^c J_1(u)/u\,du \geq \pi^{-2}, \tag{1.6}$$

which can be shown as follows:

$$\begin{aligned}
2^{-1}\pi\int_0^c J_1(u)/u\,du &= \int_0^1 \sqrt{1-t^2}\,t^{-1}\sin(ct)\,dt \\
&\geq \int_0^{2\pi/c} = \left(\int_0^{\pi/c} + \int_{\pi/c}^{2\pi/c}\right) \\
&= c^{-1}\int_0^{\pi}\left[\sqrt{c^2-t^2}\,t^{-1} - \sqrt{c^2-(\pi+t)^2}\,(\pi+t)^{-1}\right]\sin t\,dt \\
&\geq c^{-1}\int_0^{\pi}\sqrt{c^2-t^2}\left(t^{-1}-(\pi+t)^{-1}\right)\sin t\,dt \\
&= \pi c^{-1}\int_0^{\pi}\sqrt{c^2-t^2}\,(t(\pi+t))^{-1}\sin t\,dt \\
&\geq \pi c^{-1}\int_0^{\pi}\sqrt{c^2-\pi^2}\,(\pi 2\pi)^{-1}\sin t\,dt \\
&\geq (2\pi)^{-1}.
\end{aligned}$$

In fact, it can be shown that $\lim_{c\to\infty}\int_0^c J_1(u)/u\,du = 1$.

2. Statements of the main results. Since the data in (0.2) are unordered, we will naturally use them in a symmetric way. As a first step, we will construct from (0.2) a function $X_n(z)$ such that $X_n(z)$ depends on the data symmetrically and $\lim_n E(X_n(z)) = 0$ iff $z \notin L$. We will then use $X_n(\cdot)$ to construct a random set T_n that converges strongly in the Hausdorff sense to L.

Introduce the following function on the complex plane $\mathbb{C}$:

$$G_{\lambda,\delta}(z) = \int_{|\xi|\leq\lambda}\left[\exp(i\langle z,\xi\rangle + \sigma^2|\xi|^2/2)\right] J_1(\delta|\xi|)/|\xi|\,d\xi, \tag{2.1}$$

where $\langle z,\xi\rangle = (\operatorname{Re} z)\cdot(\operatorname{Re}\xi) + (\operatorname{Im} z)\cdot(\operatorname{Im}\xi)$ and J_1 is the Bessel function of order one. Note that the integrand is meaningful at $\xi = 0$, since (Section 1a) J_1 has a zero of order one at the origin.

We claim that $G_{\lambda,\delta}$ is in fact a real-valued function. Let $z = |z|\exp(i\alpha)$. By changing the variable ξ in (2.1) to $\xi\exp(i\alpha)$, it is easy to check that $G_{\lambda,\delta}(z)$

depends only on $|z|$. So the number z in the integrand may be replaced by $|z|$. Then changing the integral to the polar coordinate and using (1.4),

$$\begin{aligned} G_{\lambda,\delta}(z) &= \int_0^{\lambda}\int_0^{2\pi}\left[\exp\left(i|z|r\cos\theta + \sigma^2r^2/2\right)\right] J_1(\delta r)\, d\theta\, dr \\ &= 2\pi\int_0^{\lambda}\left[\exp\left(\sigma^2r^2/2\right)\right] J_0(|z|r)J_1(\delta r)\, dr, \end{aligned} \tag{2.2}$$

which verifies the claim, because (Section 1a) $J_0(z)$ and $J_1(z)$ are real when z is real.

Since $E(\exp(i\langle e_{n,k}, \xi\rangle)) = \exp(-\sigma^2|\xi|^2/2)$, by first taking the expectation and then using the same argument, we can show that

$$\begin{aligned} E\{G_{\lambda,\delta}(z - Z_{n,k})\} &= 2\pi\int_0^{\lambda} J_0\left(|z - f(k/n)|r\right)J_1(\delta r)\, dr \\ &= 2\pi\delta^{-1}\int_0^{\lambda} rJ_0\left(|z - f(k/n)|r\right) dr \int_0^{\delta} uJ_0(ru)\, du. \end{aligned} \tag{2.3}$$

Note that in the second equality we have used (1.3).

The following heuristic argument will show how the function $G_{\lambda,\delta}$ can be used to estimate the trajectory L. For brevity, we assume that f is one-to-one and $\min_{0\le\theta\le1}|f'(\theta)| > 0$. Consider the occupation measure μ_z, which is defined for any Borel set S on the real line as $\mu_z(S) = m\{\theta \in [0,1]: |z - f(\theta)| \in S\}$, where m is Lebesgue measure. Then μ_z has a compact support $[m(z), M(z)]$ with $M(z) = \max_{0\le\theta\le1}|z - f(\theta)|$ and $m(z) = \min_{0\le\theta\le1}|z - f(\theta)|$. Furthermore, $m(z) = 0$ iff $z \in L$. Geometrically speaking, $\mu_z([0,t])$ is a measure of that portion of L, which lies inside the circle with radius t and center at z. In most cases, μ_z has a piecewise smooth derivative g_z unless L is a circle with center at z. But in all cases $g_z(0^+)$ exists: When $z = f(\theta) \in L$

$$g_z(0^+) = \begin{cases} 2|f'(\theta)|^{-1}, & \text{if } L \text{ is a closed curve or } \theta \neq 0,1, \\ |f'(\theta)|^{-1}, & \text{otherwise} \end{cases}$$

and $g_z(0^+) = 0$ if $z \notin L$. From (2.3) and Lebesgue's dominated convergence theorem,

$$\begin{aligned} &\lim_n E\left\{n^{-1}\sum_{k=1}^{n} G_{\lambda,\delta}(z - Z_{n,k})\right\} \\ &\quad= \lim_n 2\pi\delta^{-1}\int_0^{\lambda} rn^{-1}\sum_{k=1}^{n} J_0\left(|z - f(k/n)|r\right) dr \int_0^{\delta} uJ_0(ru)\, du \\ &\quad= 2\pi\delta^{-1}\int_0^{\lambda} r\, dr\left(\int_0^1 J_0\left(|z - f(\theta)|r\right) d\theta\right)\int_0^{\delta} uJ_0(ru)\, du \\ &\quad= 2\pi\delta^{-1}\int_{m(z)}^{M(z)} d\mu_z([0,t])\int_0^{\lambda} rJ_0(tr)\, dr\int_0^{\delta} uJ_0(ru)\, du. \end{aligned}$$

Now apply the Hankel inversion theorem (Section 1g) to the indicator function $I_{[0,\delta]}$,

$$\begin{aligned}
&\lim_{\delta\to 0^+}\lim_{\lambda\to\infty}\lim_{n} E\left\{n^{-1}\sum_{k=1}^{n} G_{\lambda,\delta}(z - Z_{n,k})\right\}\\
&= \lim_{\delta\to 0^+} 2\pi\delta^{-1}\int_{m(z)}^{M(z)} d\mu_z([0,t])\int_0^\infty rJ_0(tr)\,dr\int_0^\delta uJ_0(ru)\,du\\
&= \lim_{\delta\to 0^+} 2\pi\delta^{-1}\int_{m(z)}^{M(z)} I_{[0,\delta]}(t)\,d\mu_z([0,t])\\
&= \begin{cases} 0, & \text{if } z \notin L,\\ 2\pi g_z(0^+), & \text{if } z \in L.\end{cases}
\end{aligned}$$

Therefore, we might seek $\lambda_n \to \infty$ and $\delta_n \to 0$ such that the triple limit in the previous equation can be replaced by a single one,

$$\lim E\left\{n^{-1}\sum_{k=1}^{n} G_{\lambda_n,\delta_n}(z - Z_{n,k})\right\} = \begin{cases} 0, & \text{if } z \notin L,\\ 2\pi g_z(0^+), & \text{if } z \in L.\end{cases} \tag{2.4}$$

Because f is continuously differentiable,

$$\min_{z\in L} g_z(0^+) \geq \left(\max_{0\leq\theta\leq 1}|f'(\theta)|\right)^{-1} > 0.$$

Furthermore, the real random variable

$$X_n(z) \equiv n^{-1}\sum_{k=1}^{n} G_{\lambda_n,\delta_n}(z - Z_{n,k}), \qquad z \notin \mathbb{C}, \tag{2.5}$$

as sort of an average of those n data in (0.1), is independent of the labelling of them and thus, depends only on the random set (0.2). Since (2.4) implies that in some sense $X_n(z)$ "detects" the trajectory L at least when f is one-to-one and $\min_{0\leq\theta\leq 1}|f'(\theta)| > 0$, it is reasonable to expect that $X_n(z)$ is what we want. In fact, a careful calculation shows:

THEOREM 1. *Let f be a continuously differentiable function from $[0,1]$ to the complex plane $\mathbb{C}$ with*

$$\min_{0\leq\theta\leq 1}|f'(\theta)| > 0, \tag{2.6}$$

and let the real random variable $X_n(z)$ be constructed from the unlabelled data (0.2) via (2.5). If λ_n, δ_n, ε and η are positive constants and satisfy

$$\begin{gathered}\frac{1}{3} < \varepsilon < \frac{1}{(2\eta)} < \frac{1}{2}, \qquad \lambda_n\delta_n^{1-\varepsilon} = 1,\\ \lim\delta_n = \lim\delta_n^{\varepsilon-1}/n = 0,\end{gathered} \tag{2.7}$$

then for n large enough

$$L \subseteq E_n \subseteq \left\{z\colon \operatorname{dist}(z; L) \leq (4\pi M^2)^{2/3}\delta_n^{(1-2\eta\varepsilon)/3}\right\}, \tag{2.8}$$

where $M = \sup_{u \geq 0}[\max(\sqrt{u}\,|J_0(u)|, \sqrt{u}\,|J_1(u)|)]$ *is positive and finite and* $E_n = \{z\colon\ E(X_n(z)) \geq \delta_n^{\eta\varepsilon}\}$. *In particular,* E_n *converges to the trajectory* L *in the Hausdorff sense.*

Let us remark that the assumption (2.6) implies that for each $z \in L$, the equation $z = f(\theta)$ has at most a finite number of solutions in $[0,1]$. Then (2.4) still holds except that now $g_z(0^+)$ is probably replaced by a finite sum. This means that under (2.6), f can be treated as if it is one-to-one, as we did in the heuristic argument.

Theorem 1 can be strengthened to strong consistency by a technique similar to that used in Geman and Hwang (1982).

Denote the random set $\{z\colon\ X_n(z) \geq 2\delta_n^{\eta\varepsilon}\}$ by T_n.

THEOREM 2. *Assume that in addition to* (2.6) *and* (2.7), *there exist positive constants* a_n, d_n *with*

$$\lim a_n \delta_n^{-\eta\varepsilon} = 0, \qquad \lim d_n = \infty, \tag{2.9}$$

$$\sum_n d_n^2\left[a_n^2\sigma^4\exp\left(-\sigma^2\delta_n^{2(\varepsilon-1)}\right)\right]^{-1} \times\left[1-(1152\pi^2)^{-1}a_n^2\sigma^4\exp\left(-\sigma^2\delta_n^{2(\varepsilon-1)}\right)\right]^n < \infty. \tag{2.10}$$

Then the random set T_n *satisfies*

$$P\Big(L \subseteq T_n \subseteq \left\{z\colon \operatorname{dist}(z; L) \leq (4\pi M^2)^{2/3}\delta_n^{(1-2\eta\varepsilon)/3}\right\},\ \textit{for } n \textit{ large}\Big) = 1, \tag{2.11}$$

where M *is defined in Theorem* 1. *In particular, with probability one the random set* T_n *converges to the trajectory* L *in the Hausdorff sense.*

(2.6) in the previous two theorems can be dropped. By Sard's theorem [Spivak (1965) and Sternberg (1983)] $f(\{\theta \in [0,1]\colon\ f'(\theta) = 0\})$ is a set of measure 0 on the trajectory L. This is the same as to say that for almost all $z \in L$,

$$|f'(\theta)| > 0, \tag{2.12}$$

for each $\theta \in [0,1]$ satisfying $f(\theta) = z$. In view of Theorems 1 and 2 we would expect to recover in the proper sense almost all $z \in L$, and then, the whole L. Thus we have:

THEOREM 3. *Assume that* f *is a continuously differentiable function from* $[0,1]$ *to the complex plane* $\mathbb{C}$, *and assume* (2.7), (2.9) *and* (2.10) *hold. Then*

$$\lim_n E_n = L \quad \textit{in the Hausdorff sense}. \tag{2.13}$$

Furthermore, the random set T_n *converges with probability one to the trajectory* L *in the Hausdorff sense.*

Note that the first set-inclusion formula in both (2.8) and (2.11) may no longer be true without assuming (2.6).

Now it is easy to construct examples fulfilling the requirements of Theorem 2. For instance, let α, β, η, c and ε be positive constants independent of f satisfying $\frac{1}{3} < \varepsilon < 1/(2\eta) < \frac{1}{2}$, $0 < 2\alpha(1-\varepsilon) < 1$, and $\alpha\eta\varepsilon < \beta$. Define

$$\delta_n = (c \log n)^{-\alpha}, \qquad \lambda_n = \delta_n^{\varepsilon - 1}, \qquad d_n = n \quad \text{and} \quad a_n = (\log n)^{-\beta}.$$

Since $\log(1-x) \le -x$ for $x \ge 0$, and $2\alpha(1-\varepsilon) < 1$,

$$\begin{aligned}
&\left[1 - (1152\pi^2)^{-1} a_n^2 \sigma^4 \exp\left(-\sigma^2 \delta_n^{2(\varepsilon-1)}\right)\right]^n \\
&\quad \le \exp\left[-n(1152\pi^2)^{-1}\sigma^4 (\log n)^{-2\beta} \exp\left(-\sigma^2 (c\log n)^{2\alpha(1-\varepsilon)}\right)\right] \\
&\quad \le \exp\left[-(1152\pi^2)^{-1}\sigma^4 \exp\left[\log n - 2\beta\log(\log n) - \sigma^2(c\log n)^{2\alpha(1-\varepsilon)}\right]\right] \\
&\quad \le \exp\left[-(1152\pi^2)^{-1}\sigma^4 \exp[(\log n)/2]\right] \\
&\quad = \exp\left[-(1152\pi^2)^{-1}\sigma^4 n^{1/2}\right]
\end{aligned}$$

holds for n large. Then (2.7), (2.9) and (2.10) can be checked easily. In particular, we may let $\varepsilon = \Delta + \frac{1}{3}$, $\eta = 1 + \Delta$, $\beta = 1$ and $\alpha = 1/(4\eta\varepsilon)$ with Δ a very small positive number. The power $\alpha(1 - 2\eta\varepsilon)/3$ of $(\log n)^{-1}$ in (2.11) is then roughly $-\Delta + \frac{1}{12}$. By choosing $\varepsilon' < \Delta$ we have the following consequence of Theorems 1 and 2:

COROLLARY 4. *Assume that f satisfies the assumptions in Theorem 1. Let c_3, c_4 and ε' be positive constants with $\varepsilon' < \frac{1}{12}$. Then*

$$\lim_n \left\{z\colon E(X_n(z)) \ge c_3 (\log n)^{-1/4}\right\} = L \quad \textit{in the Hausdorff sense}$$

and the random set $\{z\colon X_n(z) \ge 2c_3(\log n)^{-1/4}\}$, denote it by $\tilde{T}_n$, satisfies

$$P\left(L \subseteq \tilde{T}_n \subseteq \left\{z\colon \operatorname{dist}(z; L) \le c_4 (\log n)^{\varepsilon' - 1/12}\right\}, \textit{ for } n \textit{ large}\right) = 1.$$

In particular, $\tilde{T}_n$ converges with probability one to the trajectory L in the Hausdorff sense.

Theorems 1, 2 and 3 will be proved in Sections 3, 4 and 5, respectively. In Section 6 we shall discuss briefly some questions and possible extensions. There we also give some remarks on the differences between these three theorems.

3. Proof of Theorem 1. It is obvious that we only have to prove (2.8). To this end the following two lemmas will be needed.

LEMMA 3.1. *If $0 < 2\delta \le s$, then for all $\lambda > 0$*

$$\left| 2\pi\delta^{-1} \int_0^\lambda r J_0(sr)\, dr \int_0^\delta u J_0(ru)\, du \right| \le 4\pi M^2 \delta^{1/2} s^{-3/2},$$

where $M = \sup_{u \ge 0} \max(\sqrt{u}\,|J_0(u)|, \sqrt{u}\,|J_1(u)|)$ is positive and finite.

LEMMA 3.2. *Under* (2.6) *and* (2.7), *there exists a constant* $\Gamma = \Gamma(f) > 0$ *such that when* n *is large enough*

$$E(X_n(z)) \geq \Gamma \delta_n^{\varepsilon}$$

holds uniformly for $z \in L$.

From the derivation of (2.4) it is clear that the right-hand side of (2.3) should be estimated first. Lemma 3.1, which serves this purpose, implies

$$|E(X_n(z))| \leq 4\pi M^2 \delta_n^{1/2} s^{-3/2}, \tag{3.1}$$

provided that $s \equiv \text{dist}(z; L) \geq 2\delta_n$. Now we may verify the second half of (2.8) as follows: For each $z \in E_n$ either $\text{dist}(z; L) < 2\delta_n$, which is smaller than $(4\pi M^2)^{2/3}\delta_n^{(1-2\eta\varepsilon)/3}$ as $n \to \infty$, or else $\text{dist}(z; L) \geq 2\delta_n$. In the latter case (3.1) is applicable, and solving for s we see that $s = \text{dist}(z; L) \leq (4\pi M^2)^{2/3}\delta_n^{(1-2\eta\varepsilon)/3}$. Hence $E_n \subseteq \{z: \text{dist}(z; L) \leq (4\pi M^2)^{2/3}\delta_n^{(1-2\eta\varepsilon)/3}\}$.

Because $\eta > 1$, the other half of (2.8) is true in view of Lemma 3.2. This completes the proof of (2.8).

What remains are the proofs of the previous two lemmas.

PROOF OF LEMMA 3.1. The finiteness of M is already proved in Section 1f. By using (1.2)

$$\begin{aligned} &2\pi\delta^{-1}u(s^2 - u^2)^{-1}\left[d\left[srJ_1(sr)J_0(ur) - J_0(sr)urJ_1(ur)\right]/dr\right] \\ &\quad = 2\pi\delta^{-1}urJ_0(sr)J_0(ur). \end{aligned}$$

Integrating the last equality over the region $0 \leq r \leq \lambda$, $0 \leq u \leq \delta$, we obtain

$$\begin{aligned} &2\pi\delta^{-1}\int_0^{\lambda} rJ_0(sr)\,dr \int_0^{\delta} u J_0(ru)\,du \\ &\quad = \left[2\pi\delta^{-1}s\lambda J_1(s\lambda)\int_0^{\delta}(s^2 - u^2)^{-1}u J_0(\lambda u)\,du\right] \\ &\qquad - \left[2\pi\delta^{-1}\lambda J_0(s\lambda)\int_0^{\delta}(s^2 - u^2)^{-1}u^2 J_1(\lambda u)\,du\right] \\ &\quad \equiv I_1 - I_2. \end{aligned} \tag{3.2}$$

By the definition of M

$$\begin{aligned} |I_1| &\leq 2\pi\delta^{-1}s^{1/2}M\int_0^{\delta}(s^2 - u^2)^{-1}u^{1/2}M\,du \\ &= 2\pi\delta^{-1}M^2\int_0^{\delta/s}(1 - u^2)^{-1}u^{1/2}\,du \\ &\leq 2\pi\delta^{-1}M^2\tfrac{4}{3}\int_0^{\delta/s}u^{1/2}\,du \\ &\leq 2\pi M^2\delta^{1/2}s^{-3/2}. \end{aligned}$$

Similarly,

$$\begin{aligned}
|I_2| &\leq 2\pi\delta^{-1}s^{-1/2}M\int_0^{\delta}(s^2-u^2)^{-1}u^{3/2}M\,du \\
&= 2\pi\delta^{-1}M^2\int_0^{\delta/s}(1-u^2)^{-1}u^{3/2}\,du \\
&\leq 2\pi\delta^{-1}M^2\tfrac{4}{3}\int_0^{\delta/s}u^{3/2}\,du \\
&\leq 2\pi\delta^{-1}M^2(\delta/s)^{5/2} \\
&\leq 2\pi\delta^{-1}M^2(\delta/s)^{3/2} \\
&= 2\pi M^2\delta^{1/2}s^{-3/2}.
\end{aligned}$$

Thus $|I_1 - I_2| \leq 4\pi M^2\delta^{1/2}s^{-3/2}$ and the proof is complete. □

Proof of Lemma 3.2. According to the remark after the statement of Theorem 1, we may assume for brevity that f is one-to-one. Furthermore, we assume that $f(0) = f(1)$, i.e., L is a closed curve, to avoid discussing separately the two endpoints and the other points of L.

Under the assumptions of Theorem 1, there are constants $A_0 = A_0(f)$ and $A_1 = A_1(f)$ such that $0 < 2A_0 \leq |f'(\theta)| \leq A_1$ for all $\theta \in [0,1]$. Then for all θ_1, θ_2 in $[0,1]$

$$|f(\theta_1) - f(\theta_2)| \leq A_1|\theta_1 - \theta_2|, \tag{3.3}$$

and there is a positive constant $\tau = \tau(f)$ such that

$$|f(\theta_1) - f(\theta_2)| \geq A_0|\theta_1 - \theta_2(\operatorname{mod} 1)| \tag{3.4}$$

holds for all $\theta_1, \theta_2 \in [0,1]$ with $|f(\theta_1) - f(\theta_2)| \leq \tau$.

Fix $\theta \in [0,1]$ and $z = f(\theta)$. We estimate $E(X_n(z))$ by first dividing the summation in (2.5) into two parts,

$$\begin{aligned}
E(X_n(z)) &= \sum_{k=1}^{n} n^{-1}E\{G_{\lambda_n,\delta_n}(z - Z_{n,k})\} \\
&= \sum_{\substack{k \\ s_k \leq c_1\delta_n^{1-\varepsilon}}} + \sum_{\substack{k \\ s_k > c_1\delta_n^{1-\varepsilon}}} \\
&\equiv \alpha_n + \beta_n,
\end{aligned} \tag{3.5}$$

where $s_k = |z - f(k/n)|$ and $c_1 \leq 1$ is a constant to be determined later.

We claim that if n is large, then

$$\alpha_n \geq c_1\delta_n^{\varepsilon}/(2A_1) \quad \text{uniformly in } \theta. \tag{3.6}$$

By using the results in Sections 1d and 1e, it is clear from (2.7) and the first equation of (2.3) that each term in α_n is no less than $2\pi n^{-1}\int_0^{\lambda_n}J_0(1)J_1(\delta_n r)\,dr \geq \pi\delta_n^{2\varepsilon-1}/(8n)$. If $|\theta - k/n| \leq c_1\delta_n^{1-\varepsilon}/A_1$, then (3.3) implies $|z - f(k/n)| \leq c_1\delta_n^{1-\varepsilon}$. The number of terms in α_n is roughly $2nc_1\delta_n^{1-\varepsilon}/A_1$, which, under the condition $\lim \delta_n^{\varepsilon-1}/n = 0$, is a big number when n is large. Now (3.6) can be easily checked.

As to β_n, we represent each term by the second equation of (2.3) and decompose it in the same way as we did in (3.2). Since $s^2(s^2-u^2)^{-1} = 1 + u^2(s^2-u^2)^{-1}$, we can write I_1 in (3.2) as

$$\begin{aligned} I_1 &= 2\pi\delta^{-1}s^{-1}\lambda J_1(s\lambda)\left[\int_0^\delta uJ_0(\lambda u)\,du + \int_0^\delta (s^2-u^2)^{-1}u^3J_0(\lambda u)\,du\right] \\ &\equiv I_3 + I_4. \end{aligned}$$

Then, with λ, δ, s and I_i replaced by λ_n, δ_n, s_k and $I_{i,k}$, respectively,

$$\begin{aligned} \beta_n &= n^{-1}\left\{\sum_{\substack{k \\ c_1\delta_n^{1-\varepsilon} < s_k \le c_2\delta_n^{1-\varepsilon}}} I_{3,k} + \sum_{\substack{k \\ c_2\delta_n^{1-\varepsilon} < s_k}} I_{3,k} + \sum_{\substack{k \\ c_1\delta_n^{1-\varepsilon} < s_k}} (I_{4,k} - I_{2,k})\right\} \qquad (3.7) \\ &\equiv \beta_{n,1} + \beta_{n,2} + \beta_{n,3}, \end{aligned}$$

where the constant c_2 will be determined later. We claim that when n is large

$$|\beta_{n,3}| \le \pi\left[c_1^{-2} + Mc_1^{-7/2}\right]\delta_n^{4\varepsilon-1}, \qquad (3.8)$$

$$|\beta_{n,2}| \le \pi M\tau^{-3/2}\delta_n^{(1+\varepsilon)/2} + 8\pi MA_0^{-3/2}A_1^{1/2}c_2^{-1/2}\delta_n^{\varepsilon}, \qquad (3.9)$$

and with $z = f(\theta) \in L$,

$$\lim \beta_{n,1}\delta_n^{-\varepsilon} = 2\pi|f'(\theta)|^{-1}\int_{c_1}^{c_2} J_1(t)/t\,dt \qquad (3.10)$$

holds uniformly with respect to $\theta \in [0,1]$.

Assume temporarily that (3.8)–(3.10) hold. By (1.6) and (3.10)

$$\beta_{n,1} \ge (\pi A_1)^{-1}\delta_n^{\varepsilon},$$

if n is large, c_1 is small and $c_2 \ge 2\pi$. Now fix c_1 sufficiently small. If c_2 is chosen such that $c_2 \ge \max(2\pi, [32\pi M(A_1/A_0)^{3/2}/c_1]^2)$, then by (3.6), (3.8) and (3.9)

$$|\beta_{n,2} + \beta_{n,3}| \le 2^{-1}\alpha_n, \quad \text{as } n \to \infty,$$

because $3^{-1} < \varepsilon < 1$ under (2.7). In view of (3.5)–(3.7) we obtain Lemma 3.2 with $\Gamma = c_1(4A_1)^{-1} + (\pi A_1)^{-1}$, i.e., depending on f.

It remains to check (3.8)–(3.10). Since (Sections 1c and 1d) $|J_0(u)| \le 1$ for u real and $0 \le J_1(u) \le u/2$ on $[0,1]$, (3.8) can be easily verified via the two estimates

$$\begin{aligned} |I_{2,k}| &\le 2\pi\delta_n^{-1}\lambda_n\int_0^{\delta_n} 2s_k^{-2}u^3\lambda_n/2\,du \\ &= 2^{-1}\pi\delta_n^3\lambda_n^2 s_k^{-2'} \\ &\le 2^{-1}\pi\delta_n^{1+2\varepsilon}\left(c_1\delta_n^{1-\varepsilon}\right)^{-2} \\ &\le \pi c_1^{-2}\delta_n^{4\varepsilon-1}, \\ |I_{4,k}| &\le 2\pi\delta_n^{-1}s_k^{-3/2}\lambda_n^{1/2}M\int_0^{\delta_n} 2s_k^{-2}u^3\,du \\ &\le \pi Mc_1^{-7/2}\delta_n^{4\varepsilon-1}, \end{aligned}$$

provided that $\lambda_n\delta_n = \delta_n^{\varepsilon} \le 1$ and $c_1\delta_n^{1-\varepsilon} \ge 2\delta_n$, which are true as $n \to \infty$.

To verify (3.9) we have, similarly,

$$|I_{3,k}| \le 2\pi\delta_n^{-1}s_k^{-3/2}\lambda_n^{1/2}M\int_0^{\delta_n} u\,du$$
$$= \pi M\delta_n^{(1+\varepsilon)/2}s_k^{-3/2}.$$

Hence, for those terms in $\beta_{n,2}$ with $s_k > \tau$

$$\left| n^{-1} \sum_{\substack{c_2\delta_n^{1-\varepsilon} < s_k \\ \tau < s_k}} I_{3,k} \right| \le \pi M\tau^{-3/2}\delta_n^{(1+\varepsilon)/2}. \tag{3.11}$$

If $s_k = |z - f(k/n)| = |f(\theta) - f(k/n)| \le \tau$, then by (3.3) and (3.4)

$$A_0|\theta - k/n| \le s_k \le A_1|\theta - k/n|.$$

By using the integral test and noting $\lim \delta_n^{\varepsilon - 1}/n = 0$,

$$\begin{aligned}
&\left| n^{-1} \sum_{c_2\delta_n^{1-\varepsilon} < s_k \le \tau} I_{3,k} \right| \\
&\le \pi M\delta_n^{(1+\varepsilon)/2}\left\{ \sum_{c_2\delta_n^{1-\varepsilon} \le A_1|\theta - k/n|} n^{-1}\left(A_0|\theta - k/n|\right)^{-3/2} \right\} \\
&\le \pi M A_0^{-3/2}\delta_n^{(1+\varepsilon)/2}\int_{c_2\delta_n^{1-\varepsilon}A_1^{-1} - n^{-1}}^{\infty} t^{-3/2}\,dt \\
&\le 4\pi M A_0^{-3/2}\delta_n^{(1+\varepsilon)/2}\left[c_2\delta_n^{1-\varepsilon}/(2A_1)\right]^{-1/2} \\
&\le 8\pi M A_0^{-3/2}A_1^{1/2}c_2^{-1/2}\delta_n^{\varepsilon}.
\end{aligned} \tag{3.12}$$

Note that the factor 2 in the second inequality is due to the fact that we have to count k on both sides of θ. Then (3.9) follows from (3.11) and (3.12).

As to (3.10) let us define the measure M_n as

$$M_n(t) = \#\{k: 1 \le k \le n,\ s_k\lambda_n \le t\}/\left(n\delta_n^{1-\varepsilon}\right).$$

Since $\lambda_n\delta_n^{1-\varepsilon} = 1$,

$$\beta_{n,1} = n^{-1}2\pi\delta_n^{-1}\lambda_n^2\int_0^{\delta_n} uJ_0(\lambda_n u)\,du\left[\sum_{c_1\delta_n^{1-\varepsilon} < s_k \le c_2\delta_n^{1-\varepsilon}} J_1(s_k\lambda_n)/(s_k\lambda_n)\right]$$
$$= \left[2\pi\delta_n^{\varepsilon - 2}\int_0^{\delta_n} uJ_0(\lambda_n u)\,du\right]\left[\int_{c_1}^{c_2} J_1(t)/t\,dM_n(t)\right].$$

Because s_k satisfies $s_k\lambda_n \le t$ in the definition of $M_n(t)$ and $\delta_n \to 0$, $s_k \le t/\lambda_n = t\delta_n^{1-\varepsilon} \to 0$. By (3.4) $|\theta - k/n| \to 0$ and $s_k = |f(\theta) - f(k/n)| \approx |f'(\theta)|\,|\theta - k/n|$. By a simple counting $M_n(t) \approx 2|f'(\theta)|^{-1}tn\lambda_n^{-1}/(n\delta_n^{1-\varepsilon}) = 2|f'(\theta)|^{-1}t$, which implies that M_n converges weakly to $2|f'(\theta)|^{-1}m$, $m =$ Lebesgue measure. Therefore,

$$\lim_n \int_{c_1}^{c_2} J_1(t)/t\,dM_n(t) = 2|f'(\theta)|^{-1}\int_{c_1}^{c_2} J_1(t)/t\,dt.$$

Then (3.10) can be proved easily by noting that $\lambda_n\delta_n = \delta_n^{\varepsilon} \to 0$ and $J_0(0) = 1$. □

4. Proof of Theorem 2. It is enough to prove (2.11). To this end, let us define

$$Y_n(z) = X_n(z) - E(X_n(z)),$$

$$R_n = \{z\colon |z| \le d_n\}, \qquad A_n = \{z \in R_n\colon |Y_n(z)| \ge a_n\}.$$

We shall show that with probability one $A_n = \phi$ eventually. Once this is done, $X_n(z)$ would be close to its expectation $E(X_n(z))$ due to $\lim a_n = 0$. Then (2.11) can be checked in view of (2.8) and (2.9).

First, we need some estimates on $G_{\lambda,\delta}$ and its first-order derivatives. Let $z = x + iy$. By using (2.7) and (2.1) it can be shown that when n is so large that $\delta_n \le 1$, then for all $z \in \mathbf{C}$,

$$\max\left(\left|G_{\lambda_n,\delta_n}(z)\right|, \left|\partial G_{\lambda_n,\delta_n}(z)/\partial x\right|, \left|\partial G_{\lambda_n,\delta_n}(z)/\partial y\right|\right) \le k_n, \tag{4.1}$$

where $k_n = (\pi/\sigma^2)\exp(\sigma^2\delta_n^{2(\varepsilon-1)}/2)$. For example,

$$\left|\partial G_{\lambda_n,\delta_n}(z)/\partial x\right| = \left|\int_{|\xi|\le\lambda_n} (i \operatorname{Re}\xi)\left[\exp(i\langle z, \xi\rangle + \sigma^2|\xi|^2/2)\right] J_1(\delta_n|\xi|)/|\xi|\, d\xi\right|$$

$$\le \int_{|\xi|\le\lambda_n} |\xi|\left[\exp(\sigma^2|\xi|^2/2)\right]/(2|\xi|)\, d\xi,$$

because $\delta_n|\xi| \le \delta_n\lambda_n = \delta_n^\varepsilon \le 1$ implies $0 \le J_1(\delta_n|\xi|) \le \delta_n|\xi|/2 \le \frac{1}{2}$ (see Section 1d). Now changing the integral to polar coordinates,

$$\begin{aligned}\left|\partial G_{\lambda_n,\delta_n}(z)/\partial x\right| &\le \pi\int_0^{\delta_n^{\varepsilon-1}} r\left[\exp(\sigma^2 r^2/2)\right] dr\\ &= (\pi/\sigma^2)\left[\exp(\sigma^2\delta_n^{2(\varepsilon-1)}/2)\right]\\ &= k_n.\end{aligned}$$

Note that the stronger inequality $0 \le J_1(\delta_n|\xi|) \le \delta_n|\xi|/2$ is used for estimating $|G_{\lambda_n,\delta_n}(z)|$.

Cover R_n by b_n closed balls $B_1, B_2, \ldots, B_{b_n}$ with common radius r_n and their centers $z_1, z_2, \ldots, z_{b_n}$ at the lattice point of the form (pr_n, qr_n), where p and q are integers. Remember that $\lim d_n = \infty$ and we will choose r_n such that $\lim r_n = 0$. By considering the circumscribed square we see that

$$b_n \le (2d_n/r_n)^2. \tag{4.2}$$

If r_n is defined by

$$a_n = 8r_nk_n, \tag{4.3}$$

then $\lim r_n = 0$, because $\lim a_n = \lim \delta_n = 0$ implies $\lim k_n = \infty$.

We claim that for each $z \in B_j = \{z\colon \operatorname{dist}(z, z_j) \le r_n\}$, $1 \le j \le b_n$,

$$\left|Y_n(z) - Y_n(z_j)\right| \le a_n/2. \tag{4.4}$$

Let $w = \operatorname{Re} z_j + i(\operatorname{Im} z)$. By using (4.1) and the mean value theorem for one real variable,

$$\begin{aligned}
&\left|G_{\lambda_n,\delta_n}(z - Z_{n,k}) - G_{\lambda_n,\delta_n}(z_j - Z_{n,k})\right| \\
&\quad \le \left|G_{\lambda_n,\delta_n}(z - Z_{n,k}) - G_{\lambda_n,\delta_n}(w - Z_{n,k})\right| \\
&\qquad + \left|G_{\lambda_n,\delta_n}(w - Z_{n,k}) - G_{\lambda_n,\delta_n}(z_j - Z_{n,k})\right| \\
&\quad \le 2r_n k_n = a_n/4.
\end{aligned}$$

Hence, by taking the expectation,

$$\left|E\big(G_{\lambda_n,\delta_n}(z - Z_{n,k})\big) - E\big(G_{\lambda_n,\delta_n}(z_j - Z_{n,k})\big)\right| \le a_n/4$$

and (4.4) follows from these two inequalities and (2.5).

Since $X_n(z)$ real implies $Y_n(z)$ is real, by (4.4)

$$\begin{aligned}
P(A_n \ne \phi) &= P\big(\text{there exists } z \in R_n \text{ such that } |Y_n(z)| \ge a_n\big) \\
&\le \sum_{j=1}^{b_n} P\Big(\sup_{B_j} |Y_n(z)| \ge a_n\Big) \\
&\le \sum_{j=1}^{b_n} P\big(|Y_n(z_j)| \ge a_n/2\big) \qquad (4.5) \\
&= \sum_{j=1}^{b_n} \big[P\big(Y_n(z_j) \ge a_n/2\big) + P\big(Y_n(z_j) \le -a_n/2\big)\big].
\end{aligned}$$

We follow Geman and Hwang (1982) to estimate the right-hand side of (4.5). Let

$$c_{n,k}(t,z) = E\Big(\exp\Big(t\big[G_{\lambda_n,\delta_n}(z - Z_{n,k}) - E\big(G_{\lambda_n,\delta_n}(z - Z_{n,k})\big) - a_n/2\big]\Big)\Big)$$

be the moment generating function of

$$\big[G_{\lambda_n,\delta_n}(z - Z_{n,k}) - E\big(G_{\lambda_n,\delta_n}(z - Z_{n,k})\big) - a_n/2\big].$$

By Chebyshev's inequality, for each $t \ge 0$ and $z \in \mathbb{C}$,

$$\begin{aligned}
P\big(Y_n(z) \ge a_n/2\big) &= P\big(\exp[nt(Y_n(z) - a_n/2)] \ge 1\big) \\
&\le E\big(\exp[nt(Y_n(z) - a_n/2)]\big) \qquad (4.6) \\
&= \prod_{k=1}^{n} c_{n,k}(t,z).
\end{aligned}$$

It is clear that $c_{n,k}(0,z) = 1$ and $c'_{n,k}(0,z) = -a_n/2$. We claim that when n is large,

$$c_{n,k}(t,z) \le 1 - a_n t/4 \qquad (4.7)$$

holds for $0 \le t \le a_n/(288k_n^2)$ and $z \in \mathbb{C}$.

By (4.1) and (4.3),

$$\begin{aligned} c''_{n,k}(t,z) &\le (2k_n + a_n/2)^2 \exp(t(2k_n - a_n/2)) \\ &\le (2k_n + 4r_n k_n)^2 \exp(2tk_n). \end{aligned}$$

For $0 \le t \le a_n/(288k_n^2)$, (4.3) implies $\lim r_n = 0$ and then

$$\lim_n \exp(2tk_n) \le \lim_n \exp(a_n/(144k_n)) = \lim_n \exp(r_n/16) = 1.$$

Thus for n large such that $r_n \le 1$,

$$\begin{aligned} c'_{n,k}(t,z) &= c'_{n,k}(0,z) + \int_0^t c''_{n,k}(s,z)\,ds \\ &\le -(a_n/2) + 2(6k_n)^2 t \\ &\le -(a_n/2) + 72k_n^2 a_n/(288k_n^2) = -a_n/4, \end{aligned}$$

and then

$$\begin{aligned} c_{n,k}(t,z) &= c_{n,k}(0,z) + \int_0^t c'_{n,k}(s,z)\,ds \\ &\le 1 - a_n t/4. \end{aligned}$$

This proves (4.7).

Since (4.6) holds for all $t \ge 0$ we may take $t = a_n/(288k_n^2)$ and use (4.7) to get for all $z \in \mathbb{C}$

$$P(Y_n(z) \ge a_n/2) \le \left[1 - a_n^2/(1152k_n^2)\right]^n \equiv m_n,$$

where $m_n = [1 - (1152\pi^2)^{-1} a_n^2 \sigma^4 \exp(-\sigma^2 \delta_n^{2(\varepsilon-1)})]^n$.

It is clear from the previous proof that the same bound holds for $P(Y_n(z) \le -a_n/2)$. Combine together with (4.5), (4.2), (4.3) and (2.10)

$$\begin{aligned} \sum_n P(A_n \ne \phi) &\le \sum_n 2b_n m_n \\ &\le \sum_n 2(2d_n/r_n)^2 m_n \\ &= 8^3\pi^2 \sum d_n^2 \left[a_n^2\sigma^4 \exp\left(-\sigma^2\delta_n^{2(\varepsilon-1)}\right)\right]^{-1} m_n \\ &< \infty. \end{aligned}$$

Thus, by the Borel–Cantelli lemma,

$$P(A_n \ne \phi \text{ i.o.}) = 0.$$

Or, equivalently,

$$P\big(\text{for } n \text{ large } |Y_n(z)| < a_n \text{ holds for all } |z| \le d_n\big) = 1.$$

Note that by definition $Y_n(z) = X_n(z) - E(X_n(z))$.

Under (2.9) $a_n = o(\delta_n^{\eta\varepsilon})$ and R_n eventually contains the trajectory L. By using the triangular inequality (2.11) follows immediately from (2.8). This completes the proof. □

5. Proof of Theorem 3. Since (2.6) is not used in the proof of Lemma 3.1 the second set-inclusion formula

$$E_n \subseteq \left\{z\colon \operatorname{dist}(z; L) \le (4\pi M^2)^{2/3}\delta_n^{(1-2\eta\varepsilon)/3}\right\}, \tag{5.1}$$

in (2.8) is true as in Theorem 1. In order to prove (2.13) it is then enough to show that L becomes "contained" in E_n.

If L is degenerate to a point, say z_0, the computation leading to (3.6) shows that for n large

$$E(X_n(z_0)) \ge \pi\delta_n^{2\varepsilon-1}/8 \ge \pi\delta_n^0/8 \ge \delta_n^{\eta\varepsilon}.$$

Thus $L \subseteq E_n$ in this case. Otherwise, Sard's theorem is applicable and (2.12) holds. Then

$$\lim_{\nu\downarrow 0} D_\nu = L \quad \text{in the Hausdorff sense,} \tag{5.2}$$

where $D_\nu = \{z \in L\colon |f'(\theta)| \ge \nu \text{ for each } \theta \in f^{-1}(z)\}$.

Since $|f'(\theta)| \ge \nu$ for each $\theta \in f^{-1}(D_\nu)$, it can be seen from the proof of Theorem 1 that for fixed $\nu > 0$,

$$D_\nu = f\left(f^{-1}(D_\nu)\right) \subseteq E_n \tag{5.3}$$

holds for all n larger than some constant which might depend on ν. Now (2.13) follows from (5.1), (5.2), (5.3) and the fact that $D_\nu \subseteq L$.

Once again (2.13) can be improved to probability one convergence by the technique in Geman and Hwang (1982). The proof is the same as that of Theorem 2 and is thus omitted. □

6. Remarks. (i) *Random observation points.* From the proof of Theorems 1 and 2 it is clear that similar results hold if the observation points are chosen randomly. More precisely, we may replace the random set (0.2) by the random set

$$\{f(u_{n,k}) + e_{n,k}\colon 1 \le k \le n\},$$

where $e_{n,k}$ are as before, $u_{n,k}$ are i.i.d. with a positive density $p(\theta)$ on $[0,1]$, and $\{e_{n,k}\}$, $\{u_{n,k}\}$ are independent. Note that $p(\theta)$ need not to be known.

(ii) *One-dimensional analogue.* Problem (0.2) has the following one-dimensional analogue. Let f be a nondecreasing, right continuous function on $(0,1)$. Suppose that at time n we are given the random set

$$\{f(k/n) + e_{n,k}\colon 1 \le k < n\},$$

where $e_{n,k}$ are i.i.d. and $N(0,\sigma^2)$. We do not know how to restore f by using the empirical distribution approach. But the Hankel transform applies again. It has been shown in Chow (1980) that with probability one its distribution function

$$F(t) = m\{\theta \in (0,1)\colon f(\theta) \le t\}$$

can be found, where m is Lebesgue measure. This in turn determines f.

(iii) *Higher-dimensional case.* Similar to (2.4) it can be shown formally that Theorem 3 should be true if f is a continuously differentiable function from $[0,1]^k$ to $\mathbb{R}^m$ with $k \le m$. The case $k = 2$ has been treated. Since $f([0,1]^k)$ can be found with probability one, so can some related quantities, like arc length (for $k = 1$) or area (for $k = 2$). These will appear in a forthcoming paper.

(iv) *Questions.* Throughout the paper we assumed that σ^2 is a known constant. It is clear from (2.2) and (2.5) that our detector function $X_n(z)$ depends critically on σ. This is an unpleasant assumption on the random error $e_{n,k}$. Can it be removed? Besides $X_n(z)$ is very complicated. Does there exist a simple one? In view of (2.4) our results, namely (2.8) and (2.11), are not so satisfactory. Can one find λ_n and δ_n such that (2.4) is reached?

(v) *A comment on Theorems* 1, 2 *and* 3. Under (2.6) it is clear from (2.8) that $L \subseteq \liminf E_n \subseteq \limsup E_n \subseteq L$. Hence

$$\lim E_n = L \quad \text{in the point-set sense} \tag{6.1}$$

as well as in the Hausdorff sense.

In case that (2.6) is not true, we conclude from (5.1) and (5.3) that

$$\bigcup_{\nu>0} D_\nu \subseteq \liminf E_n \subseteq \limsup E_n \subseteq L. \tag{6.2}$$

Since $L - \bigcup_{\nu>0} D_\nu = \{z \in L\colon\ f'(\theta) = 0 \text{ for some } \theta \in f^{-1}(z)\}$, (6.1) may no longer be valid in the present case. Because L is a compact subset in $\mathbb{C}$, Sard's theorem and (6.2) guarantee that L can be recovered after knowing all E_n.

Similar remarks hold for T_n.

(vi) *Epilogue.* The case that f is a conformal mapping on the closed unit disk was treated in Chow (1980). There the data set (0.2) was replaced by the random set

$$\{f(\exp(2\pi ik/n)) + e_{n,k}\colon 1 \le k \le n\}.$$

Being conformal implies that $f(\exp(i2\pi\theta))$, as a function of $\theta \in [0,1]$, is one-to-one and satisfies (2.6). Therefore, the boundary $f(\{z\colon |z| = 1\})$ can be found. Then one version of f can be constructed by using the orthogonal polynomials on the boundary $f(\{z\colon |z| = 1\})$ [see Smirnov and Lebedev (1968)].

Acknowledgment. This paper is essentially based on my Ph.D. thesis. My deepest gratitude goes to Professor U. Grenander for his many valuable suggestions.

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