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# REPRODUCIBILITY AND NATURAL EXPONENTIAL FAMILIES WITH POWER VARIANCE FUNCTIONS

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Let $X_1, \dots, X_n$ be i.i.d. r.v.'s having common distribution belonging to a family $\mathscr{F} = \{F_\theta\colon \theta \in \Theta \subset R\}$ indexed by a parameter $\theta$. $\mathscr{F}$ is said to be reproducible if there exists a sequence $\{\alpha(n)\}$ such that $\mathscr{L}(\alpha(n)\sum_{i=1}^n X_i) \in \mathscr{F}$ for all $\theta \in \Theta$ and $n = 1, 2, \dots$. This property is investigated in connection with linear exponential families of order 1 and its intimate relationship to such families having a power variance function is demonstrated. Moreover, the role of such families is examined, in a unified approach, with respect to properties relative to infinite divisibility, steepness, convolution, stability, self-decomposability, unimodality, and cumulants.

**1. Introduction.** The notion of a distribution function (d.f.) $F$ being "reproductive" with respect to a parameter $\theta$ was introduced by Wilks (1963) as follows: Suppose $X_1$ and $X_2$ are independent random variables (r.v.'s) with d.f.'s $F(\cdot : \theta_1)$ and $F(\cdot : \theta_2)$, respectively, where $\theta_1$ and $\theta_2$ are values of a parameter $\theta$. Let $Z$ denote the r.v. $X_1 + X_2$. Then, if the d.f. of $Z$ is $F(\cdot : \theta_1 + \theta_2)$, the d.f. $F(\cdot : \theta)$ is said to be reproductive with respect to $\theta$.

It is both interesting and surprising to note that very little attention has been directed to this notion in the literature. Not only has the descriptive terminology "reproductive" not been adopted by many authors to describe this phenomenon, but, more significantly, although a number of specific distributions (such as the normal, chi-square, and Poisson) are known to satisfy this property, there do not appear to be any systematic investigations of what broad families of distributions possess such a property.

Here we consider a variation of this property, which we term "reproducibility," as applied to a linear exponential family of order 1. [Morris (1982) uses the terminology natural exponential family for such a family and, hereafter, we use the abbreviation NEF to denote this family.] We define reproducibility as follows:

Definition 1.1. Let $\mathscr{F} = \{F_\theta\colon \theta \in \Theta \subset R\}$ be a family of distributions indexed by a parameter $\theta$. Let $X_1, \dots, X_n$ be i.i.d. r.v.'s with $\mathscr{L}(X_1) \in \mathscr{F}$ and $\alpha(n)$, $n = 1, 2, \dots$, nonnull constants. [$\mathscr{L}(Z)$ signifies the law of the r.v. $Z$.] $\mathscr{F}$ is said to be reproducible in $\theta$ (or reproducible) if, for all $\theta \in \Theta$ and $n = 1, 2, \dots$ there exists a mapping $g_n$ from $\Theta$ *onto* $\Theta$ such that $\mathscr{L}(\alpha(n)\sum_{i=1}^n X_i) = F_{g_n(\theta)}$. The constants $\alpha(n)$ are referred to as the stabilizing constants of $\mathscr{F}$.

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An extension of Definition 1.1 to the multi-parameter situation is straightforward. Indeed, the present investigation was motivated by a result of Bar-Lev and Reiser (1982) obtained for the two-parameter case. They considered i.i.d. r.v.'s $X_1, \dots, X_n$ with common distribution a member of a special exponential family of order 2 with $\theta = (\theta_1, \theta_2)$ as the natural parameter. They showed that, for this family, the distribution of $\sum_{i=1}^{n} X_i/n$ is as that of $X_1$ with parameter $(n\theta_1, n\theta_2)$. Thus, in terms of the multi-parameter extension of Definition 1.1, this family is reproducible with stabilizing constants $\alpha(n) = 1/n$. Barndorff-Nielsen and Blaesild (1983) generalized to a higher dimensional parameter space the results of Bar-Lev and Reiser for this special case, $\alpha(n) = 1/n$. Baringhaus, Davies, and Plachky (1976) considered reproducibility with stabilizing constants $\alpha(n) = 1$, in the case of NEF's. They showed that the only reproducible family with such stabilizing constants is the family of Poisson-type distributions [defined following Equation (2.7)]. Using a different method of proof, Bar-Lev and Enis (1985) independently obtained the same result. This result is a special case of our Theorem 2.2.

In this paper, we show that the class of reproducible NEF's with stabilizing constants of the form $n^{\beta}$ coincides (with one exception) with the NEF class having power variance functions. Before defining the latter family, we introduce some notation and assumptions. Throughout the sequel, we consider $\mathscr{F} = \{F_\theta\colon \theta \in \Theta \subset R\}$ as being a minimal and steep NEF with members

$$dF_\theta(x) = h(x)\exp\{\theta x + c(\theta)\}\, d\nu(x), \tag{1.1}$$

where $\nu$ is a $\sigma$-finite measure on some Borel set of the real line. Denote by $C$ the convex support of (1.1), and for $\theta \in \operatorname{int}\Theta$, let $\mu = \mu(\theta) = -dc(\theta)/d\theta$ and $\Omega = \mu(\operatorname{int}\Theta)$ be the mean value and mean parameter space, respectively. By Barndorff-Nielsen (1978), Theorem 9.2, $\Omega = \operatorname{int} C$. We also denote by $(V(\mu), \Omega)$, $\phi(t\colon \theta) = \exp\{\Psi(t\colon \theta)\}$, and $\Psi(t\colon \theta) = c(\theta) - c(\theta - t)$ the variance function (VF), Laplace transform (LT), and cumulant transform, respectively, corresponding to (1.1).

DEFINITION 1.2. A NEF is said to have a power variance function (abbreviated NEF-PVF) if its variance function is of the form $V(\mu) = \alpha\mu^{\gamma}$, $\mu \in \Omega$, for some constants $\alpha \neq 0$ and $\gamma$, called the scale and power parameters, respectively.

In addition to its intimate relationship with reproducibility, the class of NEF-PVF's is shown to be a broad family with interesting properties and would seem to have a potentially important role in statistical inference.

In Section 2, we derive some basic properties of NEF-PVF's and reproducible NEF's and demonstrate the above-mentioned equivalence. In Section 3, we classify NEF-PVF's into classes and show that these classes are closed under convolutions and positive scale transformations. We prove that all NEF-PVF's are infinitely divisible with a self-generating property (the meaning of which is given in Section 3). In Section 4, we find all NEF-PVF's (see Figure 1 for an illustration). It is proved that there exist no steep NEF-PVF's corresponding to $\gamma$-values in the intervals $(-\infty, 0)$ or $(0, 1)$. The $\gamma$-values 0, 1, and 2 correspond to normal, Poisson-type, and gamma distributions, respectively. For the remaining $\gamma$-values we show that, (i) for any $1 < \gamma < 2$, the corresponding NEF-PVF is composed of compound Poisson distributions generated by gamma variates and, (ii) for any $\gamma > 2$, the corresponding NEF-PVF is generated by a stable distribution with characteristic exponent $(2-\gamma)/(1-\gamma)$. Densities of NEF-PVF distributions with $\gamma > 2$ are derived in Section 5. Section 6 is devoted to derivations of general expressions for cumulants and relationships between cumulants and moments of different orders. Unimodality is discussed in Section 7. It is shown that NEF-PVF's with $\gamma > 2$ are self-decomposable and thus are unimodal. For NEF-PVF's with $1 < \gamma < 2$, we present a special case which demonstrates nonunimodality.

**2. Reproducibility and NEF-PVF's.** Morris (1982, 1983) established a general framework of NEF's by means of their variance functions and showed that the VF $(V(\mu), \Omega)$ characterizes a NEF within the class of NEF's. He defined and discussed a particular class of NEF's possessing a quadratic variance function (QVF) and showed that only six families of univariate distributions have such a property. Members in the class of NEF-PVF's with $\gamma = 0, 1, 2$ are also members in the class of NEF-QVF's and correspond to the normal, Poisson-type, and gamma distributions, respectively.

In order to investigate the relationship between NEF's that are reproducible and those with power variance functions, we derive some basic properties of these families. We begin with NEF-PVF's and determine the structure of $\Omega$, $\Theta$, $c(\theta)$, and $\Psi(t:\theta)$.

In general, the structure of the set $\Omega$ is determined as follows. Given $V(\cdot)$ and $\mu_0$ with $0 < V(\mu_0) < \infty$, $\Omega$ is defined as the largest open interval containing $\mu_0$ such that $0 < V(m) < \infty$ all $m \in \Omega$ (Morris, 1982). Thus, for $V(\mu) = \alpha\mu^{\gamma}$, $\Omega = R$ if and only if $\gamma = 0$; for $\gamma \neq 0$, $\Omega$ is either $R^+$ or $R^-$. For $\gamma \neq 0$, the two models, one with $\Omega = R^-$ and the other with $\Omega = R^+$, characterize two different NEF's. However, the former model can be considered as the reflection of the latter about the origin, in the sense that if $X_1$ has a distribution corresponding to (1.1) and has VF $(V(\mu) = \alpha\mu^{\gamma}, \Omega = R^+)$, then the distribution of $X_1^* = -X_1$ belongs to a NEF-PVF with mean $\mu^* = -\mu$ and VF $(V^*(\mu^*), \Omega^*)$, where $V^*(\mu^*) = V(-\mu^*) = \alpha(-\mu^*)^{\gamma}$ and $\Omega^* = -\Omega = R^-$. The corresponding changes, with respect to the natural parameter, resulting from such a transformation are obtained by replacing $\theta$, $\Theta$, and $c(\theta)$, for $X_1$, by $\theta^* = -\theta$, $\Theta^* = -\Theta$, and $c^*(\theta^*) = c(-\theta^*)$, respectively, for $X_1^*$. Accordingly, we can restrict ourselves to deriving results for the case $\Omega = R^+$, as the results for the case $\Omega = R^-$ can be obtained by a suitable change of sign (see Remark 2.1).

Assuming that $\Omega$ is either $R$ ($\gamma = 0$) or $R^+$ with $\gamma \neq 1, 2$ (the case where $\gamma = 1, 2$ will be treated separately), then clearly $\alpha > 0$ and the corresponding forms of $\theta$ and $c(\theta)$ are derived as follows. Since $d\theta/d\mu = (\alpha\mu^{\gamma})^{-1}$, we immediately obtain that $\theta = (\alpha(1-\gamma))^{-1}\mu^{1-\gamma} + m$, $\mu = \{\alpha(1-\gamma)(\theta - m)\}^{1/(1-\gamma)}$, and $c(\theta) = -(\alpha(2-\gamma))^{-1}\{\alpha(1-\gamma)(\theta - m)\}^{(2-\gamma)/(1-\gamma)} + d$, for some constants $m$ and $d$. A reparameterization of the natural parameter from $\theta$ to $\theta_1 = \theta - m$ yields an appropriate change in $c(\theta)$ into $c_1(\theta_1) = c(\theta_1 + m)$. [For convenience, we continue to use $\theta$ and $c(\theta)$ rather than $\theta_1$ and $c_1(\theta_1)$.] Under such a reparameterization which does not affect $\Omega$, the above results can be written as

$$\theta = \frac{1}{\alpha(1-\gamma)}\mu^{1-\gamma}, \tag{2.1}$$

$$\mu = \{\alpha(1-\gamma)\theta\}^{1/(1-\gamma)}, \tag{2.2}$$

$$c(\theta) = -\frac{1}{\alpha(2-\gamma)}\{\alpha(1-\gamma)\theta\}^{(2-\gamma)/(1-\gamma)} + d, \tag{2.3}$$

and the corresponding cumulant transform is given by

$$\begin{aligned}\Psi(t:\theta) = \frac{1}{\alpha(2-\gamma)}\{&[\alpha(1-\gamma)(\theta-t)]^{(2-\gamma)/(1-\gamma)} \\ &-[\alpha(1-\gamma)\theta]^{(2-\gamma)/(1-\gamma)}\}.\end{aligned} \tag{2.4}$$

As $\mu$ ranges over $\Omega$, the range of $\theta = \theta(\mu)$ is $\operatorname{int}\Theta$ and is determined by (2.1) as:

(i) for $\Omega = R$ $(\gamma = 0)$: $\operatorname{int}\Theta = \Theta = R$

and

(ii) for $\Omega = R^+$ $(\gamma \neq 1,2)$:

$$\operatorname{int}\Theta = \begin{cases} R^-, & \text{if } 1-\gamma < 0, \\ R^+, & \text{if } 1-\gamma > 0. \end{cases} \tag{2.5}$$

Similarly, for $\Omega = R^+$ and $\gamma = 1,2$, we obtain, respectively,

$$\Psi(t:\theta) = (1/\alpha)\{e^{\alpha(\theta-t)} - e^{\alpha\theta}\}, \qquad \Theta = R, \tag{2.6}$$

and

$$\Psi(t:\theta) = (1/\alpha)\log\{\theta/(\theta-t)\}, \qquad \operatorname{int}\Theta = R^-. \tag{2.7}$$

(2.6) is the cumulant transform of a Poisson type distribution, which is obtained by a (positive) scale transformation of the standard ($\alpha = 1$) Poisson distribution, and (2.7) corresponds to a gamma distribution.

REMARK 2.1. For $\Omega = R^-$, the corresponding cumulant transforms are simply obtained by replacing, in (2.4), (2.6), and (2.7), $\theta$ and $t$ by $-\theta$ and $-t$, respectively. The corresponding parameter space is not affected if $\gamma = 0, 1$. However, if $\gamma \neq 0, 1$, $\operatorname{int}\Theta$ is changed [see (2.5) above] to $R^+$ for $1-\gamma < 0$ and to $R^-$ for $1-\gamma > 0$.

We now discuss reproducible NEF's. The following lemma presents some properties of such families.

LEMMA 2.1. *Let $\mathscr{F}$ be a reproducible NEF with stabilizing constants $\alpha(n)$. Then*

(i) *for all $n \geq 1$ and $\mu \in \Omega$,*

$$n\alpha^2(n)V(\mu) = V(n\alpha(n)\mu); \tag{2.8}$$

(ii) $\alpha(n) \neq n^{-1}$, *for all $n \geq 2$;*
(iii) $g_n(\text{int}\,\Theta) = \text{int}\,\Theta$;
(iv) $\Omega$ *is either $R^+$, $R^-$, or $R$.*

PROOF. (i) Let $X_1, \dots, X_n$ be i.i.d. r.v.'s with common distribution belonging to $\mathscr{F}$. Then $E(\alpha(n)\Sigma_{i=1}^n X_i) = n\alpha(n)\mu$ and $\text{Var}(\alpha(n)\Sigma_{i=1}^n X_i) = n\alpha^2(n)V(\mu)$. Since $\mathscr{F}$ is reproducible, the distribution of $\alpha(n)\Sigma_{i=1}^n X_i$ is in the same NEF as $X_1$. This implies (2.8).

(ii) Assume $\alpha(n_0) = n_0^{-1}$ for some $n_0 \geq 2$. Substituting this in (2.8), we obtain $n_0^{-1}V(\mu) = V(\mu)$. Since $V(\mu) > 0$, it follows that $n_0 = 1$, a contradiction.

(iii) We assume that $\Theta$ is not open (otherwise, the statement is clearly valid), and show that $g_n(\text{int}\,\Theta) \subset \text{int}\,\Theta$. The latter result and the fact that $g_n(\cdot)$ is an onto mapping imply, by straightforward argumentation, that $g_n(\text{int}\,\Theta) = \text{int}\,\Theta$. Accordingly, let $\theta_0 \in \text{int}\,\Theta$ and $X_1, \dots, X_n$ be i.i.d. r.v.'s with common distribution $F_{\theta_0}$. Since $E_{\theta_0}|X_i| < \infty$ and $\mathscr{L}(\alpha(n)\Sigma_{i=1}^n X_i) = F_{g_n(\theta_0)}$, we have $\int |x|\, dF_{g_n(\theta_0)}(x) < \infty$. Brown (1982), Proposition 3.3, shows that a minimal NEF (with canonical statistic $X$) is steep iff $E_\theta|X| = \infty$ for all $\theta \in \Theta - \text{int}\,\Theta$. Applying this proposition to our case, we obtain that $g_n(\theta_0) \in \text{int}\,\Theta$, since $\mathscr{F}$ is steep by assumption.

(iv) Since $\mathscr{F}$ is reproducible, we have $n\alpha(n)\mu(\theta) = \mu(g_n(\theta))$, for $n \geq 1$ and $\theta \in \text{int}\,\Theta$. As $g_n(\text{int}\,\Theta) = \text{int}\,\Theta$ and $n\alpha(n)\mu(\text{int}\,\Theta) = \mu(g_n(\text{int}\,\Theta))$, we obtain $n\alpha(n)\Omega = \Omega$ for all $n$. But $n\alpha(n) \neq 1$ for all $n \geq 2$, hence $\Omega$ must be either $R^+$, $R^-$, or $R$. □

The next two results relate reproducible NEF's with NEF-PVF's. As $\Omega$ must be either $R$, $R^+$, or $R^-$ for both of the latter families, the first of these results concerns the case where $\Omega = R$ and the second deals with the case $\Omega = R^+$. The case where $\Omega = R^-$ can be treated in a manner analogous to that of $\Omega = R^+$.

THEOREM 2.1. *For $\Omega = R$, $\mathscr{F}$ is the family of normal distributions (i.e., $\mathscr{F}$ is NEF-PVF with $\gamma = 0$) iff $\mathscr{F}$ is reproducible. In this case, the only two possible stabilizing constants are $\alpha(n) = \pm n^{-1/2}$ and these correspond to $g_n(\theta) = \pm n^{1/2}\theta$.*

PROOF. Let $\mathscr{F}$ be the family of normal distributions. Its cumulant transform, given by (2.4) with $\gamma = 0$ and $\Omega = \Theta = R$, satisfies $n\Psi(\pm n^{-1/2}t : \theta) = \Psi(t : \pm n^{1/2}\theta)$ for all $n$ and $\theta \in R$. This implies that $\mathscr{F}$ is reproducible with $\alpha(n) = \pm n^{-1/2}$ and $g_n(\theta) = \pm n^{1/2}\theta$. Now, assume that $\mathscr{F}$ is reproducible. By substituting $\mu = 0$ in (2.8), we obtain $n\alpha^2(n)V(0) = V(0)$, which implies $\alpha(n) = \pm n^{-1/2}$. Thus (2.8) becomes $V(\mu) = V(\pm n^{1/2}\mu)$, which holds for all $n \geq 1$ and

$\mu \in R$. The proof that $V(\mu)$ is identically constant on $R$ can be completed in a manner analogous to that of Theorem 2.2, for $\beta = -\frac{1}{2}$ [see the lines following Equation (2.9)]. □

THEOREM 2.2. *For $\Omega = R^+$, $\mathscr{F}$ is a NEF-PVF with power parameter $\gamma \neq 2$ iff $\mathscr{F}$ is reproducible with stabilizing constants $\alpha(n) = n^{(1-\gamma)/(\gamma-2)}$. Here, $g_n(\theta) = \theta n^{(\gamma-1)/(\gamma-2)}$ for $\gamma \neq 1$, and $g_n(\theta) = \theta + (1/\alpha)\log n$ for $\gamma = 1$.*

PROOF. For convenience, let $\beta = (1-\gamma)/(\gamma-2)$ and note that as $\gamma$ ranges over $R - \{2\}$, $\beta$ ranges over $R - \{-1\}$. Let $\mathscr{F}$ be a NEF-PVF with $\gamma \neq 2$. For $\gamma \neq 1$, the natural parameter space and the cumulant transform corresponding to $\mathscr{F}$ are given by (2.5) and (2.4), respectively. The latter satisfies $n\Psi(n^{\beta}t\colon \theta) = \Psi(t\colon \theta n^{-\beta})$ for all $n \geq 1$ and $\theta \in \Theta$. For $\gamma = 1$, the corresponding cumulant transform is given by (2.6) with $\Theta = R$; here, we have $n\Psi(t\colon \theta) = \Psi(t\colon \theta + (1/\alpha)\log n)$ for all $n$ and $\theta$. Thus, $\mathscr{F}$ is reproducible with $\alpha(n)$ and $g_n(\theta)$ as stated. [If $\gamma = 2$, corresponding to the case where $\mathscr{F}$ is the family of gamma distributions, then (2.8) is satisfied only for $n = 1$. Thus, this family is not reproducible. However, in Section 3, it is shown to be reproducible when considered as being a two-parameter family.]

Now, let $\mathscr{F}$ be reproducible with stabilizing constants $\alpha(n) = n^{\beta}$, $\beta \neq -1$. Substituting these in (2.8), we obtain

$$n^{2\beta+1}V(\mu) = V(n^{\beta+1}\mu), \quad \text{for } n \geq 1,\ \mu > 0, \tag{2.9}$$

from which it follows that

$$(1/n^{2\beta+1})V(\mu) = V(\mu/n^{\beta+1}), \quad \text{for } n \geq 1,\ \mu > 0. \tag{2.10}$$

Multiplying both sides of (2.10) by $m^{2\beta+1}$, where $m$ is an arbitrary positive integer, yields

$$(m/n)^{2\beta+1}V(\mu) = m^{2\beta+1}V(\mu/n^{\beta+1}) = V\big((m/n)^{\beta+1}\mu\big), \quad \text{for } m, n \geq 1,\ \mu > 0. \tag{2.11}$$

In (2.11), set $\mu = 1$ and $\alpha = V(1)$ to obtain

$$\alpha x^{2\beta+1} = V(x^{\beta+1}), \tag{2.12}$$

which holds for any rational $x > 0$. Since $V(\cdot)$ is continuous, it follows that (2.12) holds for any real $x > 0$. Setting $y = x^{\beta+1}$ in (2.12) yields $V(y) = \alpha y^{\gamma}$, $y > 0$, the desired result. □

We end this section by demonstrating Theorem 2.2 with the following two examples; in both, $\nu$ is the Lebesgue measure over $(0, \infty)$ and $\Theta = R^- \cup \{0\}$. In Examples 2.1 and 2.2 with $\theta = 0$, the corresponding distributions are stable with characteristic exponents $\frac{1}{2}$ and $\frac{1}{3}$, respectively. These two examples are special cases of steep (but nonregular) NEF-PVF's generated by stable distributions (see Section 4).

EXAMPLE 2.1. Inverse Gaussian distribution.

$$dF(x) = (2\pi)^{-1/2}x^{-3/2}\lambda^{1/2}\exp\left\{-\delta x/2 - \lambda/(2x) + (\delta\lambda)^{1/2}\right\}dx, \qquad \delta \geq 0, \quad \lambda > 0.$$

Here, $\theta = -\delta/2$, $\operatorname{int}\Theta = R^-$, and $c(\theta) = (-2\lambda\theta)^{1/2} + \frac{1}{2}\log\lambda$. $c(\theta)$ is of the form (2.3) with $\gamma = 3$, $\alpha = 1/\lambda$, and $\alpha(n) = n^{-2}$.

EXAMPLE 2.2. Modified Bessel-type distribution.

$$dF(x) = (2\pi)^{-1}3^{1/2}\lambda x^{-3/2}K_{1/3}(\lambda x^{-1/2})\exp\left\{-px + 3(\lambda^2 p/4)^{1/3}\right\}dx, \qquad \lambda > 0, \quad p \geq 0,$$

where $K_{1/3}(z)$ is the modified Bessel function of the second kind of order $\frac{1}{3}$ with argument $z$. For reference to this distribution, see Oberhettinger and Badii (1973), page 155, and also Zolotarev (1954), who considers the special case $p = 0$. Here, $\theta = -p$, $\operatorname{int}\Theta = R^-$, and $c(\theta)$ is of the form (2.3) with $\gamma = 2.5$, $\alpha = 4/3\lambda$, and $\alpha(n) = n^{-3}$.

**3. NEF-PVF's considered as two-parameter families.** Let $\mathscr{F}$ be a NEF-PVF (with $\Omega$ either $R$ or $R^+$) having a PVF $V(\mu) = \alpha\mu^\gamma$, $\mu \in \Omega$, for some constants $\alpha > 0$ and $\gamma$, and $X$ be a r.v. with $\mathscr{L}(X) \in \mathscr{F}$. The transformation $X \to bX$ with $b > 0$, yields a distribution belonging to a NEF-PVF, say $\mathscr{F}^*$. In this case, $\mu^* = E(bX) = b\mu$, $V^*(\mu^*) = \operatorname{Var}(bX) = \alpha b^{2-\gamma}(\mu^*)^\gamma$, and $\Omega^* = b\Omega = \Omega$, so that $\mathscr{F}$ and $\mathscr{F}^*$ possess the same power parameter $\gamma$ and the same convex support. However, unless $\gamma = 2$ (in which case $\mathscr{F} = \mathscr{F}^*$), $\mathscr{F}$ and $\mathscr{F}^*$ differ by the value of the scale parameter.

As $b$ ranges over $R^+$ ($\gamma \neq 2$), $\alpha b^{2-\gamma}$, considered as a function of $b$, ranges over $R^+$ too. Accordingly, we can classify NEF-PVF's by the values of $\gamma$ and the form of $\Omega$, as follows: Two NEF-PVF's are said to belong to the same class if their power parameters and convex supports are identical. Obviously, any member in some class can be obtained by a positive scale transformation of any other member in the same class. For $\gamma = 2$ and $\Omega = R^+$ we define the corresponding class to include all gamma distributions with varying shape parameter $1/\alpha$, $\alpha > 0$, and mean $\mu > 0$.

By this definition of classes, each class contains all NEF-PVF's with equal $\gamma$ and $\Omega$. Each such class can equivalently be considered as being a two-parameter family of distributions indexed by the parameters $\alpha$ and $\mu$ (or equivalently by $\alpha$ and $\theta$). The parameter space of this two-parameter family is the Cartesian product $R^+ \times \Omega$ (or $R^+ \times \Theta$).

Distributions in the same class, although posessing the same convex support, may have different supports. However, as will be apparent from the results of Section 4, each class, excluding the case $\gamma = 1$, possesses a common support.

The normal class ($\gamma = 0$) is closed under linear transformations. However, each of the other classes, although closed under positive scale transformations, is not closed under linear ones: $X \to bX + c$, $b > 0$, $c \neq 0$. Under the latter transformation, the distribution of $bX + c$ will belong to a NEF with mean

$\mu^* = b\mu + c$ and VF $(\alpha b^{2-\gamma}(\mu^* - c)^\gamma,\ \Omega^* = b\Omega + c)$. This NEF loses its PVF property and also possesses different convex support.

All classes are closed under the convolution operation. As if $X_1, \dots, X_n$ are i.i.d. with common NEF-PVF distribution having VF $(\alpha\mu^\gamma, \Omega)$, then the law of $\sum_{i=1}^n X_i$ belongs to a NEF-PVF in the same class as $X_1$ with scale parameter $\alpha n^{1-\gamma}$. This result implies that any such class of distributions, parameterized by $\alpha$ and $\mu$, is reproducible with stabilizing constants $\alpha(n) \equiv 1$. In particular, the family of gamma distributions, although not reproducible when parameterized by $\mu$ ($\alpha$ fixed), is reproducible when considered as a two-parameter family.

By Definition 1.1 and Theorems 2.1 and 2.2, it follows that NEF-PVF's with $\gamma \neq 2$ are composed of infinitely divisible members. We now use the above properties of classes to show that *all* NEF-PVF distributions are infinitely divisible with a self-generating property. By this, we mean that any NEF-PVF distribution $F$ can be represented as the $n$-fold convolution of a distribution within the same class as $F$. Let $X$ be a r.v. having a NEF-PVF distribution with VF $(\alpha\mu^\gamma, \Omega)$. We have to show that, for every $n = 1, 2, \dots$, $\mathscr{L}(X) = \mathscr{L}(\sum_{i=1}^n Y_{i,n})$, where the $Y_{i,n}$'s, $i = 1, \dots, n$, are i.i.d. with common distribution belonging to the same class as $X$. To verify this, consider, for each $n$, the NEF-PVF in the same class as that of $X$ with scale parameter $\alpha n^{\gamma-1}$. Within this NEF-PVF, choose the member with mean $\mu^* = \mu/n$ as the common distribution for the $Y_{i,n}$'s. We immediately obtain

$$\operatorname{Var}\left(\sum_{i=1}^n Y_{i,n}\right) = nV^*(\mu^*) = n\alpha n^{\gamma-1}(\mu^*)^\gamma, \qquad \mu^* \in \Omega,$$

which establishes the desired result.

**4. Classifications of NEF-PVF's by sets of γ-values.** In this section, we identify all NEF-PVF distributions. We determine the values of $\gamma$ for which no NEF-PVF's exist. Permissible values of $\gamma$ are divided into two main sets including NEF-PVF's with similar properties. Figure 1 illustrates and summarizes the results of this section.

NEF-PVF's with power parameters 0, 1, and 2 have already been characterized as corresponding to normal, Poisson-type, and gamma distributions, respectively. We henceforth assume that $\gamma \neq 0, 1, 2$ and $\Omega = R^+$. In the following discussion, it is convenient to let $\rho = (2-\gamma)/(1-\gamma)$. With this notation, (2.5), (2.3), and (2.4) become, respectively,

$$\operatorname{int}\Theta = \begin{cases} R^-, & \text{if } \rho < 1, \\ R^+, & \text{if } \rho > 1, \end{cases} \tag{4.1}$$

$$c(\theta) = -\frac{\rho-1}{\alpha\rho}\left\{\frac{\alpha\theta}{\rho-1}\right\}^\rho + d, \tag{4.2}$$

and

$$\Psi(t:\theta) = \frac{\rho-1}{\alpha\rho}\left\{\left[\frac{\alpha(\theta-t)}{\rho-1}\right]^\rho - \left[\frac{\alpha\theta}{\rho-1}\right]^\rho\right\}. \tag{4.3}$$

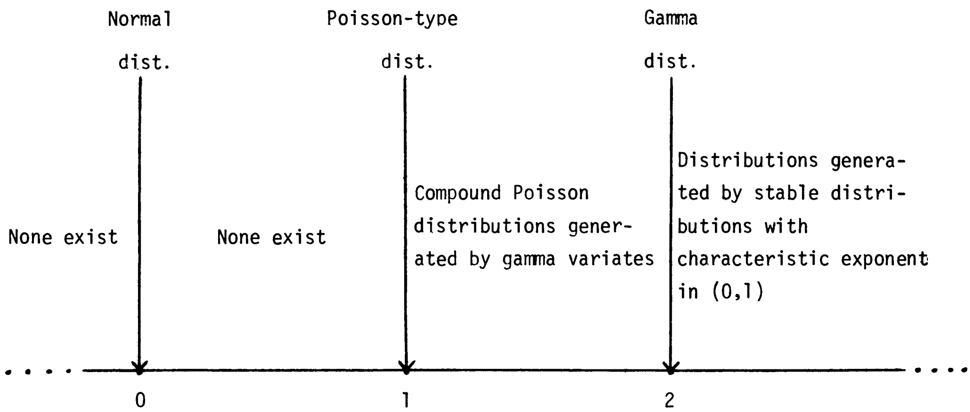


FIG. 1. *Classifications of steep NEF-PVF distributions by sets of* $\gamma$*-values.*

The examined $\gamma$-values are partitioned into $\gamma$-sets in the intervals $(-\infty, 0)$, $(0,1)$, $(1,2)$, and $(2,\infty)$. These correspond, respectively, to $\rho$-sets in the intervals $(1,2)$, $(2,\infty)$, $(-\infty,0)$, and $(0,1)$.

The following theorem establishes the results illustrated by Figure 1.

THEOREM 4.1. *Let* $\mathscr{F}$ *be a NEF-PVF with power parameter* $\gamma$. *Then*

(i) *it is necessary that* $\gamma \notin (-\infty,0) \cup (0,1)$ [*equivalently,* $\rho \notin (1,2) \cup (2,\infty)$];

(ii) *for any fixed* $\gamma \in (1,2)$ $(\rho < 0)$, $\mathscr{F}$ *is a family of compound Poisson distributions generated by a gamma distribution;*

(iii) *for any fixed* $\gamma \in (2,\infty)$ $(0 < \rho < 1)$, $\mathscr{F}$ *is the family generated by a stable distribution concentrated on* $[0,\infty)$ *and possessing a characteristic exponent* $\rho$.

PROOF. (i) We give an indirect proof. Fix $\gamma \in (-\infty,0) \cup (0,1)$ and assume that $\mathscr{F}$ is a NEF-PVF with VF $(\alpha\mu^{\gamma}, R^{+})$. As $\mu$ ranges over $\Omega = R^{+}$, $\theta = \theta(\mu)$ ranges over $\operatorname{int}\Theta$. Since $\rho > 1$ then, by (4.1), $\operatorname{int}\Theta = R^{+}$. The latter is the interior of the natural parameter space $\Theta$ of the steep (and thus full) exponential model (1.1). Let $S$ be the common support of $\mathscr{F}$. By definition, $\Theta$ is the effective domain of $e^{-c(\theta)} = \int_S h(x) e^{\theta x}\, d\nu(x)$ and has been shown above to be at most $R^{+} \cup \{0\}$. However, $\Omega = \operatorname{int} C = R^{+}$ so that $S \subset R^{+} \cup \{0\}$, and this implies that the effective domain of $e^{-c(\theta)}$ also contains $R^{-}$, a contradiction.

(ii) Let $\rho \in (-\infty, 0)$ be fixed, then by (4.1) $\operatorname{int}\Theta = R^{-}$. For $\theta \in R^{-}$, let $\lambda = -\theta$ and write $\xi(t:\lambda)$ for $\Psi(t:\theta)$ expressed in terms of $\lambda$. Then (4.3) can be rewritten as

$$\xi(t:\lambda) = \frac{\rho - 1}{\alpha\rho}\left(\frac{\alpha\lambda}{1-\rho}\right)^{\rho}\left\{(1 + t/\lambda)^{\rho} - 1\right\}, \qquad t > -\lambda.$$

Let

$$p = \frac{\rho - 1}{\alpha\rho}\left(\frac{\alpha\lambda}{1-\rho}\right)^{\rho} \quad \text{and} \quad g_\lambda(t) = (1 + t/\lambda)^\rho.$$

Then, $\xi(t:\lambda)$ can be written as $\xi(t:\lambda) = p\{g_\lambda(t) - 1\}$, where $p > 0$ (since $\lambda > 0$ and $\rho < 0$) and $g_\lambda(t)$ is the LT corresponding to a gamma distribution with scale parameter $1/\lambda$ and shape parameter $-\rho$. But $e^{p\{g_\lambda(t)-1\}}$ is the LT corresponding to the random sum $S_N = \Sigma_{i=1}^N Y_i$ ($S_0 \equiv 0$), where the $Y_i$'s are i.i.d. with common LT $g_\lambda(t)$ and the r.v. $N$ (independent of the $Y_i$'s) has a Poisson distribution with mean $p$. It is an immediate consequence of the present case (i.e., $\rho < 0$) that int $\Theta = \Theta = R^-$. [*Note*: There does not appear to be a universally adopted terminology to describe the relationship above. The terminology used here (i.e., "compound Poisson distribution generated by a gamma distribution") is that of Feller (1966), page 501.]

(iii) Let $G_\rho$ be a stable distribution with a characteristic exponent $\rho \in (0,1)$, as considered in Feller (1966), Theorem 1, page 424. $G_\rho$ is absolutely continuous relative to a Lebesgue measure on $(0,\infty)$ and possesses a LT of the form $\exp\{-t^\rho\}$, $t \ge 0$. Let $G_\rho^*(y) = G_\rho(y/a)$ where $a = \{(1-\rho)^{1-\rho}/(\rho\alpha^{1-\rho})\}^{1/\rho}$ for some positive constant $\alpha$. Clearly, $G_\rho^*$ is also a stable distribution concentrated on $[0,\infty)$ (since $a > 0$) with $\exp\{-(at)^\rho\}$ as its LT. We use $G_\rho^*$ to generate a NEF with cumulant transform of the form (4.3), as follows. Define $c(\theta)$ on $\Theta$, the largest interval for which the integral below exists, by

$$c(\theta) = -\log\int_{[0,\infty)} e^{\theta x}\, dG_\rho^*(x), \qquad \theta \in \Theta.$$

Clearly, $\Theta = (-\infty, 0]$ and $c(\theta)$ has the form

$$c(\theta) = (-a\theta)^\rho = -\frac{(\rho-1)}{\alpha\rho}\left(\frac{\alpha\theta}{\rho-1}\right)^\rho.$$

We generate a NEF with members $F_\theta$, $\theta \in \Theta$, by defining

$$dF_\theta(x) = \exp\{\theta x + c(\theta)\}\, dG_\rho^*(x). \tag{4.4}$$

Obviously, $F_\theta$ is a distribution function with cumulant transform (4.3).

The NEF just defined is nonregular since $\Theta$ contains 0 as a boundary point. However, it is steep as is shown by the following argument. $G_\rho^*$ is stable with characteristic exponent $0 < \rho < 1$, and has a finite $r$th moment only for $r \in (0,\rho)$ [see Feller (1966), page 215]; i.e., $\int_{[0,\infty)}|x|\, dG_\rho^*(x) = \infty$. Since $F_0 \equiv G_\rho^*$, the desired result is established by applying Proposition 3.3 of Brown (1982). □

**5. Densities of NEF-PVF's generated by stable distributions.** Steepness is a property of exponential models defined and discussed by Barndorff-Nielsen (1978). This property has an essential importance in connection with maximum likelihood estimation and other related topics. The most common and classic example provided in the literature for a steep exponential model is the inverse Gaussian one (Example 2.1). The family of modified Bessel-type distributions (Example 2.2) is a further example of this type.

Theorem 4.1(iii) provides us with a class of (nonregular) steep exponential models, composed of NEF-PVF's with power parameter $\gamma > 2$ $(0 < \rho < 1)$. The above two examples are special cases with $\rho = \frac{1}{2}$ and $\rho = \frac{1}{3}$, respectively.

Any NEF-PVF in this class, with members $F_\theta$, $\theta \in (-\infty, 0]$, is generated by a stable distribution $G_\rho^*$ defined in Theorem 4.1. $G_\rho^*$ itself differs only by a scale factor from $G_\rho$. Accordingly, the problem of obtaining the density of $F_\theta$ (relative to a Lebesgue measure on $R^+$) is solved by determining the density of $G_\rho$. As is known [Lukacs (1970), Section 5.8], apart from the inverse Gaussian distribution, no other stable distributions with characteristic exponent $\rho \in (0,1)$ are known, whose densities are given by elementary functions. However, series expansions of such densities, which we consider below, are available. In some special cases as $\rho = \frac{1}{3}, \frac{2}{3}$, and some other rational $\rho$, these series expansions can be expressed in terms of higher transcendental functions (Zolotarev, 1954).

Let $g_\rho$, $g_\rho^*$, and $f_\theta$ denote the densities corresponding to $G_\rho$, $G_\rho^*$, and $F_\theta$, respectively. Pollard (1946) showed that the density $g_\rho$ corresponding to the LT, $\exp\{-t^\rho\}$, $\rho \in (0,1)$, can be expressed as

$$g_\rho(x) = -\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\sin(\pi\rho k)\frac{\Gamma(\rho k+1)}{x^{\rho k+1}}, \qquad x > 0.$$

Since $g_\rho^*(x) = (1/a)g_\rho(x/a)$, where $a = \{(1-\rho)^{1-\rho}/(\rho\alpha^{1-\rho})\}^{1/\rho}$, we obtain by use of (4.4)

$$\begin{aligned} f_\theta(x) &= (1/a)g_\rho(x/a)\exp\{\theta x + c(\theta)\} \\ &= \left\{-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\sin(\pi\rho k)\frac{(1-\rho)^{k(1-\rho)}\Gamma(\rho k+1)}{\rho^k\alpha^{k(1-\rho)}x^{\rho k+1}}\right\} \\ &\quad\times\exp\left\{\theta x + \frac{(1-\rho)}{\alpha\rho}\left[\frac{\alpha\theta}{\rho-1}\right]^\rho\right\}, \\ &\qquad x > 0, \quad \theta \le 0, \quad \alpha > 0, \quad 0 < \rho < 1. \end{aligned}$$

*Note*: $g_\rho$ is exactly $p_{\alpha\alpha}$ in the notation of Lukacs (1970), page 140.

The following example treats the case $\rho = \frac{2}{3}$ for which $f_\theta$ can be expressed in terms of the Whittaker function.

EXAMPLE 5.1. Whittaker-type distribution. Pollard (1946) [see also Zolotarev (1954)] showed that $g_{2/3}$ can be given the form

$$g_{2/3}(x) = -\frac{1}{2(3\pi)^{1/2}}\frac{1}{x}e^{-2/(27x^2)}W_{-1/2,-1/6}\left(\frac{-4}{27x^2}\right),$$

where $W_{\alpha,\beta}(z)$ is the Whittaker function. For this case, we have

$$\begin{aligned} f_\theta(x) &= \left\{-\frac{1}{2(3\pi)^{1/2}}\frac{1}{x}e^{-1/(12\alpha x^2)}W_{-1/2,-1/6}\left(-\frac{1}{6\alpha x^2}\right)\right\} \\ &\quad\times\exp\left\{\theta x + (1/2\alpha)(-3\alpha\theta)^{2/3}\right\}. \end{aligned}$$

This result is used in Section 6 for evaluating certain integrals involved with Whittaker functions.

**6. Moments, cumulants, and upper bounds for tail probabilities.** In this section, as a consequence of the special structure of the variance functions for NEF-PVF distributions, we obtain simple expressions relating moments and central moments with cumulants and obtain bounds for tail probabilities for these distributions.

Consider NEF-PVF classes as defined in Section 3. For given $\gamma$ and $\Omega$, any such class is parameterized by $(\alpha, \mu)$ or $(\alpha, \theta)$. Other useful parameterizations can be given in terms of $(\alpha, \kappa_2)$, $(\kappa_1, \kappa_2)$ or, in general, in terms of $(\kappa_p, \kappa_q)$, $p < q$, where $\kappa_r$, $r = 1, 2, \dots$, denotes the $r$th cumulant. For all such possible kinds of parameterizations, we express, in a unified manner, cumulants of all orders. The resulting expressions, which are interesting by themselves, are useful for characterization problems concerned with NEF-PVF distributions.

For the derivations in the sequel, we make use of the following relation (Morris, 1982), which holds between cumulants of any NEF,

$$C_{r+1}(\kappa_1) = V(\kappa_1)C_r'(\kappa_1), \qquad r = 1, 2, \dots. \tag{6.1}$$

Here, $\kappa_1 = \mu$, $\kappa_r = C_r(\kappa_1)$ is the $r$th cumulant expressed in terms of $\kappa_1$, and prime denotes derivative wrt $\kappa_1$. Relation (6.1) implies that, for the normal case ($\gamma = 0$), $\kappa_r = 0$ for all $r \geq 3$. For the remainder of this section, we discuss only the case $\Omega = R^+$ and $\gamma \geq 1$. Accordingly, for $\gamma \geq 1$ and $\Omega = R^+$, we denote

$$\delta_r(\gamma) = \prod_{j=0}^{r} [j\gamma - (j-1)], \qquad r \geq 0.$$

Then, as is easily verified by induction, we obtain the following expressions for cumulants in terms of $(\alpha, \kappa_1)$ and $(\alpha, \kappa_2)$:

$$\kappa_{r+2} = \alpha^{r+1}\delta_r(\gamma)\kappa_1^{(r+1)\gamma - r}, \qquad r \geq 0, \quad \gamma \geq 1, \tag{6.2}$$

$$\kappa_{r+2} = \alpha^{r/\gamma}\delta_r(\gamma)\kappa_2^{r+1-r/\gamma}, \qquad r \geq 0, \quad \gamma \geq 1. \tag{6.3}$$

It is interesting to note that, for $\gamma \geq 1$ and $\Omega = R^+$, NEF-PVF cumulants of all orders are positive. This result follows from (6.2) by noting that $\delta_r(\gamma) \geq 1$ for all $\gamma \geq 1$ and $r \geq 0$.

By substituting $\kappa_1 = -dc(\theta)/d\theta$ into (6.2), we can express the cumulants in terms of $(\alpha, \theta)$ as

$$\kappa_{r+2} = \begin{cases} \alpha^{r+1}\delta_r(\gamma)[\alpha(1-\gamma)\theta]^{\gamma/(1-\gamma)-r}, & \gamma > 1, \quad \theta \in R^-, \quad r \geq 0, \\ \alpha^{r+1}e^{\alpha\theta}, & \gamma = 1, \quad \theta \in R, \quad r \geq 0. \end{cases} \tag{6.4}$$

A representation of cumulants in terms of the mean ($\kappa_1$) and variance ($\kappa_2$) can be made, by substituting $\alpha = \kappa_2/\kappa_1^{\gamma}$ into (6.2), to obtain

$$\kappa_{r+2} = \delta_r(\gamma)\kappa_2^{r+1}\kappa_1^{-r}, \qquad r \geq 0. \tag{6.5}$$

A more general expression for $\kappa_{r+2}$, in terms of any pair of cumulants of smaller orders, say $(\kappa_{p+2}, \kappa_{q+2})$, is found to be

$$\kappa_{r+2} = \delta_r(\gamma)\left(\frac{\delta_p(\gamma)}{\kappa_{p+2}}\right)^{(r-q)/(q-p)}\left(\frac{\kappa_{q+2}}{\delta_q(\gamma)}\right)^{(r-p)/(q-p)}, \qquad 0 \leq p < q < r. \tag{6.6}$$

The relations given by (6.6) are useful in the characterization of NEF-PVF distributions by zero regression properties (Bar-Lev and Stramer, 1984).

Moments and central moments of NEF-PVF distributions in terms of any of the parameters introduced above can be calculated, recursively, by use of formulas (7.1) and (7.2) of Morris (1982). For an illustration, we express the central moments $\mu_j$, $j = 1, 2, \ldots,$ in terms of the parameter $(\kappa_1, \kappa_2)$. Formula (7.2) of Morris (1982) can be rewritten as

$$\mu_{r+2} = \kappa_{r+2} + \sum_{j=2}^{r} \binom{r+1}{j} \mu_j \kappa_{r-j+2}, \qquad r \geq 2.$$

Using this and relation (6.5) yields

$$\mu_{r+2} = \kappa_2^{r+1} \kappa_1^{-r} \sum_{j=0}^{r} \binom{r+1}{j} \delta_{r-j}(\gamma) \mu_j (\kappa_1/\kappa_2)^j, \qquad r \geq 2. \tag{6.7}$$

Further, since (for $\Omega = R^+$) $\kappa_1, \kappa_2 > 0$ and $\delta_r(\gamma) \geq 1$ for $\gamma \geq 1$ and $r \geq 0$, it follows from (6.7) that all central moments $\mu_r$, $r \geq 2$, are positive.

The coefficients of skewness, $\gamma_1 = \kappa_3/\kappa_2^{3/2}$, and kurtosis, $\gamma_2 = \kappa_4/\kappa_2^2$, of NEF-PVF distributions, expressed in terms of $(\kappa_1, \kappa_2)$, have the forms:

$$\gamma_1 = \gamma \kappa_2^{1/2} \kappa_1^{-1} \quad \text{and} \quad \gamma_2 = \gamma(2\gamma - 1)\kappa_2 \kappa_1^{-2}.$$

Since $\gamma \geq 1$ and $\kappa_1 > 0$, we obtain $\gamma_2 > 0$. This implies that the corresponding NEF-PVF distributions are leptokurtic [Kendall and Stuart (1977), page 88]. Note that $\gamma_2/\gamma_1^2 = 2 - 1/\gamma$ so that the set of possible values for $(\gamma_1, \gamma_2)$ is a parabola. Also, by defining $\gamma_{2n+1} = \mu_3 \mu_{2n+3}/\mu_2^{n+3}$, $n \geq 1$, as general measures of "skewness" [Kendall and Stuart (1977), Section 3.31], we obtain that these quantities are always positive since $\mu_{2n+3} > 0$, $n \geq 1$.

NEF-PVF moments can be computed similarly by use of formula (7.1) of Morris (1982). Such expressions for moments can be used to evaluate integrals of certain higher transcendental functions, which are not available in frequently used tables of integrals [e.g., Gradshteyn and Ryzhik (1965)]. For example, let $\mu'_r$ denote the $r$th moment of the Whittaker-type distribution considered in Example 5.1. Here $\gamma = 4$ and we immediately obtain

$$\int_0^\infty x^{r-1} \exp\{\theta x - 1/(12\alpha x^2)\} W_{-1/2, -1/6}(-1/(6\alpha x^2))\, dx$$

$$= -\mu'_r 2(3\pi)^{1/2} \exp\left\{ \frac{(-3\alpha\theta)^{2/3}}{2\alpha} \right\}, \qquad r \geq 1, \quad \alpha > 0, \quad \theta < 0.$$

We conclude this section by presenting an expression for an upper bound for (right) tail probabilities for NEF-PVF distributions. Morris (1982, Section 9) showed that if $X$ has a NEF distribution with VF $(V(\mu), \Omega)$, then

$$P[X \geq x] \leq \exp\{-B((x-\mu)/\sigma)\},$$

for $x \geq \mu$, where $\sigma^2 = V(\mu)$ and

$$B(t) = \sigma^2 \int_0^t [(t-w)/V(\mu + \sigma w)]\, dw.$$

When applied to NEF-PVF distributions, this expression for an upper bound can be given a very simple closed form. Excluding the cases $\gamma = 0, 1, 2$ which have been treated by Morris, one can obtain by elementary computations that if $X$ has a NEF-PVF distribution with $\Omega = R^+$ and $\gamma \in (1,2) \cup (2,\infty)$, then

$$P[X \geq x] \leq \exp\left\{\frac{1}{\sigma^2}\left[\frac{\mu x}{1-\gamma} - \frac{\mu^\gamma x^{2-\gamma}}{(1-\gamma)(2-\gamma)} - \frac{\mu^2}{2-\gamma}\right]\right\}, \qquad x \geq \mu,$$

where the term in braces is a strictly concave function of $x$. This is not necessarily the sharpest upper bound that can be achieved for a particular NEF-PVF distribution. However, it does provide a unified upper bound for general NEF-PVF's.

**7. Unimodality.** The normal, Poisson-type, and gamma distributions are known to be unimodal. [For the unimodality of the gamma and Poisson-type distributions, see Examples 6.2 and 6.9 in Barndorff-Nielsen (1978).] A natural question which arises concerns the unimodality of the remaining NEF-PVF distributions. In this section, we show that NEF-PVF distributions, generated by stable distributions ($0 < \rho < 1$) are self-decomposable, a property which implies unimodality [Lukacs (1983), Theorem 5.4.3]. We also discuss unimodality of NEF-PVF distributions with $\rho < 0$, and present a special case according to which the given distributions are not unimodal.

We first present a lemma which is used to prove unimodality of NEF-PVF distributions with $\rho \in (0,1)$.

LEMMA 7.1. *For any* $\beta > 0$, $\theta < 0$, $0 < \rho < 1$, *and* $0 < c < 1$, *the function*

$$\varphi_c(t) \equiv \exp\{-\beta[(t-\theta)^\rho - (ct-\theta)^\rho]\}, \tag{7.1}$$

*defined on* $[0,\infty)$, *is a LT of a distribution concentrated on* $[0,\infty)$.

PROOF. Let $\beta$, $c$, $\rho$, and $\theta$ be as in the statement of the lemma and denote

$$m_c(t) \equiv \beta[(t-\theta)^\rho - (ct-\theta)^\rho].$$

Then, $\varphi_c(t)$ can be written as $\varphi_c(t) = \exp\{-m_c(t)\}$. For $t > 0$, $t > ct$, and thus $(t-\theta)^\rho > (ct-\theta)^\rho$, which implies that $m_c(t)$ is positive. Since $\varphi_c(0) = 1$, the statement of the lemma is established if one can show that $m_c(t)$ possesses a completely monotone derivative [Feller (1966), Theorem 1 and Criterion 2, pages 415–417]. For this purpose, let

$$g_c(t) \equiv dm_c(t)/dt = \beta\rho\left[(t-\theta)^{\rho-1} - c(ct-\theta)^{\rho-1}\right], \qquad t > 0,$$

and $g_c^{(n)}(t)$ denote the $n$th derivative of $g_c(t)$ wrt $t$. It can be easily shown that

$$(-1)^n g_c^{(n)}(t) = (-1)^n \beta\rho_n\left[(t-\theta)^{\rho-(n+1)} - c^{n+1}(ct-\theta)^{\rho-(n+1)}\right],$$

where $\rho_n \equiv \prod_{j=0}^{n}(\rho - j)$, $n = 0, 1, \dots$ .

For $0 < \rho < 1$, $\rho_n > 0$ if $n$ is even or zero, and $\rho_n < 0$ if $n$ is odd. Hence, $(-1)^n\beta\rho_n > 0$ for all $n$. Thus, for proving complete monotonicity of $g_c(t)$, it remains to show that

$$\left[(t-\theta)^{\rho-(n+1)} - c^{n+1}(ct-\theta)^{\rho-(n+1)}\right] \geq 0, \tag{7.2}$$

for all $t > 0$ and $n \geq 0$. Now, for $\rho$ and $c$ belonging to $(0,1)$, $c > c^{(n+1)/(n+1-\rho)}$ for all $n \geq 0$. Therefore, for any $t > 0$ and $\theta < 0$, we have

$$t\big(c - c^{(n+1)/(n+1-\rho)}\big) + \theta\big(c^{(n+1)/(n+1-\rho)} - 1\big) > 0,$$

from which inequality (7.2) immediately follows. □

THEOREM 7.1. *All NEF-PVF distributions generated by stable distributions are unimodal.*

PROOF. Consider NEF-PVF distributions with fixed $\rho \in (0,1)$. Their LT's are given by $\phi(t:\theta) = \exp\{\Psi(t:\theta)\}$, $\theta \leq 0$, where $\Psi(t:\theta)$ is given by (4.3). For $\theta = 0$, $\phi(t:0)$ is the LT of a stable distribution with characteristic exponent $\rho$, and thus the corresponding distribution is unimodal [Lukacs (1983), corollary to Theorem 4.2.1 and Theorem 5.4.3]. Fix $\theta < 0$ and let $X$ be a r.v. with LT $\phi(t:\theta)$. For every $0 < c < 1$, let $Y_c$ be a r.v., independent of $X$, having a LT of the form (7.1) with $\beta = [(1-\rho)/\alpha]^{\rho}(1/\rho)$. Using Laplace transform techniques, it can be easily shown that $X =_D cX + Y_c$. That is, $X$ has a self-decomposable distribution [for a definition see Lukacs (1983), Section 4.2]. A result due to Yamazato (1975) shows that all self-decomposable distributions are unimodal. This completes the proof. [Yamazato's result is also quoted and proved in Lukacs (1983), Theorem 5.4.3.] □

We now discuss the question of unimodality of NEF-PVF distributions with $\rho < 0$. Let $F$ denote such a distribution. According to (ii) of Theorem 4.1, $F$ coincides with the distribution of the random sum $\sum_{i=1}^{N} Y_i$, where $N$ has a Poisson distribution with mean

$$p = \frac{\rho - 1}{\alpha\rho}\left(\frac{\alpha\theta}{\rho - 1}\right)^{\rho}, \qquad \theta < 0, \quad \alpha > 0, \quad \rho < 0,$$

and the $Y_i$'s, independent of $N$, are i.i.d. having a common gamma distribution with scale and shape parameters $-1/\theta$ and $-\rho$, respectively.

Wolfe (1971), Theorem 3, considered the problem of unimodality for the special case of $F$ where the $Y_i$'s have a common exponential distribution with mean equal to one. This special case of $F$ corresponds to values $(\theta, \rho, p) = (-1, -1, 4/\alpha^2)$. Wolfe showed that $F$ is unimodal with single mode at 0 if $\alpha \geq \sqrt{2}$, and otherwise is not unimodal.

For a more general case, with various values of $\theta < 0$ and $\rho < 0$, $F$ can be given the form [cf. Wolfe (1971), page 917]

$$F(x) = e^{-p}E(x) + e^{-p}\int_0^x \left[e^{\theta t}\sum_{n=1}^{\infty}\frac{b^n t^{-n\rho-1}}{n!\,\Gamma(-n\rho)}\right]dt,$$

where $E(x)$ is a distribution function degenerate at 0, $b = -(1/\rho)\{(1-\rho)/\alpha\}^{1-\rho}$, $\rho < 0$, $\theta < 0$, and $\alpha > 0$. It seems that similar results can be obtained for such $F$ by adopting Wolfe's approach and analyzing the behavior of $d^2F(x)/dx^2$ for $x > 0$. We do not pursue this matter further and leave it as an open question.

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